Quantum gravity corrections to the Schwarzschild mass

Marco Cavaglià*
Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut, Am Mühlenberg 1, D-14476 Golm, Germany
and Departamento de Fisica, Universidade da Beira Interior, R. Marquês d'Ávila e Bolama, 6200 Covilhã, Portugal

Carlo Ungarelli†
Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut, Am Mühlenberg 1, D-14476 Golm, Germany
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Vacuum spherically symmetric Einstein gravity in $N \geq 4$ dimensions can be cast in a two-dimensional conformal nonlinear sigma model form by first integrating on the $(N-2)$-dimensional (hyper)sphere and then performing a canonical transformation. The conformal sigma model is described by two fields which are related to the Arnowitt-Deser-Misner (ADM) mass and to the radius of the $(N-2)$-dimensional (hyper)sphere, respectively. By quantizing perturbatively the theory we estimate the quantum corrections to the ADM mass of a black hole.

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I. INTRODUCTION

Classically, a neutral, nonrotating, spherically symmetric black hole in vacuum is completely identified by the value of its Arnowitt-Deser-Misner (ADM) mass M_{ADM} (see e.g. [1]). Since gravity does not couple to any matter field—and we impose *ab initio* spherical symmetry—M_{ADM} is constant and the geometry possesses one extra Killing vector in addition to the Killing vectors which are associated with the spherical symmetry (Birkhoff theorem). The general solution of Einstein equations is the famous Schwarzschild metric. It describes an *eternal black hole*.

Naively, we would expect both properties—the Birkhoff theorem and $M_{ADM}=\text{const}$—to be broken at the quantum level. The validity of the Birkhoff theorem in the quantum canonical theory of spherically symmetric gravity has been investigated in Refs. [2–4]. It has been shown that the Birkhoff theorem holds at the quantum level; i.e., the quantum theory of spherically symmetric gravity in vacuum is a quantum mechanical system with a finite number of degrees of freedom (*quantum Birkhoff theorem*) [2]. Moreover, the Hilbert space of the quantum theory is completely determined by the eigenstates of the (gauge invariant) mass operator.

The aim of this paper is to explore whether the other classical property ($M_{ADM}=\text{const}$) holds in the quantum gravity regime as well. The result of our investigation is that quantum gravity corrections to the Schwarzschild mass appear at the second order in the curvature perturbative expansion. For instance, quantum fluctuations of the mass of a four-dimensional black hole are, for distances much greater than the horizon radius,

$$\Delta M_{ADM} \lesssim m_{pl}\left(\frac{l_{pl}}{R}\right)^2, \tag{1}$$

where l_{pl} and m_{pl} are the Planck length and the Planck mass, respectively. (Notations: Here and throughout the paper we use natural units.)

A number of approximations are needed to obtain Eq. (1). We will discuss them in detail in the following sections. Here let us just emphasize two important points concerning Eq. (1). Firstly, the quantum theory breaks down on the horizon(s) where the coupling constants of the perturbative expansion diverge. Therefore, Eq. (1) is strictly valid for distances much greater than the Schwarzschild radius of the black hole. Secondly, quantum fluctuations vanish for large radii, i.e., at large distances from the black hole. In the asymptotic regime the black hole behaves classically and the mass is constant. Quantum fluctuations of the Schwarzschild mass due to pure quantum gravity effects become manifest when the black hole horizon is approached.

Equation (1) is obtained in the context of the nonlinear sigma model approach to spherically symmetric gravity whose basic ingredients are described in Ref. [2]. First, N-dimensional spherically symmetric gravity is cast in a dilaton gravity form by integrating over the $(N-2)$ spherical coordinates. Then, by a canonical field redefinition the action is transformed in a two-dimensional conformal nonlinear sigma model with a fixed target metric. The new fields are the dilaton and a gauge invariant field M which is constant on the classical solutions of the field equations and can be identified with the ADM mass of the black hole.

The new action can be quantized perturbatively by expanding the metric of the target space in normal Riemann coordinates [5]. Since the expansion parameter is proportional to the curvature of the manifold, the theory is a free field theory far away from the black hole horizon and for large ADM mass in Planck units. The perturbative theory turns out to be infrared and ultraviolet divergent. Infrared divergences are eliminated by the introduction of an infrared regulator m. The theory is regularized by usual dimensional regularization techniques. The consistency of the procedure

*Email address: cavaglia@mercury.ubi.pt
†Present address: School of Computer Science and Mathematics, University of Portsmouth, Mercantile House, Hampshire Terrace, Portsmouth P01 2EG, United Kingdom; email address: ungarell@aei-potsdam.mpg.de

is verified *a posteriori* by calculating the one-loop β function. The theory becomes asymptotically free at large energy scales, where the perturbative regime is valid and the infrared regulator can be neglected.

The amplitude of the quantum fluctuations of the ADM mass at a given order in the perturbative expansion can be read straightforwardly from the two-point Green functions of the theory. This is possible because the nonlinear sigma model fields are the ADM mass and the dilaton. Since the fields have a direct geometrical meaning any problem related to their interpretation in terms of physical quantities disappears.

The outline of the paper is as follows. In the next section we illustrate the classical theory of N-dimensional ($N\geqslant 4$) spherically symmetric gravity. We start with the dilaton gravity description and then introduce the nonlinear sigma model picture. Although a part of this section reviews previous work (see [2] and references therein), its content is useful to make the paper self-contained. Sections III and IV are devoted to the classical expansion in normal Riemann coordinates and to the perturbative quantization of the theory, respectively. (The evaluation of the relevant Feynman diagrams is briefly outlined in the Appendix.) Finally, in Sec. V we state our conclusions.

