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Quantum Hermite–Hadamard inequality by means of a Green function

Muhammad Adil Khan¹, Noor Mohammad¹, Eze R. Nwaeze² and Yu-Ming Chu^{3,4*}

*Correspondence:

chuyuming2005@126.com

³Department of Mathematics,
Huzhou University, Huzhou, China

⁴School of Mathematics and
Statistics, Changsha University of
Science & Technology, Changsha,
China

Full list of author information is
available at the end of the article

Abstract

The purpose of this work is to present the quantum Hermite–Hadamard inequality through the Green function approach. While doing this, we deduce some novel quantum identities. Using these identities, we establish some new inequalities in this direction. We contemplate the possibility of expanding the method, outlined herein, to recast the proofs of some known inequalities in the literature.

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1 Introduction

Let $\psi : [b_1, b_2] \rightarrow \mathbb{R}$ be a convex function. Then the following double inequality holds:

$$\psi\left(\frac{b_1 + b_2}{2}\right) \leq \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \psi(x) dx \leq \frac{\psi(b_1) + \psi(b_2)}{2}. \quad (1.1)$$

It is known in the literature as the Hermite–Hadamard inequality. This inequality has instigated plethora of papers. Results concerning generalization, refinement, and extension of (1.1) are also found; see [1–9, 12, 14, 15, 17–20, 23, 27–30] and the references therein.

In the early 16th century, the concept of q -calculus was introduced. Ever since, integral inequalities of the trapezoid, Ostrowski, Cauchy–Bunyakovsky–Schwarz, Grüss, Hölder, Grüss–Čebyšev, and other types have been established in the q -calculus sense. In 2014, Tariboon and Ntouyas [33] obtained the following q -calculus version of (1.1).

Theorem 1.1 *Let $\psi : [b_1, b_2] \rightarrow \mathbb{R}$ be a convex continuous function on (b_1, b_2) , and let $0 < q < 1$. Then*

$$\psi\left(\frac{b_1 + b_2}{2}\right) \leq \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \psi(x) {}_{b_1}d_q x \leq \frac{q\psi(b_1) + \psi(b_2)}{1 + q}. \quad (1.2)$$

In 2016, Kunt and İşcan [21] observed, via a counterexample, that the left-hand side of inequality (1.2) is not necessarily true. Subsequently, Alp et al. [11] proved the following correct version of (1.2).

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Theorem 1.2 ([11]) Let $\psi : [b_1, b_2] \rightarrow \mathbb{R}$ be a convex differentiable function on (b_1, b_2) , and let $0 < q < 1$. Then

$$\psi\left(\frac{qb_1 + b_2}{1+q}\right) \leq \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \psi(x)_{b_1} d_q x \leq \frac{q\psi(b_1) + \psi(b_2)}{1+q}. \quad (1.3)$$

Remark 1.3 It is important to note that the inequality in Theorem 1.2 was first established by Marinković et al. [24, Theorem 5.3].

The aim of this work is to recast inequality (1.3) in Theorem 1.2 via another approach different from that presented in [11]. Specifically, we do this using a Green function. In the process, we establish some identities that are also used to obtain more results in this direction.

We organize this paper as follows. Section 2 contains a brief introduction of the quantum calculus. Our main results are then framed and proved in Sect. 3.

2 Preliminaries

Quantum calculus is known as the calculus without limits. In this section, we present a quick overview of the theory of q -calculus. The interested reader is invited to the book [16] for an in-depth study of this subject. We begin with these basic definitions.

Definition 2.1 ([32]) Let $\psi : [b_1, b_2] \rightarrow \mathbb{R}$ be a continuous function, and let $w \in [b_1, b_2]$. Then the expression

$${}_{{b_1}}D_q \psi(w) = \frac{\psi(w) - \psi(qw + (1-q)b_1)}{(1-q)(w - b_1)}, \quad w \neq b_1, \quad {}_{{b_1}}D_q \psi(b_1) = \lim_{w \rightarrow b_1} {}_{{b_1}}D_q \psi(w) \quad (2.1)$$

is called the q -derivative on $[b_1, b_2]$ of the function at w .

We call ψ q -differentiable on $[b_1, b_2]$ if ${}_{{b_1}}D_q \psi(w)$ exists for all $w \in [b_1, b_2]$.

Definition 2.2 ([32]) Let $\psi : [b_1, b_2] \rightarrow \mathbb{R}$ be a continuous function. Then the q -integral on $[b_1, b_2]$ is defined as

$$\int_{b_1}^w \psi(x)_{b_1} d_q x = (1-q)(w - b_1) \sum_{k=0}^{\infty} q^k \psi(q^k w + (1-q^k)b_1) \quad (2.2)$$

for $w \in [b_1, b_2]$. Moreover, if $c \in (b_1, w)$, then the q -integral on $[b_1, b_2]$ is defined as

$$\int_c^w \psi(x)_{b_1} d_q x = \int_{b_1}^w \psi(x)_{b_1} d_q x - \int_{b_1}^c \psi(x)_{b_1} d_q x. \quad (2.3)$$

Remark 2.3 In light of Definitions 2.1 and 2.2, we make the following remarks:

1. By taking $b_1 = 0$ expression (2.1) boils down to the well-known q -derivative $D_q \psi(w)$ of the function $\psi(w)$ defined by

$$D_q \psi(w) = \frac{\psi(w) - \psi(qw)}{(1-q)w}.$$

2. Also, if $b_1 = 0$, then (2.2) reduces to the classical q -integral of a function $\psi : [0, \infty) \rightarrow \mathbb{R}$ defined by

$$\int_0^w \psi(x)_0 d_q x = (1-q)w \sum_{k=0}^{\infty} q^k \psi(q^k w).$$

Some known results in continuous calculus have been extended to the q -calculus framework as follows.

