Quantum Grothendieck rings and derived Hall algebras

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Abstract

Let $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ be a simple Lie algebra over $\mathbb{C}$ of type A, D, E, and let $U_q(L\mathfrak{g})$ be the associated quantum loop algebra. Following Nakajima [**N3**], Varagnolo-Vasserot [**VV1**], and the first author [**He2**], we study a t-deformation $\mathscr{K}_t$ of the Grothendieck ring of a tensor category $\mathscr{C}_\mathbb{Z}$ of finite-dimensional $U_q(L\mathfrak{g})$-modules. We obtain a presentation of $\mathscr{K}_t$ by generators and relations.

Let Q be a Dynkin quiver of the same type as $\mathfrak{g}$. Let $DH(Q)$ be the derived Hall algebra of the bounded derived category $D^b(\mathrm{mod}(FQ))$ over a finite field F, introduced by Toën [**T**]. Our presentation shows that the specialization of $\mathscr{K}_t$ at $t = \sqrt{|F|}$ is isomorphic to $DH(Q)$. Under this isomorphism, the classes of fundamental $U_q(L\mathfrak{g})$-modules are mapped to scalar multiples of the classes of indecomposable objects in $DH(Q)$.

Our presentation of $\mathscr{K}_t$ is deduced from the preliminary study of a tensor subcategory $\mathscr{C}_Q$ of $\mathscr{C}_\mathbb{Z}$ analogous to the heart $\mathrm{mod}(FQ)$ of the triangulated category $D^b(\mathrm{mod}(FQ))$. We show that the t-deformed Grothendieck ring $\mathscr{K}_{t,Q}$ of $\mathscr{C}_Q$ is isomorphic to the positive part of the quantum enveloping algebra of $\mathfrak{g}$, and that the basis of classes of simple objects of $\mathscr{K}_{t,Q}$ corresponds to the dual of Lusztig's canonical basis. The proof relies on the algebraic characterizations of these bases, but we also give a geometric approach in the last section.

It follows that for every orientation Q of the Dynkin diagram, the category $\mathscr{C}_Q$ gives a new categorification of the coordinate ring $\mathbb{C}[N]$ of a unipotent group N with Lie algebra $\mathfrak{n}$, together with its dual canonical basis.

Contents

1 Introduction

1.1 Let $\mathfrak{g}$ be a simple Lie algebra of type A, D, E over $\mathbb{C}$. We denote by $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ a triangular decomposition of $\mathfrak{g}$. Let v be an indeterminate, and let

$$U_v(\mathfrak{g}) = U_v(\mathfrak{n}) \otimes U_v(\mathfrak{h}) \otimes U_v(\mathfrak{n}_-)$$

be the corresponding Drinfeld-Jimbo quantum enveloping algebra over $\mathbb{C}(v)$, defined via a v-analogue of the Chevalley-Serre presentation of $U(\mathfrak{g})$. Using a geometric realization of $U_v(\mathfrak{n})$ in terms of perverse sheaves on varieties of representations of a quiver Q of the same Dynkin type as $\mathfrak{g}$, Lusztig [**Lu1**] has defined a canonical basis $\mathbf{B}$ of $U_v(\mathfrak{n})$ with favorable positivity properties. This was inspired by a seminal work of Ringel [**Ri2**], showing that the twisted Hall algebra of the category $\mathrm{mod}\,(FQ)$ of representations of Q over a finite field F, is isomorphic to the specialization of $U_v(\mathfrak{n})$ at $v = \sqrt{|F|}$.

1.2 One can associate with $\mathfrak{g}$ another quantum algebra. Let $L\mathfrak{g} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}$ be the loop algebra of $\mathfrak{g}$. Let q be a nonzero complex number, which is not a root of unity. Via a q-analogue of the loop presentation of $U(L\mathfrak{g})$, Drinfeld [**D**] has defined the quantum loop algebra $U_q(L\mathfrak{g})$, an algebra over $\mathbb{C}$. The finite-dimensional representations of $U_q(L\mathfrak{g})$ have attracted a lot of attention, because of their connection with the trigonometric solutions of the quantum Yang-Baxter equation with spectral parameter. In this paper we focus on a tensor subcategory $\mathscr{C}_\mathbb{Z}$ of the category of finite-dimensional $U_q(L\mathfrak{g})$-modules, whose simple objects are parametrized by a discrete set (for the precise definition of $\mathscr{C}_\mathbb{Z}$ see [**HL**, §3.7] or §5.2 below). Denote by $\mathscr{R}$ the complexified Grothendieck ring of $\mathscr{C}_\mathbb{Z}$. Let t be another indeterminate. By works of Nakajima [**N3**] and Varagnolo-Vasserot [**VV1**], the $\mathbb{C}$-algebra $\mathscr{R}$ has an interesting t-deformation $\mathscr{R}_t$ over $\mathbb{C}(t)$. The first author [**He2**] has introduced a slightly different deformation $\mathscr{K}_t$. These t-deformations are important because they contain for every simple object L of $\mathscr{C}_\mathbb{Z}$ a "class" $[L]_t$ which can be characterized by axioms similar to those of Lusztig for the canonical basis $\mathbf{B}$. As a consequence, Nakajima [**N3**] has shown that one can calculate algorithmically the character of L.

1.3 Surprisingly, these deformed Grothendieck rings have not been much studied from the ring theoretic point of view, and for instance, to the best of our knowledge, there is no available presentation by generators and relations in the literature. One of the main results of this paper (Theorem 7.3) is a presentation of $\mathscr{K}_t$, with a similar flavor as the familiar Drinfeld-Jimbo presentation of $U_v(\mathfrak{n})$. More precisely, this presentation shows that $\mathscr{K}_t$ is obtained by taking an infinite number of copies of $U_t(\mathfrak{n})$ labelled by $m \in \mathbb{Z}$, and then imposing t-boson relations between generators of copies sitting at adjacent integers, and t-commutation relations between generators of non-adjacent copies.

1.4 Let $D^b(\mathrm{mod}(FQ))$ be the bounded derived category of $\mathrm{mod}(FQ)$. Toën [**T**] has attached to this triangulated category an associative algebra called the derived Hall algebra of $D^b(\mathrm{mod}(FQ))$ (see also [**XX**]). Let $DH(Q)$ denote the twisted derived Hall algebra obtained by twisting Toën's multiplication by means of the Ringel form, as in [**S**]. It follows from our presentation of $\mathscr{K}_t$ that:

Theorem 1.1 *(a) The specialization of $\mathscr{K}_t$ at $t = \sqrt{|F|}$ is isomorphic to $DH(Q)$.*

(b) Under this isomorphism, the classes of fundamental $U_q(L\mathfrak{g})$-modules are mapped to scalar multiples of the classes of indecomposable stalk complexes in $DH(Q)$, and the basis of classes of standard $U_q(L\mathfrak{g})$-modules is mapped to a rescaling of the natural basis of $DH(Q)$ indexed by isoclasses of objects of $D^b(\mathrm{mod}(FQ))$.

There is a similar result for the t-deformed Grothendieck ring $\mathscr{R}_t$ of [**N3**, **VV1**], but the twisted derived Hall algebra should be replaced by a non-twisted one (Remark 8.4).

1.5 To obtain our presentation of $\mathscr{K}_t$ we first consider a tensor subcategory $\mathscr{C}_Q$ of $\mathscr{C}_\mathbb{Z}$ which "looks like $\mathrm{mod}(FQ)$ inside $D^b(\mathrm{mod}(FQ))$". Recall that in [**HL**] we have introduced an increasing sequence $(\mathscr{C}_\ell)_{\ell>0}$ of subcategories of $\mathscr{C}_\mathbb{Z}$. When Q is a bipartite orientation of the Dynkin diagram and the Coxeter number h is even, $\mathscr{C}_Q$ is just the subcategory $\mathscr{C}_\ell$ with $\ell = h/2-1$. The general definition of $\mathscr{C}_Q$ for an arbitrary orientation Q will be given in §5.11 below. Let $\mathscr{K}_{t,Q}$ be the subalgebra of $\mathscr{K}_t$ spanned by the elements $[L]_t$ associated with the simple objects L of $\mathscr{C}_Q$. Note that $\mathscr{K}_t$ and $\mathscr{K}_{t,Q}$ are algebras over $\mathbb{C}(t^{1/2})$, where $t^{1/2}$ is a square root of t.

The quantum algebra $U_v(\mathfrak{n})$ is endowed with a distinguished scalar product. Let $\mathbf{B}^*$ be the basis of $U_v(\mathfrak{n})$ adjoint to the canonical basis $\mathbf{B}$ with respect to this scalar product. Let $v^{1/2}$ be a square root of v, and set $\mathscr{U}_v(\mathfrak{n}) := \mathbb{C}(v^{1/2})\otimes U_v(\mathfrak{n})$. The main step for obtaining the presentation of $\mathscr{K}_t$ is:

Theorem 1.2 *(a) There is a $\mathbb{C}$-algebra isomorphism $\Phi\colon \mathscr{K}_{t,Q} \xrightarrow{\sim} \mathscr{U}_v(\mathfrak{n})$ with $\Phi(t^{1/2}) = v^{1/2}$.*

(b) For every simple object L of $\mathscr{C}_Q$, the image $\Phi([L]_t)$ belongs to $\mathbf{B}^*$ (up to some half-integral power of v).

Nakajima obtained in [**N4**] similar results for the first subcategory $\mathscr{C}_1$ of [**HL**]. Namely, he showed that the classes $[L]_t$ of simple objects of $\mathscr{C}_1$ can be identified with a subset of the basis $\widetilde{\mathbf{B}}^*$ of $U_v(\widetilde{\mathfrak{n}})$. Here $\widetilde{\mathfrak{n}}$ denotes the positive part of the Kac-Moody algebra of rank $2\,\mathrm{rk}(\mathfrak{g})$ attached to the decorated Dynkin diagram of $\mathfrak{g}$, and $\widetilde{\mathbf{B}}$ is Lusztig's canonical basis of $U_v(\widetilde{\mathfrak{n}})$. For example, if $\mathfrak{g}$ has type A_3, $\widetilde{\mathfrak{g}}$ has type E_6. Note that in Theorem 1.2, we do not use $\widetilde{\mathfrak{n}}$, but only $\mathfrak{n}$.

1.6 Let $A_v(\mathfrak{n})$ be the graded dual of the vector space $U_v(\mathfrak{n})$. It can be endowed with a multiplication coming from the comultiplication of $U_v(\mathfrak{g})$, and regarded as the quantum coordinate ring of the unipotent group N with Lie algebra $\mathfrak{n}$ (see *e.g.* [**GLS**]). The basis $\mathbf{B}^*$ can be identified with a basis of $A_v(\mathfrak{n})$ called the dual canonical basis. It specializes when $v\mapsto 1$ to a basis $\mathscr{B}$ of the coordinate ring $\mathbb{C}[N]$.

By specializing $v^{1/2}$ and $t^{1/2}$ to 1 in Theorem 1.2, we see that the complexified Grothendieck ring $\mathscr{R}_Q$ of $\mathscr{C}_Q$ can be identified with the coordinate ring $\mathbb{C}[N]$ in such a way that the basis of $\mathscr{R}_Q$ consisting of the classes of simple objects becomes Lusztig's dual canonical basis $\mathscr{B}$ of $\mathbb{C}[N]$. We can therefore state:

Theorem 1.3 *The tensor category $\mathscr{C}_Q$ is a categorification of the ring $\mathbb{C}[N]$ and its dual canonical basis $\mathscr{B}$.*

Note that, by work of Khovanov-Lauda [**KL**], Rouquier [**Ro**], and Varagnolo-Vasserot [**VV2**], $(\mathbb{C}[N],\mathscr{B})$ has another categorification in terms of KLR-algebras. In type A_n, KLR-algebras are isomorphic to blocks of affine Hecke algebras, and the category $\mathscr{C}_Q$ for an equi-oriented quiver Q is related to a category of representations of affine Hecke algebras through the quantum affine Schur-Weyl duality. It would be interesting to find for other Dynkin quivers Q similar functors between $\mathscr{C}_Q$ and the module categories of the corresponding KLR-algebras.

1.7 The first author [**He4**] has shown that tensor products of simple objects of $\mathscr{C}_\mathbb{Z}$ have the following remarkable property: a tensor product $L_1\otimes\cdots\otimes L_k$ of simple objects L_i is simple if and only if for every pair $1\le i<j\le k$ the tensor product $L_i\otimes L_j$ is simple. Using Theorem 1.2 this yields the following:

Corollary 1.4 *A product $b_1 \cdots b_k$ of elements b_i of the dual canonical basis $\mathbf{B}^*$ of $U_v(\mathfrak{n})$ belongs to $\mathbf{B}^*$ up to a power of v if and only if for every pair $1 \le i < j \le k$ the product $b_i b_j$ belongs to $\mathbf{B}^*$ up to a power of v.*

Corollary 1.4 was expected in relation with the program of Berenstein and Zelevinsky [**BZ1**, **BZ2**] of describing $\mathbf{B}^*$ in terms of quantum cluster algebras. But it was only known in a few low rank cases.

1.8 Theorem 1.2 also gives new supporting evidence for some conjectures formulated in [**GLS**] and [**HL**]. It was conjectured in [**HL**, §13] that for every $\ell \in \mathbb{N}$, the Grothendieck ring $\mathscr{R}_\ell$ of $\mathscr{C}_\ell$ has a particular cluster algebra structure for which all cluster monomials are classes of simple objects. In [**GLS**], it is shown that $U_v(\mathfrak{n})$ has a quantum cluster algebra structure, and it is conjectured that all quantum cluster monomials belong to $\mathbf{B}^*$. Suppose that Q is bipartite and h is even. Set $h' = h/2 - 1$. By comparing initial seeds, one sees that the quantum cluster structure of $\mathscr{K}_{t,Q}$ obtained by transporting via Φ^{-1} the quantum cluster structure of $U_v(\mathfrak{n})$ is a t-analogue of the cluster structure of $\mathscr{R}_{h'}$ conjectured in [**HL**]. Thus, by Theorem 1.2, the two conjectures of [**GLS**] for $U_v(\mathfrak{n})$ and of [**HL**] for $\mathscr{R}_{h'}$ are essentially equivalent.

In [**HL**] and [**N4**], the conjecture for $\mathscr{R}_\ell$ was proved in the first non trivial case $\ell = 1$. (In [**HL**] some combinatorial steps of the proof were only verified for $\mathfrak{g}$ of type A_n and D_4; the proof of [**N4**] is general and uses geometric representation theory.) Since $\mathscr{C}_1$ is a tensor subcategory of $\mathscr{C}_Q$ (for every $\mathfrak{g}$ except $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$), $\mathscr{K}_{t,Q}$ contains a subring $\mathscr{K}_{t,1}$ corresponding to $\mathscr{C}_1$. It is easy to see that $\Phi(\mathscr{K}_{t,1})$ is equal to the subalgebra $\mathscr{U}_v(\mathfrak{n}(w))$ of [**GLS**] where $w = c^2$ is the square of the Coxeter element of the Weyl group of $\mathfrak{g}$ corresponding to the bipartite quiver Q. This is a quantum cluster algebra of finite cluster type, equal to the Dynkin type of $\mathfrak{g}$ in the classification of Fomin and Zelevinsky. Thus, using [**HL**, **N4**, **Q**], Theorem 1.2 readily implies:

Corollary 1.5 *Let $w = c^2$ be as above. Then $\mathbf{B}^* \cap U_v(\mathfrak{n}(w))$ is equal to the set of quantum cluster monomials of $U_v(\mathfrak{n}(w))$.*

For $\mathfrak{g}$ of type A_n, Lampe [**La**] has given a direct proof of the fact that the quantum cluster *variables* of $U_v(\mathfrak{n}(w))$ belong to $\mathbf{B}^*$.

1.9 Since the bases $\mathbf{B}^*$ and $\{[L]_t\}$ have geometric origin, it is natural to ask for a geometric explanation of Theorem 1.2 (b). In the final part of the paper, we show (Theorem 9.11) that the quiver representation spaces $E_\mathbf{d}$ used by Lusztig to define the canonical basis of $U_v(\mathfrak{n})$ are isomorphic to some particular graded quiver varieties $\mathfrak{M}_0^\bullet(W^\mathbf{d})$ used by Nakajima for describing the classes $[L]_t$ of the simple objects L of $\mathscr{C}_Q$. Moreover the intersection cohomology sheaves of closures of $G_\mathbf{d}$-orbits in $E_\mathbf{d}$ can be identified with the intersection cohomology sheaves of closures of strata in $\mathfrak{M}_0^\bullet(W^\mathbf{d})$. This is inspired by a similar result of Nakajima [**N4**] for the category $\mathscr{C}_1$.

1.10 We now give an overview of the structure of the paper. In Section 2, we set up our notation and introduce an important bijection φ between the set of fundamental modules of $\mathscr{C}_\mathbb{Z}$ and the vertices of the Auslander-Reiten quiver of $D^b(KQ)$. We use this bijection to express the entries of the inverse of the quantum Cartan matrix of $\mathfrak{g}$ in terms of the Ringel form of Q, or in terms of the scalar product of the weight lattice of $\mathfrak{g}$ (Proposition 2.5). By construction, the quantum Grothendieck ring $\mathscr{K}_t$ is a subring of a quantum torus $\mathscr{Y}_t$ over $\mathbb{C}(t^{1/2})$. The t-commutation relations between generators of $\mathscr{Y}_t$ are expressed in terms of entries of the inverse of the quantum Cartan matrix of $\mathfrak{g}$ [**He2**], hence by Proposition 2.5, in terms of scalar products of weights of $\mathfrak{g}$. The quantum Grothendieck ring $\mathscr{K}_{t,Q}$ is a subring of a subtorus $\mathscr{Y}_{t,Q}$ of $\mathscr{Y}_t$, of rank r equal to the number of positive roots of $\mathfrak{g}$. On the other hand, by [**GLS**], $\mathscr{U}_v(\mathfrak{n})$ has an explicit embedding into a quantum torus $\mathscr{T}_{v,Q}$ of rank r over $\mathbb{C}(v^{1/2})$, whose generators are certain unipotent quantum flag minors. The explicit v-commutation relations between these minors involve scalar products of roots and weights of $\mathfrak{g}$. Comparing these two presentations, we show that there is an isomorphism $\Phi\colon \mathscr{Y}_{t,Q} \to \mathscr{T}_{v,Q}$ mapping $t^{1/2}$ to $v^{1/2}$ (Proposition 3.8).

The proof that Φ restricts to an isomorphism from $\mathscr{K}_{t,Q}$ to $\mathscr{U}_v(\mathfrak{n})$ is based on some explicit systems of algebraic identities satisfied by the generators of both algebras. In Section 4, we recall from [**GLS**] a system of quantum determinantal identities occuring in $U_v(\mathfrak{n})$, and in Section 5 we derive a quantum T-system for the (q,t)-characters of the Kirillov-Reshetikhin modules. (In [**N2**, §4], a quantum T-system was already obtained for the t-deformed product used in [**N3**, **VV1**]. A quantum cluster algebra related to the quantum T-system of type A_1 is also studied in [**DFK**].) Comparing these two systems we obtain that Φ maps the classes of the Kirillov-Reshetikhin modules of $\mathscr{C}_Q$ to certain quantum minors of $\mathscr{U}_v(\mathfrak{n})$ (multiplied by explicit half-integral powers of v). In particular, Φ maps the classes of the fundamental modules of $\mathscr{C}_Q$ in $\mathscr{K}_{t,Q}$ to the generators of the dual PBW-basis of $\mathscr{U}_v(\mathfrak{n})$ associated with Q (up to powers of $v^{1/2}$). This proves the first part of Theorem 1.2. The second part is deduced from the algebraic characterizations of $\mathbf{B}^*$ and of the classes $[L]_t$ (Section 6). After some examples, we give the proof of Corollary 1.4.

The above-mentioned presentation of $\mathscr{K}_t$ (Theorem 7.3) is deduced from Theorem 1.2 in Section 7, and in Section 8 we prove the isomorphism with the derived Hall algebra $DH(Q)$ stated in Theorem 1.1. Finally, in Section 9, we explain our geometric approach to Theorem 1.2 (b) (Theorem 9.11).

2 Cartan matrices and Auslander-Reiten quivers

2.1 Cartan matrix

Let $\mathfrak{g}$ be a simple Lie algebra of type A, D, E. We denote by I the set of vertices of its Dynkin diagram, and we put $n = |I|$. The *Cartan matrix* of $\mathfrak{g}$ is the $I \times I$ matrix C with entries

$$C_{ij} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i \text{ and } j \text{ are adjacent vertices of the Dynkin diagram,} \\ 0 & \text{otherwise.} \end{cases}$$

We shall often use the shorthand notation $i \sim j$ to say that $C_{ij} = -1$.

We denote by P the weight lattice of $\mathfrak{g}$, and by ϖ_i $(i \in I)$ its basis of fundamental weights. The simple roots are defined by

$$\alpha_i = \sum_{j \in I} C_{ij} \varpi_j, \qquad (i \in I).$$

The set of simple roots is denoted by $\Pi := \{\alpha_i \mid i \in I\}$. We denote by $(\cdot,\cdot)$ the scalar product of P defined by $(\alpha_i, \varpi_j) = \delta_{ij}$. Equivalently $(\alpha_i, \alpha_j) = C_{ij}$. The Weyl group W is generated by the reflexions s_i acting on P by

$$s_i(\lambda) = \lambda - (\lambda, \alpha_i)\alpha_i, \qquad (\lambda \in P,\ i \in I).$$

The root system of $\mathfrak{g}$ is $\Delta := W\Pi$. It decomposes as $\Delta = \Delta_+ \sqcup \Delta_-$, where $\Delta_+ = \Delta \cap (\oplus_{i \in I} \mathbb{N}\alpha_i)$ and $\Delta_- = -\Delta_+$. We write $r := |\Delta_+|$.

A *Coxeter element* of W is a product of the form $c = s_{i_1} \cdots s_{i_n}$ where $(i_1, \ldots, i_n)$ is an arbitrary ordering of I. All Coxeter elements are conjugate in W. Their common order is called the *Coxeter number* and denoted by h. We have $hn = 2r$.

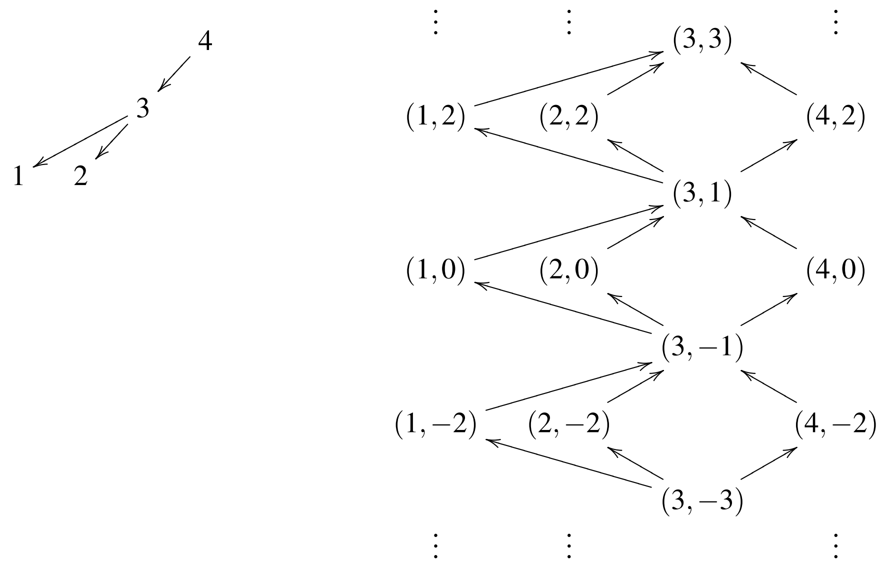

Figure 1: *A quiver Q of type D_4 and its repetition quiver $\widehat{Q}$.*

2.2 Quivers

Let Q be an orientation of the Dynkin diagram of $\mathfrak{g}$. In other words, Q is a Dynkin quiver of the same Dynkin type as $\mathfrak{g}$.

For $i \in I$, we denote by $s_i(Q)$ the quiver obtained from Q by changing the orientation of every arrow with source i or target i. Let $w = s_{i_1} \cdots s_{i_k} \in W$ be a reduced decomposition. We say that $\mathbf{i} = (i_1, \ldots, i_k)$ is *adapted* to Q if i_1 is a source of Q, i_2 is a source of $s_{i_1}(Q)$, ..., i_k is a source of $s_{i_{k-1}} \cdots s_{i_1}(Q)$. There is a unique Coxeter element having reduced expressions adapted to Q. We shall denote it by τ.