II. CLASSICAL THEORY

It is well known [6–9] that for spherically symmetric metrics the N-dimensional Einstein-Hilbert action (the Ricci tensor is defined as in [10])

$$S^{(N)}=\frac{1}{16\pi l_{pl}^{N-2}}\int d^N y\sqrt{-G}R^{(N)}(G) \tag{2}$$

can be cast, upon integration on the $N-2$ spherical coordinates, in the dilaton gravity form

$$S_{DG}=\int d^2x\sqrt{-g}\left[\phi R^{(2)}(g)-\frac{d}{d\phi}\ln[W(\phi)](\nabla\phi)^2+V(\phi)\right], \tag{3}$$

where the dilaton field is related to the radius of the $(N-2)$-dimensional sphere and $W(\phi)$ and $V(\phi)$ are given functions whose form depends on the N-dimensional metric ansatz. (We neglect surface terms as they are irrelevant for the following discussion. For a detailed discussion about the role of boundary terms see e.g. [2,11,12] and references therein.)

Theories of the form (3) admit the existence of the (gauge invariant) quantity [2,13]

$$M=N(\phi)-W(\phi)(\nabla\phi)^2,\quad N(\phi)=\int^{\phi}d\phi'[W(\phi')V(\phi')], \tag{4}$$

which is locally conserved, i.e.,

$$\nabla_\mu M=0. \tag{5}$$

Equation (5) can be easily proved by differentiating Eq. (4) and using the field equations

$$\nabla_{(\mu}\nabla_{\nu)}\phi-g_{\mu\nu}\nabla^2\phi+\frac{1}{2}g_{\mu\nu}V(\phi)+\frac{d}{d\phi}\ln[W(\phi)]\left[\nabla_{(\mu}\nabla_{\nu)}\phi-\frac{1}{2}g_{\mu\nu}(\nabla\phi)^2\right]=0, \tag{6}$$

$$R^{(2)}(g)+2\nabla^2\ln[W(\phi)]+\frac{dV(\phi)}{d\phi}=0. \tag{7}$$

A further property of M is conformal (Weyl) invariance [14]. Indeed, by rescaling the two-dimensional metric [15]

$$g_{\mu\nu}(x)\to g_{\mu\nu}(x)A(\phi), \tag{8}$$

$V(\phi)$ and $W(\phi)$ transform as

$$V(\phi)\to V(\phi)/A(\phi),$$
$$W(\phi)\to W(\phi)A(\phi). \tag{9}$$

Equation (4) is clearly invariant under Eqs. (8),(9). Using Eqs. (8),(9), the action (3) can be cast in a simpler form by a suitable choice of $A(\phi)$. Here and throughout the paper we will set $A(\phi)=1/W(\phi)$ which corresponds to choosing the spherically symmetric ansatz ($\alpha,\beta=0,\ldots,N-1,\ \mu,\nu=0,1$) [9]

$$\begin{aligned}ds_N^2&=G_{\alpha\beta}dy^\alpha dy^\beta\\&=[\phi(x)]^{-(N-3)/(N-2)}g_{\mu\nu}(x)dx^\mu dx^\nu\\&\quad+[\gamma\phi(x)]^{2/(N-2)}d\Omega_{N-2}^2,\quad \phi>0.\end{aligned} \tag{10}$$

With this choice $W(\phi)\to 1$ and the dilaton gravity action (3) becomes

$$S_{DG}=\int d^2x\sqrt{-g}[\phi R^{(2)}(g)+V(\phi)], \tag{11}$$

where

$$V(\phi)=(N-2)(N-3)(\gamma^2\phi)^{-1/(N-2)}. \tag{12}$$

Here $\gamma=16\pi l_{pl}^{N-2}/V_{N-2}$ and $V_{N-2}=2\pi^{(N-1)/2}/\Gamma[(N-1)/2]$ is the volume of the $(N-2)$-dimensional unit sphere $d\Omega_{N-2}^2$. On the gauge shell the quantity M coincides, apart from some numerical factors, with the ADM [1] mass

$$M_{\rm ADM}=\frac{\gamma^{1/(N-2)}}{N-2}M. \tag{13}$$

This property will be essential in the following.

The dilaton gravity action (3) can be cast in a nonlinear conformal sigma model form. Here our treatment follows closely [2]. In two-dimensions the Ricci scalar $R^{(2)}(g)$ can be locally written as

$$R^{(2)}(g)=2\nabla_\mu A^\mu, \quad A^\mu=\frac{\nabla^\mu\nabla^\nu\chi\nabla_\nu\chi-\nabla_\nu\nabla^\nu\chi\nabla^\mu\chi}{\nabla_\rho\chi\nabla^\rho\chi}, \tag{14}$$

where χ is an arbitrary, nonconstant, function of the coordinates. Equation (14) can be easily checked using conformal coordinates. Since Eq. (14) is a generally covariant expression, and any two-dimensional metric can be locally cast in the conformal form by a coordinate transformation [16], Eq. (14) is valid in any system of coordinates.