Theorem 2.4 ([32]) *Let $\psi : [b_1, b_2] \rightarrow \mathbb{R}$ be a continuous function. Then we have*

$$\int_{\delta}^w {}_{b_1}D_q \psi(w)_{b_1} d_q x = \psi(w) - \psi(\delta) \quad \text{for } \delta \in (b_1, w).$$

Theorem 2.5 ([13]) *Let $\psi, \phi : [b_1, b_2] \rightarrow \mathbb{R}$ be two continuous functions and suppose $\psi(x) \leq \phi(x)$ for all $x \in [b_1, b_2]$. Then*

$$\int_{b_1}^w \psi(x)_{b_1} d_q x \leq \int_{b_1}^w \phi(x)_{b_1} d_q x.$$

Theorem 2.6 ([32]) *Let $\psi : [b_1, b_2] \rightarrow \mathbb{R}$ be a continuous function. Then*

$$\begin{aligned} {}_{b_1}D_q \int_{b_1}^w \psi(x)_{b_1} d_q x &= \psi(w); \\ \int_c^w {}_{b_1}D_q \psi(x)_{b_1} d_q x &= \psi(w) - \psi(c), \quad \text{for } c \in (b_1, w). \end{aligned}$$

Theorem 2.7 ([32]) *Let $\psi, \phi : [b_1, b_2] \rightarrow \mathbb{R}$ be continuous functions, and let $\alpha \in \mathbb{R}$. Then, for $w \in [b_1, b_2]$ and $c \in (b_1, w)$, we have*

$$\begin{aligned} \int_{b_1}^w [\psi(x) + \phi(x)]_{b_1} d_q x &= \int_{b_1}^w \psi(x)_{b_1} d_q x + \int_{b_1}^w \phi(x)_{b_1} d_q x; \\ \int_{b_1}^w \alpha \psi(x)_{b_1} d_q x &= \alpha \int_{b_1}^w \psi(x)_{b_1} d_q x; \\ \int_c^w \psi(x)_{b_1} {}_{b_1}D_q \phi(x)_{b_1} d_q x \\ &= \psi(w)\phi(w) - \psi(c)\phi(c) - \int_c^w \phi(qx + (1-q)b_1)_{b_1} {}_{b_1}D_q \psi(x)_{b_1} d_q x. \end{aligned}$$

3 Main results

We will prove our fundamental results with the help of the following lemma.

Lemma 3.1 ([10, 25]) *Let G be the Green function defined on $[b_1, b_2] \times [b_1, b_2]$ by*

$$G(x, u) = \begin{cases} b_1 - u, & b_1 \leq u \leq x; \\ b_1 - x, & x \leq u \leq b_2. \end{cases}$$

Then any $\psi \in C^2([b_1, b_2])$ can be expressed as

$$\psi(x) = \psi(b_1) + (x - b_1)\psi'(b_2) + \int_{b_1}^{b_2} G(x, \mu)\psi''(\mu) d\mu. \quad (3.1)$$

We now state and justify our main results.

Theorem 3.2 *Let $\psi : [b_1, b_2] \rightarrow \mathbb{R}$ be a convex twice differentiable function on (b_1, b_2) . If $0 < q < 1$, then*

$$\psi\left(\frac{qb_1 + b_2}{q + 1}\right) \leq \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \psi(x)_{b_1} d_q x \leq \frac{q\psi(b_1) + \psi(b_2)}{q + 1}. \quad (3.2)$$

Proof If we set $x = \frac{qb_1 + b_2}{q + 1}$ in (3.1), then we get

$$\begin{aligned} \psi\left(\frac{qb_1 + b_2}{q + 1}\right) &= \psi(b_1) + \left(\frac{qb_1 + b_2}{q + 1} - b_1\right)\psi'(b_2) + \int_{b_1}^{b_2} G\left(\frac{qb_1 + b_2}{q + 1}, u\right)\psi''(u) du \\ &= \psi(b_1) + \frac{b_2 - b_1}{q + 1}\psi'(b_2) + \int_{b_1}^{b_2} G\left(\frac{qb_1 + b_2}{q + 1}, u\right)\psi''(u) du. \end{aligned} \quad (3.3)$$

By computing we obtain that

$$\begin{aligned} &\frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \psi(x)_{b_1} d_q x \\ &= \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \left\{ \psi(b_1) + (x - b_1)\psi'(b_2) + \int_{b_1}^{b_2} G(x, u)\psi''(u) du \right\}_{b_1} d_q x \\ &= \psi(b_1) + \frac{b_2 - b_1}{q + 1}\psi'(b_2) + \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \int_{b_1}^{b_2} G(x, u)\psi''(u) du_{b_1} d_q x. \end{aligned} \quad (3.4)$$