We denote by Q_1 the set of arrows of Q. A *height function* $\xi \colon I \to \mathbb{Z}$ on Q is a function satisfying

$$\xi_j = \xi_i - 1 \quad \text{if} \quad i \to j \in Q_1.$$

Since Q is connected, two height functions differ by a constant. We fix such a function ξ. Define

$$\widehat{I} := \{(i,p) \in I \times \mathbb{Z} \mid p - \xi_i \in 2\mathbb{Z}\}.$$

We attach to Q the infinite *repetition quiver* $\widehat{Q}$, defined as the oriented graph with vertex set $\widehat{I}$ and two types of arrows:

(i) if there is an arrow $i \to j$ in Q we have arrows $(i,p) \to (j,p+1)$ in $\widehat{Q}$ for all $(i,p) \in \widehat{I}$;

(ii) if there is an arrow $i \to j$ in Q we have arrows $(j,q) \to (i,q+1)$ in $\widehat{Q}$ for all $(j,q) \in \widehat{I}$.

Note that $\widehat{Q}$ depends only on the Dynkin diagram, and not on the choice of orientation Q. In fact, it is well known that $\widehat{Q}$ is the quiver of a $\mathbb{Z}$-covering of the preprojective algebra associated with Q. In the literature, this quiver is often denoted by $\mathbb{Z}Q$. An example is shown in Figure 1, where the height function is $\xi_1 = \xi_2 = 0,\ \xi_3 = 1,\ \xi_4 = 2$.

Let $\widehat{\Delta} := \Delta_+ \times \mathbb{Z}$. We now describe a natural labelling of the vertices of $\widehat{Q}$ by $\widehat{\Delta}$. For $i \in I$, let $B(i)$ be the subset of I consisting of all j's such that there is a path from j to i in Q. Define

$$\gamma_i := \sum_{j \in B(i)} \alpha_j, \qquad (i \in I).$$

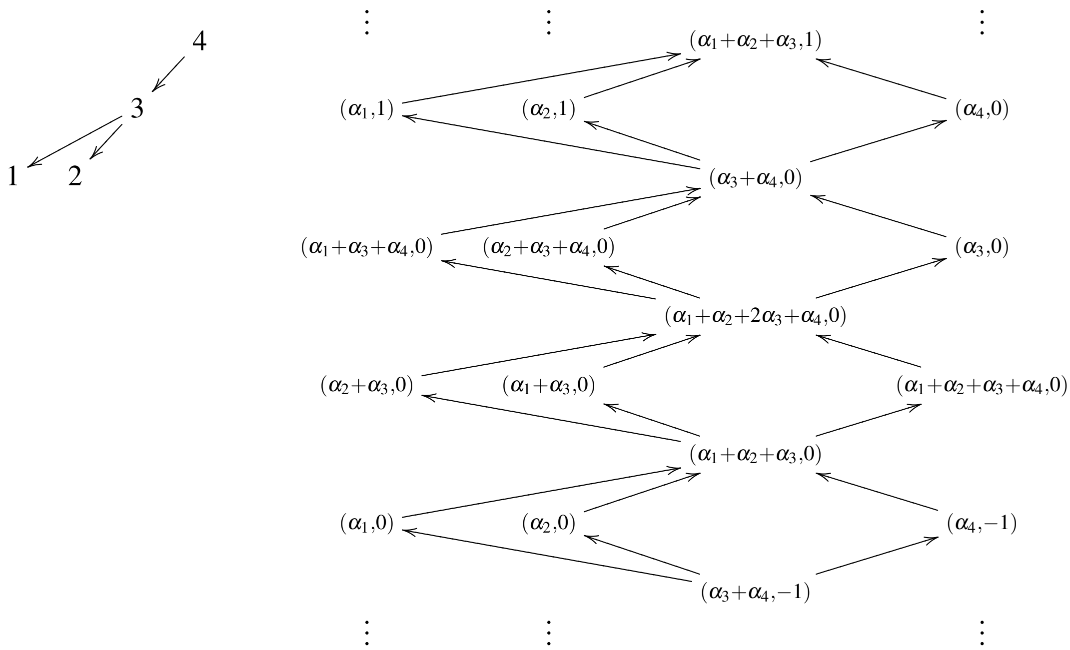

Figure 2: *The labelling of* $\widehat{Q}$ *by* $\widehat{\Delta}$ *for* Q *of type* D_4.

We have $\gamma_i \in \Delta_+$. There is a unique bijection $\varphi\colon \widehat{I} \to \widehat{\Delta}$ defined inductively as follows:

(a) $\varphi(i,\xi_i) = (\gamma_i, 0)$ for $i \in I$;

(b) suppose that $\varphi(i,p) = (\beta,m)$; then

- $\varphi(i,p-2) = (\tau(\beta),m)$ if $\tau(\beta) \in \Delta_+$;
- $\varphi(i,p-2) = (-\tau(\beta),m-1)$ if $\tau(\beta) \in \Delta_-$;
- $\varphi(i,p+2) = (\tau^{-1}(\beta),m)$ if $\tau^{-1}(\beta) \in \Delta_+$;
- $\varphi(i,p+2) = (-\tau^{-1}(\beta),m+1)$ if $\tau^{-1}(\beta) \in \Delta_-$.

Note that this second labelling of $\widehat{Q}$ depends on Q. This is illustrated in Figure 2.

2.3 Auslander-Reiten theory

The quiver $\widehat{Q}$ with its labelling by $\widehat{\Delta}$ arises in the representation theory of the path algebra KQ of Q over a field K, as we shall now recall. We refer the reader to [**ARS**, **ASS**, **GR**, **Ri1**] for background on quiver representations and Auslander-Reiten theory.

Let $\mathrm{mod}(KQ)$ be the abelian category of representations of Q over K. For an object X of $\mathrm{mod}(KQ)$ we write $\underline{\dim}(X)$ for its dimension vector. We define the Ringel bilinear form

$$\langle X,Y\rangle := \dim(\mathrm{Hom}(X,Y)) - \dim(\mathrm{Ext}^1(X,Y)), \qquad (X,Y \in \mathrm{mod}(KQ)),$$

and the symmetric form $(X,Y) := \langle X,Y\rangle + \langle Y,X\rangle$. It is known that these forms depend only on the dimension vectors $\underline{\dim}(X)$ and $\underline{\dim}(Y)$. Moreover, if we identify in the standard way $\underline{\dim}(X)$ and $\underline{\dim}(Y)$ with elements of the root lattice of $\mathfrak{g}$, then (X,Y) coincides with the natural scalar product $(\underline{\dim}(X),\underline{\dim}(Y))$. In this picture, α_i is the dimension vector of the simple KQ-module S_i supported on vertex i, and γ_i is the dimension vector of its injective envelope I_i. Recall that, by Gabriel's theorem, the isoclasses of indecomposable KQ-modules are in natural bijection with Δ_+. They form the vertices of the Auslander-Reiten quiver Γ_Q of $\mathrm{mod}(KQ)$. The map $\beta \mapsto (\beta,0)$ identifies Γ_Q with the full subgraph of $\widehat{Q}$ with set of vertices $\Delta_+ \times \{0\}$. The map τ restricted to the dimension vectors in Δ_+ of non projective KQ-modules is the Auslander-Reiten translation of $\mathrm{mod}(KQ)$ [**ARS**].

Let $D^b(\mathrm{mod}(KQ))$ be the bounded derived category of KQ. Its indecomposable objects are the stalk complexes $X[i]$, consisting of an indecomposable object X of $\mathrm{mod}(KQ)$ sitting in degree $i \in \mathbb{Z}$, and zero objects in all other degrees. Thus, the isoclasses of indecomposable objects of $D^b(\mathrm{mod}(KQ))$ are naturally labelled by $\widehat{\Delta}$. Using this labelling, the quiver $\widehat{Q}$ is identified with the Auslander-Reiten quiver of the triangulated category $D^b(\mathrm{mod}(KQ))$ [**Ha**].

2.4 Quantum Cartan matrix

Let z be an indeterminate, and let $C(z)$ be the matrix with entries

$$C_{ij}(z) = \begin{cases} z+z^{-1} & \text{if } i=j, \\ -1 & \text{if } i \sim j, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $C(1)$ is just the Cartan matrix C of $\mathfrak{g}$. Since $\det(C) \neq 0$, $\det(C(z)) \neq 0$. We denote by $\widetilde{C}(z)$ the inverse of the matrix $C(z)$. This is a matrix with entries $\widetilde{C}_{ij}(z) \in \mathbb{Q}(z)$. Denoting by A the adjacency matrix of the Dynkin diagram we have

$$C(z) = (z+z^{-1})I - A,$$

therefore

$$\widetilde{C}(z) = \sum_{k\geq 0} (z+z^{-1})^{-k-1}A^k.$$

Hence the entries of $\widetilde{C}(z)$ have power series expansions in z of the form

$$\widetilde{C}_{ij}(z) = \sum_{m\geq 1} \widetilde{C}_{ij}(m)\, z^m,$$

where $\widetilde{C}_{ij}(m) \in \mathbb{Z}$. Note that since $C(z)$ is a symmetric matrix, we have $\widetilde{C}_{ij}(m) = \widetilde{C}_{ji}(m)$.

2.5 Formula for $\widetilde{C}_{ij}(m)$

We will now give several equivalent expressions for the coefficients $\widetilde{C}_{ij}(m)$. For other expressions of $\widetilde{C}_{ij}(z)$ in type A_n and D_n, see [**FR1**, Appendix C].

Fix an orientation Q of the Dynkin diagram, and recall from §2.2 and §2.3 the associated notation ξ_i, γ_i, the Coxeter transformation τ, and the Ringel form $\langle\cdot,\cdot\rangle$.

Proposition 2.1 *Let $m \geq 1$. If $m+\xi_i-\xi_j-1$ is odd then $\widetilde{C}_{ij}(m) = 0$. Otherwise*

$$\widetilde{C}_{ij}(m) = \left(\tau^{(m+\xi_i-\xi_j-1)/2}(\gamma_i), \varpi_j\right). \tag{1}$$

Equivalently,

$$\widetilde{C}_{ij}(m) = \left\langle \tau^{(m+\xi_i-\xi_j-1)/2}(I_i), I_j \right\rangle. \tag{2}$$

Proof — Let us denote temporarily by $D_{ij}(m)$ the value of $\widetilde{C}_{ij}(m)$ predicted by the proposition. We want to show that

$$\sum_{k\in I,\ m\geq 1} C_{ik}(z) D_{kj}(m) z^m = \delta_{ij}, \qquad (i,j\in I).$$

Using the definition of $C_{ik}(z)$, this is equivalent to show that

$$\sum_{m\geq 1}\left(\left(z+z^{-1}\right)D_{ij}(m)-\sum_{k\sim i}D_{kj}(m)\right)z^m=\delta_{ij}, \qquad (i,j\in I). \tag{3}$$

The coefficient of z^0 in the left-hand side is equal to $D_{ij}(1)$. If $\xi_i-\xi_j$ is odd then by definition $D_{ij}(1)=0$. Otherwise, if $\xi_i-\xi_j=2l$, then $D_{ij}(1)=(\tau^l(\gamma_i),\varpi_j)$ is the coefficient of α_j in $\tau^l(\gamma_i)$. Let (β,m) be the vertex of $\widehat{Q}$ in the column of $(\gamma_i,0)$ and at the same height as $(\gamma_j,0)$. Such a vertex exists because $\xi_i-\xi_j$ is even, and clearly $\beta=\pm\tau^l(\gamma_i)$. Now it is a well-known fact from the combinatorics of Auslander-Reiten quivers that for all vertices (γ,s) of $\widehat{Q}$ at the same height as $(\gamma_j,0)$ the coefficient of α_j in γ is 0, except if $(\gamma,s)=(\gamma_j,0)$ in which case it is equal to 1. Hence we have $D_{ij}(1)=\delta_{ij}$.

Consider now the coefficient of z^m $(m\geq 1)$ in (3). We need to show that

$$D_{ij}(m+1)+D_{ij}(m-1)-\sum_{k:\,k\sim i}D_{kj}(m)=0, \qquad (i,j\in I,\ m\geq 1). \tag{4}$$

Note that for $k\sim i$ we have $\xi_k=\xi_i\pm 1$, hence if $m+\xi_i-\xi_j$ is odd, all summands of the left-hand side are zero. Otherwise, writing $m+\xi_i-\xi_j=2l$, the left-hand side of (4) is

$$\left(\tau^l(\gamma_i)+\tau^{l-1}(\gamma_i)-\sum_{k\sim i}\tau^{l+(\xi_k-\xi_i+1)/2}(\gamma_k)\ ,\ \varpi_j\right).$$

Now it is again a familiar fact from the combinatorics of Auslander-Reiten quivers that

$$\tau^l(\gamma_i)+\tau^{l-1}(\gamma_i)=\sum_{k\sim i}\tau^{l+(\xi_k-\xi_i+1)/2}(\gamma_k),$$

since the roots $\tau^l(\gamma_i)$, $\tau^{l-1}(\gamma_i)$, and $\tau^{l+(\xi_k-\xi_i+1)/2}(\gamma_k)$ with $k\sim i$, form a mesh. This proves (1).

Finally, if $\beta=\underline{\dim}X$ then (β,ϖ_j) is equal to the coefficient of α_j in β, hence

$$(\beta,\varpi_j)=\dim(\mathrm{Hom}(X,I_j))=\langle X,I_j\rangle,$$

because I_j is injective. This proves (2). $\Box$

Example 2.2 Take $\mathfrak{g}$ of type A_4. One has for instance

$$\begin{aligned}
\widetilde{C}_{11}(z) &= z^1-z^9+z^{11}-z^{19}+\cdots\\
\widetilde{C}_{12}(z) &= z^2-z^8+z^{12}-z^{18}+\cdots\\
\widetilde{C}_{13}(z) &= z^3-z^7+z^{13}-z^{17}+\cdots\\
\widetilde{C}_{14}(z) &= z^4-z^6+z^{14}-z^{16}+\cdots\\
\widetilde{C}_{21}(z) &= z^2-z^8+z^{12}-z^{18}+\cdots\\
\widetilde{C}_{22}(z) &= z^1+z^3-z^7-z^9+z^{11}+z^{13}-z^{17}-z^{19}+\cdots\\
\widetilde{C}_{23}(z) &= z^2+z^4-z^6-z^8+z^{12}+z^{14}-z^{16}-z^{18}\cdots\\
\widetilde{C}_{24}(z) &= z^3-z^7+z^{13}-z^{17}+\cdots
\end{aligned}$$

Let us choose the sink-source orientation Q with height function $\xi_1 = 0, \xi_2 = 1, \xi_3 = 0, \xi_4 = 1$. Then $\tau = s_2 s_4 s_1 s_3$, and since $\tau^5 = 1$, the roots $\tau^l(\gamma_i)$ are all determined by:

$$
\begin{array}{llll}
\gamma_1 = \alpha_1 + \alpha_2, & \gamma_2 = \alpha_2, & \gamma_3 = \alpha_2 + \alpha_3 + \alpha_4, & \gamma_4 = \alpha_4, \\
\tau(\gamma_1) = \alpha_3 + \alpha_4, & \tau(\gamma_2) = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, & \tau(\gamma_3) = \alpha_1 + \alpha_2 + \alpha_3, & \tau(\gamma_4) = \alpha_2 + \alpha_3, \\
\tau^2(\gamma_1) = -\alpha_4, & \tau^2(\gamma_2) = \alpha_3, & \tau^2(\gamma_3) = -\alpha_2, & \tau^2(\gamma_4) = \alpha_1, \\
\tau^3(\gamma_1) = -\alpha_2 - \alpha_3, & \tau^3(\gamma_2) = -\alpha_2 - \alpha_3 - \alpha_4, & \tau^3(\gamma_3) = -\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4, & \tau^3(\gamma_4) = -\alpha_1 - \alpha_2, \\
\tau^4(\gamma_1) = -\alpha_1, & \tau^4(\gamma_2) = -\alpha_1 - \alpha_2 - \alpha_3, & \tau^4(\gamma_3) = -\alpha_3, & \tau^4(\gamma_4) = -\alpha_3 - \alpha_4.
\end{array}
$$

For instance by Proposition 2.1, $\widetilde{C}_{23}(6)$ is equal to the coefficient of α_3 in $\tau^3(\gamma_2) = -\alpha_2 - \alpha_3 - \alpha_4$, namely to -1.

Corollary 2.3 *For $i, j \in I$ and $m \geq 1$ we have*

$$\widetilde{C}_{ij}(m + 2h) = \widetilde{C}_{ij}(m).$$

Proof — Since $\tau^h = 1$, this follows immediately from Proposition 2.1. □

3 Quantum tori

3.1 The quantum torus $\mathbf{Y}_t$

Recall from §2.2 the labelling set $\widehat{I}$ of $\widehat{Q}$. Define

$$\mathscr{Y} := \mathbb{C}\left[Y_{i,p}^{\pm 1} \mid (i,p) \in \widehat{I}\right]$$

to be the Laurent polynomial ring generated by a collection of commutative variables $Y_{i,p}$ labelled by $\widehat{I}$. This ring is related to a tensor subcategory $\mathscr{C}_{\mathbb{Z}}$ of the category of finite-dimensional $U_q(L\mathfrak{g})$-modules considered in [**HL**] (see below §5.2).

Let t be an indeterminate. Following [**He2**] we introduce a t-deformed version $(\mathbf{Y}_t, *)$ of $\mathscr{Y}$, with noncommutative multiplication denoted by $*$. This is the $\mathbb{C}(t)$-algebra generated by variables still denoted by $Y_{i,p}$, subject to the t-commutation relations

$$Y_{i,p} * Y_{j,s} := t^{\mathscr{N}(i,p;j,s)} Y_{j,s} * Y_{i,p}, \qquad ((i,p), (j,s) \in \widehat{I}), \tag{5}$$

where

$$\mathscr{N}(i,p;j,s) := \widetilde{C}_{ij}(p-s-1) - \widetilde{C}_{ij}(p-s+1) - \widetilde{C}_{ij}(s-p-1) + \widetilde{C}_{ij}(s-p+1). \tag{6}$$

Here we have extended the definition of $\widetilde{C}_{ij}(m)$ to every $m \in \mathbb{Z}$ by setting $\widetilde{C}_{ij}(m) = 0$ if $m \leq 0$. Note that, since $\widetilde{C}(z)$ is symmetric, we have

$$\mathscr{N}(i,p;j,s) = -\mathscr{N}(j,s;i,p), \qquad (i,j \in I,\ p,s \in \mathbb{Z}). \tag{7}$$

If $p = s$ then $\mathscr{N}(i,p;j,s) = 0$. Otherwise, without loss of generality we can assume that $p < s$. Then, (6) simplifies as

$$\mathscr{N}(i,p;j,s) = \widetilde{C}_{ij}(s-p+1) - \widetilde{C}_{ij}(s-p-1), \qquad (p < s). \tag{8}$$

We regard the noncommutative ring $(\mathbf{Y}_t, *)$ as a quantum torus of infinite rank.

Remark 3.1 In [**VV1**] and [**N3**], the construction of a t-deformed Grothendieck ring is based on a slightly different quantum torus. Namely, in these papers the product is defined by:

$$Y_{i,p} \cdot Y_{j,s} := t^{\mathscr{N}'(i,p;j,s)} Y_{j,s} \cdot Y_{i,p}, \qquad ((i,p),(j,s) \in \widehat{I}), \tag{9}$$

where instead of (6) the following exponent is used:

$$\mathscr{N}'(i,p;j,s) := -2\left(\widetilde{C}_{ij}(p-s-1) - \widetilde{C}_{ij}(s-p-1)\right). \tag{10}$$

For instance, in type A_3, we have

$$Y_{1,0} * Y_{2,1} = tY_{2,1} * Y_{1,0}, \qquad \text{whereas} \qquad Y_{1,0} \cdot Y_{2,1} = Y_{2,1} \cdot Y_{1,0}.$$

In [**N3**, **VV1**], the definition of the product comes from a convolution operation for certain perverse sheaves on quiver varieties, and the deformation parameter t encodes the natural grading of complexes of sheaves. Our product $*$ comes from [**He2**] and the original construction of q-characters. Indeed in [**FR2**], the variables $Y_{i,p} \in \mathscr{Y}$ are defined as formal power series in elements of $U_q(L\mathfrak{g})$, and they pairwise commute. In [**He2**], these formal power series are replaced by certain infinite sums $\tilde{Y}_{i,p}$ in elements of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ (with non trivial central charge c), which can be seen as vertex operators. The original variable $Y_{i,p}$ is just one factor of the complete variable $\tilde{Y}_{i,p}$. The relations of the quantum affine algebra then give rise to t-commutation relations between the $\tilde{Y}_{i,p}$, where the parameter t appears as a formal power series with coefficients in $\mathbb{C}[c^{\pm 1}]$ [**He2**, Theorem 3.11]. The defining relations (5) (6) of $*$ are obtained by replacing t by t^{-1} in those t-commutation relations.

Recall from §2.2 the bijection $\varphi\colon \widehat{I} \to \widehat{\Delta}$.

Proposition 3.2 *Let (i,p) and (j,s) be elements of $\widehat{I}$ with $p < s$. There holds*

$$\mathscr{N}(i,p;j,s) = \left(\tau^{(s-p+\xi_i-\xi_j)/2}(\gamma_i),\, \gamma_j\right).$$

Moreover, if $\varphi(i,p) = (\beta, m)$ and $\varphi(j,s) = (\delta, l)$, then

$$\mathscr{N}(i,p;j,s) = (-1)^{l-m}(\beta,\delta).$$

Proof — First note that the definition of $\widehat{I}$ implies that $s - p + \xi_i - \xi_j \in 2\mathbb{Z}$. By Proposition 2.1, we have

$$\begin{aligned}\mathscr{N}(i,p;j,s) &= \widetilde{C}_{ij}(s-p+1) - \widetilde{C}_{ij}(s-p-1)\\ &= \left\langle \tau^{(s-p+\xi_i-\xi_j)/2}(I_i),\, I_j\right\rangle - \left\langle \tau^{(s-p+\xi_i-\xi_j)/2-1}(I_i),\, I_j\right\rangle.\end{aligned}$$

Now recall the classical formula

$$\langle \tau^{-1}(X), Y\rangle = -\langle Y, X\rangle, \qquad (X,Y \in \mathrm{mod}(FQ)).$$

It follows that

$$\begin{aligned}\mathscr{N}(i,p;j,s) &= \left\langle \tau^{(s-p+\xi_i-\xi_j)/2}(I_i),\, I_j\right\rangle + \left\langle I_j, \tau^{(s-p+\xi_i-\xi_j)/2}(I_i)\right\rangle\\ &= \left(\tau^{(s-p+\xi_i-\xi_j)/2}(I_i),\, I_j\right)\\ &= \left(\tau^{(s-p+\xi_i-\xi_j)/2}(\gamma_i),\, \gamma_j\right).\end{aligned}$$

This proves the first equality. The second equality is immediately deduced from the first one if we note that, by definition of the bijection φ,

$$\tau^{(\xi_i-p)/2}(\gamma_i) = (-1)^m\beta, \qquad \tau^{(\xi_j-s)/2}(\gamma_i) = (-1)^l\delta.$$

□

Remark 3.3 For the product of [**VV1**] and [**N3**], we have, for $p<s$, a similar expression

$$\mathscr{N}'(i,p;j,s) = 2\left\langle \tau^{(s-p+\xi_i-\xi_j)/2-1}(\gamma_i),\ \gamma_j\right\rangle = -2\left\langle \gamma_j, \tau^{(s-p+\xi_i-\xi_j)/2}(\gamma_i)\right\rangle = (-1)^{l-m+1}2\langle\delta,\beta\rangle,$$

in which the symmetric scalar product $(\cdot,\cdot)$ is replaced by the non-symmetric Ringel form $\langle\cdot,\cdot\rangle$.

3.2 Commutative monomials

Let us adjoin a square root $t^{1/2}$ of t and extend the quantum torus $(\mathbf{Y}_t,*)$ to

$$(\mathscr{Y}_t,*) := \mathbb{C}(t^{1/2})\otimes_{\mathbb{C}(t)}(\mathbf{Y}_t,*).$$

We notice that the expression

$$t^{\frac{1}{2}\mathscr{N}(j,s;i,p)}\,Y_{i,p}*Y_{j,s} = t^{\frac{1}{2}\mathscr{N}(i,p;j,s)}\,Y_{j,s}*Y_{i,p}\in\mathscr{Y}_t$$

is invariant under permutation of (i,p) and (j,s). We can then denote it as a *commutative* monomial $Y_{i,p}Y_{j,s} = Y_{j,s}Y_{i,p}$, and write

$$Y_{i,p}*Y_{j,s} = t^{\frac{1}{2}\mathscr{N}(i,p;j,s)}\,Y_{i,p}Y_{j,s}.$$

More generally, for a family $(u_{i,p}\mid(i,p)\in\widehat{I})$ of integers with finitely many nonzero components, the expression

$$t^{\frac{1}{2}\sum_{(i,p)<(j,s)}u_{i,p}u_{j,s}\mathscr{N}(j,s;i,p)}\ \overrightarrow{\mathop{*}_{(i,p)\in\widehat{I}}}\ Y_{i,p}^{u_{i,p}}$$

does not depend on the chosen ordering of $\widehat{I}$ used to define it. We will denote it as a commutative monomial $\prod_{(i,p)\in\widehat{I}}Y_{i,p}^{u_{i,p}}$, and write

$$\overrightarrow{\mathop{*}_{(i,p)\in\widehat{I}}}\ Y_{i,p}^{u_{i,p}} = t^{\frac{1}{2}\sum_{(i,p)<(j,s)}u_{i,p}u_{j,s}\mathscr{N}(i,p;j,s)}\prod_{(i,p)\in\widehat{I}}Y_{i,p}^{u_{i,p}}.$$

The commutative monomials form a basis of the $\mathbb{C}(t^{1/2})$-vector space $\mathscr{Y}_t$. It will be convenient to denote commutative monomials by

$$m = \prod_{(i,p)\in\widehat{I}}Y_{i,p}^{u_{i,p}(m)}.$$

A commutative monomial m is said to be *dominant* if $u_{i,p}(m)\geq 0$ for every $(i,p)\in\widehat{I}$.