Differentiating Eq. (4), and using Eq. (14) with $\chi=\phi$, the action (3) can be written as a functional of M and ϕ. The result is

$$S=\int_\Sigma d^2x\sqrt{-g}\,\frac{\nabla_\mu\phi\nabla^\mu M}{N(\phi)-M}+S_\partial, \tag{15}$$

where S_∂ is the surface term

$$S_\partial=2\int_\Sigma d^2x\sqrt{-g}\nabla_\mu[\nabla^\mu\phi+\phi A^\mu]. \tag{16}$$

Let us investigate the classical solutions of Eq. (15). Varying Eq. (15) with respect to M and ϕ we find

$$\nabla_\mu\nabla^\mu\phi-V(\phi)=0, \tag{17}$$

$$\nabla_\mu M\nabla^\mu M+\nabla_\nu\phi\nabla^\nu\phi\nabla_\mu\nabla^\mu M=0. \tag{18}$$

Equations (17) and (18) must be complemented by the constraints

$$\nabla_{(\mu}\phi\nabla_{\nu)}M-\frac{1}{2}g_{\mu\nu}\nabla_\sigma\phi\nabla^\sigma M=0, \tag{19}$$

which are obtained by varying Eq. (15) with respect to the metric $g_{\mu\nu}$. The general solution of Eqs. (17)–(19) can be easily obtained using conformal coordinates. Setting

$$g_{\mu\nu}=\rho\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\rightarrow ds^2=2\rho(u,v)du\,dv, \tag{20}$$

Eqs. (17),(18) and the constraints (19) read

$$\partial_u\partial_v\phi-\frac{\rho}{2}V(\phi)=0,$$

$$\partial_u\partial_v M+\frac{\partial_u M\partial_v M}{N-M}=0, \tag{21}$$

$$\partial_u\phi\partial_u M=0, \quad \partial_v\phi\partial_v M=0.$$

From Eqs. (21) and (12) it follows that M is constant, $M=M_0$. Using Eqs. (21) and (4) the general solution can be written

$$M=M_0, \quad \phi=\phi(\Psi), \quad \frac{d\phi}{d\Psi}=N[\phi(\Psi)]-M_0, \tag{22}$$

where $\Psi=U(u)+V(v)$, U and V being arbitrary functions. (The arbitrariness in the choice of Ψ reflects the residual coordinate reparametrization invariance in the conformal gauge. Given U and V correspond to a particular choice of conformal coordinates.) The two-dimensional metric is

$$\begin{aligned}ds^2&=4[N(\phi)-M_0]\partial_u\Psi\partial_v\Psi du\,dv\\&=4[N(\phi)-M_0]dU\,dV,\end{aligned} \tag{23}$$

or, using the coordinates $[\phi\equiv\phi(U+V), T\equiv U-V]$,

$$ds^2=-[N(\phi)-M_0]dT^2+[N(\phi)-M_0]^{-1}d\phi^2. \tag{24}$$

The general solution depends on the single variable ϕ. This result is usually known as the *generalized Birkhoff theorem* (see e.g. [13,17–19]). Finally, substituting Eq. (12) in Eq. (24) and using Eq. (10) we have

$$\begin{aligned}ds_N^2=&-[1-J/R^{N-3}]d\tau^2+[1-J/R^{N-3}]^{-1}dR^2\\&+R^2d\Omega_{N-2},\end{aligned} \tag{25}$$

where

$$\tau=(N-2)\gamma^{-1/(N-2)}t,$$

$$R=(\gamma\phi)^{1/(N-2)}, \tag{26}$$

$$J=\frac{\gamma^{(N-1)/(N-2)}}{(N-2)^2}M_0=\frac{\gamma}{N-2}M_{ADM}.$$

Let us conclude this section with a couple of remarks. We have seen that two-dimensional dilaton gravity can be described by a two-dimensional nonlinear sigma model with a given target space metric. In particular, for N-dimensional spherically symmetric gravity the fields appearing in the conformal sigma model are the dilaton and the ADM mass, i.e. quantities which have a direct physical interpretation. The description of spherically symmetric gravity in terms of geometrical variables is essential for the quantization of the model since the quantum fields can be directly related to the original spacetime geometry and problems related to their interpretation do not show up. The equivalence between the nonlinear sigma model action (15) and the dilaton gravity action (11) can be proved at the canonical level as well. This has been done in Ref. [2]. The general canonical transformation includes, as particular cases, the canonical transformations discussed in Ref. [20] for the Callan-Giddings-Harvey-Strominger (CGHS) [21] model and Ref. [12] for the four-dimensional black hole.

III. SIGMA MODEL CURVATURE EXPANSION

The nonlinear sigma model (15) can be quantized perturbatively by expanding the target space metric in Riemann normal coordinates [5]. Let us define the adimensional mass $\mathcal{M}=\gamma^{2/(N-2)}M$. The bulk term of the action (15) in the conformal gauge can be cast in the form $[\sigma\equiv(u,v)]$

$$S=\int d^2\sigma\, G_{ij}(X)\partial_\mu X^i\partial^\mu X^j, \quad (27)$$

where $X^0\equiv\mathcal{M}$, $X^1\equiv\phi$ and the metric of the target space is

$$G_{ij}(X)=\frac{1}{\mathcal{N}(X^1)-X^0}\begin{pmatrix}0 & 1/2\\ 1/2 & 0\end{pmatrix},\quad \mathcal{N}(X^1)=\gamma^{2/(N-2)}N(\phi). \quad (28)$$

Now we expand the target metric (28) in Riemann normal coordinates around a point $X(0)$ (vacuum expectation value). At the second order in the Riemann expansion the metric is [22]

$$G_{ij}(X)=G_{ij}[X(0)]-\frac{1}{3}R_{ijkl}[X(0)]x^kx^l -\frac{1}{3!}R_{ijkl;m}[X(0)]x^kx^lx^m+O(x^4), \quad (29)$$

where $X^i=X^i(0)+x^i$, $x^i\equiv\delta X^i$. Using Eq. (28) and substituting Eq. (29) in Eq. (27) the action at the second order in the Riemann expansion is

$$S=\int d^2\sigma\,\mathcal{L},$$

$$\mathcal{L}=\frac{1}{2}[\partial_\mu y^i\partial^\mu y_i +g\varepsilon_{ij}\varepsilon_{kl}\partial_\mu y^i\partial^\mu y^k y^j y^l(1+\overline{g}A_q y^q+O(y^2))]. \quad (30)$$