Subtracting (3.4) from (3.3), we get:

$$\begin{aligned} &\psi\left(\frac{qb_1 + b_2}{q + 1}\right) - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \psi(x)_{b_1} d_q x \\ &= \psi(b_1) + \frac{b_2 - b_1}{q + 1}\psi'(b_2) + \int_{b_1}^{b_2} G\left(\frac{qb_1 + b_2}{q + 1}, u\right)\psi''(u) du \\ &\quad - \psi(b_1) - \frac{b_2 - b_1}{q + 1}\psi'(b_2) - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \int_{b_1}^{b_2} G(x, u)\psi''(u) du_{b_1} d_q x \\ &= \int_{b_1}^{b_2} G\left(\frac{qb_1 + b_2}{q + 1}, u\right)\psi''(u) du - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \int_{b_1}^{b_2} G(x, u)\psi''(u) du_{b_1} d_q x \\ &= \int_{b_1}^{b_2} \left\{ G\left(\frac{qb_1 + b_2}{q + 1}, u\right) - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} G(x, u)_{b_1} d_q x \right\} \psi''(u) du \\ &= \int_{b_1}^{b_2} \left[G\left(\frac{qb_1 + b_2}{q + 1}, u\right) + \frac{1}{b_2 - b_1} \left\{ \frac{(u - b_1)^2}{q + 1} + (b_2 - u)(u - b_1) \right\} \right] \psi''(u) du. \end{aligned} \quad (3.5)$$

Next, we consider the function

$$f(u) = G\left(\frac{qb_1 + b_2}{q+1}, u\right) + \frac{1}{b_2 - b_1} \left\{ \frac{(u - b_1)^2}{q+1} + (b_2 - u)(u - b_1) \right\}. \quad (3.6)$$

For this, the following cases are possible.

Case 1. If $b_1 \leq u \leq \frac{qb_1 + b_2}{q+1}$, then

$$f(u) = b_1 - u + \frac{1}{b_2 - b_1} \left\{ \frac{(u - b_1)^2}{q+1} + (b_2 - u)(u - b_1) \right\}.$$

Therefore

$$\begin{aligned} f'(u) &= -1 + \frac{1}{b_2 - b_1} \left\{ \frac{2(u - b_1)}{q+1} + (b_2 - u) - (u - b_1) \right\}; \\ f''(u) &= \frac{1}{b_2 - b_1} \left\{ \frac{2}{q+1} - 2 \right\} = \frac{-2q}{(1+q)(b_2 - b_1)} < 0. \end{aligned}$$

This implies that f' is decreasing and $f'(b_1) = 0$, which shows that $f'(u) \leq 0$. Thus f is also decreasing, and $f(b_1) = 0$, that is, $f(u) \leq 0$ for all $u \in [b_1, \frac{qb_1 + b_2}{q+1}]$.

Case 2. If $\frac{qb_1 + b_2}{q+1} \leq u \leq b_2$, then

$$\begin{aligned} f(u) &= \frac{b_1 - b_2}{q+1} + \frac{1}{b_2 - b_1} \left\{ \frac{(u - b_1)^2}{q+1} + (b_2 - u)(u - b_1) \right\}; \\ f'(u) &= \frac{1}{b_2 - b_1} \left\{ \frac{u - b_1}{q+1} (1 - q) + (b_2 - u) \right\} > 0. \end{aligned}$$

Hence f is increasing and $f(b_2) = 0$. So, $f(u) \leq 0$ for all $u \in [\frac{qb_1 + b_2}{q+1}, b_2]$.

Now, using (3.5) and the fact that $\psi''(u) \geq 0$ for all $u \in [b_1, b_2]$, since ψ is convex, we obtain the first inequality:

$$\psi\left(\frac{qb_1 + b_2}{q+1}\right) \leq \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \psi(x)_{b_1} d_q x.$$

For the right-hand side inequality, we recall that

$$\begin{aligned} \psi(x) &= \psi(b_1) + (x - b_1)\psi'(b_2) + \int_{b_1}^{b_2} G(x, u)\psi''(u) du; \\ \psi(b_2) &= \psi(b_1) + (b_2 - b_1)\psi'(b_2) + \int_{b_1}^{b_2} G(b_2, u)\psi''(u) du; \\ \frac{q\psi(b_1) + \psi(b_2)}{q+1} &= \psi(b_1) + \frac{b_2 - b_1}{q+1}\psi'(b_2) + \frac{1}{q+1} \int_{b_1}^{b_2} G(b_2, u)\psi''(u) du. \end{aligned} \quad (3.7)$$

Subtracting (3.4) from (3.7), we get

$$\frac{q\psi(b_1) + \psi(b_2)}{q+1} - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \psi(x)_{b_1} d_q x$$

$$= \int_{b_1}^{b_2} \left[\frac{G(b_2, u)}{q+1} + \frac{1}{b_2 - b_1} \left\{ \frac{(u - b_1)^2}{q+1} + (b_2 - u)(u - b_1) \right\} \right] \psi''(u) du. \quad (3.8)$$

Let

$$F(u) = \frac{G(b_2, u)}{q+1} + \frac{1}{b_2 - b_1} \left\{ \frac{(u - b_1)^2}{q+1} + (b_2 - u)(u - b_1) \right\}.$$

Then

$$\begin{aligned} F(u) &= \frac{b_1 - u}{q+1} + \frac{1}{b_2 - b_1} \left\{ \frac{(u - b_1)^2}{q+1} + (b_2 - u)(u - b_1) \right\}; \\ F'(u) &= \frac{-1}{q+1} + \frac{1}{b_2 - b_1} \left\{ \frac{2(u - b_1)}{q+1} + (b_2 - u) - (u - b_1) \right\}; \\ F''(u) &= \frac{1}{b_2 - b_1} \left\{ \frac{2}{q+1} - 2 \right\} = \frac{-2q}{(1+q)(b_2 - b_1)} < 0. \end{aligned}$$

Here we also observe two cases.

Case 3. If $b_1 \leq u \leq \frac{b_1+b_2}{2}$, then $F''(u) < 0$. Therefore F' is decreasing, and also $F'(\frac{b_1+b_2}{2}) = 0$, which shows that $F'(u) \geq 0$. Moreover, F is increasing, and $F(b_1) = 0$. Hence $F(u) \geq 0$ for all $u \in [b_1, \frac{b_1+b_2}{2}]$.