The noncommutative product of two commutative monomials m_1 and m_2 is given by

$$m_1*m_2 = t^{\frac{1}{2}D(m_1,m_2)}m_1m_2 = t^{D(m_1,m_2)}m_2*m_1, \tag{11}$$

where

$$D(m_1,m_2) = \sum_{(i,p),(j,s)\in\widehat{I}}u_{i,p}(m_1)u_{j,s}(m_2)\mathscr{N}(i,p;j,s),$$

and

$$m_1m_2 = \prod_{(i,p)\in\widehat{I}}Y_{i,p}^{u_{i,p}(m_1)+u_{i,p}(m_2)},$$

denotes the commutative product.

3.3 The quantum torus $\mathbf{Y}_{t,Q}$

Recall the bijection $\varphi\colon \widehat{I} \to \widehat{\Delta}$ of §2.2. Define

$$\widehat{I}_Q := \varphi^{-1}(\Delta_+ \times \{0\}) \subset \widehat{I},$$

and let $\mathbf{Y}_{t,Q}$ be the the $\mathbb{C}(t)$-subalgebra of $(\mathbf{Y}_t, *)$ generated by the variables $Y_{i,p}$ $((i,p) \in \widehat{I}_Q)$. This is a quantum torus of rank $r = |\Delta_+|$. We will also use the extended torus

$$(\mathscr{Y}_{t,Q}, *) := \mathbb{C}(t^{1/2}) \otimes_{\mathbb{C}(t)} (\mathbf{Y}_{t,Q}, *).$$

Example 3.4 We take $\mathfrak{g}$ of type D_4 and choose Q as in Figure 1 and Figure 2. Comparing the two figures we see that $\mathbf{Y}_{t,Q}$ is generated by

$$Y_{1,0}^{\pm 1},\ Y_{1,-2}^{\pm 1},\ Y_{1,-4}^{\pm 1},\ Y_{2,0}^{\pm 1},\ Y_{2,-2}^{\pm 1},\ Y_{2,-4}^{\pm 1},\ Y_{3,1}^{\pm 1},\ Y_{3,-1}^{\pm 1},\ Y_{3,-3}^{\pm 1},\ Y_{4,2}^{\pm 1},\ Y_{4,0}^{\pm 1},\ Y_{4,-2}^{\pm 1}.$$

3.4 The quantum torus $\mathbf{T}_{v,Q}$

Let w_0 be the longest element of W. Let $\mathbf{i} = (i_1, \ldots, i_r)$ be a reduced expression of w_0 adapted to Q (see §2.2). Following [**GLS**, §11], we introduce a quantum torus $\mathbf{T}_{v,Q}$ of rank r over $\mathbb{C}(v)$. (The indeterminate v is denoted by q in [**GLS**]). Its generators are certain unipotent quantum minors

$$D_{\varpi_{i_k}, \lambda_k}, \qquad (1 \le k \le r)$$

in the quantum coordinate ring $A_v(\mathfrak{n})$. Here λ_k is the weight given by

$$\lambda_k = s_{i_1} \cdots s_{i_k}(\varpi_{i_k}), \qquad (1 \le k \le r).$$

The definition of $A_v(\mathfrak{n})$ will be recalled in §4.1 below. At this stage we only need to know the explicit v-commutation relations satisfied by these minors. It is shown in [**GLS**, Lemma 11.2] that for $k < l$ there holds

$$D_{\varpi_{i_k},\lambda_k} D_{\varpi_{i_l},\lambda_l} = v^{(\varpi_{i_k} - \lambda_k,\ \varpi_{i_l} + \lambda_l)} D_{\varpi_{i_l},\lambda_l} D_{\varpi_{i_k},\lambda_k}. \tag{12}$$

For $1 \le k \le r$, set $k^- := \max(\{s < k \mid i_s = i_k\} \cup \{0\})$. Define

$$Z_k := D_{\varpi_{i_k},\lambda_k} \left(D_{\varpi_{i_{k^-}},\lambda_{k^-}} \right)^{-1}, \tag{13}$$

where if $k^- = 0$ we understand $D_{\varpi_{i_{k^-}},\lambda_{k^-}} = 1$. Clearly, Z_k $(1 \le k \le r)$ is another set of generators of $\mathbf{T}_{v,Q}$. Let

$$\beta_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}), \qquad (1 \le k \le r). \tag{14}$$

Note that we have

$$\lambda_k = \lambda_{k^-} - \beta_k, \qquad (1 \le k \le r), \tag{15}$$

where if $k^- = 0$ we use the convention $\lambda_{k^-} = \varpi_{i_k}$.

Proposition 3.5 *For $1 \le k < l \le r$, we have:*

$$Z_k Z_l = v^{-(\beta_k, \beta_l)} Z_l Z_k. \tag{16}$$

Proof — Let us introduce the integers μ_{kl} and ν_{kl} such that

$$D_{\varpi_{i_k},\lambda_k} D_{\varpi_{i_l},\lambda_l} = v^{\mu_{kl}} D_{\varpi_{i_l},\lambda_l} D_{\varpi_{i_k},\lambda_k}, \qquad Z_k Z_l = v^{\nu_{kl}} Z_l Z_k, \qquad (1 \le k,\ l \le r).$$

By definition of Z_k we have

$$\nu_{kl} = (\mu_{kl} - \mu_{k^-l}) - (\mu_{kl^-} - \mu_{k^-l^-}),$$

where we use the convention that $\mu_{k^-l} = 0$ if $k^- = 0$, $\mu_{kl^-} = 0$ if $l^- = 0$, and $\mu_{k^-l^-} = 0$ if $k^- = 0$ or $l^- = 0$. Since $k^- < k < l$, we have

$$\mu_{kl} - \mu_{k^-l} = (\varpi_{i_k} - \lambda_k,\ \varpi_{i_l} + \lambda_l) - (\varpi_{i_k} - \lambda_{k^-},\ \varpi_{i_l} + \lambda_l) = (\beta_k, \varpi_{i_l} + \lambda_l). \tag{17}$$

(a) If $k < l^-$ we have similarly

$$\mu_{kl^-} - \mu_{k^-l^-} = ((\beta_k, \varpi_{i_l} + \lambda_{l^-})$$

and so

$$\nu_{kl} = (\beta_k, \varpi_{i_l} + \lambda_l) - (\beta_k, \varpi_{i_l} + \lambda_{l^-}) = -(\beta_k, \beta_l),$$

as required.

(b) If $k = l^-$ then $\mu_{kl^-} = 0$ and $\mu_{k^-l} = (\varpi_{i_k} - \lambda_{k^-},\ \varpi_{i_k} + \lambda_k)$. Hence

$$\begin{aligned}
\nu_{kl} &= (\beta_k, \varpi_{i_l} + \lambda_l) + (\varpi_{i_k} - \lambda_{k^-},\ \varpi_{i_k} + \lambda_k) \\
&= (\beta_k, \varpi_{i_k} + \lambda_k) - (\beta_k, \beta_l) + (\varpi_{i_k} - \lambda_{k^-},\ \varpi_{i_k} + \lambda_k) \\
&= -(\beta_k, \beta_l) + (\varpi_{i_k} - \lambda_{k^-} + \beta_k,\ \varpi_{i_k} + \lambda_k) \\
&= -(\beta_k, \beta_l) + (\varpi_{i_k} - \lambda_k,\ \varpi_{i_k} + \lambda_k) \\
&= -(\beta_k, \beta_l)
\end{aligned}$$

as required, because $(\varpi_{i_k} - \lambda_k,\ \varpi_{i_k} + \lambda_k) = (\varpi_{i_k}, \varpi_{i_k}) - (\lambda_k, \lambda_k) = 0$.

(c) If $k > l^-$ then $\mu_{kl^-} = -\mu_{l^-k} = -(\varpi_{i_l} - \lambda_{l^-},\ \varpi_{i_k} + \lambda_k)$. Hence

$$\begin{aligned}
-\mu_{kl^-} + \mu_{k^-l^-} &= (\varpi_{i_l} - \lambda_{l^-},\ \varpi_{i_k} + \lambda_k) + (\varpi_{i_k} - \lambda_{k^-},\ \varpi_{i_l} + \lambda_{l^-}) \\
&= -(\beta_k,\ \varpi_{i_l} + \lambda_{l^-}) + (\varpi_{i_l} - \lambda_{l^-},\ \varpi_{i_k} + \lambda_k) + (\varpi_{i_k} - \lambda_k,\ \varpi_{i_l} + \lambda_{l^-}) \\
&= -(\beta_k,\ \varpi_{i_l} + \lambda_{l^-}) + 2(\varpi_{i_k}, \varpi_{i_l}) - 2(\lambda_k, \lambda_{l^-}) \\
&= -(\beta_k,\ \varpi_{i_l} + \lambda_{l^-}).
\end{aligned}$$

Indeed, since $l^- < k < l$, we have

$$(\lambda_k, \lambda_{l^-}) = (s_{i_{l^-+1}} \cdots s_{i_k}(\varpi_{i_k}),\ \varpi_{i_l}) = (\varpi_{i_k}, s_{i_k} \cdots s_{i_{l^-+1}}(\varpi_{i_l})) = (\varpi_{i_k}, \varpi_{i_l}).$$

It follows that again, $\nu_{kl} = (\beta_k, \varpi_{i_l} + \lambda_l) - (\beta_k, \varpi_{i_l} + \lambda_{l^-}) = -(\beta_k, \beta_l)$, as required. □

3.5 An isomorphism

It is well known that the roots β_k $(1 \le k \le r)$ give an enumeration of Δ_+. Therefore, for every $(i,p) \in \widehat{I}_Q$ there is a unique k such that $\varphi(i,p) = (\beta_k, 0)$.

Proposition 3.6 *The assignment*

$$t \mapsto v,$$

$$Y_{i,p} \mapsto Z_k, \quad \text{where } (i,p) \in \widehat{I}_Q, \text{ and } \varphi(i,p) = (\beta_k, 0),$$

extends to an isomorphism of quantum tori from $\mathbf{Y}_{t,Q}$ *to* $\mathbf{T}_{v,Q}$.

Proof — This follows immediately from Proposition 3.2 and Proposition 3.5 if we note that when $\varphi(i,p) = (\beta_k, 0)$ and $\varphi(j,s) = (\beta_l, 0)$, $p < s$ implies that $k > l$. □

3.6 The involution σ and the rescaled generators X_k

Let $\mathscr{T}_{v,Q} := \mathbb{C}(v^{1/2}) \otimes_{\mathbb{C}(v)} \mathbf{T}_{v,Q}$. For $\gamma = \sum_i c_i \alpha_i$ in the root lattice of $\mathfrak{g}$, we set

$$\deg \gamma := \sum_i c_i, \qquad N(\gamma) := \frac{(\gamma,\gamma)}{2} - \deg \gamma. \tag{18}$$

Following [**GLS**], we introduce an involution σ of $\mathscr{T}_{v,Q}$, defined as the $\mathbb{C}$-algebra anti-automorphism satisfying

$$\sigma(v^{1/2}) = v^{-1/2}, \qquad \sigma\left(D_{\varpi_{i_k},\lambda_k}\right) = v^{N(\varpi_{i_k}-\lambda_k)} D_{\varpi_{i_k},\lambda_k}. \tag{19}$$

We rescale the generators Z_k of $\mathbf{T}_{v,Q}$ by defining

$$X_k := \begin{cases} v^{N(\beta_k)/2} Z_k & \text{if } 1 \le k \le n, \\ v^{N(\beta_k)/2 + (\varpi_{i_k} - \lambda_{k-n},\, \beta_k)} Z_k & \text{if } n+1 \le k \le r. \end{cases} \tag{20}$$

Note that these elements live in $\mathscr{T}_{v,Q}$.

Lemma 3.7 *For* $1 \le k \le r$ *we have:*

$$\sigma(X_k) = X_k.$$

Proof — For convenience, we set $\lambda_{k-n} = \varpi_{i_k}$ if $k - n \le 0$. Using (13), (12), and the definition of σ, we have

$$\sigma(Z_k) = v^{N(\varpi_{i_k}-\lambda_k) - N(\varpi_{i_k}-\lambda_{k-n}) - (\varpi_{i_k}-\lambda_{k-n},\, \varpi_{i_k}+\lambda_k)} Z_k.$$

A simple calculation using (15) shows that

$$N(\varpi_{i_k}-\lambda_k) - N(\varpi_{i_k}-\lambda_{k-n}) - (\varpi_{i_k}-\lambda_{k-n},\, \varpi_{i_k}+\lambda_k) = N(\beta_k) + 2(\varpi_{i_k}-\lambda_{k-n},\, \beta_k),$$

and the lemma follows. □

Clearly, the rescaled generators X_k satisfy the same commutation relations as the Z_k. Therefore, if we define for $\mathbf{a} := (a_1, \ldots, a_r) \in \mathbb{Z}^r$,

$$X^{\mathbf{a}} := v^{\frac{1}{2}\sum_{i<j} a_i a_j (\beta_i, \beta_j)} X_1^{a_1} \cdots X_r^{a_r}, \tag{21}$$

we have by Proposition 3.5,

$$\sigma(X^{\mathbf{a}}) = v^{-\frac{1}{2}\sum_{i<j} a_i a_j (\beta_i, \beta_j)} X_r^{a_r} \cdots X_1^{a_1} = X^{\mathbf{a}}. \tag{22}$$

Thus, $X^{\mathbf{a}}$ is σ-invariant, and more generally an element of $\mathscr{T}_{v,Q}$ is σ-invariant if and only if all the coefficients of its expansion with respect to the basis $\{X^{\mathbf{a}} \mid \mathbf{a} \in \mathbb{Z}^r\}$ are invariant under the map $v^{1/2} \mapsto v^{-1/2}$. Moreover, one checks easily that

$$X^{\mathbf{a}} X^{\mathbf{b}} = v^{\frac{1}{2}\sum_{i<j}(a_j b_i - a_i b_j)(\beta_i,\beta_j)} X^{\mathbf{a}+\mathbf{b}} = v^{\sum_{i<j}(a_j b_i - a_i b_j)(\beta_i,\beta_j)} X^{\mathbf{b}} X^{\mathbf{a}}. \tag{23}$$

3.7 The isomorphism Φ

We can now state the main result of this section, which follows immediately from Proposition 3.6 and Equations (11), (23).

Proposition 3.8 *There is a $\mathbb{C}$-algebra isomorphism $\Phi\colon \mathscr{Y}_{t,Q} \to \mathscr{T}_{v,Q}$ given by*

$$\Phi(t^{1/2}) = v^{1/2}, \qquad \Phi(Y_{i,p}) = X_k \quad \textit{for} \quad (i,p) \in \widehat{I}_Q \quad \textit{and} \quad \varphi(i,p) = (\beta_k, 0).$$

More generally, let

$$m = \prod_{(i,p)\in\widehat{I}_Q} Y_{i,p}^{u_{i,p}(m)}$$

be a commutative *monomial in $\mathscr{Y}_{t,Q}$, as in §3.2, and let $\mathbf{a} = (a_1,\dots,a_r)$ where $a_k = u_{(i,p)}(m)$ if $\varphi(i,p) = (\beta_k, 0)$. Then we have*

$$\Phi(m) = X^{\mathbf{a}}.$$

□

4 Quantum groups

4.1 Background

Let $\mathfrak{n}$ denote a maximal nilpotent subalgebra of $\mathfrak{g}$. Let $U_v(\mathfrak{n})$ be the Drinfeld-Jimbo quantum enveloping algebra of $\mathfrak{n}$ over $\mathbb{C}(v)$, with Chevalley generators e_i $(i \in I)$ subject to the quantum Serre relations:

$$\begin{aligned} &e_i e_j - e_j e_i = 0 && \text{if } C_{ij} = 0,\\ &e_i^2 e_j - (v+v^{-1}) e_i e_j e_i + e_j e_i^2 = 0 && \text{if } C_{ij} = -1. \end{aligned}$$

It is endowed with a natural scalar product $(\cdot,\cdot)$ which we normalize by $(e_i, e_i) = 1$ (see *e.g.* [**GLS**, §4.3]). We denote by $A_v(\mathfrak{n})$ the graded dual vector space of $U_v(\mathfrak{n})$. The map $x \mapsto (x,\cdot)$ is a vector space isomorphism from $U_v(\mathfrak{n})$ to $A_v(\mathfrak{n})$, which allows to define a multiplication on $A_v(\mathfrak{n})$ by transporting the multiplication of $U_v(\mathfrak{n})$.

Thus, $U_v(\mathfrak{n})$ and $A_v(\mathfrak{n})$ are isomorphic algebras, but they have dual integral forms and therefore they specialize differently at $v = 1$. One should regard $A_v(\mathfrak{n})$ as a quantum coordinate ring of the unipotent group N with Lie algebra $\mathfrak{n}$. For example, the elements $D_{\varpi_{i_k},\lambda_k}$ of §3.4 are quantum analogues of certain generalized minors on N. We set

$$\mathscr{U}_v(\mathfrak{n}) := \mathbb{C}(v^{1/2}) \otimes_{\mathbb{C}(v)} U_v(\mathfrak{n}), \qquad \mathscr{A}_v(\mathfrak{n}) := \mathbb{C}(v^{1/2}) \otimes_{\mathbb{C}(v)} A_v(\mathfrak{n}).$$

Since the basis involved in Theorem 1.2 (b) is the dual canonical basis $\mathbf{B}^*$, it is more natural to think of the quantum algebra of Theorem 1.2 (a) as being $\mathscr{A}_v(\mathfrak{n})$ rather than $\mathscr{U}_v(\mathfrak{n})$.

The algebra $U_v(\mathfrak{n})$ has a natural grading by the root lattice of $\mathfrak{g}$, given by $\deg(e_i) = \alpha_i$. The above isomorphism allows to transfer this grading to $A_v(\mathfrak{n})$.

4.2 Determinantal identities

In [**GLS**], it is shown that $A_v(\mathfrak{n})$ has a quantum cluster algebra structure. In particular, an explicit realization of $A_v(\mathfrak{n})$ as a subalgebra of the quantum torus $\mathbf{T}_{v,Q}$ is given. This goes as follows.

For $u,w \in W$ and $\lambda \in P_+$, one has unipotent quantum minors $D_{u(\lambda),w(\lambda)} \in A_v(\mathfrak{n})$ (see [**GLS**, §5.2]). They satisfy

$$D_{u(\lambda),w(\lambda)} = \begin{cases} 1 & \text{if } u(\lambda) = w(\lambda), \\ 0 & \text{if } u(\lambda) \not\leq w(\lambda). \end{cases}$$

Let $\mathbf{i} = (i_1,\ldots,i_r)$ be as in §3.4. In [**GLS**, §5.4], a system of identities relating the quantum minors

$$D(b,d) := D_{s_{i_1}\cdots s_{i_b}(\varpi_{i_b}),\ s_{i_1}\cdots s_{i_d}(\varpi_{i_b})}, \qquad (0 \leq b \leq d \leq r,\ \ i_b = i_d \in I), \tag{24}$$

is derived, which we now recall. By convention, we write $D(0,b) = D_{\varpi_{i_b},\ s_{i_1}\cdots s_{i_b}(\varpi_{i_b})}$. Note that the minors $D(0,b)$ $(1 \leq b \leq r)$ form by definition a system of generators of $\mathbf{T}_{v,Q}$. We will also use the following shorthand notation:

$$b^-(j) := \max\left(\{s < b \mid i_s = j\} \cup \{0\}\right), \tag{25}$$

$$b^- := \max\left(\{s < b \mid i_s = i_b\} \cup \{0\}\right), \tag{26}$$

$$\mu(b,j) := s_{i_1}\cdots s_{i_b}(\varpi_j). \tag{27}$$

In (27) we understand that $\mu(0,j) = \varpi_j$.

Proposition 4.1 ([GLS]) *Let $1 \leq b < d \leq r$ be such that $i_b = i_d = i$. There holds*

$$v^A D(b,d)D(b^-,d^-) = v^{-1+B} D(b,d^-)D(b^-,d) \; + \; v^C \overrightarrow{\prod_{j\sim i}} D(b^-(j),d^-(j)) \tag{28}$$

where

$$A = (\mu(d,i),\, \mu(b^-,i) - \mu(d^-,i)), \qquad B = (\mu(d^-,i),\, \mu(b^-,i) - \mu(d,i)),$$

and

$$C = \sum_{\substack{j<k \\ j\sim i,\ k\sim i}} (\mu(d,j),\, \mu(b,k) - \mu(d,k))\,.$$

This system of identities allows to express inductively every minor $D(b,d)$ as a rational function of the flag minors $D(0,c)$. Moreover, it follows from [**GLS**, Theorem 12.3] that all these rational functions belong in fact to $\mathbf{T}_{v,Q}$, and that $A_v(\mathfrak{n})$ is the subalgebra of $\mathbf{T}_{v,Q}$ generated by the minors $D(b^-,b)$ $(1 \leq b \leq r)$.

4.3 The dual canonical basis $\mathbf{B}^*$

Let us write

$$E^*(\beta_k) := D(k^-,k), \qquad (1 \leq k \leq r), \tag{29}$$

and for $\mathbf{a} = (a_1,\ldots,a_r) \in \mathbb{Z}^r$,

$$E^*(\mathbf{a}) = v^{-\sum_{k=1}^r a_k(a_k-1)/2} E^*(\beta_1)^{a_1} \cdots E^*(\beta_r)^{a_r}. \tag{30}$$

Then $\mathbf{E}^* = \{E^*(\mathbf{a}) \mid \mathbf{a} \in \mathbb{Z}^r\}$ is a $\mathbb{C}(v)$-basis of $A_v(\mathfrak{n})$, dual to a basis of $U_v(\mathfrak{n})$ of PBW-type, as defined by Lusztig. The basis $\mathbf{E}^*$ is called the *dual PBW-basis* of $A_v(\mathfrak{n})$.

The involution σ of $\mathscr{T}_{v,Q}$ (see §3.6) can be restricted to $A_v(\mathfrak{n})$. Lusztig [**Lu1**] has constructed a canonical basis $\mathbf{B}$ of $U_v(\mathfrak{n})$. The dual basis $\mathbf{B}^* = \{B^*(\mathbf{a}) \mid \mathbf{a} \in \mathbb{N}^r\}$ of $A_v(\mathfrak{n})$ can be characterized as follows (see *e.g.* [**GLS**]).

Proposition 4.2 *For* $\mathbf{a} = (a_1, \ldots, a_r) \in \mathbb{N}^r$*, the vector* $B^*(\mathbf{a})$ *is uniquely determined by the following conditions:*

(a) $B^*(\mathbf{a}) \in E^*(\mathbf{a}) + \sum_{\mathbf{c} \neq \mathbf{a}} v^{-1}\mathbb{Z}[v^{-1}]E^*(\mathbf{c})$*;*

(b) let $\beta(\mathbf{a}) := \sum_{1 \leq k \leq r} a_k \beta_k$*. Then* $\sigma(B^*(\mathbf{a})) = v^{N(\beta(\mathbf{a}))} B^*(\mathbf{a})$*.*

The integer $N(\gamma)$ of (b) is defined in (18). Note that $\beta(\mathbf{a})$ is just the weight of $B^*(\mathbf{a})$ or $E^*(\mathbf{a})$ in the natural grading of $A_v(\mathfrak{n})$ by the root lattice of $\mathfrak{g}$. The basis $\mathbf{B}^*$ is called the *dual canonical basis* of $A_v(\mathfrak{n})$.

5 Quantum Grothendieck rings

5.1 Background

For recent surveys on the representation theory of quantum loop algebras, we invite the reader to consult [**CH**] or [**Le**].

Let $L\mathfrak{g}$ be the loop algebra attached to $\mathfrak{g}$, and let $U_q(L\mathfrak{g})$ be the associated quantum enveloping algebra. We assume that the deformation parameter $q \in \mathbb{C}^*$ is not a root of unity.