Here ε_{ij} is the two-dimensional completely antisymmetric Levi Civita tensor, $y^i=(x^0\pm x^1)/\sqrt{2}$, $A_q=(1+\tilde{g},1-\tilde{g})$, and g, $\overline{g}$, $\tilde{g}$ are the adimensional coupling constants

$$g=-\frac{1}{3}\frac{\mathcal{V}[X^1(0)]}{\mathcal{N}[X^1(0)]-X^0(0)}=-\frac{1}{3}\frac{V(\phi_0)}{N(\phi_0)-M_0}, \quad (31)$$

$$\overline{g}=\frac{1}{2\sqrt{2}}\frac{1}{\mathcal{N}[X^1(0)]-X^1(0)}=\frac{1}{2\sqrt{2}}\frac{\gamma^{-2/(N-2)}}{N(\phi_0)-M_0}, \quad (32)$$

$$\tilde{g}=\left[\frac{\mathcal{V}'[X^1(0)]}{\mathcal{V}[X^1(0)]}[\mathcal{N}[X^1(0)]-X^0(0)] -\mathcal{V}[X^1(0)]\right][\mathcal{N}[X^1(0)]-X^0(0)] =\left[\frac{V'(\phi_0)}{V(\phi_0)}[N(\phi_0)-M_0]-V(\phi_0)\right] \times[N(\phi_0)-M_0]\gamma^{4/(N-2)}, \quad (33)$$

where $\mathcal{V}=d\mathcal{N}/dX^1$. Using Eqs. (12), (13), and (26) g, $\overline{g}$, and $\tilde{g}$ read

$$g=-\frac{\kappa}{3}\frac{N-3}{N-2}\left(\frac{l_{pl}}{R}\right)^{N-2}\frac{1}{1-\frac{\kappa}{N-2}\frac{M_{ADM}}{m_{pl}}\left(\frac{l_{pl}}{R}\right)^{N-3}}, \quad (34)$$

$$\overline{g}=\frac{1}{2\sqrt{2}}\frac{\kappa^{(N-3)/(N-2)}}{(N-2)^2}\left(\frac{l_{pl}}{R}\right)^{N-3}\frac{1}{1-\frac{\kappa}{N-2}\frac{M_{ADM}}{m_{pl}}\left(\frac{l_{pl}}{R}\right)^{N-3}}, \quad (35)$$

$$\tilde{g}=-(N-2)^4\kappa^{-(N-4)/(N-2)}\left(\frac{l_{pl}}{R}\right)^{4-N}\left[1-\frac{\kappa}{(N-2)^2} \times\frac{M_{ADM}}{m_{pl}}\left(\frac{l_{pl}}{R}\right)^{N-3}\right]\left[1-\frac{\kappa}{N-2}\frac{M_{ADM}}{m_{pl}}\left(\frac{l_{pl}}{R}\right)^{N-3}\right], \quad (36)$$

where $\kappa=16\pi/V_{N-2}$. Let us investigate the behavior of the coupling constants. For $R\to\infty$ and fixed M_{ADM}, i.e., in the asymptotically flat region far away from a black hole of given (classical) mass M_{ADM}, Eqs. (34)–(36) read

$$g_\infty\sim-\frac{\kappa}{3}\frac{N-3}{N-2}\left(\frac{l_{pl}}{R}\right)^{N-2}\left[1+O\left(\frac{l_{pl}}{R}\right)^{N-3}\right],$$

$$\overline{g}_\infty\sim\frac{1}{2\sqrt{2}}\frac{\kappa^{(N-3)/(N-2)}}{(N-2)^2}\left(\frac{l_{pl}}{R}\right)^{N-3}\left[1+O\left(\frac{l_{pl}}{R}\right)^{N-3}\right], \quad (37)$$

$$\tilde{g}_\infty\sim-(N-2)^4\kappa^{-(N-4)/(N-2)}\left(\frac{l_{pl}}{R}\right)^{4-N} \times\left[1+O\left(\frac{l_{pl}}{R}\right)^{N-3}\right].$$

As expected, the Riemann expansion is an expansion in powers of the curvature, i.e., in powers of l_{pl}/R. The theory becomes free in the asymptotically flat region where the first order correction to the free theory is of order $O((l_{pl}/R)^{N-2})$. The perturbative expansion fails on the black hole horizon, the coupling constants g and $\overline{g}$ blowing up when $R^{N-3}\to J$. The perturbative Riemann expansion is also valid for large values of M_{ADM}/m_{pl} at distances

$$R-J^{1/(N-3)}\sim l_{pl}\left(\frac{M_{ADM}}{m_{pl}}\right)^{1/(N-3)}. \quad (38)$$

In this regime the dimensional coupling constants (34)–(36) read

$$g\sim\left(\frac{m_{pl}}{M_{ADM}}\right)^{(N-2)/(N-3)},$$

$$\overline{g}\sim\left(\frac{m_{pl}}{M_{ADM}}\right), \quad (39)$$

$$\tilde{g} \sim \left(\frac{m_{pl}}{M_{ADM}} \right)^{(4-N)/(N-3)}$$

and the Riemann expansion is an expansion in powers of m_{pl}/M_{ADM}. The theory becomes a free field theory when $m_{pl}/M_{ADM} \ll 1$.

IV. PERTURBATIVE QUANTIZATION

In this section we quantize the theory at one loop and at the first order in the curvature expansion ($\bar{g} = \tilde{g} = 0$). Since the target space is not Ricci flat, the conformal symmetry is not preserved at the quantum level. Conformal symmetry breaking implies running coupling constants and effective terms in the action that depend on the conformal factor. Since quantum corrections to the ADM mass due to these terms are subdominant we will postpone their discussion at the end of the section and work in the unit gauge.