Case 4. Also, if $\frac{b_1+b_2}{2} \leq u \leq b_2$, then $F''(u) < 0$. So, F' is decreasing, and $F'(\frac{b_1+b_2}{2}) = 0$, which implies that $F'(u) \leq 0$. Hence F is decreasing, and $F(b_2) = 0$, and then $F(u) \geq 0$ for all $u \in [\frac{b_1+b_2}{2}, b_2]$.

Combining these two cases, we conclude that $F(u) \geq 0$ for all $u \in [b_1, b_2]$. Applying (3.8) and the convexity of ψ , we establish the right-hand side of the desired inequality. The proof is complete. \square

Next, we prove new quantum Hermite–Hadamard inequalities for the class of monotone and convex functions.

Theorem 3.3 *Let $\psi \in C^2([b_1, b_2])$ and $0 < q < 1$. Then:*

(i). *If $|\psi''|$ is an increasing function, then*

$$\left| \frac{q\psi(b_1) + \psi(b_2)}{1+q} - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \psi(x) {}_{b_1}d_q x \right| \leq |\psi''(b_2)| \left[\frac{q(b_2 - b_1)^2}{6(1+q)} \right].$$

(ii). *If $|\psi''|$ is a decreasing function, then*

$$\left| \frac{q\psi(b_1) + \psi(b_2)}{1+q} - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \psi(x) {}_{b_1}d_q x \right| \leq |\psi''(b_1)| \left[\frac{q(b_2 - b_1)^2}{6(1+q)} \right].$$

(iii). *If $|\psi''|$ is a convex function, then*

$$\begin{aligned} & \left| \frac{q\psi(b_1) + \psi(b_2)}{1+q} - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \psi(x) {}_{b_1}d_q x \right| \\ & \leq \max\{|\psi''(b_1)|, |\psi''(b_2)|\} \left[\frac{q(b_2 - b_1)^2}{6(1+q)} \right]. \end{aligned}$$

Proof To prove (i), by (3.8) we get:

$$\begin{aligned}
 & \left| \frac{q\psi(b_1) + \psi(b_2)}{1+q} - \frac{1}{b_2-b_1} \int_{b_1}^{b_2} \psi(x) {}_{b_1}d_q x \right| \\
 &= \left| \int_{b_1}^{b_2} \left[\frac{G(b_2, u)}{q+1} + \frac{1}{b_2-b_1} \left\{ \frac{(u-b_1)^2}{q+1} + (b_2-u)(u-b_1) \right\} \right] \psi''(u) du \right| \\
 &= \left| \int_{b_1}^{b_2} \left[\frac{b_1-u}{q+1} + \frac{1}{b_2-b_1} \left\{ \frac{(u-b_1)^2}{q+1} - b_1b_2 + (b_1+b_2)u - u^2 \right\} \right] \psi''(u) du \right| \\
 &\leq |\psi''(b_2)| \left[\frac{b_1}{1+q} \int_{b_1}^{b_2} 1 du - \frac{1}{1+q} \int_{b_1}^{b_2} u du + \frac{1}{b_2-b_1} \left\{ \int_{b_1}^{b_2} \frac{(u-b_1)^2}{1+q} du \right. \right. \\
 &\quad \left. \left. - b_1b_2 \int_{b_1}^{b_2} 1 du + (b_1+b_2) \int_{b_1}^{b_2} u du - \int_{b_1}^{b_2} u^2 du \right\} \right] \\
 &= |\psi''(b_2)| \left[\frac{b_1(b_2-b_1)}{1+q} - \frac{u^2}{2(1+q)} \Big|_{b_1}^{b_2} + \frac{1}{b_2-b_1} \left\{ \frac{(u-b_1)^3}{3(1+q)} \Big|_{b_1}^{b_2} - b_1b_2(b_2-b_1) \right. \right. \\
 &\quad \left. \left. + (b_1+b_2) \frac{u^2}{2} \Big|_{b_1}^{b_2} - \frac{u^3}{3} \Big|_{b_1}^{b_2} \right\} \right] \\
 &= |\psi''(b_2)| \left[\frac{b_1(b_2-b_1)}{1+q} - \frac{b_2^2-b_1^2}{2(1+q)} + \frac{1}{b_2-b_1} \left\{ \frac{(b_2-b_1)^3}{3(1+q)} - b_1b_2(b_2-b_1) \right. \right. \\
 &\quad \left. \left. + (b_1+b_2) \frac{b_2^2-b_1^2}{2} - \frac{b_2^3-b_1^3}{3} \right\} \right] \\
 &= |\psi''(b_2)| \left[\frac{q(b_2-b_1)^2}{6(1+q)} \right],
 \end{aligned}$$

which proves the inequality in (i).