By [**FR2**], every finite-dimensional $U_q(L\mathfrak{g})$-module M (of type 1) has a q-character $\chi_q(M)$. These q-characters generate a commutative $\mathbb{C}$-algebra isomorphic to the complexified Grothendieck ring of the category of finite-dimensional irreducible $U_q(L\mathfrak{g})$-modules. Nakajima [**N3**], Varagnolo and Vasserot [**VV1**], and Hernandez [**He2**], have studied t-deformations of the q-characters of the standard modules and of the simple modules, as well as corresponding t-deformations of the Grothendieck ring. Although slightly different, these t-deformed Grothendieck rings are essentially equivalent, and in particular they give rise to the same (q,t)-characters for the simple modules. In what follows, we will use the version of [**He2**]. Its definition will be recalled in the next sections.

5.2 The subcategory $\mathscr{C}_{\mathbb{Z}}$

The simple finite-dimensional irreducible $U_q(L\mathfrak{g})$-modules (of type 1) are usually labelled by Drinfeld polynomials. Here we shall use an alternative labelling by dominant monomials (see [**FR2**]). Moreover, as in [**HL**], we shall restrict our attention to a certain tensor subcategory $\mathscr{C}_{\mathbb{Z}}$ of the category of finite-dimensional $U_q(L\mathfrak{g})$-modules. The simple modules in $\mathscr{C}_{\mathbb{Z}}$ are labelled by the dominant monomials in $\mathscr{Y}$, or equivalently, by the dominant commutative monomials in $(\mathscr{Y}_{t^{1/2}}, *)$ (see §3.2), and their q-characters belong to $\mathscr{Y}$. We shall denote by $L(m)$ the simple module labelled by the dominant monomial m. When $m = Y_{i,p}$ is reduced to a single variable, $L(m)$ is called a *fundamental module*. When m is the only dominant monomial occuring in $\chi_q(L(m))$, $L(m)$ is said to be *minuscule*. Fundamental modules are examples of minuscule modules [**FM**].

5.3 Standard modules

To a dominant commutative monomial m is also attached a tensor product of fundamental modules called a *standard module* $M(m)$ defined by

$$M(m) := \overrightarrow{\bigotimes_{(i,p) \in \widehat{I}}} L(Y_{i,p})^{\otimes u_{i,p}(m)}, \tag{31}$$

where the product is ordered according to the following partial order on $\widehat{I}$:

$$(i,p) < (j,s) \iff p < s.$$

Note that for any fixed $p \in \mathbb{Z}$ and any total order on I, the tensor product

$$\overrightarrow{\bigotimes_{i\in I}} L(Y_{i,p})^{\otimes u_{i,p}(m)}$$

is irreducible, and its isomorphism class $L\left(\prod_{i\in I} Y_{i,p}^{k}\right)$ does not depend on the order of the factors, hence (31) is well defined up to isomorphism (see [**FM**, Proposition 6.15]). The classes $[M(m)]$ of the standard modules $M(m)$ form a second basis of the Grothendieck group of $\mathscr{C}_{\mathbb{Z}}$.

5.4 The ring $\mathscr{K}_t$

We introduce the commutative monomials [**FR2**]

$$A_{i,p} = Y_{i,p+1} Y_{i,p-1} \prod_{j\sim i} Y_{j,p}^{-1}, \qquad ((i,p-1) \in \widehat{I}). \tag{32}$$

Recall from §3.2 that commutative monomials in $\mathscr{Y}$ can be regarded as elements of $(\mathscr{Y}_t, *)$. More generally, the commutative polynomials

$$Y_{i,p}\left(1 + A_{i,p+1}^{-1}\right) = Y_{i,p} \,+\, Y_{i,p+2}^{-1} \prod_{j\sim i} Y_{j,p+1}, \qquad ((i,p) \in \widehat{I})$$

can be regarded as elements of $(\mathscr{Y}_t, *)$. For $i \in I$, let $\mathscr{K}_{i,t}$ be the $\mathbb{C}(t^{1/2})$-subalgebra of $\mathscr{Y}_t$ (for the noncommutative product $*$) generated by

$$Y_{i,p}\left(1 + A_{i,p+1}^{-1}\right), \quad Y_{j,s}^{\pm 1}, \qquad \left((i,p),(j,s) \in \widehat{I},\ j \neq i\right).$$

(In [**He1**], $\mathscr{K}_{i,t}$ is identified with the kernel of a t-deformed screening operator.) Define

$$\mathscr{K}_t := \bigcap_{i\in I} \mathscr{K}_{i,t}.$$

It is shown in [**He2**] that an element of $\mathscr{K}_t$ is uniquely determined by the coefficients of its dominant monomials. Moreover, for any dominant monomial m, there is a unique $F(m) \in \mathscr{K}_t$ such that m occurs in $F(m)$ with multiplicity 1 and no other dominant monomial occurs in $F(m)$. These $F(m)$ form a $\mathbb{C}(t^{1/2})$-basis of $\mathscr{K}_t$.

5.5 Comparison with other t-deformations

The product $*$ used in this paper is the same as that of [**He2**], except that we have replaced t by t^{-1}. The product of [**He2**] is slightly different from the products of [**N3**] and [**VV1**] (see Remark 3.1). However, as shown in [**He2**, Proposition 3.16], for every $(i,p),(j,s) \in \widehat{I}$ the pairs $(Y_{i,p}, A_{j,s})$ and $(A_{i,p}, A_{j,s})$ are t-commutative with the *same exponents of t* for the three products of [**N3**, **VV1**, **He2**]. This implies that the t-deformations of the Grothendieck ring $\mathscr{R}$ of $\mathscr{C}_{\mathbb{Z}}$ associated with the three products are essentially equivalent, as will be explained below.

5.6 (q,t)-characters of standard modules

For a dominant commutative monomial $m \in \mathscr{Y}_{t^{1/2}}$, define

$$[M(m)]_t := t^{\alpha(m)} \overleftarrow{\underset{p\in\mathbb{Z}}{*}} F\left(\prod_{i\in I} Y_{i,p}^{u_{i,p}(m)}\right) \in \mathscr{K}_t. \tag{33}$$

Here, $\alpha(m) \in \frac{1}{2}\mathbb{Z}$ is chosen so that m occurs with coefficient 1 in the expansion of $[M(m)]_t$ on the basis of commutative monomials of $\mathscr{Y}_{t^{1/2}}$. The coefficients of $[M(m)]_t$ on this basis belong to $\mathbb{Z}[t^{\pm 1}]$ and may therefore be specialized at $t = 1$. The obtained specialization of $[M(m)]_t$ at $t = 1$ is equal to $\chi_q(M(m))$, the q-character of the standard module $M(m)$. Therefore we may use the alternative notation

$$\chi_{q,t}(M(m)) := [M(m)]_t,$$

and call this element of $\mathscr{K}_t$ the *(q,t)-character of $M(m)$*.

5.7 The bar involution

One shows that there is a unique $\mathbb{C}$-algebra anti-automorphism of $(\mathscr{Y}_t, *)$ such that

$$\overline{t^{1/2}} = t^{-1/2}, \quad \overline{Y_{i,p}} = Y_{i,p}, \qquad ((i,p) \in \widehat{I}).$$

Clearly, the *commutative* monomials are bar-invariant, as in [**N3**, **VV1**]. The subring $\mathscr{K}_t$ is stable under the bar involution, since each $\mathscr{K}_{i,t}$ is obviously stable. It follows that the elements $F(m)$ are bar-invariant (since m is the unique dominant monomial of $\overline{F(m)}$). Hence the coefficients of the expansion of $F(m)$ on the basis of commutative monomials are unchanged under the replacement of t by t^{-1}. Therefore, $F(m)$ is the same as in [**He2**]. Since we have used in (33) the reverse product $\overleftarrow{*}$, the elements $\chi_{q,t}(M(m))$ also coincide with the corresponding elements of [**N3**, **VV1**, **He2**], *i.e.* the coefficients of their expansion on the basis of commutative monomials are the same.

5.8 (q,t)-characters of simple modules

Proposition 5.1 ([N3]) *For every dominant monomial m, there is a unique element $[L(m)]_t$ of $\mathscr{K}_t$ satisfying*

(a) $\overline{[L(m)]_t} = [L(m)]_t$,

(b) $[L(m)]_t \in [M(m)]_t + \sum_{m'<m} t^{-1}\mathbb{Z}[t^{-1}]\,[M(m')]_t$.

Here $m' \leq m$ means that $m(m')^{-1}$ is a product of elements $A_{i,p}$ in $\mathscr{Y}$.

By §5.7, the elements $[L(m)]_t$ coincide with the corresponding elements of [**N3**, **VV1**, **He2**]. Using the geometry of quiver varieties, Nakajima has shown:

Theorem 5.2 ([N3]) *The specialization of $[L(m)]_t$ at $t = 1$ is equal to $\chi_q(L(m))$, and the coefficients of the expansion of $[L(m)]_t$ as a linear combination of monomials in the $Y_{i,p}$'s belong to $\mathbb{N}[t^{\pm 1}]$.*

Therefore we may use the alternative notation

$$\chi_{q,t}(L(m)) := [L(m)]_t,$$

and call this element of $\mathscr{K}_t$ the *(q,t)-character of $L(m)$*.

Corollary 5.3 *(a) If $L(m)$ is minuscule, $\chi_{q,t}(L(m)) = F(m)$.*

(b) If $\chi_q(L(m))$ is multiplicity-free, then $\chi_{q,t}(L(m)) = \chi_q(L(m))$ does not depend on t when expressed on the basis of commutative monomials.

Proof — By the positivity statement of Theorem 5.2, every monomial occuring in $\chi_{q,t}(L(m))$ already occurs in $\chi_q(L(m))$. Thus, if $L(m)$ is minuscule then $\chi_{q,t}(L(m))$ is an element of $\mathscr{K}_t$ containing the unique dominant monomial m, which proves (a). If $\chi_q(L(m))$ is multiplicity-free, then the coefficient of every commutative monomial in $\chi_{q,t}(L(m))$ is of the form t^k for some $k \in \mathbb{Z}$. But since $\chi_{q,t}(L(m))$ is bar-invariant, we must have $k = 0$, which proves (b). □

5.9 Multiplicative structure

We shall regard the noncommutative ring $(\mathscr{K}_t, *)$ as a *t-deformed version of the Grothendieck ring $\mathscr{R}$*. But one should be aware that only the simple modules $L(m)$ and the standard modules $M(m)$ have well-defined "classes" $\chi_{q,t}(L(m))$ and $\chi_{q,t}(M(m))$ in $\mathscr{K}_t$.

For any dominant monomials m_1 and m_2, write

$$\chi_{q,t}(L(m_1)) * \chi_{q,t}(L(m_2)) = \sum_m c^m_{m_1,m_2}(t^{1/2})\, \chi_{q,t}(L(m)).$$

Note that every irreducible (q,t)-character is of the form $\chi_{q,t}(L(m)) = m(1 + \sum_k M_k)$, where the M_k are monomials in the $A^{-1}_{i,p}$ with coefficients in $\mathbb{N}[t, t^{-1}]$ (see [**He2**]). So, by §5.5, the above coefficients $c^m_{m_1,m_2}(t^{1/2})$ are obtained from the corresponding ones in [**N3**, **VV1**] by multiplying by some t^k with $k \in \mathbb{Z}/2$. Varagnolo and Vasserot have shown the following positivity result:

Theorem 5.4 ([VV1]) *The structure constants $c^m_{m_1,m_2}(t^{1/2})$ belong to $\mathbb{N}[t^{1/2}, t^{-1/2}]$.*

Corollary 5.5 *$L(m_1) \otimes L(m_2) \simeq L(m)$ is a simple module if and only if*

$$\chi_{q,t}(L(m_1)) * \chi_{q,t}(L(m_2)) = t^{2k} \chi_{q,t}(L(m_2)) * \chi_{q,t}(L(m_1)) = t^k \chi_{q,t}(L(m))$$

for some $k \in \mathbb{Z}/2$.

Proof — If $L(m_1) \otimes L(m_2) \simeq L(m)$ then $\chi_q(L(m_1)) * \chi_q(L(m_2)) = \chi_q(L(m))$. Hence $c^m_{m_1,m_2}(1) = 1$, and it follows from Theorem 5.4 that $\chi_{q,t}(L(m_1)) * \chi_{q,t}(L(m_2)) = t^k \chi_{q,t}(L(m))$ for some $k \in \mathbb{Z}/2$. Applying the bar involution, we get $\chi_{q,t}(L(m_2)) * \chi_{q,t}(L(m_1)) = t^{-k} \chi_{q,t}(L(m))$. If conversely $\chi_{q,t}(L(m_1)) * \chi_{q,t}(L(m_2)) = t^k \chi_{q,t}(L(m))$, then specializing t to 1 we get

$$\chi_q(L(m_1))\chi_q(L(m_2)) = \chi_q(L(m_1) \otimes L(m_2)) = \chi_q(L(m)),$$

hence $L(m_1) \otimes L(m_2) \simeq L(m)$. □

5.10 Quantum T-system

For $(i,p)\in\widehat{I}$ and $k\in\mathbb{N}$, let $m^{(i)}_{k,p} := Y_{i,p}Y_{i,p+2}\cdots Y_{i,p+2k-2}$. The simple $U_q(L\mathfrak{g})$-module

$$W^{(i)}_{k,p} := L\left(m^{(i)}_{k,p}\right)$$

is called a *Kirillov-Reshetikhin module*. (By convention, if $k=0$ then $W^{(i)}_{k,p}$ is the trivial one-dimensional module.) The q-characters of the Kirillov-Reshetikhin modules satisfy the following system of algebraic identities called *T-system* [**KNS**, **N2**, **He3**]. For every $(i,p)\in\widehat{I}$ and $k>0$, there holds

$$\chi_q\left(W^{(i)}_{k,p}\right)\chi_q\left(W^{(i)}_{k,p+2}\right) = \chi_q\left(W^{(i)}_{k-1,p+2}\right)\chi_q\left(W^{(i)}_{k+1,p}\right) + \prod_{j\neq i}\chi_q\left(W^{(j)}_{k,p+1}\right)^{-C_{ij}}.$$

This can be lifted to a t-deformed T-system in $\mathscr{K}_t$, as shown by the next proposition (see also [**N2**, §4], where a different t-deformed product is used, as explained in Remark 3.1 and §5.5). Before stating it, we note that $\bigotimes_{j\sim i} W^{(j)}_{k,p+1}$ is a simple module, hence by Corollary 5.5 the (q,t)-characters $\chi_{q,t}(W^{(j)}_{k,p+1})$ pairwise t-commute in $\mathscr{K}_t$. Moreover, it is easy to check that, since $\widetilde{C}(z)$ is symmetric, for any $j\sim i$ and $j'\sim i$, one has $m^{(j)}_{k,p+1} * m^{(j')}_{k,p+1} = m^{(j')}_{k,p+1} * m^{(j)}_{k,p+1}$, hence the (q,t)-characters $\chi_{q,t}(W^{(j)}_{k,p+1})$ do in fact pairwise commute in $\mathscr{K}_t$. So we may write $\mathop{*}_{j\sim i}\chi_{q,t}\left(W^{(j)}_{k,p+1}\right)$ without specifying an ordering of the factors.

Proposition 5.6 *In $\mathscr{K}_t$ there holds:*

$$\chi_{q,t}\left(W^{(i)}_{k,p}\right) * \chi_{q,t}\left(W^{(i)}_{k,p+2}\right) = t^{\alpha(i,k)}\chi_{q,t}\left(W^{(i)}_{k-1,p+2}\right) * \chi_{q,t}\left(W^{(i)}_{k+1,p}\right) + t^{\gamma(i,k)} \mathop{*}_{j\sim i}\chi_{q,t}\left(W^{(j)}_{k,p+1}\right),$$

where

$$\alpha(i,k) = -1 + \frac{1}{2}\left(\widetilde{C}_{ii}(2k-1) + \widetilde{C}_{ii}(2k+1)\right), \qquad \gamma(i,k) = \alpha(i,k)+1. \tag{34}$$

Proof — Using Theorem 5.4, we see that the claimed identity holds for some integers $\alpha(i,k)$ and $\gamma(i,k)$, and we only have to check (34). To do so it is enough to compare the coefficients of some particular monomials on both sides. We have $m^{(i)}_{k,p} * m^{(i)}_{k,p+2} = t^{\alpha} m^{(i)}_{k-1,p+2} * m^{(i)}_{k+1,p}$, where

$$\begin{aligned}
\alpha &= \sum_{a=1}^{k-1}\mathscr{N}(i,p;i,p+2a) + \frac{1}{2}\mathscr{N}(i,p;i,p+2k)\\
&= \sum_{a=1}^{k-1}\left(\widetilde{C}_{ii}(2a+1) - \widetilde{C}_{ii}(2a-1)\right) + \frac{1}{2}\left(\widetilde{C}_{ii}(2k+1) - \widetilde{C}_{ii}(2k-1)\right)\\
&= -\widetilde{C}_{ii}(1) + \frac{1}{2}\left(\widetilde{C}_{ii}(2k-1) + \widetilde{C}_{ii}(2k+1)\right).
\end{aligned}$$

Thus $\alpha(i,k) = \alpha = -1 + (\widetilde{C}_{ii}(2k-1) + \widetilde{C}_{ii}(2k+1))/2$, as claimed.

Similarly, $\chi_q(W^{(i)}_{k,p})$ contains the monomial $m := m^{(i)}_{k,p}A^{-1}_{i,p+2k-1}\cdots A^{-1}_{i,p+3}A^{-1}_{i,p+1}$ with coefficient 1, and we have $m\, m^{(i)}_{k,p+2} = \prod_{j\sim i} m^{(j)}_{k,p+1}$. Now

$$m * m^{(i)}_{k,p+2} = \left(\left(m^{(i)}_{k,p+2}\right)^{-1}\prod_{j\sim i} m^{(j)}_{k,p+1}\right) * m^{(i)}_{k,p+2} = t^{\gamma}\prod_{j\sim i} m^{(j)}_{k,p+1},$$

where

$$\begin{aligned}
\gamma &= \frac{1}{2}\sum_{j\sim i}\sum_{a=1}^{k}\sum_{b=1}^{k}\mathscr{N}(j,p+2a-1;i,p+2b) \\
&= \frac{1}{2}\sum_{j\sim i}\sum_{a=1}^{k}\sum_{b=1}^{k}\Big(\widetilde{C}_{ji}(2(a-b)-2)-\widetilde{C}_{ji}(2(a-b))-\widetilde{C}_{ji}(2(b-a))+\widetilde{C}_{ji}(2(b-a)+2)\Big) \\
&= \frac{1}{2}\sum_{j\sim i}\sum_{a=1}^{k}\Big(\widetilde{C}_{ji}(2(a-k)-2)-\widetilde{C}_{ji}(2(a-1))-\widetilde{C}_{ji}(2(1-a))+\widetilde{C}_{ji}(2(k-a)+2)\Big) \\
&= \frac{1}{2}\sum_{j\sim i}\sum_{a=1}^{k}\Big(-\widetilde{C}_{ji}(2(a-1))+\widetilde{C}_{ji}(2(k-a)+2)\Big) \\
&= \frac{1}{2}\sum_{j\sim i}\widetilde{C}_{ji}(2k).
\end{aligned}$$

Thus $\gamma(i,k)=\gamma=\Big(\sum_{j\sim i}\widetilde{C}_{ji}(2k)\Big)/2=\Big(\sum_{j\sim i}\widetilde{C}_{ij}(2k)\Big)/2=(\widetilde{C}_{ii}(2k-1)+\widetilde{C}_{ii}(2k+1))/2$, as claimed. Here, the last equality comes from the definition of $\widetilde{C}(z)$ (see the proof of Proposition 2.5). □

Example 5.7 (a) Take $\mathfrak{g}$ of type A_1. We have

$$\widetilde{C}(z)=z-z^3+z^5-z^7+z^9-\cdots,$$

hence $\alpha(k)=-1$ for all $k>0$. Thus we get

$$\chi_{q,t}\left(W_{k,p}\right)*\chi_{q,t}\left(W_{k,p+2}\right)=t^{-1}\chi_{q,t}\left(W_{k-1,p+2}\right)*\chi_{q,t}\left(W_{k+1,p}\right)+1.$$

(b) Take $\mathfrak{g}$ of type A_3. Choose $i=1$, $k=1$, and $p=0$. Using for example Proposition 2.5, we can calculate

$$\widetilde{C}_{11}(z)=z-z^7+z^9-z^{15}+\cdots,\qquad \widetilde{C}_{12}(z)=z^2-z^6+z^{10}-z^{14}+\cdots,$$

hence

$$\alpha(1,1)=-1+\frac{1}{2}\Big(\widetilde{C}_{11}(1)+\widetilde{C}_{11}(3)\Big)=-\frac{1}{2},\qquad \gamma(1,1)=\frac{1}{2}\widetilde{C}_{12}(2)=\frac{1}{2}.$$

Thus Proposition 5.6 gives

$$\chi_{q,t}\left(W_{1,0}^{(1)}\right)*\chi_{q,t}\left(W_{1,2}^{(1)}\right)=t^{-1/2}\chi_{q,t}\left(W_{2,0}^{(1)}\right)+t^{1/2}\chi_{q,t}\left(W_{1,1}^{(2)}\right).$$

5.11 The subcategory $\mathscr{C}_Q$

Recall the quantum torus $\mathscr{Y}_{t,Q}$ of §3.3. The dominant commutative monomials in $\mathscr{Y}_{t,Q}$ parametrize the simple objects of an abelian subcategory $\mathscr{C}_Q$ of $\mathscr{C}_{\mathbb{Z}}$. More precisely, we define $\mathscr{C}_Q$ as the full subcategory of $\mathscr{C}_{\mathbb{Z}}$ whose objects have all their composition factors of the form $L(m)$ where m is a dominant commutative monomial in $\mathscr{Y}_{t,Q}$. When Q is a sink-source orientation of the Dynkin diagram and the Coxeter number h is even, $\mathscr{C}_Q$ is one of the subcategories $\mathscr{C}_\ell$ introduced in [**HL**]; namely, $\mathscr{C}_Q=\mathscr{C}_{h'}$ where $h'=h/2-1$.

Lemma 5.8 *$\mathscr{C}_Q$ is closed under tensor products, hence is a tensor subcategory of $\mathscr{C}_{\mathbb{Z}}$.*

Proof — This is a slight modification of the proof of [**HL**, Proposition 3.2]. Let $L(m)$ and $L(m')$ be in $\mathscr{C}_Q$. This means that m and m' are monomials in the variables $Y_{i,p}$, $(i,p) \in \hat{I}_Q$. If $L(m'')$ is a composition factor of $L(m) \otimes L(m')$ then m'' is a product of monomials of $\chi_q(L(m))$ and $\chi_q(L(m'))$. So we have $m'' = mm'M$ where M is a monomial in the $A_{j,r}^{-1}$. Then it is checked as in [**HL**, Section 5.2.4] that, for m'' to be dominant, these (j,r) have to satisfy $(j,r-1) \in \hat{I}_Q$ and $(j,r+1) \in \hat{I}_Q$. It follows that m'' depends only on the variables $Y_{i,p}$, $(i,p) \in \hat{I}_Q$, because $\widehat{I}_Q$ is a "convex slice" of $\widehat{I}$, that is, it satisfies:

(i) if $(i,p), (i,p+2k) \in \widehat{I}_Q$ for $i \in I$, $p \in \mathbb{Z}$, $k > 0$, then $(i,p+2j) \in \widehat{I}_Q$ for $1 \leq j \leq k-1$;

(ii) if $(i,p), (i,p+2) \in \widehat{I}_Q$ for $i \in I$, $p \in \mathbb{Z}$, then for every $j \sim i$ we have $(j,p+1) \in \widehat{I}_Q$.

Hence the result. □

Example 5.9 We continue Example 3.4. We take $\mathfrak{g}$ of type D_4 and choose Q as in Figure 1. The simple objects of $\mathscr{C}_Q$ are of the form $L(m)$, where

$$m = Y_{1,0}^{u_{1,0}} Y_{1,-2}^{u_{1,-2}} Y_{1,-4}^{u_{1,-4}} Y_{2,0}^{u_{2,0}} Y_{2,-2}^{u_{2,-2}} Y_{2,-4}^{u_{2,-4}} Y_{3,1}^{u_{3,1}} Y_{3,-1}^{u_{3,-1}} Y_{3,-3}^{u_{3,-3}} Y_{4,2}^{u_{4,2}} Y_{4,0}^{u_{4,0}} Y_{4,-2}^{u_{4,-2}}.$$

and $u_{i,p} \in \mathbb{N}$.

5.12 The ring $\mathscr{K}_{t,Q}$ and the truncated (q,t)-characters

We denote by $\mathscr{K}_{t,Q}$ the $\mathbb{C}(t^{1/2})$-subalgebra of $\mathscr{K}_t$ spanned by the (q,t)-characters $\chi_{q,t}(L(m))$ of the simple objects $L(m)$ in $\mathscr{C}_Q$. We call $\mathscr{K}_{t,Q}$ the *t-deformed Grothendieck ring of $\mathscr{C}_Q$*.