The vacuum-to-vacuum amplitude is (for notations see Ref. [23])

$$W[J] = N \int \mathcal{D}[y^i] e^{iS[y,J]}, \quad S[y,J] = \int d^2\sigma [\mathcal{L} + J_i y^i]. \tag{40}$$

The free two-points Green function (propagator) is

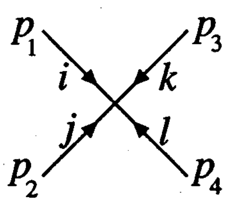
$$\langle y_i(\sigma_1) y_j(\sigma_2) \rangle = i\,\eta_{ij} \int \frac{d^2p}{(2\pi)^2} \frac{1}{p^2 + i\epsilon} e^{-ip(\sigma_1 - \sigma_2)},$$

$$= -\eta_{ij} \frac{1}{4\pi} \ln[(\sigma_1 - \sigma_2)^2]. \tag{41}$$

At the first order in the Riemann expansion the perturbative potential is

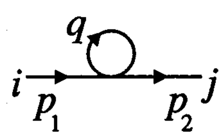
$$V_1(y) = \frac{g}{2} \varepsilon_{ij} \varepsilon_{kl} \partial_\mu y^i \partial^\mu y^k y^j y^l. \tag{42}$$

The corresponding Feynman rule for the interaction vertex is



$$= -i \frac{g}{4!} (2\pi)^2 \delta \left(\sum p_i \right) [\varepsilon_{ij} \varepsilon_{kl} (p_1 - p_2)(p_3 - p_4) + \varepsilon_{ik} \varepsilon_{jl} (p_1 - p_3)(p_2 - p_4) + \varepsilon_{il} \varepsilon_{kj} (p_1 - p_4)(p_3 - p_2)]. \tag{43}$$

The one-loop correction of the two-point Green function is

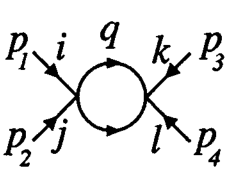


$$= (2\pi)^2 \delta \left(\sum p_i \right) \prod \frac{1}{p_i^2 + i\epsilon} \Gamma^{(2)}_{ij}, \tag{44}$$

where

$$\Gamma^{(2)}_{ij} = -g\,\eta_{ij} \int \frac{d^2q}{(2\pi)^2} \frac{q^2 + p_1^2}{q^2 + i\epsilon}. \tag{45}$$

The (on-shell) one-loop correction of the four-point Green function (s channel) is



$$= (2\pi)^2 \delta \left(\sum p_i \right) \prod \frac{1}{p_i^2 + i\epsilon} \Gamma^{(4)}_{ijkl}. \tag{46}$$

where

$$\Gamma^{(4)}_{ijkl} = g^2 \int \frac{d^2q}{(2\pi)^2} \frac{1}{(q^2 + i\epsilon)[(p_1 + p_2 - q)^2 + i\epsilon]} \times (\eta_{ik}\eta_{jl} A_1 + \eta_{il}\eta_{jk} A_2 + \eta_{ij}\eta_{kl} A_3), \tag{47}$$

$$A_1 = 4(p_1 p_2)[(p_1 p_2) - q^2 + q(p_1 + p_2)] + q^2[q^2 - 2q(p_1 + p_2)] - 2[(qp_2)(qp_3) + (qp_1)(qp_4)],$$

$$A_2 = 4(p_1 p_2)[(p_1 p_2) - q^2 + q(p_1 + p_2)] + q^2[q^2 - 2q(p_1 + p_2)]$$

$$-2[(qp_1)(qp_3)+(qp_2)(qp_4)],$$

$$A_3=8[q(p_1-p_2)][q(p_3-p_4)].$$

The two- and four-point Green functions (44) and (46) are infrared and ultraviolet divergent. The infrared divergence can be eliminated by inserting an infrared regulator. We will check *a posteriori* the consistency of this procedure by proving that the theory is asymptotically free in the ultraviolet region, i.e., that the theory is perturbative for large values of the energy. In order to regularize the theory we have to compute the ultraviolet divergences. Using dimensional regularization the divergence of the two-point Green function (44) is (details of the calculation are given in the Appendix)

$$[\text{divergence } \Gamma^{(2)}_{ij}]=i\frac{g}{2\pi\epsilon}\eta_{ij}p^2, \tag{48}$$

where $\epsilon=2-d$. The divergence above is eliminated by inserting in the Lagrangian density the counterterm (minimal subtraction)

$$\mathcal{L}^{(2)}=\frac{1}{2}\left(-\frac{g}{2\pi\epsilon}\right)\partial_\mu y^i\partial^\mu y_i. \tag{49}$$

The divergence of the four-point Green function (46) is ($s+t+u$ channels)

$$[\text{divergence } \Gamma^{(4)}_{ijkl}]=i\frac{11}{2\pi\epsilon}\mu^\epsilon g^2[\eta_{ij}\eta_{kl}s+\eta_{ik}\eta_{jl}t+\eta_{il}\eta_{jk}u], \tag{50}$$

where $s=(p_1+p_2)^2$, $t=(p_1+p_3)^2$ and $u=(p_1+p_4)^2$ are the Mandelstam variables. The divergence (50) is eliminated by inserting in the Lagrangian density the counterterm

$$\mathcal{L}^{(4)}=\frac{1}{2}g\mu^\epsilon\left(-\frac{11}{6\pi}\frac{g}{\epsilon}\right)\varepsilon_{ij}\varepsilon_{kl}\partial_\mu y^i\partial^\mu y^k y^j y^l. \tag{51}$$