Part (ii) can be proved in a similar fashion. For part (iii), using (3.8) and the fact that $|\psi''|$ is bounded above, on the interval $[b_1, b_2]$, by $\max\{|\psi''(b_1)|, |\psi''(b_2)|\}$ as a convex function, we obtain:

$$\begin{aligned}
 & \left| \frac{q\psi(b_1) + \psi(b_2)}{1+q} - \frac{1}{b_2-b_1} \int_{b_1}^{b_2} \psi(x) {}_{b_1}d_q x \right| \\
 &\leq \max\{|\psi''(b_1)|, |\psi''(b_2)|\} \left[\frac{q(b_2-b_1)^2}{6(1+q)} \right]. \quad \square
 \end{aligned}$$

Theorem 3.4 Let $\psi \in C^2([b_1, b_2])$, and let $|\psi''|$ be a concave function. Then, for $0 < q < 1$,

$$\begin{aligned}
 & \left| \frac{q\psi(b_1) + \psi(b_2)}{1+q} - \frac{1}{b_2-b_1} \int_{b_1}^{b_2} \psi(x) {}_{b_1}d_q x \right| \\
 &\leq (b_2-b_1)^2 \left[\frac{1}{2(q+1)} \left| \psi''\left(\frac{b_1+2b_2}{3}\right) \right| \right. \\
 &\quad \left. + \frac{1}{3(q+1)} \left| \psi''\left(\frac{b_1+3b_2}{4}\right) \right| + \frac{1}{6} \left| \psi''\left(\frac{b_1+b_2}{2}\right) \right| \right].
 \end{aligned}$$

Proof Employing identity (3.8), we have:

$$\begin{aligned} & \left| \frac{q\psi(b_1) + \psi(b_2)}{1+q} - \frac{1}{b_2-b_1} \int_{b_1}^{b_2} \psi(x) {}_{b_1}d_q x \right| \\ &= \left| \int_{b_1}^{b_2} \left[\frac{G(b_2, u)}{q+1} + \frac{1}{b_2-b_1} \left\{ \frac{(u-b_1)^2}{q+1} + (b_2-u)(u-b_1) \right\} \right] \psi''(u) du \right| \\ &= \left| \int_{b_1}^{b_2} \left[\frac{b_1-u}{q+1} + \frac{1}{b_2-b_1} \left\{ \frac{(u-b_1)^2}{q+1} + (b_2-u)(u-b_1) \right\} \right] \psi''(u) du \right|. \end{aligned}$$

Suppose $u = (1-t)b_1 + tb_2$ with $t \in [0, 1]$. Then

$$\begin{aligned} &= \left| \int_0^1 \left[\frac{b_1 - (1-t)b_1 - tb_2}{q+1} + \frac{1}{b_2-b_1} \left\{ \frac{((1-t)b_1 + tb_2 - b_1)^2}{q+1} \right. \right. \right. \\ &\quad \left. \left. + (b_2 - (1-t)b_1 - tb_2)(-tb_1 + tb_2) \right\} \right] \psi''((1-t)b_1 + tb_2)(b_2 - b_1) dt \Big| \\ &= \left| \int_0^1 \left[\frac{-t(b_2 - b_1)}{q+1} + \frac{1}{b_2-b_1} \left\{ \frac{t^2(b_2 - b_1)^2}{q+1} \right. \right. \right. \\ &\quad \left. \left. + t(1-t)(b_2 - b_1)(b_2 - b_1) \right\} \right] \psi''((1-t)b_1 + tb_2)(b_2 - b_1) dt \Big| \\ &\leq (b_2 - b_1)^2 \left[\frac{1}{q+1} \left| \int_0^1 t \psi''((1-t)b_1 + tb_2) dt \right| \right. \\ &\quad \left. + \frac{1}{q+1} \left| \int_0^1 t^2 \psi''((1-t)b_1 + tb_2) dt \right| + \left| \int_0^1 t(1-t) \psi''((1-t)b_1 + tb_2) dt \right| \right]. \quad (3.9) \end{aligned}$$

Now, using the Jensen integral inequality, we get the following estimates:

$$\begin{aligned} & \left| \int_0^1 t \psi''((1-t)b_1 + tb_2) dt \right| \\ &\leq \int_0^1 t dt \left| \psi'' \left(\frac{\int_0^1 t((1-t)b_1 + tb_2) dt}{\int_0^1 t dt} \right) \right| \\ &= \frac{1}{2} \left| \psi'' \left(\frac{b_1 \int_0^1 (t-t^2) dt + b_2 \int_0^1 t^2 dt}{\frac{1}{2}} \right) \right| \\ &= \frac{1}{2} \left| \psi'' \left(\frac{b_1 + 2b_2}{3} \right) \right|, \quad (3.10) \end{aligned}$$

$$\begin{aligned} & \left| \int_0^1 t^2 \psi''((1-t)b_1 + tb_2) dt \right| \\ &\leq \int_0^1 t^2 dt \left| \psi'' \left(\frac{\int_0^1 t^2((1-t)b_1 + tb_2) dt}{\int_0^1 t^2 dt} \right) \right| \\ &= \frac{1}{3} \left| \psi'' \left(\frac{b_1 \int_0^1 (t^2 - t^3) dt + b_2 \int_0^1 t^3 dt}{\frac{1}{3}} \right) \right| \\ &= \frac{1}{3} \left| \psi'' \left(\frac{b_1 + 3b_2}{4} \right) \right|, \quad (3.11) \end{aligned}$$

and

$$\begin{aligned}
 & \left| \int_0^1 (t-t^2) \psi''((1-t)b_1 + tb_2) dt \right| \\
 & \leq \int_0^1 (t-t^2) dt \left| \psi'' \left(\frac{\int_0^1 (t-t^2)((1-t)b_1 + tb_2) dt}{\int_0^1 (t-t^2) dt} \right) \right| \\
 & = \frac{1}{6} \left| \psi'' \left(\frac{b_1 \int_0^1 (t-t^2)(1-t) dt + b_2 \int_0^1 (t-t^2)t dt}{\frac{1}{6}} \right) \right| \\
 & = \frac{1}{6} \left| \psi'' \left(\frac{b_1 + b_2}{2} \right) \right|. \tag{3.12}
 \end{aligned}$$