The (q,t)-character of a simple object $L(m)$ of $\mathscr{C}_Q$ contains in general many monomials m' which do not belong to $\mathscr{Y}_{t,Q}$. By discarding these monomials we obtain a *truncated (q,t)-character*. We shall denote by $\widetilde{\chi}_{q,t}(L(m))$ the truncated (q,t)-character of $L(m)$. One checks that for a simple object $L(m)$ of $\mathscr{C}_Q$, all the dominant monomials occuring in $\chi_{q,t}(L(m))$ belong to the truncated (q,t)-character $\widetilde{\chi}_{q,t}(L(m))$ (the proof is similar to that of [**HL**] for the category $\mathscr{C}_1$, as for the proof of Lemma 5.8 above). Therefore the truncation map

$$\chi_{q,t}(L(m)) \mapsto \widetilde{\chi}_{q,t}(L(m))$$

extends to an injective algebra homomorphism from $\mathscr{K}_{t,Q}$ to $\mathscr{Y}_{t,Q}$. In the sequel we shall identify $\mathscr{K}_{t,Q}$ with the subalgebra of $\mathscr{Y}_{t,Q}$ given by the image of this homomorphism.

6 An isomorphism between quantum Grothendieck rings and quantum groups

6.1 The isomorphism between $\mathscr{K}_{t,Q}$ and $\mathscr{A}_v(\mathfrak{n})$

Recall the isomorphism $\Phi\colon \mathscr{Y}_{t,Q} \to \mathscr{T}_{v,Q}$ of Proposition 3.8, and the notation

$$\mathscr{A}_v(\mathfrak{n}) := \mathbb{C}(v^{1/2}) \otimes A_v(\mathfrak{n}).$$

Define the *rescaled dual canonical basis* of $\mathscr{A}_v(\mathfrak{n})$:

$$\widetilde{\mathbf{B}}^* := \left\{ \widetilde{B}^*(\mathbf{a}) := v^{N(\beta(\mathbf{a}))/2} B^*(\mathbf{a}) \mid B^*(\mathbf{a}) \in \mathbf{B}^* \right\}.$$

Clearly, the elements of $\widetilde{\mathbf{B}}^*$ are invariant under the involution σ. The next theorem is Theorem 1.2 in a slightly more precise formulation.

Theorem 6.1 *(a) Φ restricts to an isomorphism*

$$\mathscr{K}_{t,Q} \xrightarrow{\sim} \mathscr{A}_v(\mathfrak{n}).$$

(b) The basis of $\mathscr{K}_{t,Q}$ consisting of the irreducible truncated (q,t)-characters $\widetilde{\chi}_{q,t}(L(m))$ is mapped by Φ onto $\widetilde{\mathbf{B}}^*$.

Proof — We introduce some necessary notation. For $1 \le k \le r$, let $k_{\min} := \min\{1 \le s \le r \mid i_s = i_k\}$. Set $k^{(0)} := k$ and, for a negative integer j, define $k^{(j)} = (k^{(j+1)})^-$, where the notation b^- is as in Equation (26). We also note that, by definition of λ_k and β_k, if $k^- \neq 0$ then $\tau^{-1}(\lambda_k) = \lambda_{k^-}$ and $\tau^{-1}(\beta_k) = \beta_{k^-}$.

Let us fix some $(i,p) \in \widehat{I}_Q$. By definition of Φ, we have:

$$\Phi(Y_{i,p}) = X_k \quad \text{for} \quad (i,p) \in \widehat{I}_Q \quad \text{and} \quad \varphi(i,p) = (\beta_k, 0).$$

Note that this relation between (i,p) and β_k implies in particular that $i_k = i$. Since if $k^- \neq 0$,

$$\varphi(i,p+2) = (\tau^{-1}(\beta_k),0) = (\beta_{k^-},0),$$

we deduce that $\Phi\left(Y_{i,p}Y_{i,p+2}Y_{i,p+4}\cdots Y_{i,\xi_i}\right)$ is equal up to a power of v to $X_k X_{k^-} X_{k^{(-2)}} \cdots X_{k_{\min}}$, that is, up to a power of v, to $D(0,k)$. Since the commutative monomial $Y_{i,p}Y_{i,p+2}\cdots Y_{i,\xi_i}$ is bar-invariant, its image is σ-invariant, so it has to be equal to $v^{N(\varpi_{i_k}-\lambda_k)/2}D(0,k)$. Now $Y_{i,p}Y_{i,p+2}\cdots Y_{i,\xi_i}$ is equal to the truncated (q,t)-character of the Kirillov-Reshetikhin module $W^{(i)}_{1+(\xi_i-p)/2,\,p}$. Hence we have shown that

$$\Phi\left(\widetilde{\chi}_{(q,t)}\left(W^{(i)}_{1+(\xi_i-p)/2,\,p}\right)\right) = v^{N(\varpi_{i_k}-\lambda_k)/2}D(0,k).$$

We now want to show that, more generally, for $1 \le s \le (\xi_i - p)/2 + 1$ we have

$$\Phi\left(\widetilde{\chi}_{(q,t)}\left(W^{(i)}_{s,p}\right)\right) = v^{N(\lambda_{k^{(-s)}}-\lambda_k)/2}D(k^{(-s)},k). \tag{35}$$

This will be proved by comparing Proposition 4.1 and Proposition 5.6. Let us denote by

$$\widetilde{D}(b,d) := v^{N(\lambda_b-\lambda_d)/2}D(b,d)$$

the rescaled quantum minors. Note that

$$N(\lambda_b-\lambda_d) = \frac{1}{2}(\lambda_b-\lambda_d,\, \lambda_b-\lambda_d) - \deg(\lambda_b-\lambda_d) = (\lambda_b,\, \lambda_b-\lambda_d) - \deg(\lambda_b-\lambda_d).$$

We can rewrite Proposition 4.1 as

$$\widetilde{D}(b,d)\widetilde{D}(b^-,d^-) = v^X\,\widetilde{D}(b,d^-)\widetilde{D}(b^-,d) \;+\; v^Y \overrightarrow{\prod_{j\sim i}}\widetilde{D}(b^-(j),d^-(j)) \tag{36}$$

where

$$X := -1 + B - A + \frac{1}{2}\left((\lambda_b,\, \lambda_b-\lambda_d) + (\lambda_{b^-},\, \lambda_{b^-}-\lambda_{d^-}) - (\lambda_{b^-},\, \lambda_{b^-}-\lambda_d) - (\lambda_b,\, \lambda_b-\lambda_{d^-})\right),$$

and

$$Y := C - A + \frac{1}{2}\left((\lambda_b, \lambda_b - \lambda_d) + (\lambda_{b^-}, \lambda_{b^-} - \lambda_{d^-}) - \sum_{j\sim i} \left(\lambda_{b^-(j)}, \lambda_{b^-(j)} - \lambda_{d^-(j)}\right) \right).$$

Replacing A and B by their values from Proposition 4.1, and simplifying the resulting expression, we easily get

$$X = -1 + \frac{1}{2}(\lambda_b + \lambda_{b^-}, \lambda_{d^-} - \lambda_d).$$

Now, writing $i_b = i_d = i$ and $b = d^{(-s)}$,

$$(\lambda_b + \lambda_{b^-}, \lambda_{d^-} - \lambda_d) = (\lambda_b, \lambda_{d^-}) - (\lambda_{b^-}, \lambda_d) = (\varpi_i, \tau^{s-1}(\varpi_i)) - (\varpi_i, \tau^{s+1}(\varpi_i)).$$

Hence, using that $\tau^{s-1}(\varpi_i) - \tau^{s+1}(\varpi_i) = \tau^s(\gamma_i) + \tau^{s-1}(\gamma_i)$, by Proposition 2.1 we get

$$X = -1 + \frac{1}{2}\left(\widetilde{C}_{ii}(2s-1) + \widetilde{C}_{ii}(2s+1)\right).$$

Similarly, replacing A and C by their values from Proposition 4.1, and simplifying the resulting expression, we get

$$Y = (\varpi_i, \tau(\varpi_i) - \tau^{s+1}(\varpi_i)) + (\varpi_i, \varpi_i - \tau^s(\varpi_i)) - \frac{1}{2}\sum_{j\sim i}\sum_{k\sim i}(\tau^{(\xi_j-\xi_k)/2}\varpi_j, \varpi_k - \tau^s(\varpi_k)).$$

Using the identities

$$\varpi_i - \tau^s(\varpi_i) = \sum_{l=0}^{s-1}\tau^l(\gamma_i), \qquad \tau^l(\gamma_i) + \tau^{l+1}(\gamma_i) = \sum_{k\sim i}\tau^{l+(1+\xi_k-\xi_i)/2}(\gamma_k),$$

we get

$$(\varpi_i, \tau(\varpi_i) - \tau^{s+1}(\varpi_i)) + (\varpi_i, \varpi_i - \tau^s(\varpi_i)) = \sum_{k\sim i}\left(\tau^{(\xi_i-\xi_k-1)/2}(\varpi_i), \varpi_k - \tau^s(\varpi_k)\right),$$

hence,

$$Y = \frac{1}{2}\sum_{k\sim i}\left(\tau^{(\xi_i-\xi_k-1)/2}\left(2\varpi_i - \sum_{j\sim i}\tau^{(\xi_j-\xi_i+1)/2}(\varpi_j)\right), \varpi_k - \tau^s(\varpi_k)\right).$$

Now,

$$2\varpi_i - \sum_{j\sim i}\tau^{(\xi_j-\xi_i+1)/2}(\varpi_j) = 2\varpi_i - \sum_{j\sim i;\ \xi_j-\xi_i=1}\tau(\varpi_j) - \sum_{j\sim i;\ \xi_j-\xi_i=-1}\varpi_j = \alpha_i + \sum_{j\sim i;\ \xi_j-\xi_i=1}\gamma_j = \gamma_i.$$

Hence

$$\begin{aligned}
Y &= \frac{1}{2}\sum_{k\sim i}\left(\tau^{(\xi_i-\xi_k-1)/2}(\gamma_i), \varpi_k - \tau^s(\varpi_k)\right)\\
&= -\frac{1}{2}\sum_{k\sim i}\left(\tau^{-s+(\xi_i-\xi_k-1)/2}(\gamma_i), \varpi_k\right)\\
&= \frac{1}{2}\sum_{k\sim i}\left(\tau^{s+(\xi_i-\xi_k-1)/2}(\gamma_i), \varpi_k\right)\\
&= \frac{1}{2}\sum_{k\sim i}\widetilde{C}_{ik}(2s).
\end{aligned}$$

Therefore $X = \alpha(i,s)$, $Y = \gamma(i,s)$, and by Proposition 5.6 we see that, for any $(i,p') \in \widehat{I}_Q$, there holds in $\mathscr{K}_t$:

$$\chi_{q,t}\left(W^{(i)}_{s,p'}\right) * \chi_{q,t}\left(W^{(i)}_{s,p'+2}\right) = t^X \chi_{q,t}\left(W^{(i)}_{s-1,p'+2}\right) * \chi_{q,t}\left(W^{(i)}_{s+1,p'}\right) + t^Y \mathop{*}_{j\sim i} \chi_{q,t}\left(W^{(j)}_{s,p'+1}\right). \quad (37)$$

It was shown in [**GLS**] that one can express every quantum minor $D(b,d)$ as a (noncommutative) Laurent polynomial in the quantum flag minors $D(0,k)$, by means of an explicit sequence of applications of Proposition 4.1. Equivalently, every rescaled quantum minor $\widetilde{D}(b,d)$ can be expressed as a Laurent polynomial in the $\widetilde{D}(0,k)$'s with coefficients in $\mathbb{Z}[v^{\pm 1/2}]$, by means of an explicit sequence of applications of (36). By comparing (36) and (37), we see that the (q,t)-character of $W^{(i)}_{s,\,p}$ (where $\varphi(i,p) = (\beta_d, 0)$ and $b = d^{(-s)}$) can be expressed by the *same* Laurent polynomial in the (q,t)-characters

$$\chi_{q,t}\left(W^{(j)}_{1+(\xi_j-p')/2,\,p'}\right), \qquad ((j,p') \in \widehat{I}_Q),$$

where $v^{1/2}$ is replaced by $t^{1/2}$. This proves (35). In particular, we have

$$\Phi\left(\widetilde{\chi}_{q,t}\left(L(Y_{i,p})\right)\right) = \Phi\left(\widetilde{\chi}_{q,t}\left(W^{(i)}_{1,p}\right)\right) = \widetilde{D}(d^-,d) = v^{N(\beta_d)/2} E^*(\beta_d), \qquad ((i,p) \in \widehat{I}_Q).$$

Thus, Φ maps the truncated (q,t)-characters of the fundamental modules of $\mathscr{C}_Q$, that is, a set of algebra generators of $\mathscr{K}_{t,Q}$, to the rescaled dual PBW generators of $\mathscr{A}_v(\mathfrak{n})$. This proves (a).

It follows that Φ maps the truncated (q,t)-characters of the standard modules of $\mathscr{C}_Q$ to the elements of the dual PBW-basis of $\mathscr{A}_v(\mathfrak{n})$ up to some power of v. Let us calculate this power of v. By Proposition 5.1, we have that $[M(m)]_t - [L(m)]_t$ is a linear combination of $[L(m')]_t$ with coefficients in $t^{-1}\mathbb{Z}[t^{-1}]$, where $[L(m)]_t$ and the $[L(m')]_t$ are bar-invariant. On the other hand, note that the rescaling factor $v^{N(\beta(\mathbf{a}))/2}$ of the dual canonical basis depends only on the weight of the vector $B^*(\mathbf{a})$. Hence if we write $\widetilde{E}^*(\mathbf{a}) = v^{N(\beta(\mathbf{a}))/2}E^*(\mathbf{a})$, the transition matrix between the rescaled dual PBW-basis $\{\widetilde{E}^*(\mathbf{a})\}$ and the rescaled dual canonical basis $\{\widetilde{B}^*(\mathbf{a})\}$ is identical to the transition matrix between the original bases. Thus, by Proposition 4.2, $\widetilde{E}^*(\mathbf{a}) - \widetilde{B}^*(\mathbf{a})$ is a linear combination of $\widetilde{B}^*(\mathbf{a}')$ with coefficients in $v^{-1}\mathbb{Z}[v^{-1}]$, where $\widetilde{B}^*(\mathbf{a})$ and the $\widetilde{B}^*(\mathbf{a}')$ are σ-invariant. By Proposition 3.8, Φ maps the set of bar-invariant elements of $\mathscr{K}_{t,Q}$ to the set of σ-invariant elements of $\mathscr{A}_v(\mathfrak{n})$. This implies that Φ maps the basis of $\mathscr{K}_{t,Q}$ given by the truncated (q,t)-characters of the standard modules of $\mathscr{C}_Q$, to the rescaled dual PBW-basis $\{\widetilde{E}^*(\mathbf{a})\}$ of $\mathscr{A}_v(\mathfrak{n})$. Finally, using again Proposition 4.2 and Proposition 5.1, this yields (b). $\square$

Example 6.2 Let $\mathfrak{g}$ be of type A_2. Let Q be the quiver of type A_2 with height function $\xi_1 = 2$ and $\xi_2 = 1$. We have $\mathbf{i} = (1,2,1)$, and

$$D(0,1) = D_{\varpi_1, s_1(\varpi_1)}, \quad D(0,2) = D_{\varpi_2, s_1 s_2(\varpi_2)}, \quad D(1,3) = D_{s_1(\varpi_1), s_1 s_2 s_1(\varpi_1)} = D_{\varpi_2, s_2(\varpi_2)}.$$

Let e_1 and e_2 be the Chevalley generators of $U_v(\mathfrak{n})$. In the identification $A_v(\mathfrak{n}) \equiv U_v(\mathfrak{n})$ we have $D(0,1) \equiv e_1$ and $D(1,3) \equiv e_2$.

In this case the quantum torus $\mathscr{Y}_{t,Q}$ is generated by $Y_{1,0}, Y_{1,2}, Y_{2,1}$, so $\mathscr{K}_{t,Q}$ is generated by $\widetilde{\chi}_{q,t}(L(Y_{1,0})), \widetilde{\chi}_{q,t}(L(Y_{1,2})), \widetilde{\chi}_{q,t}(L(Y_{2,1}))$. The isomorphism Φ of Theorem 6.1 satisfies

$$\Phi(\widetilde{\chi}_{q,t}(L(Y_{1,2}))) = D(0,1), \quad \Phi(\widetilde{\chi}_{q,t}(L(Y_{1,0}))) = D(1,3).$$

Thus Theorem 6.1 implies that $\widetilde{\chi}_{q,t}(L(Y_{1,2}))$ and $\widetilde{\chi}_{q,t}(L(Y_{1,0}))$ generate $\mathscr{K}_{t,Q}$ and satisfy the quantum Serre relations.

This can easily be checked by means of the quantum T-system. Indeed we have by Proposition 5.6:

$$\chi_{q,t}(L(Y_{1,0}))*\chi_{q,t}(L(Y_{1,2})) = t^{-1/2}\chi_{q,t}(L(Y_{1,0}Y_{1,2})) + t^{1/2}\chi_{q,t}(L(Y_{2,1})),$$

and by applying the bar-involution

$$\chi_{q,t}(L(Y_{1,2}))*\chi_{q,t}(L(Y_{1,0})) = t^{1/2}\chi_{q,t}(L(Y_{1,0}Y_{1,2})) + t^{-1/2}\chi_{q,t}(L(Y_{2,1})).$$

Eliminating $\chi_{q,t}(L(Y_{1,0}Y_{1,2}))$ we get

$$(t^{-1/2}-t^{3/2})\chi_{q,t}(L(Y_{2,1})) = \chi_{q,t}(L(Y_{1,2}))*\chi_{q,t}(L(Y_{1,0})) - t\chi_{q,t}(L(Y_{1,0}))*\chi_{q,t}(L(Y_{1,2})),$$

which shows that $\mathscr{K}_{t,Q}$ is generated by $\widetilde{\chi}_{q,t}(L(Y_{1,2}))$ and $\widetilde{\chi}_{q,t}(L(Y_{1,0}))$. Finally, using that

$$\chi_{q,t}(L(Y_{2,1}))*\chi_{q,t}(L(Y_{1,0})) = t^{-1}\chi_{q,t}(L(Y_{1,0}))*\chi_{q,t}(L(Y_{2,1}))$$

we obtain that

$$\begin{aligned}\chi_{q,t}(L(Y_{1,2}))*\chi_{q,t}(L(Y_{1,0}))^2 - t\chi_{q,t}(L(Y_{1,0}))*\chi_{q,t}(L(Y_{1,2}))*\chi_{q,t}(L(Y_{1,0}))\\ = t^{-1}\chi_{q,t}(L(Y_{1,0}))*\chi_{q,t}(L(Y_{1,2}))*\chi_{q,t}(L(Y_{1,0})) - \chi_{q,t}(L(Y_{1,0}))^2*\chi_{q,t}(L(Y_{1,2})),\end{aligned}$$

which is the first quantum Serre relation. The second one is obtained similarly.

Example 6.3 In this example, we illustrate the calculations behind the proof of Theorem 6.1. Let $\mathfrak{g}$ be of type A_3. Let Q be the quiver of type A_3 with height function $\xi_1 = \xi_3 = 2$ and $\xi_2 = 3$. Thus Q has source 2 and sinks 1, 3. We take $\mathbf{i} = (2,1,3,2,1,3)$, hence

$$\beta_1 = \alpha_2, \quad \beta_2 = \alpha_1+\alpha_2, \quad \beta_3 = \alpha_2+\alpha_3, \quad \beta_4 = \alpha_1+\alpha_2+\alpha_3, \quad \beta_5 = \alpha_3, \quad \beta_6 = \alpha_1,$$

and

$$\begin{aligned}&\lambda_1 = \varpi_2-\alpha_2, && \lambda_2 = \varpi_1-\alpha_1-\alpha_2, && \lambda_3 = \varpi_3-\alpha_2-\alpha_3,\\ &\lambda_4 = \varpi_2-\alpha_1-2\alpha_2-\alpha_3, && \lambda_5 = \varpi_1-\alpha_1-\alpha_2-\alpha_3, && \lambda_6 = \varpi_3-\alpha_1-\alpha_2-\alpha_3.\end{aligned}$$

Note that in this case $w_0 = c^2$ where $c = s_2s_1s_3$ is a Coxeter element. Thus, this example illustrates also Corollary 1.5. The quantum unipotent minors generating $\mathscr{T}_{v,Q}$ are

$$\begin{aligned}&D(0,1) = D_{\varpi_2,s_2(\varpi_2)}, && D(0,2) = D_{\varpi_1,s_2s_1(\varpi_1)}, && D(0,3) = D_{\varpi_3,s_2s_1s_3(\varpi_3)},\\ &D(0,4) = D_{\varpi_2,s_2s_1s_3s_2(\varpi_2)}, && D(0,5) = D_{\varpi_1,s_2s_1s_3s_2s_1(\varpi_1)}, && D(0,6) = D_{\varpi_3,w_0(\varpi_3)}.\end{aligned}$$

The generators of the dual PBW-basis are

$$\begin{aligned}&E^*(\beta_1) = D(0,1), && E^*(\beta_2) = D(0,2), && E^*(\beta_3) = D(0,3),\\ &E^*(\beta_4) = D_{s_2(\varpi_2),s_2s_1s_3s_2(\varpi_2)}, && E^*(\beta_5) = D_{s_2s_1(\varpi_1),s_2s_1s_3s_2s_1(\varpi_1)}, && E^*(\beta_6) = D_{s_2s_1s_3(\varpi_3),w_0(\varpi_3)}.\end{aligned}$$

The new generators X_i of $\mathscr{T}_{v,Q}$ are

$$\begin{aligned}&X_1 = D(0,1), && X_2 = v^{-1/2}D(0,2), && X_3 = v^{-1/2}D(0,3),\\ &X_4 = v^{-1}D(0,4)D(0,1)^{-1}, && X_5 = v^{-1}D(0,5)D(0,2)^{-1}, && X_6 = v^{-1}D(0,6)D(0,3)^{-1}.\end{aligned}$$

Let us define integers λ_{ij} and μ_{ij} by

$$D(0,i)D(0,j) = v^{\lambda_{ij}}D(0,j)D(0,i), \qquad X_iX_j = v^{\mu_{ij}}X_jX_i, \qquad (1 \le i,j \le 6).$$

The matrices $L = [\lambda_{ij}]$ and $M = [\mu_{ij}]$ are given by

$$L = \begin{bmatrix} 0 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix}, \qquad M = \begin{bmatrix} 0 & -1 & -1 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 & 1 & -1 \\ 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 1 & 0 & -1 & -1 \\ -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 1 & 0 & 0 \end{bmatrix}.$$

The generators of $\mathscr{Y}_{t,Q}$ are $Y_{1,0}, Y_{3,0}, Y_{2,1}, Y_{1,2}, Y_{3,2}, Y_{2,3}$. The isomorphism Φ is defined by

$$\Phi(Y_{1,0}) = X_5, \quad \Phi(Y_{3,0}) = X_6, \quad \Phi(Y_{2,1}) = X_4, \quad \Phi(Y_{1,2}) = X_2, \quad \Phi(Y_{3,2}) = X_3, \quad \Phi(Y_{2,3}) = X_1.$$

The truncated (q,t)-characters of the fundamental modules of $\mathscr{C}_Q$ are expressed in terms of commutative monomials by

$$\begin{aligned} \widetilde{\chi}_{q,t}(Y_{1,2}) &= Y_{1,2}, & \widetilde{\chi}_{q,t}(Y_{1,0}) &= Y_{1,0} + Y_{1,2}^{-1}Y_{2,1} + Y_{2,3}^{-1}Y_{3,2}, \\ \widetilde{\chi}_{q,t}(Y_{2,1}) &= Y_{2,1} + Y_{1,2}Y_{2,3}^{-1}Y_{3,2}, & \widetilde{\chi}_{q,t}(Y_{2,3}) &= Y_{2,3}, \\ \widetilde{\chi}_{q,t}(Y_{3,2}) &= Y_{3,2}, & \widetilde{\chi}_{q,t}(Y_{3,0}) &= Y_{3,0} + Y_{3,2}^{-1}Y_{2,1} + Y_{2,3}^{-1}Y_{1,2}, \end{aligned}$$

Here, we have used the shorthand notation $\widetilde{\chi}_{q,t}(m)$ instead of $\widetilde{\chi}_{q,t}(L(m))$. We also have

$$\widetilde{\chi}_{q,t}(Y_{1,0}Y_{1,2}) = Y_{1,0}Y_{1,2}, \qquad \widetilde{\chi}_{q,t}(Y_{2,1}Y_{2,3}) = Y_{2,1}Y_{2,3}, \qquad \widetilde{\chi}_{q,t}(Y_{3,0}Y_{3,2}) = Y_{3,0}Y_{3,2}.$$