Finally, the one-loop renormalized Lagrangian density is

$$\mathcal{L}_{ren}=\frac{1}{2}Z_1\partial_\mu y^i\partial^\mu y_i+\frac{1}{2}g\mu^\epsilon Z_2\varepsilon_{ij}\varepsilon_{kl}\partial_\mu y^i\partial^\mu y^k y^j y^l, \tag{52}$$

where

$$Z_1=1-\frac{g}{2\pi\epsilon},\quad Z_2=1-\frac{11g}{6\pi\epsilon}. \tag{53}$$

Now we can calculate the β function and the anomalous dimension $\gamma(g)$ of the y fields at one loop. The result is

$$\beta(g)=-\frac{5}{6\pi}g^2+O(g^3), \tag{54}$$

$$\gamma(g)=\frac{g}{2\pi}+O(g^2). \tag{55}$$

Integrating Eq. (54) we obtain

$$g=g_s\frac{1}{1+\dfrac{5}{6\pi}g_s\ln\dfrac{\mu}{\mu_s}}, \tag{56}$$

where g_s is the value of the coupling constant g at the renormalization scale μ_s. From Eq. (56) we see that $g\to 0$ for $\mu\to\infty$, i.e., the theory becomes free at high energy scales (asymptotic freedom). The perturbative regime of the theory is realized at short distances, where the theory itself exhibits an ultraviolet stable fixed point. Since the model is asymptotically free in the ultraviolet region, it is possible to neglect the dependence of the Green functions on the infrared regulator. Solving the renormalization group equation at one loop for the N-point Green function, we obtain

$$\langle y_1y_2\dots y_N;g,\mu\rangle=\left(\frac{g}{g_s}\right)^{(3/10)N}\langle y_1y_2\dots y_N;g_s,\mu_s\rangle. \tag{57}$$

Now let us evaluate the two-point Green function at one loop. We have

$$\langle y_i(\sigma_1)y_j(\sigma_2)\rangle=-\eta_{ij}\frac{1}{4\pi}\left[1-\frac{g}{4\pi}\ln\left(\frac{\mu}{m}\right)^2+O(g^2)\right]\times\ln[(\sigma_1-\sigma_2)^2]. \tag{58}$$

In term of the fields x^i the only Green function different from zero is

$$\langle x^0(\sigma_1)x^1(\sigma_2)\rangle=-\frac{1}{4\pi}\left[1-\frac{g}{4\pi}\ln\left(\frac{\mu}{m}\right)^2+O(g^2)\right]\times\ln[(\sigma_1-\sigma_2)^2]. \tag{59}$$

As a result, at the first order in the curvature expansion, the one-loop quantum correction to the Schwarzschild mass is identically zero

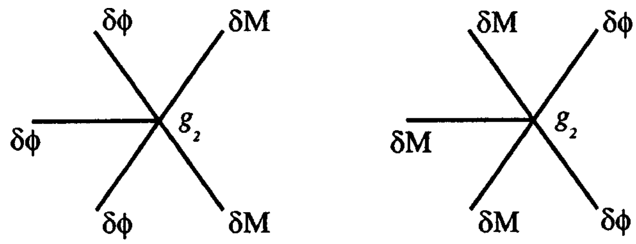
$$\langle\delta\mathcal{M}(\sigma_1)\delta\mathcal{M}(\sigma_2)\rangle=O(g^2). \tag{60}$$

Actually, a simple observation shows that the two-point Green function at first order in the curvature expansion is zero at any loop. This result follows from the invariance of the interaction Lagrangian density, Eq. (42), under Poincaré group transformations in the y field space. Since the interaction vertex (43) has two $\delta\mathcal{M}$ and two $\delta\phi$ legs, and the propagator is antidiagonal in the fields ($\delta\mathcal{M},\delta\phi$), any two-point Green function is necessarily diagonal [antidiagonal] in the y^i [$\delta\mathcal{M},\delta\phi$] fields, respectively.

Let us now consider the perturbative potential at second order in the curvature expansion

$$V_2(y)=\frac{1}{2}g\bar{g}\varepsilon_{ij}\varepsilon_{kl}A_q\partial_\mu y^i\partial^\mu y^k y^j y^l y^q. \tag{61}$$

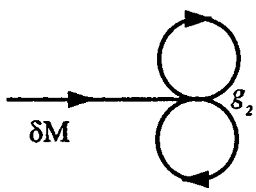
This interaction breaks the Poincaré invariance in the y field space. We have two different vertices:



(62)

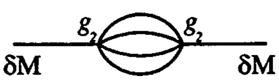
where $g_2 \sim g\overline{g}\tilde{g}$.

Since $V_2(y)$ is an odd functional of the y fields we expect the interaction Lagrangian density (61) to give a nonvanishing two-loop contribution to the one-point Green function $\langle \delta\mathcal{M} \rangle$



(63)

However, a straightforward computation of this diagram shows that $\langle \delta\mathcal{M} \rangle$ is identically zero. The first nonvanishing two-point Green function for the $\delta\mathcal{M}$ field at the second order in the curvature expansion is given by the Feynman diagram



(64)

The above diagram gives a nondiagonal logarithmic divergence to the propagator of the y fields. By adding the appropriate counterterm in the Lagrangian, the first finite, nonzero, two point Green function for the $\delta\mathcal{M}$ field is

$$\langle \delta\mathcal{M}(\sigma_1)\delta\mathcal{M}(\sigma_2) \rangle = -\frac{1}{4\pi} A \ln\left(\frac{\mu}{m}\right)(1+B\tilde{g}^2)(g\overline{g})^2 \times \ln[(\sigma_1-\sigma_2)^2], \tag{65}$$

where A,B are constant adimensional factors.