Putting (3.10), (3.11), and (3.12) into (3.9), we get

$$\begin{aligned}
 & \left| \frac{q\psi(b_1) + \psi(b_2)}{1+q} - \frac{1}{b_2-b_1} \int_{b_1}^{b_2} \psi(x) {}_{b_1}d_q x \right| \\
 & \leq (b_2-b_1)^2 \left[\frac{1}{2(q+1)} \left| \psi'' \left(\frac{b_1+2b_2}{3} \right) \right| \right. \\
 & \quad \left. + \frac{1}{3(q+1)} \left| \psi'' \left(\frac{b_1+3b_2}{4} \right) \right| + \frac{1}{6} \left| \psi'' \left(\frac{b_1+b_2}{2} \right) \right| \right]. \quad \square
 \end{aligned}$$

Theorem 3.5 Suppose $\psi \in C^2([b_1, b_2])$ and $q \in (0, 1)$. Then:

(i). If $|\psi''|$ is an increasing function, then

$$\begin{aligned}
 & \left| \psi'' \left(\frac{qb_1 + b_2}{1+q} \right) - \frac{1}{b_2-b_1} \int_{b_1}^{b_2} \psi(x) {}_{b_1}d_q x \right| \\
 & \leq |\psi''(b_2)| \left[\frac{(qb_1 + b_2)^2}{2(1+q)^2} - b_1 \frac{qb_1 + b_2}{1+q} + \frac{q(b_2-b_1)^2}{(1+q)^2} - \frac{(b_2-b_1)^2}{3(q+1)} \right] \\
 & \quad - \frac{|\psi''(b_2)|}{6(b_2-b_1)} [2b_1^3 + b_2^3 - 3b_1b_2^2].
 \end{aligned}$$

(ii). If $|\psi''|$ is a decreasing function, then

$$\begin{aligned}
 & \left| \psi'' \left(\frac{qb_1 + b_2}{1+q} \right) - \frac{1}{b_2-b_1} \int_{b_1}^{b_2} \psi(x) {}_{b_1}d_q x \right| \\
 & \leq |\psi''(b_1)| \left[\frac{(qb_1 + b_2)^2}{2(1+q)^2} - b_1 \frac{qb_1 + b_2}{1+q} + \frac{q(b_2-b_1)^2}{(1+q)^2} - \frac{(b_2-b_1)^2}{3(q+1)} \right] \\
 & \quad - \frac{|\psi''(b_2)|}{6(b_2-b_1)} [2b_1^3 + b_2^3 - 3b_1b_2^2].
 \end{aligned}$$

(iii). If $|\psi''|$ is a convex function, then

$$\begin{aligned}
 & \left| \psi'' \left(\frac{qb_1 + b_2}{1+q} \right) - \frac{1}{b_2-b_1} \int_{b_1}^{b_2} \psi(x) {}_{b_1}d_q x \right| \\
 & \leq \max \{ |\psi''(b_2)|, |\psi''(b_1)| \} \left[\frac{(qb_1 + b_2)^2}{2(1+q)^2} - b_1 \frac{qb_1 + b_2}{1+q} + \frac{q(b_2-b_1)^2}{(1+q)^2} \right. \\
 & \quad \left. - \frac{(b_2-b_1)^2}{3(q+1)} \right] - \min \{ |\psi''(b_1)|, |\psi''(b_2)| \} \left[\frac{2b_1^3 + b_2^3 - 3b_1b_2^2}{6(b_2-b_1)} \right].
 \end{aligned}$$

Proof To prove item (i), we use (3.5) and the fact that $|\psi''|$ is an increasing function to obtain:

$$\begin{aligned}
& \left| \psi'' \left(\frac{qb_1 + b_2}{1+q} \right) - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \psi(x) {}_{b_1}d_q x \right| \\
&= \left| \int_{b_1}^{b_2} \left[G \left(\frac{qb_1 + b_2}{q+1}, u \right) + \frac{1}{b_2 - b_1} \left\{ \frac{(u - b_1)^2}{q+1} + (b_2 - u)(u - b_1) \right\} \right] \psi''(u) du \right| \\
&\leq |\psi''(b_2)| \int_{b_1}^{b_2} \left| G \left(\frac{qb_1 + b_2}{q+1}, u \right) + \frac{1}{b_2 - b_1} \left\{ \frac{(u - b_1)^2}{q+1} + (b_2 - u)(u - b_1) \right\} \right| du \\
&\leq |\psi''(b_2)| \left[\int_{b_1}^{\frac{qb_1 + b_2}{1+q}} \left| b_1 - u + \frac{1}{b_2 - b_1} \left\{ \frac{(u - b_1)^2}{q+1} - u^2 + (b_1 + b_2)u - b_1 b_2 \right\} \right| du \right. \\
&\quad \left. + \int_{\frac{qb_1 + b_2}{1+q}}^{b_2} \left| \frac{b_1 - b_2}{q+1} + \frac{1}{b_2 - b_1} \left\{ \frac{(u - b_1)^2}{q+1} - u^2 + (b_1 + b_2)u - b_1 b_2 \right\} \right| du \right] \\
&= |\psi''(b_2)| \left[\int_{b_1}^{\frac{qb_1 + b_2}{1+q}} \left\{ u - b_1 - \frac{1}{b_2 - b_1} \left\{ \frac{(u - b_1)^2}{q+1} - u^2 + (b_1 + b_2)u - b_1 b_2 \right\} \right\} du \right. \\
&\quad \left. + \int_{\frac{qb_1 + b_2}{1+q}}^{b_2} \left\{ \frac{b_2 - b_1}{q+1} - \frac{1}{b_2 - b_1} \left\{ \frac{(u - b_1)^2}{q+1} - u^2 + (b_1 + b_2)u - b_1 b_2 \right\} \right\} du \right] \\
&= |\psi''(b_2)| \left[\left(\frac{u^2}{2} - b_1 u \right) \Big|_{b_1}^{\frac{qb_1 + b_2}{1+q}} - \frac{1}{b_2 - b_1} \left\{ \frac{(u - b_1)^3}{3(q+1)} \Big|_{b_1}^{\frac{qb_1 + b_2}{1+q}} - \frac{u^3}{3} \Big|_{b_1}^{\frac{qb_1 + b_2}{1+q}} \right. \right. \\
&\quad \left. \left. + (b_1 + b_2) \frac{u^2}{2} \Big|_{b_1}^{\frac{qb_1 + b_2}{1+q}} - b_2 b_1 u \Big|_{b_1}^{\frac{qb_1 + b_2}{1+q}} \right\} + \frac{b_2 - b_1}{1+q} \left(b_2 - \frac{qb_1 + b_2}{1+q} \right) - \frac{1}{b_2 - b_1} \right. \\
&\quad \left. \times \left\{ \frac{(u - b_1)^3}{3(q+1)} \Big|_{\frac{qb_1 + b_2}{1+q}}^{b_2} - \frac{u^3}{3} \Big|_{\frac{qb_1 + b_2}{1+q}}^{b_2} + (b_1 + b_2) \frac{u^2}{2} \Big|_{\frac{qb_1 + b_2}{1+q}}^{b_2} - b_2 b_1 u \Big|_{\frac{qb_1 + b_2}{1+q}}^{b_2} \right\} \right] \\
&= |\psi''(b_2)| \left[\left(\frac{(\frac{qb_1 + b_2}{1+q})^2}{2} - b_1 \left(\frac{qb_1 + b_2}{1+q} \right) - \frac{b_1^2}{2} + b_1^2 \right) - \frac{1}{b_2 - b_1} \left\{ \frac{(\frac{qb_1 + b_2}{1+q} - b_1)^3}{3(q+1)} \right. \right. \\
&\quad \left. \left. - \frac{(qb_1 + b_2)^3}{3(1+q)^3} + \frac{b_1^3}{3} + (b_1 + b_2) \frac{(qb_1 + b_2)^2}{2(1+q)^2} - (b_1 + b_2) \frac{b_1^2}{2} - b_2 b_1 \left(\frac{qb_1 + b_2}{1+q} - b_1 \right) \right\} \right. \\
&\quad \left. + \frac{b_2 - b_1}{1+q} \left(b_2 - \frac{qb_1 + b_2}{1+q} \right) - \frac{1}{b_2 - b_1} \left\{ \frac{(b_2 - b_1)^3}{3(q+1)} - \frac{(\frac{qb_1 + b_2}{1+q} - b_1)^3}{3(q+1)} - \frac{b_2^3}{3} \right. \right. \\
&\quad \left. \left. + \frac{(qb_1 + b_2)^3}{3(1+q)^3} + (b_1 + b_2) \frac{b_2^2}{2} - (b_1 + b_2) \frac{(qb_1 + b_2)^2}{2(1+q)^2} - b_2 b_1 \left(b_2 - \frac{qb_1 + b_2}{1+q} \right) \right\} \right] \\
&= |\psi''(b_2)| \left[\left(\frac{(qb_1 + b_2)^2}{2(1+q)^2} - b_1 \left(\frac{qb_1 + b_2}{1+q} \right) + \frac{b_1^2}{2} \right) - \frac{1}{b_2 - b_1} \left\{ \frac{(b_2 - b_1)^3}{3(q+1)^4} \right. \right. \\
&\quad \left. \left. - \frac{(qb_1 + b_2)^3}{3(1+q)^3} + \frac{b_1^3}{3} + (b_1 + b_2) \frac{(qb_1 + b_2)^2}{2(1+q)^2} - (b_1 + b_2) \frac{b_1^2}{2} - b_2 b_1 \left(\frac{b_2 - b_1}{1+q} \right) \right\} \right. \\
&\quad \left. + \frac{q(b_2 - b_1)^2}{(1+q)^2} - \frac{1}{b_2 - b_1} \left\{ \frac{(b_2 - b_1)^3}{3(q+1)} - \frac{(b_2 - b_1)^3}{3(q+1)^4} - \frac{b_2^3}{3} + \frac{(qb_1 + b_2)^3}{3(1+q)^3} \right. \right. \\
&\quad \left. \left. + (b_1 + b_2) \frac{b_2^2}{2} - (b_1 + b_2) \frac{(qb_1 + b_2)^2}{2(1+q)^2} - b_2 b_1 \frac{q(b_2 - b_1)}{1+q} \right\} \right]
\end{aligned}$$