Using the expression of $D(0,k)$ in terms of X_j's and the definition of Φ, one checks that

$$\begin{aligned} &\Phi(Y_{2,3}) = D(0,1), && \Phi(Y_{1,2}) = v^{-1/2}D(0,2), && \Phi(Y_{3,2}) = v^{-1/2}D(0,3), \\ &\Phi(Y_{2,1}Y_{2,3}) = v^{-1}D(0,4), && \Phi(Y_{1,0}Y_{1,2}) = v^{-1}D(0,5), && \Phi(Y_{3,0}Y_{3,2}) = v^{-1}D(0,6), \end{aligned}$$

in agreement with Theorem 6.1. By Proposition 4.1, we have

$$v^{-1}D(1,4)D(0,1) = v^{-1}D(1,1)D(0,4) + D(0,2)D(0,3),$$

hence

$$v^{-1}D(1,4) = (v^{-1}D(0,4) + D(0,2)D(0,3))D(0,1)^{-1}.$$

Therefore

$$\Phi^{-1}\left(v^{-1}D(1,4)\right) = (Y_{2,1}Y_{2,3} + tY_{1,2} * Y_{3,2}) * Y_{2,3}^{-1} = Y_{2,1} + tY_{1,2} * Y_{3,2} * Y_{2,3}^{-1} = Y_{2,1} + Y_{1,2}Y_{3,2}Y_{2,3}^{-1},$$

where the last equality follows from (11). Thus we have

$$\Phi^{-1}\left(v^{-1}D(1,4)\right) = \Phi^{-1}\left(v^{-1}E^*(\beta_4)\right) = \widetilde{\chi}_{q,t}(Y_{2,1}),$$

in agreement with Theorem 6.1. Next, we have again by Proposition 4.1,

$$D(2,5)D(0,2) = v^{-1}D(0,5) + D(1,4).$$

Hence

$$\Phi^{-1}(D(2,5)) = \left(Y_{1,0}Y_{1,2} + t(Y_{2,1} + Y_{1,2}Y_{2,3}^{-1}Y_{3,2})\right) * (t^{1/2}Y_{1,2})^{-1}.$$

Now,

$$(Y_{1,0}Y_{1,2}) * Y_{1,2}^{-1} = t^{1/2}Y_{1,0} * Y_{1,2} * Y_{1,2}^{-1} = t^{1/2}Y_{1,0},$$

and similarly

$$Y_{2,1} * Y_{1,2}^{-1} = t^{-1/2} Y_{2,1} Y_{1,2}^{-1}, \quad (Y_{1,2} Y_{2,3}^{-1} Y_{3,2}) * Y_{1,2})^{-1} = t^{-1/2} Y_{2,3}^{-1} Y_{3,2}.$$

Therefore,

$$\Phi^{-1}(D(2,5)) = \Phi^{-1}(E^*(\beta_5)) = Y_{1,0} + Y_{1,2}^{-1} Y_{2,1} + Y_{2,3}^{-1} Y_{3,2} = \widetilde{\chi}_{q,t}(Y_{1,0}).$$

Similarly, starting from the minor identity

$$D(3,6)D(3,0) = v^{-1} D(0,6) + D(1,4).$$

we deduce that

$$\Phi^{-1}(D(3,6)) = \Phi^{-1}(E^*(\beta_6)) = Y_{3,0} + Y_{3,2}^{-1} Y_{2,1} + Y_{2,3}^{-1} Y_{1,2} = \widetilde{\chi}_{q,t}(Y_{3,0}).$$

Thus we have checked that Φ maps the fundamental characters

$$\widetilde{\chi}_{q,t}(Y_{1,0}), \quad \widetilde{\chi}_{q,t}(Y_{3,0}), \quad \widetilde{\chi}_{q,t}(Y_{2,1}), \quad \widetilde{\chi}_{q,t}(Y_{1,2}), \quad \widetilde{\chi}_{q,t}(Y_{3,2}), \quad \widetilde{\chi}_{q,t}(Y_{2,3}),$$

to the rescaled dual PBW generators $v^{N(\beta_k)/2} E^*(\beta_k)$, in agreement with Theorem 6.1.

6.2 Proof of Corollary 1.4

Let $b_1, \ldots, b_k \in \mathbf{B}^*$, and let $L_1, \ldots, L_k$ be the simple objects of $\mathscr{C}_Q$ such that

$$\Phi(\widetilde{\chi}_{q,t}(L_i)) \in v^{\mathbb{Z}/2} b_i, \qquad (1 \le i \le k).$$

We have $\Phi(\widetilde{\chi}_{q,t}(L_1) * \cdots * \widetilde{\chi}_{q,t}(L_k)) \in v^{\mathbb{Z}/2} b_1 \cdots b_k$, thus, by Theorem 6.1, $b_1 \cdots b_k \in v^{\mathbb{Z}} \mathbf{B}^*$ if and only if $\widetilde{\chi}_{q,t}(L_1) * \cdots * \widetilde{\chi}_{q,t}(L_k)$ is the (q,t)-character of a simple module up to a power of v, that is by Corollary 5.5, if and only if $L_1 \otimes \cdots \otimes L_k$ is simple. Hence Corollary 1.4 follows from [**He4**].

7 A presentation of quantum Grothendieck rings

In the remaining sections we drop the symbol $*$ for the t-deformed product of $\mathscr{K}_t$, and simply write xy instead of $x * y$.

7.1 The generators

Fix an orientation Q of the Dynkin diagram of $\mathfrak{g}$. Define an involution ν of I by $w_0(\alpha_i) = -\alpha_{\nu(i)}$. For $i \in I$ write $\varphi^{-1}(\alpha_i, 0) = (k_i, p_i) \in \widehat{I}_Q$. Define the following elements of $\mathscr{K}_t$:

$$x_{i,m}^Q := \chi_{q,t}\left(L\left(Y_{\nu^m(k_i), p_i + mh}\right)\right), \qquad (i \in I,\ m \in \mathbb{Z}). \tag{38}$$

The elements $x_{i,0}^Q$ belong to $\mathscr{K}_{t,Q}$ and map to the Chevalley generators $D_{\varpi_i, s_i(\varpi_i)} \equiv e_i$ of $\mathscr{A}_v(\mathfrak{n}) \equiv \mathscr{U}_v(\mathfrak{n})$ under the isomorphism Φ of Theorem 6.1. Hence $\mathscr{K}_{t,Q}$ has a presentation given by the generators $x_{i,0}^Q$ $(i \in I)$ subject to the relations (see §4.1)

$$x_{i,0}^Q x_{j,0}^Q - x_{j,0}^Q x_{i,0}^Q = 0 \qquad \text{if } C_{ij} = 0,$$
$$(x_{i,0}^Q)^2 x_{j,0}^Q - (t + t^{-1}) x_{i,0}^Q x_{j,0}^Q x_{i,0}^Q + x_{j,0}^Q (x_{i,0}^Q)^2 = 0 \qquad \text{if } C_{ij} = -1.$$

In particular, every $\chi_{q,t}(L(Y_{i,p}))$ with $(i,p) \in \widehat{I}_Q$ can be written as a noncommutative polynomial in the $x^Q_{i,0}$'s.

For $m \in \mathbb{Z}$, let $\mathscr{K}^{(m)}$ be the subalgebra of $\mathscr{K}_t$ generated by the $x^Q_{i,m}$ $(i \in I)$. Thus $\mathscr{K}^{(0)} = \mathscr{K}_{t,Q}$, and $\mathscr{K}^{(m)}$ is isomorphic to $\mathscr{K}^{(0)}$ for every $m \in \mathbb{Z}$. This comes from the fact that $\mathscr{K}_t$ is generated by the fundamental (q,t)-characters $\chi_{q,t}(L(Y_{i,p}))$ $((i,p) \in \widehat{I})$, and that the assignment

$$\chi_{q,t}(L(Y_{i,p})) \mapsto \chi_{q,t}\left(L(Y_{\nu(i),\,p+h})\right)$$

extends to an algebra automorphism Σ of $\mathscr{K}_t$. (In fact, $L(Y_{i,p})$ is the $U_q(L\mathfrak{g})$-module dual to $L(Y_{\nu(i),\,p+h})$ [**CP**, §5], see also [**FM**, Cor. 6.10].) Let $\widehat{I}_{Q,m} := \varphi^{-1}(\Delta_+ \times \{m\})$. Thus $\chi_{q,t}(L(Y_{i,p})) \in \mathscr{K}^{(m)}$ for $(i,p) \in \widehat{I}_{Q,m}$. Therefore, we have proved:

Lemma 7.1 *The elements $x^Q_{i,m}$ $(i \in I, m \in \mathbb{Z})$ generate $\mathscr{K}_t$.* □

7.2 The presentation

We start with the following:

Lemma 7.2 *Let $(i,p) \in \widehat{I}$ and $(j,p+h) \in \widehat{I}$. Write $V := L(Y_{i,p}) \otimes L(Y_{j,\,p+h})$.*

(a) If $j \neq \nu(i)$ then V is simple.

(b) If $j = \nu(i)$ then $\chi_q(V) = \chi_q\left(L(Y_{i,p}Y_{\nu(i),\,p+h})\right) + 1$.

(c) In general we have

$$\chi_{q,t}(L(Y_{i,p}))\,\chi_{q,t}\left(L(Y_{j,p+h})\right) = t^{-(\alpha_i,\alpha_{\nu(j)})}\chi_{q,t}\left(L(Y_{j,p+h})\right)\chi_{q,t}(L(Y_{i,p})) + \delta_{i\,\nu(j)}(1-t^{-2}),$$

where δ_{ik} is the Kronecker symbol δ.

Proof — Consider the product $\pi := \chi_q(L(Y_{i,p}))\,\chi_q\left(L(Y_{\nu(i),\,p+h})\right)$. By [**FM**, §6], $\chi_q(L(Y_{i,p}))$ contains only one dominant monomial, namely $Y_{i,p}$, one anti-dominant monomial, namely $Y^{-1}_{\nu(i),p+h}$, and all its other monomials involve only variables of the form $Y^{\pm 1}_{j,m}$ with $p < m < p+h$. It follows that, if $j \neq \nu(i)$, then π contains no other dominant monomial than $Y_{i,p}Y_{j,\,p+h}$, hence V is irreducible and isomorphic to $L(Y_{i,p}Y_{\nu(i),\,p+h})$. This proves (a).

If $j = \nu(i)$ then π contains only two dominant monomials, that is, $Y_{i,p}Y_{\nu(i),\,p+h}$ and 1. Therefore V has at most two composition factors, $L(Y_{i,p}Y_{\nu(i),\,p+h})$ and the trivial one-dimensional representation. Since $L(Y_{i,p}) = L(Y_{\nu(i),\,p+h})^*$, the trivial representation is indeed a composition factor of V because $U_q(L\mathfrak{g})$ is a Hopf algebra. This proves (b).

It follows that

$$\chi_q(L(Y_{i,p}))\,\chi_q\left(L(Y_{j,p+h})\right) = \chi_q\left(L(Y_{i,p}Y_{j,p+h})\right) + \delta_{i\,\nu(j)}.$$

In $\mathscr{K}_t$, this identity gets t-deformed as

$$\chi_{q,t}(L(Y_{i,p}))\,\chi_{q,t}\left(L(Y_{j,p+h})\right) = t^{\frac{1}{2}\mathscr{N}(i,p;j,p+h)}\chi_{q,t}\left(L(Y_{i,p}Y_{j,p+h})\right) + \delta_{i\,\nu(j)}.$$

Now using Proposition 3.2 and a sink-source orientation Q where i is a source, we see that $\mathscr{N}(i,p;j,p+h) = \mathscr{N}(i,0;j,h) = -(\alpha_i,\alpha_{\nu(j)})$. Using the bar involution, we also have

$$\chi_{q,t}\left(L(Y_{j,p+h})\right)\chi_{q,t}(L(Y_{i,p})) = t^{-\frac{1}{2}\mathscr{N}(i,p;j,p+h)}\chi_{q,t}\left(L(Y_{i,p}Y_{j,p+h})\right) + \delta_{i\,\nu(j)}.$$

Then (c) follows by eliminating $\chi_{q,t}\left(L(Y_{i,p}Y_{j,p+h})\right)$ between these two equations. □

We can now give a presentation of $\mathscr{K}_t$.

Theorem 7.3 *The algebra $\mathcal{K}_t$ is isomorphic to the $\mathbb{C}(t^{1/2})$-algebra $\mathscr{A}$ presented by generators $y_{i,m}$ $(i \in I, m \in \mathbb{Z})$ subject only to the following relations:*

(R1) for every $m \in \mathbb{Z}$,

$$y_{i,m}y_{j,m} - y_{j,m}y_{i,m} = 0 \qquad \text{if } (\alpha_i, \alpha_j) = 0,$$
$$y_{i,m}^2 y_{j,m} - (t+t^{-1})y_{i,m}y_{j,m}y_{i,m} + y_{j,m}y_{i,m}^2 = 0 \qquad \text{if } (\alpha_i, \alpha_j) = -1;$$

(R2) for every $m \in \mathbb{Z}$ and every $i, j \in I$,

$$y_{i,m}y_{j,m+1} = t^{-(\alpha_i,\alpha_j)}y_{j,m+1}y_{i,m} + \delta_{ij}(1-t^{-2});$$

(R3) for every $p > m+1$ and every $i, j \in I$,

$$y_{i,m}y_{j,p} = t^{(-1)^{p-m}(\alpha_i,\alpha_j)}y_{j,p}y_{i,m}.$$

Proof — We fix a sink-source orientation Q. We first check that the $x_{i,m}^Q$ satisfy the above relations. The relations (R1) are the Drinfeld-Jimbo relations for the subalgebra $\mathcal{K}^{(m)}$, as explained in §7.1. The relations (R2) follow from Lemma 7.2 (c) when $\xi_i = \xi_j$. If $\xi_i \neq \xi_j$, then $x_{i,m}^Q x_{j,m+1}^Q$ corresponds to a tensor product of the form $L(Y_{i,p}) \otimes L(Y_{j,p+1})$ or $L(Y_{i,p}) \otimes L(Y_{j,p+2h-1})$. These two types of tensor products are always irreducible [**FM**, Proposition 6.15]. Using Corollary 5.5, it follows that $x_{i,m}$ and $x_{j,m+1}$ t-commute, and the exponent of t is calculated by means of Proposition 3.2. For the relations (R3) we note that $L(Y_{i,p}) \otimes L(Y_{j,s})$ is irreducible if $s - p > h$ [**FM**, Proposition 6.15], and we conclude similarly.

It follows that we have a surjective homomorphism F from $\mathscr{A}$ to $\mathcal{K}_t$ given by $y_{i,m} \mapsto x_{i,m}^Q$, and we have to show that this is an isomorphism. Define $\mathscr{A}^{(m)}$ as we have defined $\mathcal{K}^{(m)}$ before. Then $\mathscr{A}^{(m)}$ is presented by the relations (1) (with $x_{i,m}^Q$ replaced by $y_{i,m}$), so F restricts to an isomorphism from $\mathscr{A}^{(m)}$ to $\mathcal{K}^{(m)}$. It follows from the relations (R2) and (R3) that every monomial M in the $y_{i,m}$'s can be rewritten as a linear combination of monomials of the form $M_{k_1}M_{k_2}\cdots M_{k_s}$ with $M_{k_j} \in \mathscr{A}^{(k_j)}$ and $k_1 > k_2 > \cdots > k_s$. So we have $\mathscr{A} = \overleftarrow{\prod_{m\in\mathbb{Z}}} \mathscr{A}^{(m)}$. Now each $\mathcal{K}^{(m)}$ has a basis $\mathscr{B}^{(m)}$ consisting of the (q,t)-characters of standard modules that it contains. Taking

$$\mathscr{B}' := \{b_{k_1}b_{k_2}\cdots b_{k_s} \mid b_{k_j} \in \mathscr{A}^{(k_j)},\ F(b_{k_j}) \in \mathscr{B}^{(k_j)},\ k_1 > \cdots > k_s,\ s \in \mathbb{N}\},$$

we get a spanning set of $\mathscr{A}$ such that $F(\mathscr{B}')$ is a basis of $\mathcal{K}_t$, consisting of the (q,t)-characters of all the standard modules of $\mathscr{C}_\mathbb{Z}$. Hence $\mathscr{B}'$ is a basis of $\mathscr{A}$ and F is an isomorphism. □

Example 7.4 Let $\mathfrak{g} = \mathfrak{sl}_2$. By Theorem 7.3, $\mathcal{K}_t$ is presented by generators $y_m := \chi_{q,t}(L(Y_{2m}))$ indexed by $m \in \mathbb{Z}$, subject to

$$\begin{aligned} y_m y_{m+1} &= t^{-2}y_{m+1}y_m + 1 - t^{-2}, \\ y_m y_p &= t^{2(-1)^{p-m}}y_p y_m, \qquad \text{if } p > m+1. \end{aligned}$$

Remark 7.5 (a) It was shown by Frenkel and Reshetikhin [**FR2**, Corollary 2] that the (classical) Grothendieck ring $\mathscr{R}$ of $\mathscr{C}_\mathbb{Z}$ is the polynomial ring in the classes of all fundamental modules $L(Y_{i,p})$ $((i,p) \in \widehat{I})$. More recently, a presentation of $\mathscr{R}$ in terms of Kirillov-Reshetikhin modules and T-systems was given in [**IIKNS**, Corollary 2.9].

Note that our presentation of $\mathscr{K}_t$ does *not* yield a new presentation of $\mathscr{R}$. Indeed, in order to obtain $\mathscr{R}$ from $\mathscr{K}_t$ by specializing t at 1, one needs to use the integral form K_t defined in §8.2 below, and the $x_{i,m}^{Q}$ are not generators of K_t if $\mathfrak{g} \neq \mathfrak{sl}_2$.

(b) For $m \in \mathbb{Z}$, let $\mathscr{K}^{(m,m+1)}$ denote the subalgebra of $\mathscr{K}_t$ generated by $y_{i,m}, y_{i,m+1}$ $(i \in I)$. It follows from Theorem 7.3 that $\mathscr{K}^{(m,m+1)}$ is isomorphic to the t-deformed boson algebra $\mathscr{B}_t(\mathfrak{g})$ introduced by Kashiwara [**K**, §3.3].

8 Derived Hall algebras

8.1 The Hall algebra $H(Q)$

Let F be a finite field, and let $u := |F|^{1/2} \in \mathbb{R}_{>0}$. Let $\mathrm{mod}(FQ)$ be the abelian category of representations of Q over F. The twisted Hall algebra $H(Q)$, introduced by Ringel, is the $\mathbb{C}$-algebra with basis $\{z_X\}$ labelled by the isoclasses of objects in $\mathrm{mod}(FQ)$, with multiplication

$$z_X z_Y = u^{\langle Y,X\rangle} \sum_W g_{X,Y}^W z_W,$$

where $g_{X,Y}^W$ is the number of submodules T of W such that $T \simeq X$ and $W/T \simeq Y$. Ringel [**Ri2**, **Ri3**, **Ri4**] has shown that $H(Q)$ is isomorphic to the $\mathbb{C}$-algebra $U_u(\mathfrak{n})$ obtained from $U_v(\mathfrak{n})$ by specializing v at u. In this isomorphism, the basis $\{z_X\}$ is mapped to a PBW-basis of $U_u(\mathfrak{n})$. In particular, if S_i denotes the 1-dimensional simple supported on $i \in I$, z_{S_i} is mapped to the Chevalley generator e_i.

8.2 The derived Hall algebra $DH(Q)$

Let $D^b(\mathrm{mod}(FQ))$ be the bounded derived category of $\mathrm{mod}(FQ)$. Toën [**T**, §7] has associated with this triangulated category an associative algebra $DH(Q)$ with the following presentation. The generators $z_X^{[m]}$ are labelled by all pairs (X,m) where X is an isoclass of $\mathrm{mod}(FQ)$ and $m \in \mathbb{Z}$. (The pair (X,m) corresponds to the stalk complex with X in degree m.) The relations are:

(D1) for every $m \in \mathbb{Z}$,

$$z_X^{[m]} z_Y^{[m]} = u^{\langle Y,X\rangle} \sum_W g_{X,Y}^W z_W^{[m]};$$

(D2) for every $m \in \mathbb{Z}$,

$$z_X^{[m]} z_Y^{[m+1]} = u^{-\langle Y,X\rangle} \sum_{W,T} u^{-\langle W,T\rangle} \gamma_{X,Y}^{T,W} z_T^{[m+1]} z_W^{[m]};$$

(D3) for $p > m+1$,

$$z_X^{[m]} z_Y^{[p]} = u^{(-1)^{p-m}(X,Y)} z_Y^{[p]} z_X^{[m]}.$$

Here, the Hall number $\gamma_{X,Y}^{T,W}$ is defined by Toën as

$$\gamma_{X,Y}^{T,W} := \frac{|\mathrm{Ex}(W,Y,X,T)|}{|\mathrm{Aut}(X)||\mathrm{Aut}(Y)|},$$

where $\mathrm{Ex}(W,Y,X,T)$ is the finite subset of $\mathrm{Hom}(W,Y) \times \mathrm{Hom}(Y,X) \times \mathrm{Hom}(X,T)$ consisting of exact sequences $0 \to W \to Y \to X \to T \to 0$. Note that, as in §8.1, we have twisted the multiplication by inserting in the original Hall product $z_X^{[m]} z_Y^{[p]}$ of [**T**] a factor $u^{(-1)^{p-m}\langle Y,X\rangle}$, see [**S**].

Consider the elements $z_{i,m} := z_{S_i}^{[m]}$ for $i \in I$ and $m \in \mathbb{Z}$. As in §7.1, we see that the $z_{i,m}$ generate $DH(Q)$. More precisely, we have:

Proposition 8.1 *The algebra $DH(Q)$ is generated by the $z_{i,m}$ $(i \in I, m \in \mathbb{Z})$ subject only to the following relations:*

(H1) for every $m \in \mathbb{Z}$,

$$z_{i,m}\, z_{j,m} - z_{j,m}\, z_{i,m} = 0 \qquad\qquad \text{if } (\alpha_i, \alpha_j) = 0,$$
$$z_{i,m}^2\, z_{j,m} - (u + u^{-1}) z_{i,m}\, z_{j,m}\, z_{i,m} + z_{j,m}\, z_{i,m}^2 = 0 \qquad \text{if } (\alpha_i, \alpha_j) = -1;$$

(H2) for every $m \in \mathbb{Z}$ and every $i, j \in I$,

$$z_{i,m}\, z_{j,m+1} = u^{-(\alpha_i, \alpha_j)} z_{j,m+1}\, z_{i,m} + \delta_{ij} \frac{u^{-1}}{u^2 - 1};$$

(H3) For every $p > m+1$ and every $i, j \in I$,

$$z_{i,m}\, z_{j,p} = u^{(-1)^{p-m}(\alpha_i, \alpha_j)}\, z_{j,p}\, z_{i,m}.$$

Proof — The relations (H1) follow immediately from (D1) and Ringel's theorem. The relations (H3) follow immediately from (D3). Let us deduce the relations (H2) from (D2).

If $i \neq j$, the only exact sequences $0 \to W \to S_j \to S_i \to T \to 0$ are of the form

$$0 \to S_j \xrightarrow{f} S_j \xrightarrow{0} S_i \xrightarrow{g} S_i \to 0$$

where 0 means the zero map, and f and g are isomorphisms. Clearly there are $(|F| - 1)^2$ such sequences, and since $|\operatorname{Aut}(S_i)| = |\operatorname{Aut}(S_j)| = |F| - 1$, we get that $\gamma_{S_i, S_j}^{S_j, S_i} = 1$. Hence

$$z_{i,m}\, z_{j,m+1} = u^{-\langle S_j, S_i \rangle} u^{-\langle S_i, S_j \rangle} z_{j,m+1}\, z_{i,m} = u^{-(\alpha_i, \alpha_j)} z_{j,m+1}\, z_{i,m}.$$

If $i = j$, we have two types of exact sequences $0 \to W \to S_i \to S_i \to T \to 0$, namely

$$0 \to S_i \xrightarrow{f} S_i \xrightarrow{0} S_i \xrightarrow{g} S_i \to 0, \quad \text{and} \quad 0 \to 0 \xrightarrow{0} S_i \xrightarrow{h} S_i \xrightarrow{0} 0 \to 0,$$

where f, g, h are isomorphisms. It follows that

$$\gamma_{S_i, S_i}^{S_i, S_i} = 1, \qquad \text{and} \qquad \gamma_{S_i, S_i}^{0,0} = \frac{1}{|F| - 1} = \frac{1}{u^2 - 1},$$

hence

$$z_{i,m}\, z_{i,m+1} = u^{-(S_i, S_i)} z_{i,m+1}\, z_{i,m} + u^{-\langle S_i, S_i \rangle} \frac{1}{u^2 - 1}.$$

This proves (H2). Finally, the proof that relations (H1), (H2), (H3) give a presentation of $DH(Q)$ is entirely similar to the proof of the analogous statement in Theorem 7.3 (the basis $\mathscr{B}^{(m)}$ is replaced by $\{z_X^{[m]} \mid X \text{ isoclass of } \operatorname{mod}(FQ)\}$), and we omit it. □

8.3 The isomorphism between $\mathscr{K}_u$ and $DH(Q)$

Define the integral form

$$K_t := \bigoplus_L \mathbb{C}[t^{1/2}, t^{-1/2}]\, \chi_{q,t}(L) \subset \mathscr{K}_t,$$

where the sum runs over all isoclasses L of simple objects in $\mathscr{C}_{\mathbb{Z}}$. By Theorem 5.4, this is a subring of $\mathscr{K}_t$. Set

$$\mathscr{K}_u := \mathbb{C} \otimes_{\mathbb{C}[t^{1/2}, t^{-1/2}]} K_t,$$

where $\mathbb{C}$ is regarded as a $\mathbb{C}[t^{1/2}, t^{-1/2}]$-module via the specialization $t^{1/2} \mapsto u^{1/2}$.