Using Eq. (65) we can estimate the upper limit of the quantum corrections of M_{ADM}. Since the perturbative expansion fails when $g\ln(\mu) \sim O(1)$ from Eq. (65) we have

$$(\Delta M_{ADM})^2 \lesssim m_{pl}^2 |(1+B\tilde{g}^2) g\overline{g}^2|. \tag{66}$$

Recalling Eqs. (37)and (39) we obtain

$$\Delta M_{ADM} \lesssim m_{pl}\left(\frac{l_{pl}}{R}\right)^{N/2}, \tag{67}$$

and

$$\Delta M_{ADM} \lesssim m_{pl}\left(\frac{m_{pl}}{M_{ADM}}\right)^{N/2(N-3)} \tag{68}$$

for $R/l_{pl} \gg 1$ and $m_{pl}/M_{ADM} \ll 1$, respectively.

Up to now we have neglected effective terms due to the dependence of the action on the conformal factor in $d=2-\epsilon$ dimensions. Equations (67) and (68) make sense only if the contributions due to these terms are subleading. This is indeed the case, as we have anticipated at the beginning of this section.

In the conformal gauge the one-loop renormalized sigma-model action at the second order in the curvature expansion is

$$S = \int d^d\sigma\, e^{-\epsilon\Psi/2}\mathcal{L}_{ren} + S_\Psi + S_{ghost}. \tag{69}$$

Here S_Ψ is the effective action due to the Weyl anomaly and $\mathcal{L}_{ren}$ is renormalized Lagrangian

$$\mathcal{L}_{ren} = \frac{1}{2}Z_1 \partial_\mu y^i \partial^\mu y_i + \frac{1}{2} g\mu^\epsilon Z_2 \varepsilon_{ij}\varepsilon_{kl}\partial_\mu y^i \partial^\mu y^k y^j y^l + \frac{1}{2} g\overline{g}\mu^\epsilon Z_3 \varepsilon_{ij}\varepsilon_{kl}\partial_\mu y^i \partial^\mu y^k y^j y^l A_q y^q, \tag{70}$$

where the divergent part of the renormalization constant $Z_3 = 1 + Cg/\epsilon$ comes from the one loop Feynman diagram with one four and one five point vertices, respectively. Note that the conformal factor Ψ is a propagating field because the Weyl anomaly action contains a kinetic term for Ψ.

Expanding around $d=2$ the leading order Ψ-dependent term in the effective Lagrangian that violates Poincaré invariance in the y field space is (recall that quantum corrections to the ADM mass are only generated by these terms)

$$\mathcal{L}_1^{CF}(\Psi) \sim g^2 \overline{g} \varepsilon_{ij}\varepsilon_{kl} A_q \partial_\mu y^i \partial^\mu y^k y^j y^l y^q \Psi. \tag{71}$$

Equation (71) gives a contribution to $\langle \delta\mathcal{M}(\sigma_1)\delta\mathcal{M}(\sigma_2) \rangle$ of order $O(g^4\overline{g}^2\tilde{g}^2)$ which is subleading to Eq. (65). Finally, at three loops Eq. (61) originates the (Poincaré breaking) term

$$\mathcal{L}_2^{CF}(\Psi) \sim (g\overline{g}\tilde{g})^2 \partial_\mu y^i \partial^\mu y^j \Sigma_{ij}\Psi, \tag{72}$$

where Σ_{ij} is a symmetric 2×2 matrix. Equation (72) gives a one-loop contribution to $\langle \delta\mathcal{M}(\sigma_1)\delta\mathcal{M}(\sigma_2) \rangle$ of order $O((g\overline{g}\tilde{g})^4)$ whose counterterm is again subleading to Eq. (65).

V. CONCLUSION

Let us summarize the main results of the paper. N-dimensional spherically symmetric gravity *in vacuo* can be reduced to a two-dimensional conformal nonlinear sigma model form, Eq. (27). The field content of the latter is given by two fields, $M(\sigma)$ and $\phi(\sigma)$. The M field is constant on the classical solutions of the field equations. It coincides, apart from a constant factor, to the ADM mass of the system. The second field, ϕ, is related to the radius of the $(N-2)$-dimensional (hyper)sphere [see Eqs. (25) and (26)]. So both fields have a direct physical meaning. This property makes the conformal nonlinear sigma model formulation very attactive. Quantization of the theory in the (M,ϕ) representation gives quantum corrections to the mass and to the radius in a direct and straightforward way. This result cannot be achieved in the usual Einstein [Eq. (2)] or dilaton gravity [Eq. (3)] approaches.

The perturbative quantization of the theory is straightforward. Firstly, the nonlinear sigma model target space is expanded in Riemann normal coordinates, i.e., in powers of the target space curvature. Then the theory is quantized—at any order—by usual quantum field theory techniques. The perturbative expansion fails on the black hole horizon(s) where the target space metric exhibits a singularity. This is not surprising: On the horizon(s) strong quantum gravity effects manifest themselves and a perturbative quantization must necessarily fail. Conversely, far away from the horizon(s) quantum gravity effects are weak, the sigma model target space is asymptotically flat and a perturbative treatment is possible. [The perturbative results hold also at distances of the order of the horizon(s) for black holes with large mass in Planck units—see Eq. (38).]

In this paper we have discussed first and second order corrections in the curvature expansion. Surprisingly, first order corrections to the ADM mass are identically zero *at any loop*. This follows from the invariance of the first order interaction under Poincaré transformations in the $(\delta M,\delta\phi)$ field space and from the antidiagonal form of the field propagator. Therefore, quantum corrections to the ADM mass of a four-dimensional black hole are not of order $\Delta M_{ADM}\sim R^{-1}$, as one would naively expect. The first nonzero quantum corrections to M_{ADM} arise (at least) at second order in the curvature expansion [see Eqs. (67) and (68)].