For $\beta \in \widehat{\Delta} = \Delta_+ \times \mathbb{Z}$, we denote by $z_\beta^{[m]}$ the basis element $z_X^{[m]}$ of $DH(Q)$ with $X \in \mathrm{mod}(FQ)$ indecomposable of dimension vector β.

The following is a slightly more precise formulation of Theorem 1.1.

Theorem 8.2 *There is a $\mathbb{C}$-algebra isomorphism $\iota \colon \mathscr{K}_u \xrightarrow{\sim} DH(Q)$ such that:*

(a) the class of the fundamental $U_q(L\mathfrak{g})$-module $L(Y_{i,p})$ of $\mathscr{C}_{\mathbb{Z}}$ is mapped by ι to a scalar multiple of $z_\beta^{[m]}$, where $(\beta, m) = \varphi(i,p)$.

(b) the basis of classes of standard $U_q(L\mathfrak{g})$-modules of $\mathscr{C}_{\mathbb{Z}}$ is mapped by ι to a rescaling of the natural basis of $DH(Q)$ labelled by all isoclasses of objects of $D^b(\mathrm{mod}(FQ))$.

Proof — We first assume, as in the proof of Theorem 7.3, that Q is a sink-source orientation of the Dynkin diagram. We can rescale the generators $x_{i,m}^Q$ of $\mathscr{K}_t$ by setting

$$\widetilde{x}_{i,m}^Q := \frac{1}{u^{1/2}(u-u^{-1})}\, x_{i,m}^Q, \qquad (i \in I,\ m \in \mathbb{Z}).$$

Clearly the new generators $\widetilde{x}_{i,m}^Q$ still satisfy the homogeneous relations (R1) and (R3) of Theorem 7.3, and the relations (R2) become

$$\widetilde{x}_{i,m}^Q\, \widetilde{x}_{j,m+1}^Q = t^{-(\alpha_i,\alpha_j)}\widetilde{x}_{j,m+1}^Q\, \widetilde{x}_{i,m}^Q + \delta_{ij}\frac{1-t^{-2}}{u(u-u^{-1})^2}.$$

Let $\overline{x}_{i,m}^Q = 1 \otimes \widetilde{x}_{i,m}^Q \in \mathscr{K}_u$. By Theorem 7.3, Proposition 8.1, the assignment $\widetilde{x}_{i,m}^Q \mapsto z_{i,m}$ extends to an algebra isomorphism ι. Indeed, in the relations (R2) we have

$$\frac{1-u^{-2}}{u(u-u^{-1})^2} = \frac{u^{-1}}{u^2-1}$$

so the generators $\overline{x}_{i,m}^Q$ of $\mathscr{K}_u$ and $z_{i,m}$ of $DH(Q)$ give rise to identical presentations.

Since the PBW-basis of $U_v(n)$ is orthogonal with respect to the bilinear form of §4.1, it only differs from the dual PBW-basis $\mathbf{E}^*$ by scalar multiples. Hence by Ringel's theorem, it follows from Theorem 6.1 that the classes of fundamental modules in $\mathscr{C}_Q$, which correspond under Φ to the elements $E^*(\beta)$ $(\beta \in \Delta_+)$ of $U_v(\mathfrak{n})$, are mapped by ι to scalar multiples of the $z_\beta^{[0]}$. So, if $\varphi(i,p) = (\beta, 0)$ we have $\iota([L(Y_{i,p})]_u) = \lambda_{i,p} z_\beta^{[0]}$ for some $\lambda_{i,p} \in \mathbb{C}$. Therefore, using on one side the automorphism of $\mathscr{K}_u$ given by $[L(Y_{i,p})]_u \mapsto [L(Y_{i,p-2})]_u$, and on the other side the corresponding automorphism of $DH(Q)$ induced by the Auslander-Reiten translation τ of $D^b(\mathrm{mod}(FQ))$, we get (a).

Since the classes of standard modules are the ordered products of the $[L(Y_{i,p})]_u$ (up to powers of u), and the basis elements of $DH(Q)$ are the ordered products of the $z_\beta^{[m]}$ (up to powers of u), we get (b).

Therefore we have proved Theorem 8.2 in the case of a sink-source orientation. But the $\mathbb{C}$-algebras $\mathscr{K}_u$ and $DH(Q)$ are both independent of Q. For $\mathscr{K}_u$ this is clear. On the other hand if Q' is another orientation of the Dynkin diagram, then $D^b(\mathrm{mod}(FQ))$ and $D^b(\mathrm{mod}(FQ'))$ are equivalent triangulated categories, so $DH(Q)$ and $DH(Q')$ are isomorphic. Thus $\mathscr{K}_u$ is isomorphic to $DH(Q)$ for an arbitrary orientation. More precisely, recall that the map $\varphi = \varphi_Q : \widehat{I} \to \widehat{\Delta}$ depends on the choice of Q. There is a triangle equivalence $F_{QQ'} : D^b(\mathrm{mod}(FQ)) \to D^b(\mathrm{mod}(FQ'))$ such that the induced isomorphism $f_{QQ'} : DH(Q) \to DH(Q')$ satisfies

$$f_{QQ'}\left(z_\beta^{[m]}\right) = z_{\beta'}^{[m']} \quad \text{where} \quad (\beta', m') = \varphi_{Q'}\varphi_Q^{-1}(\beta, m).$$

Therefore (a) and (b) hold for an arbitrary orientation. $\square$

In the proof of Theorem 7.3, we have shown that if Q is a sink-source orientation, the generators $x_{i,m}^Q$ of $\mathscr{K}_t$ satisfy the relations (R1), (R2), (R3). We can see now that this holds for any orientation Q.

Corollary 8.3 *The generators $x_{i,m}^Q$ of $\mathscr{K}_t$ satisfy the same relations for every orientation Q of the Dynkin diagram, namely the relations (R1), (R2), (R3) of Theorem 7.3.*

Proof — Let Q be any orientation, by Theorem 8.2, the elements $\bar{x}_{i,m}^Q$ of $\mathscr{K}_u$ are mapped by ι to scalar multiples of the generators $z_{i,m}$ of $DH(Q)$. Now the relations (H1), (H2), (H3) satisfied by the $z_{i,m}$ are independent of Q. Moreover, they are all homogeneous except for (H2) with $i = j$. Since scalar multiplication does not affect homogeneous relations, the elements $1 \otimes x_{i,m}^Q$ of $\mathscr{K}_u$ satisfy the relations (R1), (R2) $(i \neq j)$, (R3) with t replaced by $u = |F|^{1/2}$. Since this is true for every finite field F, it follows that the elements $x_{i,m}^Q$ of $\mathscr{K}_t$ satisfy the relations (R1), (R2) $(i \neq j)$, (R3) where t is an indeterminate.

Finally, the relations (R2) $(i = j)$ follow from Lemma 7.2 (c) with $i = \nu(j)$. $\square$

Remark 8.4 Using Remark 3.3, one can modify the presentation of $\mathscr{K}_t$ to obtain a presentation of the deformed Grothendieck ring $\mathscr{R}_t$ of [**N3**, **VV1**]. This presentation shows that the specialization of $\mathscr{R}_t$ at $t = u^{-1}$ is isomorphic to the *non-twisted* derived Hall algebra of $D^b(\mathrm{mod}(FQ))$ with the opposite product.

9 Quiver varieties

In this section we show that the variety $E_{\mathbf{d}}$ of representations of Q with dimension vector $\mathbf{d}$ can be regarded as a Nakajima graded quiver variety $\mathfrak{M}_0^\bullet(W^{\mathbf{d}})$ for an appropriate $\widehat{I}$-graded vector space $W^{\mathbf{d}}$. Moreover the stratification of $E_{\mathbf{d}}$ by $G_{\mathbf{d}}$-orbits coincides with Nakajima's stratification of $\mathfrak{M}_0^\bullet(W^{\mathbf{d}})$. It follows that the set of perverse sheaves used by Lusztig to define the (dual) canonical basis of $U_v(\mathfrak{n})$ can be identified with a subset of the set of perverse sheaves used by Nakajima for describing the classes $[L]_t$ of simple $U_q(L\mathfrak{g})$-modules. This gives a geometric way of understanding Theorem 6.1 (b).

9.1 The quiver representation space $E_{\mathbf{d}}$

Let $\mathbf{d} = (d_i)_{i \in I} \in \mathbb{N}^I$ denote a dimension vector for Q. We will identify $\mathbf{d}$ with the element $\sum_{i \in I} d_i \alpha_i$ of the root lattice of $\mathfrak{g}$. The variety $E_{\mathbf{d}}$ of representations of Q of dimension $\mathbf{d}$ is by definition

$$E_{\mathbf{d}} := \bigoplus_{i \to j} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j}),$$

the sum being over all arrows $i \to j$ of Q. This is just a $\mathbb{C}$-vector space of dimension $\sum_{i \to j} d_i d_j$, but the interesting geometry comes from the following stratification. Consider the algebraic group

$$G_{\mathbf{d}} := \prod_{i \in I} GL(d_i, \mathbb{C}).$$

It acts on $E_{\mathbf{d}}$ by base change. There are finitely many orbits in one-to-one correspondence with the isomorphism classes of representations of Q of dimension $\mathbf{d}$. Thus, using Gabriel's theorem, these orbits have a natural labelling by the set

$$I_{\mathbf{d}} := \left\{ \mathbf{a} = (a_k) \in \mathbb{N}^r \mid \sum_{k=1}^{r} a_k \beta_k = \mathbf{d} \right\},$$

where the positive roots β_k are enumerated as in (14). Let $\mathscr{O}_{\mathbf{a}}$ denote the orbit labelled by the element $\mathbf{a}$ of $I_{\mathbf{d}}$. Let $IC(\overline{\mathscr{O}_{\mathbf{a}}})$ be the intersection cohomology complex of $\overline{\mathscr{O}_{\mathbf{a}}}$, extended by zero on the complement of $\overline{\mathscr{O}_{\mathbf{a}}}$. Let $\mathscr{H}^i(IC(\overline{\mathscr{O}_{\mathbf{a}}}))$ be its ith cohomology sheaf, and $\mathscr{H}^i(IC(\overline{\mathscr{O}_{\mathbf{a}}}))_{\mathbf{c}}$ the stalk of this sheaf at a point of $\mathscr{O}_{\mathbf{c}}$.

Recall from §4.3 the dual PBW basis $\mathbf{E}^*$ and the dual canonical basis $\mathbf{B}^*$ of $A_v(\mathfrak{n})$. Write

$$E^*(\mathbf{c}) = \sum_{\mathbf{a} \in I_{\mathbf{d}}} \kappa_{\mathbf{a},\mathbf{c}}(v) B^*(\mathbf{a}).$$

Lusztig has shown:

Theorem 9.1 [**Lu1**, §9, §10] *The coefficients $\kappa_{\mathbf{a},\mathbf{c}}(v)$ are given by*

$$\kappa_{\mathbf{a},\mathbf{c}}(v) = v^{\dim \mathscr{O}_{\mathbf{c}} - \dim \mathscr{O}_{\mathbf{a}}} \sum_{i \geq 0} v^i \dim \mathscr{H}^i(IC(\overline{\mathscr{O}_{\mathbf{a}}}))_{\mathbf{c}}. \qquad (39)$$

9.2 Nakajima's variety $\mathfrak{M}_0^\bullet(W)$

Let

$$W = \bigoplus_{(i,p) \in \widehat{I}} W_i(p)$$

be a finite-dimensional $\widehat{I}$-graded $\mathbb{C}$-vector space. In his geometric construction of representations of $U_q(L\mathfrak{g})$, Nakajima [**N3**] has associated with W an affine variety $\mathfrak{M}_0^\bullet(W)$ whose definition we shall now recall.

Let $\widehat{J} := \{(i,p) \in I \times \mathbb{Z} \mid (i, p-1) \in \widehat{I}\}$, and let

$$V = \bigoplus_{(i,s) \in \widehat{J}} V_i(s)$$

be a finite-dimensional $\widehat{J}$-graded $\mathbb{C}$-vector space. Define

$$\begin{aligned} L^\bullet(V,W) &= \bigoplus_{(i,s)\in\widehat{J}} \mathrm{Hom}(V_i(s), W_i(s-1)),\\ L^\bullet(W,V) &= \bigoplus_{(i,p)\in\widehat{I}} \mathrm{Hom}(W_i(p), V_i(p-1)),\\ E^\bullet(V) &= \bigoplus_{(i,s)\in\widehat{J};\ j\sim i} \mathrm{Hom}(V_i(s), V_j(s-1)). \end{aligned}$$

Put $M^\bullet(V,W) = E^\bullet(V)\oplus L^\bullet(W,V)\oplus L^\bullet(V,W)$. An element of $M^\bullet(V,W)$ is written (B,α,β), and its components are denoted by:

$$\begin{aligned} B_{ij}(s) &\in \mathrm{Hom}(V_i(s), V_j(s-1)),\\ \alpha_i(p) &\in \mathrm{Hom}(W_i(p), V_i(p-1)),\\ \beta_i(s) &\in \mathrm{Hom}(V_i(s), W_i(s-1)). \end{aligned}$$

We denote by $\Lambda^\bullet(V,W)$ the subvariety of the affine space $M^\bullet(V,W)$ defined by the equations

$$\alpha_i(s-1)\beta_i(s) + \sum_{j\sim i} \varepsilon(i,j) B_{ji}(s-1) B_{ij}(s) = 0, \qquad ((i,s)\in\widehat{J}), \tag{40}$$

where $\varepsilon(i,j)=1$ (*resp.* $\varepsilon(i,j)=-1$) if $i\to j$ is an arrow of Q (*resp.* $i\to j$ is not an arrow of Q). The algebraic group

$$G_V := \prod_{(i,s)\in\widehat{J}} GL(V_i(s))$$

acts on $M^\bullet(V,W)$ by base change in V:

$$g\cdot(B,\alpha,\beta) = \left((g_j(s-1)B_{ij}(s)g_i(s)^{-1}),\ (g_i(p-1)\alpha_i(p)),\ (\beta_i(s)g_i(s)^{-1})\right).$$

Note that there is no action on W. This action of G_V preserves the subvariety $\Lambda^\bullet(V,W)$. One defines the affine quotient

$$\mathfrak{M}_0^\bullet(V,W) := \Lambda^\bullet(V,W) /\!\!/ G_V.$$

By definition, the coordinate ring of $\mathfrak{M}_0^\bullet(V,W)$ is the ring of G_V-invariant functions on $\Lambda^\bullet(V,W)$, and $\mathfrak{M}_0^\bullet(V,W)$ parametrizes the closed G_V-orbits. If $V_i(s)\subseteq V_i'(s)$ for every $(i,s)\in\widehat{J}$, then we have a natural closed embedding $\mathfrak{M}_0^\bullet(V,W)\subset\mathfrak{M}_0^\bullet(V',W)$. Finally, one defines

$$\mathfrak{M}_0^\bullet(W) := \bigcup_V \mathfrak{M}_0^\bullet(V,W).$$

This is an affine variety, acted upon by the algebraic group

$$G_W := \prod_{(i,p)\in\widehat{I}} GL(W_i(p)).$$

Let $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W)$ be the open subset of $\mathfrak{M}_0^\bullet(V,W)$ parametrizing the closed *free* G_V-orbits. For a given W, we have $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W)\neq\emptyset$ only for a finite number of V's. Nakajima has shown that this gives a stratification of $\mathfrak{M}_0^\bullet(W)$:

$$\mathfrak{M}_0^\bullet(W) = \bigsqcup_V \mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W).$$

A necessary condition for $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W)$ to be nonempty is that

$$\dim W_i(p) - \dim V_i(p+1) - \dim V_i(p-1) + \sum_{j\sim i} \dim V_j(p) \geq 0$$

for every $(i,p) \in \widehat{I}$. In this case we say that (V,W) is a *dominant pair*. Equivalently, by (32), the pair (V,W) is dominant if and only if the monomial $Y^W A^V \in \mathscr{Y}$ is dominant, where we use the shorthand notation

$$Y^W := \prod_{(i,p)\in\widehat{I}} Y_{i,p}^{\dim W_i(p)}, \qquad A^V := \prod_{(i,s)\in\widehat{J}} A_{i,s}^{-\dim V_i(s)}.$$

Note that this stratification of $\mathfrak{M}_0^\bullet(W)$ is G_W-invariant. Hence each stratum is a union of G_W-orbits.

Given a dominant pair (V,W) such that $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W) \neq \emptyset$, we denote by $IC_W(V)$ the intersection cohomology complex of the closure of the stratum $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W)$. Let $\mathscr{H}^i(IC_W(V))$ be its ith cohomology sheaf, and $\mathscr{H}^i(IC_W(V))_{V'}$ be the stalk of this sheaf at a point of $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V',W)$.

For a dominant monomial $m \in \mathscr{Y}$, recall from §5.6 and §5.8 the (q,t)-characters $\chi_{q,t}(M(m))$ and $\chi_{q,t}(L(m))$ of the standard and of the simple $U_q(L\mathfrak{g})$-modules labelled by m. Write

$$\chi_{q,t}(M(m')) = \sum_{m'} \zeta_{m,m'}(t)\, \chi_{q,t}(L(m)).$$

Nakajima has shown:

Theorem 9.2 [**N3**, §8] *The coefficients $\zeta_{m,m'}(t)$ are given by*

$$\zeta_{m,m'}(t) = t^{\dim \mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V',W) - \dim \mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W)} \sum_{i\geq 0} t^i \dim \mathscr{H}^i(IC_W(V))_{V'}, \tag{41}$$

for any pair of strata $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W)$ and $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V',W)$ such that $m = Y^W A^V$ and $m' = Y^W A^{V'}$.

Remark 9.3 (a) In order to make the comparison between Theorem 9.1 and Theorem 9.2 easier, we stated Nakajima's formula (41) in a different way from the original one. In [**N3**], Nakajima writes

$$t^{\dim \mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V',W)} \sum_{i\geq 0} t^{-i} \dim \mathscr{H}^i(i^!_{x_{V'}} IC_W(V))$$

for the right-hand side of (41), but in his degree convention the trivial local system on the open stratum $S = \mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W)$ appears in the intersection cohomology complex of $\overline{S}$ in degree $\dim S$, while in Lusztig's convention it appears in degree 0. Here we follow Lusztig's convention. Moreover Nakajima uses the costalk $i^!_x$ at a point x instead of the stalk i^*_x, which explains the change of t^i into t^{-i}.

(b) A dominant monomial m can be written in several ways as $m = Y^W A^V$. The fact that the right-hand side of (41) depends only on the monomials m and m', and not on the particular choices of spaces W, V, V', follows from a transversal slice argument [**N1**, §3].

9.3 An isomorphism

Let $\mathbf{d} = (d_i)$ be a dimension vector, as in §9.1. Recall the bijection $\varphi\colon \widehat{I} \to \widehat{\Delta}$ of §2.2. We define an $\widehat{I}$-graded space $W^{\mathbf{d}}$ by taking

$$W_j^{\mathbf{d}}(p) := \mathbb{C}^{d_i} \quad \text{if} \quad \varphi(j,p) = (\alpha_i, 0),$$

and $W_j^{\mathbf{d}}(p) := 0$ for all others $(j,p) \in \widehat{I}$. Clearly, the group $G_{W^{\mathbf{d}}}$ is isomorphic to $G_{\mathbf{d}}$ and we may identify $G_{W^{\mathbf{d}}} \equiv G_{\mathbf{d}}$.

Proposition 9.4 *There is a $G_{\mathbf{d}}$-equivariant closed immersion of affine varieties*

$$\Psi\colon \mathfrak{M}_0^\bullet(W^{\mathbf{d}}) \longrightarrow E_{\mathbf{d}}.$$

The proof of Proposition 9.4 will follow from the next two lemmas.

Lemma 9.5 *Let $i \neq i' \in I$ and set $(j,p) = \varphi^{-1}(\alpha_i,0)$, $(j',p') = \varphi^{-1}(\alpha_{i'},0)$. Assume that $p' \leq p$, and write $\varphi(j', \xi_j + p' - p + 2) = (\beta,0)$. Then, the coefficient of α_j in the expansion of the root β on the basis of simple roots is equal to 1 if there is an arrow $i \to i'$ in Q, and to 0 otherwise.*

Proof — By definition of (j,p), (j',p'), and of φ (see §2.2), we have

$$\alpha_i = \tau^{(\xi_j-p)/2}(\gamma_j), \qquad \alpha_{i'} = \tau^{(\xi_{j'}-p')/2}(\gamma_{j'}).$$

It follows that

$$\beta = \tau^{-1+(\xi_{j'}-\xi_j-p'+p)/2}(\gamma_{j'}) = \tau^{-1+(p-\xi_j)/2}(\alpha_{i'}).$$

Recall the Ringel bilinear form $\langle\cdot,\cdot\rangle$. It may be characterized by

$$\langle\alpha_i,\gamma_j\rangle = \delta_{ij}, \qquad (i,j\in I).$$

Hence, the coefficient of α_j in β is equal to:

$$\langle\beta,\gamma_j\rangle = \langle\tau^{-1+(p-\xi_j)/2}(\alpha_{i'}),\gamma_j\rangle = \langle\tau^{-1}(\alpha_{i'}),\tau^{(\xi_j-p)/2}(\gamma_j)\rangle = \langle\tau^{-1}(\alpha_{i'}),\alpha_i\rangle = -\langle\alpha_i,\alpha_{i'}\rangle.$$

Now

$$-\langle\alpha_i,\alpha_{i'}\rangle = -\dim(\mathrm{Hom}(S_i,S_{i'})) + \dim(\mathrm{Ext}^1(S_i,S_{i'})) = \dim(\mathrm{Ext}^1(S_i,S_{i'})),$$

and this is equal to 1 if there is an arrow from i to i' in Q, and to 0 otherwise. □

We now introduce an algebra $\widetilde{\Lambda}_Q$ defined by a quiver $\widetilde{\Gamma}_Q$ with relations. The vertices of $\widetilde{\Gamma}_Q$ are of two types:

- $w_j(p)$ for every $(j,p) = \varphi^{-1}(\alpha_i,0)$ $(i\in I)$;
- $v_j(p-1)$ for every pair $(j,p)\in\widehat{I}_Q$ such that $(j,p-2)\in\widehat{I}_Q$.

The arrows of $\widetilde{\Gamma}_Q$ are of three types:

- $a_j(p)\colon w_j(p)\to v_j(p-1)$;
- $b_j(p)\colon v_j(p)\to w_j(p-1)$;
- $\mathscr{B}_{ij}(p)\colon v_i(p)\to v_j(p-1)$ if $j\sim i$.

The relations are:

$$a_i(p-1)b_i(p) = \sum_{j\sim i}\varepsilon(i,j)\mathscr{B}_{ji}(p-1)\mathscr{B}_{ij}(p).$$

Obviously, as suggested by the notation, the definition of $\widetilde{\Lambda}_Q$ is so that the affine variety $\Lambda^\bullet(V,W^{\mathbf{d}})$ is nothing but the representation variety of $\widetilde{\Lambda}_Q$ consisting of representations for which the spaces $W_j^{\mathbf{d}}(p) = \mathbb{C}^{d_i}$ are attached to the vertices $w_j(p)$, and the spaces $V_j(p-1)$ are attached to the vertices $v_j(p-1)$ (we assume that $V_j(p-1) = 0$ if $v_j(p-1)$ is not a vertex of $\widetilde{\Gamma}_Q$).

For $i\in I$, we denote by ε_i the idempotent of $\widetilde{\Lambda}_Q$ associated with the vertex $w_j(p)$ such that $(j,p) = \varphi^{-1}(\alpha_i,0)$. We endow I with a total ordering such that $i > i'$ if $p > p'$, where as above $\varphi^{-1}(\alpha_{i'},0) = (j',p')$. It is well-known that if there is an arrow $i\to i'$ in Q then $i > i'$.

Lemma 9.6 *For $i \neq i' \in I$, we have*

$$\dim\left(\varepsilon_{i'}\widetilde{\Lambda}_Q\varepsilon_i\right) = \begin{cases} 1 & \textit{if there is a path from } i \textit{ to } i' \textit{ in } Q, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof — If $i < i'$ then $p \leq p'$ and clearly $\varepsilon_{i'}\widetilde{\Lambda}_Q\varepsilon_i = 0$. On the other hand if $i < i'$ there can be no path from i to i' in Q. Thus the lemma is clear in this case, and we may assume from now on that $i > i'$.

Let $x \in \varepsilon_{i'}\widetilde{\Lambda}_Q\varepsilon_i$, and let $i'' \in I$. Then $x \in \varepsilon_{i'}\widetilde{\Lambda}_Q\varepsilon_{i''}\widetilde{\Lambda}_Q\varepsilon_i$ if and only if x belongs to the two-sided ideal of $\widetilde{\Lambda}_Q$ generated by $a_{j''}(p-1)b_{j''}(p)$ where $\varphi(j'', p-1) = (\alpha_{i''}, 0)$. This is because $b_{j''}(p)$ is the only arrow entering in $w_{j''}(p-1)$, and $a_{j''}(p-1)$ is the only arrow exiting from $w_{j''}(p-1)$. Note that $\varepsilon_{i'}\widetilde{\Lambda}_Q\varepsilon_{i''}\widetilde{\Lambda}_Q\varepsilon_i \neq 0$ implies that $i > i'' > i'$.

Let $\mathscr{I}$ be the the two-sided ideal of $\widetilde{\Lambda}_Q$ generated by all the degree two paths:

$$a_j(p-1)b_j(p), \qquad ((j, p-1) = \varphi^{-1}(\alpha_i, 0),\ (i \in I)).$$

Then, the algebra $\widetilde{\Lambda}_Q/\mathscr{I}$ is defined by the same relations as the graded preprojective algebra $\widehat{\Lambda}$ of [**Le**, §2.8] (but we have the additional vertices $w_j(p)$ and only a finite set of vertices $v_j(p)$). It follows that $\dim\left(\left(\varepsilon_{i'}\widetilde{\Lambda}_Q\varepsilon_i\right)/(\varepsilon_{i'}\mathscr{I}\varepsilon_i)\right)$ is equal to the $v_{j'}(p'+1)$-component of the dimension vector of the indecomposable projective $\widehat{\Lambda}$-module with top $v_j(p-1)$. Now it is well-known that this dimension vector can be read off from the Auslander-Reiten quiver of Q. Namely, using our notation, the $v_j(p-1)$-component of the dimension vector of the projective with top $v_i(\xi_i - 1)$ is equal to the coefficient of α_i in the root β such that $\varphi(j,p) = (\beta, 0)$. The dimension vectors of the remaining indecomposable projectives are obtained from these particular ones by translation. It follows that we can reformulate Lemma 9.5 as follows:

$$\dim\left(\left(\varepsilon_{i'}\widetilde{\Lambda}_Q\varepsilon_i\right)/(\varepsilon_{i'}\mathscr{I}\varepsilon_i)\right) = \begin{cases} 1 & \text{if there is an arrow from } i \text{ to } i' \text{ in } Q, \\ 0 & \text{otherwise.} \end{cases} \tag{42}$$

In particular if i' is the successor of i in the descending total order defined above, then $\varepsilon_{i'}\mathscr{I}\varepsilon_i = 0$, and we have

$$\dim\left(\varepsilon_{i'}\widetilde{\Lambda}_Q\varepsilon_i\right) = \begin{cases} 1 & \text{if there is an arrow from } i \text{ to } i' \text{ in } Q, \\ 0 & \text{otherwise.} \end{cases} \tag{43}$$

Assume now that $i > i'$ are such that there is no path from i to i' in Q. Then in particular there is no arrow $i \to i'$, so by (42) we have

$$\varepsilon_{i'}\widetilde{\Lambda}_Q\varepsilon_i = \varepsilon_{i'}\mathscr{I}\varepsilon_i = \sum_{i'<j<i}\left(\varepsilon_{i'}\widetilde{\Lambda}_Q\varepsilon_j\right)\left(\varepsilon_j\widetilde{\Lambda}_Q\varepsilon_i\right).$$

For each summand, we have either no path from i to j or no path from j to i'. So we can iterate the splitting until we obtain an expression of the form

$$\varepsilon_{i'}\widetilde{\Lambda}_Q\varepsilon_i = \sum_{i'<i_1<\cdots<i_k<i}\left(\varepsilon_{i'}\widetilde{\Lambda}_Q\varepsilon_{i_k}\right)\cdots\left(\varepsilon_{i_1}\widetilde{\Lambda}_Q\varepsilon_i\right),$$

where in the right-hand side each factor $\varepsilon_k\widetilde{\Lambda}_Q\varepsilon_j$ is such that either we have a path from j to k or k is the successor of j and there is no arrow from j to k. Moreover, since we have no path from

i to i' each summand contains at least one factor of the second type, which is equal to 0 by (43). Hence we have shown that $\varepsilon_{i'}\widetilde{\Lambda}_Q\varepsilon_i = 0$.

Assume now that there is an arrow $i \to i'$ in Q. Then we have as above

$$\varepsilon_{i'}\mathscr{I}\varepsilon_i = \sum_{i'<j<i} \left(\varepsilon_{i'}\widetilde{\Lambda}_Q\varepsilon_j\right)\left(\varepsilon_j\widetilde{\Lambda}_Q\varepsilon_i\right),$$

where for each j we have either no path from i to j or no path from j to i' (because the Dynkin diagram is a tree). Thus it follows from above that $\varepsilon_{i'}\mathscr{I}\varepsilon_i = 0$, so by (42) we get $\dim\left(\varepsilon_{i'}\widetilde{\Lambda}_Q\varepsilon_i\right) = 1$.

Finally, if there is a path $i \to i_1 \to \cdots \to i_k \to i'$ in Q, with $k \geq 1$, then there is no arrow from i to i', and by (42) we have $\varepsilon_{i'}\widetilde{\Lambda}_Q\varepsilon_i = \varepsilon_{i'}\mathscr{I}\varepsilon_i$. Moreover, this path is unique, and arguing as above we can write

$$\varepsilon_{i'}\widetilde{\Lambda}_Q\varepsilon_i = \left(\varepsilon_{i'}\widetilde{\Lambda}_Q\varepsilon_{i_k}\right)\cdots\left(\varepsilon_{i_1}\widetilde{\Lambda}_Q\varepsilon_i\right),$$

where each factor has dimension 1, so again $\dim\left(\varepsilon_{i'}\widetilde{\Lambda}_Q\varepsilon_i\right) = 1$. □

Proof of Proposition 9.4 — Let V be a $\widehat{J}$-graded space, and pick $(B,\alpha,\beta) \in \Lambda^\bullet(V,W^{\mathbf{d}})$. As explained above, we can regard (B,α,β) as a representation of $\widetilde{\Lambda}_Q$. Choose two vertices i and i' of Q, and set as before

$$(j,p) = \varphi^{-1}(\alpha_i,0), \qquad (j',p') = \varphi^{-1}(\alpha_{i'},0).$$

By Lemma 9.6 we have $\dim(\varepsilon_{i'}\widetilde{\Lambda}_Q\varepsilon_i) \leq 1$. Let $\theta_{ii'}$ be a generator of $\varepsilon_{i'}\widetilde{\Lambda}_Q\varepsilon_i$. By the proof of Lemma 9.6, we can normalize the $\theta_{ii'}$ so that they verify $\theta_{i'i''}\theta_{ii'} = \theta_{ii''}$ for every $i,i',i'' \in I$. More precisely, if there is a path $i \to i_1 \to \cdots \to i_k \to i'$ in Q, then $\theta_{i,i'} = \theta_{i_ki'}\cdots\theta_{ii_1}$, and if there is no path from i to i' then $\theta_{ii'} = 0$. Evaluating $\theta_{ii'}$ in the representation (B,α,β) we obtain a linear map $\psi_{ii'} : W^{\mathbf{d}}_j(p) \to W^{\mathbf{d}}_{j'}(p')$. The collection of maps $(\psi_{ii'})$ for all arrows $i \to i'$ of Q can be regarded as a representation ψ of Q of dimension vector $\mathbf{d}$. It follows easily from the definition of the G_V-action that ψ depends only on the G_V-orbit of (B,α,β). Hence the assignment $(B,\alpha,\beta) \mapsto \psi$ induces a morphism of varieties $\Psi_V : \mathfrak{M}^\bullet_0(V,W^{\mathbf{d}}) \to E_{\mathbf{d}}$. Moreover, it follows from the known description of the generators of the coordinate ring of $\mathfrak{M}^\bullet_0(V,W^{\mathbf{d}})$ (see [**N4**, §3.1]) that this coordinate ring is generated by the matrix coefficients of the linear maps $\psi_{ii'}$ for all pairs (i,i'). Hence Ψ_V induces a surjective morphism from $\mathbb{C}[E_{\mathbf{d}}]$ to $\mathbb{C}[\mathfrak{M}^\bullet_0(V,W^{\mathbf{d}})]$, thus Ψ_V is a closed immersion. Since for V large enough we have $\mathfrak{M}^\bullet_0(V,W^{\mathbf{d}}) = \mathfrak{M}^\bullet_0(W^{\mathbf{d}})$, we obtain a closed immersion $\Psi\colon \mathfrak{M}^\bullet_0(W^{\mathbf{d}}) \to E_{\mathbf{d}}$. By construction, Ψ commutes with the actions of $G_{\mathbf{d}}$ on both varieties. □

Example 9.7 Take $\mathfrak{g}$ of type D_4. We label the Dynkin diagram so that the central node is numbered 3, and we choose $\xi_1 = \xi_2 = \xi_4 = 4$ and $\xi_3 = 5$. Thus Q has a sink-source orientation with source 3 and sinks $1,2,4$. Given a dimension vector $\mathbf{d} = (d_1,d_2,d_3,d_4)$, the corresponding graded space $W^{\mathbf{d}}$ is given by

$$W^{\mathbf{d}}_1(0) = \mathbb{C}^{d_1}, \quad W^{\mathbf{d}}_2(0) = \mathbb{C}^{d_2}, \quad W^{\mathbf{d}}_3(5) = \mathbb{C}^{d_3}, \quad W^{\mathbf{d}}_4(0) = \mathbb{C}^{d_4},$$

and the other $W_i(p)$'s are zero (see the Auslander-Reiten quiver of Q in Figure 4). An element (B,α,β) of $\Lambda(V,W^{\mathbf{d}})$ is represented in Figure 3. The defining equations (40) read

$$\begin{aligned} B_{13}(3)B_{31}(4) + B_{23}(3)B_{32}(4) + B_{43}(3)B_{34}(4) &= 0, \\ B_{31}(2)B_{13}(3) \;=\; B_{32}(2)B_{23}(3) \;=\; B_{34}(2)B_{43}(3) &= 0. \end{aligned}$$

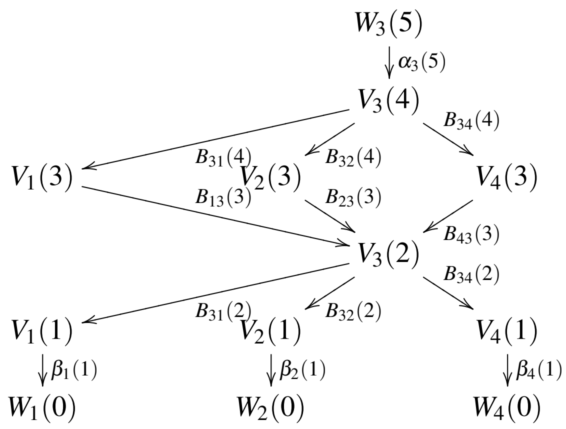

Figure 3: (B,α,β) *in type* D_4.

Thus,

$$\beta_1(1)B_{31}(2)B_{43}(3)B_{34}(4)\alpha_3(5) = -\beta_1(1)B_{31}(2)B_{23}(3)B_{32}(4)\alpha_3(5)$$

and

$$\beta_1(1)B_{31}(2)B_{13}(3)B_{31}(4)\alpha_3(5) = 0.$$

Hence we can take

$$\psi_{31} := \beta_1(1)B_{31}(2)B_{43}(3)B_{34}(4)\alpha_3(5),$$

and similarly

$$\psi_{32} := \beta_2(1)B_{32}(2)B_{13}(3)B_{31}(4)\alpha_3(5), \qquad \psi_{34} := \beta_4(1)B_{34}(2)B_{23}(3)B_{32}(4)\alpha_3(5).$$

We get a representation $\psi := (\psi_{31},\, \psi_{32},\, \psi_{34})$ of Q on the space $W^{\mathbf{d}}$.

Proposition 9.8 *There is a bijection between the set of (nonempty) strata* $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W^{\mathbf{d}})$ *and the set* $I_{\mathbf{d}}$ *of* $G_{\mathbf{d}}$*-orbits.*

Proof — Let us first consider a stratum $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W^{\mathbf{d}})$. By §9.2, the pair $(V,W^{\mathbf{d}})$ is a dominant pair. This means that we have nonnegative integers a_j $(1 \le j \le r)$ such that

$$Y^{W^{\mathbf{d}}}A^V = \prod_{j=1}^{r} Y_{i_j,p_j}^{a_j}. \tag{44}$$

Here for $1 \le j \le r$, we have put $(i_j, p_j) = \varphi^{-1}(\beta_j, 0)$. Indeed, by definition, every dominant commutative monomial of the form $Y^{W^{\mathbf{d}}}A^V$ belongs to $\mathscr{Y}_{t,Q}$. Moreover we have a natural grading of $\mathscr{Y}_{t,Q}$ by the root lattice of $\mathfrak{g}$ given by

$$\deg(Y_{i_j,p_j}) = \beta_j, \qquad (1 \le j \le r).$$

It is easy to see that for every $A_{i,s} \in \mathscr{Y}_{t,Q}$ we have $\deg A_{i,s} = 0$. Therefore

$$\sum_{k=1}^{r} a_j\beta_j = \deg\left(Y^{W^{\mathbf{d}}}\right) = \mathbf{d}.$$

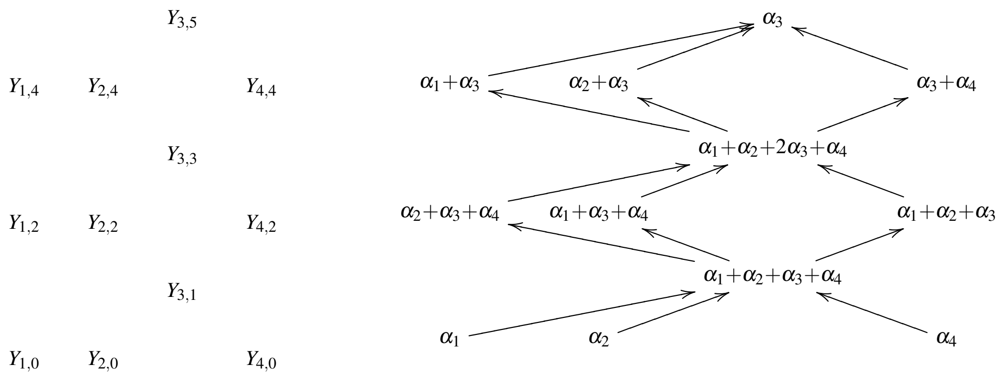

Figure 4: *The skeleton of $\mathscr{C}_Q$ and the Auslander-Reiten quiver in type D_4.*

Hence, to every stratum $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W^{\mathbf{d}})$ corresponds an element $\mathbf{a}$ of $I_{\mathbf{d}}$ given by (44).

Conversely, if $\mathbf{a}\in I_{\mathbf{d}}$, we need to show that $m_{\mathbf{a}} := \prod_{j=1}^{r} Y_{i_j,p_j}^{a_j}$ can be written in the form (44) for some nonempty stratum $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W^{\mathbf{d}})$. By [**N1**, Th. 14.3.2], this is equivalent to the fact that $m_{\mathbf{a}}$ appears in the q-character of the standard module $M(Y^{W^{\mathbf{d}}})$. For $i\in I$ write as in §7.1 $(k_i,p_i)=\varphi^{-1}(\alpha_i,0)$. Then we have by definition of $W^{\mathbf{d}}$

$$Y^{W^{\mathbf{d}}} = \prod_{i\in I} Y_{k_i,p_i}^{d_i}.$$

By §7.1 we know that the (q,t)-characters of the fundamental modules $L(Y_{k_i,p_i})$ $(i\in I)$ generate $\mathscr{K}_{t,Q}$. Hence the simple module $L(m_{\mathbf{a}})$, which is an object of $\mathscr{C}_Q$, is a composition factor of a standard module of the form $M(\prod_{i\in I} Y_{k_i,p_i}^{e_i})$ for some nonnegative integers e_i. But, as before, we must have

$$\mathbf{d} = \deg(m_{\mathbf{a}}) = \sum_{i\in I} e_i \deg(Y_{k_i,p_i}) = \sum_{i\in I} e_i\alpha_i,$$

hence $e_i = d_i$ for every i. Therefore $m_{\mathbf{a}}$ is indeed a weight of $M(Y^{W^{\mathbf{d}}})$. This proves the claim. □

Remark 9.9 The proof of Proposition 9.8 shows that $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W^{\mathbf{d}})$ is a nonempty stratum of $\mathfrak{M}_0^{\bullet}(W^{\mathbf{d}})$ if and only if $(V,W^{\mathbf{d}})$ is a dominant pair, a purely combinatorial condition. In general Nakajima [**N1**, Th. 14.3.2] only shows that this is a necessary condition. In representation-theoretic terms, this means that every dominant monomial of the form $Y^{W^{\mathbf{d}}}A^V$ occurs in the q-character of the standard module $M(Y^{W^{\mathbf{d}}})$.

Example 9.10 We continue Example 9.7. There are 12 positive roots β_k, which we identify with the vertices of the Auslander-Reiten quiver of Q represented in Figure 4. The numbering is obtained by reading this graph from top to bottom and left to right:

$$\beta_1=\alpha_3,\quad \beta_2=\alpha_1+\alpha_3,\quad \beta_3=\alpha_2+\alpha_3,\quad \beta_4=\alpha_3+\alpha_4,\quad \cdots,\quad \beta_{12}=\alpha_4.$$

The corresponding generators Y_{i_k,p_k} of $\mathscr{Y}_{t,Q}$ can be read at the corresponding place on the left side of Figure 4. Let $\mathbf{d}=(d_1,d_2,d_3,d_4)$ be a dimension vector for Q. Then

$$Y^{W^{\mathbf{d}}} = Y_{1,0}^{d_1} Y_{2,0}^{d_2} Y_{3,5}^{d_3} Y_{4,0}^{d_4}.$$

The elements of $I_{\mathbf{d}}$ are 12-tuples $\mathbf{a} \in \mathbb{N}^{12}$ encoding the decompositions of $\mathbf{d}$ into a sum of positive roots. By Proposition 9.8, they are in one-to-one correspondence with the dominant monomials of the form $Y^{W^{\mathbf{d}}}A^V$. This bijection can be read immediately from Figure 4.

For example, if $\mathbf{d} = (1,1,1,1) \equiv \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$, the correspondence is:

$$\begin{array}{lclcl}
(\alpha_1)+(\alpha_2)+(\alpha_3)+(\alpha_4) & \leftrightarrow & Y_{1,0}Y_{2,0}Y_{3,5}Y_{4,0} & \leftrightarrow & 1, \\
(\alpha_1+\alpha_3)+(\alpha_2)+(\alpha_4) & \leftrightarrow & Y_{1,4}Y_{2,0}Y_{4,0} & \leftrightarrow & A_{1,1}A_{3,2}A_{2,3}A_{4,3}A_{3,4}, \\
(\alpha_2+\alpha_3)+(\alpha_1)+(\alpha_4) & \leftrightarrow & Y_{2,4}Y_{1,0}Y_{4,0} & \leftrightarrow & A_{2,1}A_{3,2}A_{1,3}A_{4,3}A_{3,4}, \\
(\alpha_3+\alpha_4)+(\alpha_1)+(\alpha_2) & \leftrightarrow & Y_{4,4}Y_{1,0}Y_{2,0} & \leftrightarrow & A_{4,1}A_{3,2}A_{1,3}A_{2,3}A_{3,4}, \\
(\alpha_1+\alpha_2+\alpha_3)+(\alpha_4) & \leftrightarrow & Y_{4,2}Y_{4,0} & \leftrightarrow & A_{1,1}A_{2,1}A_{3,2}^2A_{1,3}A_{2,3}A_{4,3}A_{3,4}, \\
(\alpha_1+\alpha_3+\alpha_4)+(\alpha_2) & \leftrightarrow & Y_{2,2}Y_{2,0} & \leftrightarrow & A_{1,1}A_{4,1}A_{3,2}^2A_{1,3}A_{2,3}A_{4,3}A_{3,4}, \\
(\alpha_2+\alpha_3+\alpha_4)+(\alpha_1) & \leftrightarrow & Y_{1,2}Y_{1,0} & \leftrightarrow & A_{2,1}A_{4,1}A_{3,2}^2A_{1,3}A_{2,3}A_{4,3}A_{3,4}, \\
(\alpha_1+\alpha_2+\alpha_3+\alpha_4) & \leftrightarrow & Y_{3,1} & \leftrightarrow & A_{1,1}A_{2,1}A_{4,1}A_{3,2}^2A_{1,3}A_{2,3}A_{4,3}A_{3,4}.
\end{array}$$

It is obtained by replacing each root β_k in a decomposition of $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ by the corresponding variable Y_{i_k,p_k}. The third column gives the monomial $Y^{W^{\mathbf{d}}}\left(\prod_{k=1}^{r} Y_{i_k,p_k}^{a_k}\right)^{-1}$.

We can now state the main result of this section.

Theorem 9.11 (a) *We have a $G_{\mathbf{d}}$-equivariant isomorphism of varieties $\Psi\colon \mathfrak{M}_0^\bullet(W^{\mathbf{d}}) \xrightarrow{\sim} E_{\mathbf{d}}$.*

(b) *$\mathfrak{M}_0^\bullet(W^{\mathbf{d}})$ is an affine space of dimension $\sum_{i\to j} d_i d_j$.*

(c) *Lusztig's perverse sheaves $IC(\mathscr{O}_{\mathbf{a}})$ on $E_{\mathbf{d}}$ are the same as Nakajima's perverse sheaves $IC_{W^{\mathbf{d}}}(V)$ on $\mathfrak{M}_0^\bullet(W^{\mathbf{d}})$.*

Proof— In Proposition 9.4 we have constructed a $G_{\mathbf{d}}$-equivariant closed immersion Ψ of $\mathfrak{M}_0^\bullet(W^{\mathbf{d}})$ into $E_{\mathbf{d}}$. Since each stratum $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(V,W^{\mathbf{d}})$ is $G_{\mathbf{d}}$-invariant, Ψ maps every stratum to a union of orbits $\mathscr{O}_{\mathbf{a}}$. Since Ψ is injective and the number of strata is equal to the number of orbits (Proposition 9.8), it follows that Ψ maps each stratum of $\mathfrak{M}_0^\bullet(W^{\mathbf{d}})$ to a single $G_{\mathbf{d}}$-orbit in $E_{\mathbf{d}}$, so every orbit is contained in the image of Ψ. Thus Ψ is surjective. Since a surjective closed immersion between reduced schemes is an isomorphism, this proves (a). Claim (b) follows immediately from (a), and claim (c) is again a consequence of Proposition 9.8, which shows that the stratifications used for defining the perverse sheaves are the same. □

Remark 9.12 (a) By the proof of Proposition 9.8, for every dominant monomial m in $\mathscr{Y}_{t,Q}$ there is a unique pair $(V,W^{\mathbf{d}})$ such that $m = Y^{W^{\mathbf{d}}}A^V$. Hence, even if the varieties $\mathfrak{M}_0^\bullet(W^{\mathbf{d}})$ involve very particular spaces $W^{\mathbf{d}}$, the isomorphisms $\Psi\colon \mathfrak{M}_0^\bullet(W^{\mathbf{d}}) \xrightarrow{\sim} E_{\mathbf{d}}$ are enough to identify all the irreducible (q,t)-characters of $\mathscr{C}_Q$.

Thus Theorem 9.1, Theorem 9.2, and Theorem 9.11 provide a geometric explanation of part (b) of Theorem 6.1. By comparing convolution diagrams in [**Lu2**] and [**VV1**], it should also be possible to understand in a geometric manner part (a) of Theorem 6.1, that is, the multiplicative structure (see [**N4**, §3.5]).

(b) If we take for Q a quiver of type A with all arrows in the same direction, then $\mathfrak{M}_0^\bullet(W^{\mathbf{d}})$ is just a space of graded nilpotent endomorphisms as in the Ginzburg-Vasserot construction [**GV**] of type A quantum loop algebras (see *e.g.* [**Le**, §2.5.3]). So Theorem 9.11 becomes tautological in this case.

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