Equations (67) and (68) are the main contribution of the paper. Pure quantum gravity effects make the classical ADM mass of a spherically symmetric black hole fluctuate according to Eqs. (67) and (68). Hopefully, this result may help to shed light on open issues in quantum gravity and black hole physics, such as information loss, black hole thermodynamics and black hole evaporation.

Note added in proof. After this paper was accepted for publication, we became aware of an article by Kazakov and Solodukhin [25] in which quantum corrections to the Schwarzschild metric are calculated. Kazakov and Solodukhin find that for large distances the first quantum correction to the Schwarzschild metric is of order $(l_{pl}/R)^{-2}$ and does not affect the mass term. Their result is in complete agreement with the results of this paper.

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APPENDIX

In this appendix we sketch the evaluation of the Feynman diagrams (44) and (46). The integrals are evaluated by dimensional regularization.

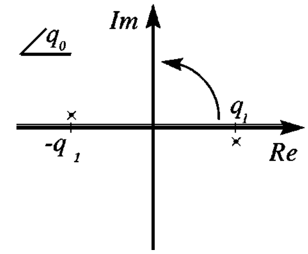

FIG. 1. Wick rotation in the Q_0 plane. The two crosses represent the poles of the integral in the Minkowski space, I_2. The latter is equal to $-iI_2^{(E)}$.

Two-point Green function

By dimensional regularization the calculation of the one-loop two-point Green function (44) is reduced to the evaluation of the (Euclidean) integral in $d=2-\epsilon$ dimensions

$$I_2^{(E)}=\mu^\epsilon\int\frac{d^{2-\epsilon}q}{(2\pi)^2}\frac{1}{q^2+m^2},\tag{A1}$$

where we have added a regulator m^2 to avoid the infrared divergence at $q=0$. Indeed, discarding the ultraviolet quadratically divergent integral (see e.g. [24]) and Wick rotating to the Euclidean space [see Fig. 1, Eq. (45)] can be written

$$\Gamma_{ij}^{(2)}=ig\,\eta_{ij}p^2I_2^{(E)}.\tag{A2}$$

The integral (A1) can be immediately evaluated (see [23], Eq. (B.16), p. 317). The result is

$$I_2^{(E)}=\frac{1}{4}\pi^{d/2-2}\Gamma(1-d/2)\left(\frac{\mu}{m}\right)^{2-d}.\tag{A3}$$

Expanding around $d=2$ we obtain

$$I_2^{(E)}=\frac{1}{4\pi}\left(\frac{\mu}{m}\right)^\epsilon\left[\frac{2}{\epsilon}+\psi(1)+\frac{\epsilon}{4}\left(\frac{\pi^2}{3}+\psi^2(1)-\psi'(1)\right)+\cdots\right],\tag{A4}$$

where ψ is the digamma function. Substituting Eq. (A4) in Eq. (A2) we obtain Eq. (48).

Four-point Green function

With a little bit of algebra Eq. (47) can be cast in the form [$(s+t+u)$-channels, $d=2-\epsilon$ dimensions]

$$\Gamma=g^2[\eta_{ij}\eta_{kl}C_1+\eta_{ik}\eta_{jl}C_2+\eta_{il}\eta_{kj}C_3],\tag{A5}$$

where

$$\begin{aligned}C_1&=-\frac{9}{2}ut[I_4(t)+I_4(u)]+(9u+2t)I_2(t)\\&\quad+(9t+2u)I_2(u),\\C_2&=-\frac{9}{2}us[I_4(s)+I_4(u)]+(9u+2s)I_2(s)\\&\quad+(9s+2u)I_2(u),\end{aligned}\tag{A6}$$

$$C_3 = -\frac{9}{2} st[I_4(s)+I_4(t)] + (9t+2s)I_2(s) + (9s+2t)I_2(t),$$

and

$$I_4(p) = \mu^\epsilon \int \frac{d^{2-\epsilon}q}{(2\pi)^2} \frac{1}{(q^2+i\epsilon)[(p-q)^2+i\epsilon]}. \qquad \text{(A7)}$$

In the reduction we have discarded the quadratically divergent integrals [24] and used the following relations:

$$\int \frac{d^dq}{(2\pi)^2} \frac{q_i q_j (d-1)}{(q^2+i\epsilon)[(p-q)^2+i\epsilon]} = \frac{p_i p_j}{2p^2}\left[(d-2)I_2(p) + \frac{dp^2}{2} I_4(p)\right] + \frac{\eta_{ij}}{2}\left[I_2(p) - \frac{p^2}{2} I_4(p)\right],$$

$$\int \frac{d^dq}{(2\pi)^2} \frac{p^i q_i}{(q^2+i\epsilon)[(p-q)^2+i\epsilon]} = \frac{p^2}{2} I_4(p),$$

$$\int \frac{d^dq}{(2\pi)^2} \frac{p^i q_i}{(p-q)^2+i\epsilon} = \frac{p^2}{2} I_2(p) + \frac{1}{2}\int \frac{d^2q}{(2\pi)^2} \frac{q^2}{(p-q)^2+i\epsilon}.$$

$I_4(p)$ can be easily evaluated by inserting an infrared regulator and performing a Wick rotation similar to the one which is described in Fig. 1. [$I_4(p)$ is infrared divergent and ultraviolet convergent.] Using [23] [Eq. (B.16) p. 317] and expanding around $d=2$ we have

$$I_4(p) = -\frac{i}{2\pi}\left(\frac{\mu}{m}\right)^\epsilon \frac{1}{p^2\sqrt{1+4m^2/p^2}} \times \ln\left|\frac{1+\sqrt{1+4m^2/p^2}}{1-\sqrt{1+4m^2/p^2}}\right|. \qquad \text{(A8)}$$

Finally, substituting Eq. (A8) and Eq. (A4) in Eq. (A5) we obtain Eq. (50).

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