$$\begin{aligned}
&= |\psi''(b_2)| \left[\frac{(qb_1 + b_2)^2}{2(1+q)^2} - b_1 \left(\frac{qb_1 + b_2}{1+q} \right) + \frac{b_1^2}{2} - \frac{1}{b_2 - b_1} \left\{ \frac{b_1^3}{3} - \frac{(b_1 + b_2)b_1^2}{2} \right. \right. \\
&\quad \left. \left. - b_2 b_1 \left(\frac{b_2 - b_1}{1+q} \right) \right\} + \frac{q(b_2 - b_1)^2}{(1+q)^2} - \frac{1}{b_2 - b_1} \left\{ -\frac{b_2^3}{3} + \frac{(b_1 + b_2)b_2^2}{2} + \frac{(b_2 - b_1)^3}{3(q+1)} \right. \right. \\
&\quad \left. \left. - b_2 b_1 \frac{q(b_2 - b_1)}{1+q} \right\} \right] \\
&= |\psi''(b_2)| \left[\frac{(qb_1 + b_2)^2}{2(1+q)^2} - b_1 \frac{qb_1 + b_2}{1+q} + \frac{b_1^2}{2} + \frac{q(b_2 - b_1)^2}{(1+q)^2} - \frac{1}{b_2 - b_1} \left\{ \frac{b_1^3}{3} - \frac{b_2^3}{3} \right. \right. \\
&\quad \left. \left. - \frac{(b_1 + b_2)b_1^2}{2} - b_2 b_1 \left(\frac{b_2 - b_1}{1+q} \right) + \frac{(b_1 + b_2)b_2^2}{2} + \frac{(b_2 - b_1)^3}{3(q+1)} - b_2 b_1 \frac{q(b_2 - b_1)}{1+q} \right\} \right] \\
&= |\psi''(b_2)| \left[\frac{(qb_1 + b_2)^2}{2(1+q)^2} - b_1 \frac{qb_1 + b_2}{1+q} + \frac{b_1^2}{2} + \frac{q(b_2 - b_1)^2}{(1+q)^2} - \frac{1}{b_2 - b_1} \left\{ \frac{b_1^3}{3} \right. \right. \\
&\quad \left. \left. - \frac{(b_1 + b_2)b_1^2}{2} - \frac{b_2^3}{3} + \frac{(b_1 + b_2)b_2^2}{2} + \frac{(b_2 - b_1)^3}{3(q+1)} - b_2 b_1 (b_2 - b_1) \right\} \right] \\
&= \frac{|\psi''(b_2)|}{b_2 - b_1} \left[\frac{(qb_1 + b_2)^2(b_2 - b_1)}{2(1+q)^2} - b_1 \frac{(qb_1 + b_2)(b_2 - b_1)}{1+q} + \frac{b_1^2(b_2 - b_1)}{2} - \frac{b_1^3}{3} \right. \\
&\quad \left. + \frac{q(b_2 - b_1)^3}{(1+q)^2} + \frac{(b_1 + b_2)b_1^2}{2} + \frac{b_2^3}{3} - \frac{(b_1 + b_2)b_2^2}{2} - \frac{(b_2 - b_1)^3}{3(q+1)} + b_2^2 b_1 - b_2 b_1^2 \right] \\
&= |\psi''(b_2)| \left[\frac{(qb_1 + b_2)^2}{2(1+q)^2} - b_1 \frac{qb_1 + b_2}{1+q} + \frac{q(b_2 - b_1)^2}{(1+q)^2} - \frac{(b_2 - b_1)^2}{3(q+1)} \right] \\
&\quad + \frac{|\psi''(b_2)|}{b_2 - b_1} \\
&\quad \times \left[\frac{3b_1^2 b_2 - 3b_1^3 - 2b_1^3 + 3b_1^3 + 3b_1^2 b_2 + 2b_2^3 - 3b_1 b_2^2 - 3b_2^3 + 6b_2^2 b_1 - 6b_2 b_1^2}{6} \right] \\
&= |\psi''(b_2)| \left[\frac{(qb_1 + b_2)^2}{2(1+q)^2} - b_1 \frac{qb_1 + b_2}{1+q} + \frac{q(b_2 - b_1)^2}{(1+q)^2} - \frac{(b_2 - b_1)^2}{3(q+1)} \right] \\
&\quad - \frac{|\psi''(b_2)|}{6(b_2 - b_1)} [2b_1^3 + b_2^3 - 3b_1 b_2^2].
\end{aligned}$$

Part (ii) can be proved in a similar way. For part (iii), using (3.5) and the fact that $|\psi''|$ is bounded above, on the interval $[b_1, b_2]$, by $\max\{|\psi''(b_1)|, |\psi''(b_2)|\}$ as a convex function, we obtain:

$$\begin{aligned}
&\left| \psi'' \left(\frac{qb_1 + b_2}{1+q} \right) - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \psi(x) {}_{b_1}d_q x \right| \\
&\leq \max \left\{ |\psi''(b_2)| \left[\frac{(qb_1 + b_2)^2}{2(1+q)^2} - b_1 \frac{qb_1 + b_2}{1+q} + \frac{q(b_2 - b_1)^2}{(1+q)^2} - \frac{(b_2 - b_1)^2}{3(q+1)} \right] \right. \\
&\quad \left. - \frac{|\psi''(b_2)|}{6(b_2 - b_1)} [2b_1^3 + b_2^3 - 3b_1 b_2^2], |\psi''(b_1)| \left[\frac{(qb_1 + b_2)^2}{2(1+q)^2} - b_1 \frac{qb_1 + b_2}{1+q} \right. \right. \\
&\quad \left. \left. + \frac{q(b_2 - b_1)^2}{(1+q)^2} - \frac{(b_2 - b_1)^2}{3(q+1)} \right] - \frac{|\psi''(b_1)|}{6(b_2 - b_1)} [2b_1^3 + b_2^3 - 3b_1 b_2^2] \right\} \\
&= \max \left\{ |\psi''(b_2)| \left[\frac{(qb_1 + b_2)^2}{2(1+q)^2} - b_1 \frac{qb_1 + b_2}{1+q} + \frac{q(b_2 - b_1)^2}{(1+q)^2} - \frac{(b_2 - b_1)^2}{3(q+1)} \right], \right. \\
&\quad \left. |\psi''(b_1)| \left[\frac{(qb_1 + b_2)^2}{2(1+q)^2} - b_1 \frac{qb_1 + b_2}{1+q} + \frac{q(b_2 - b_1)^2}{(1+q)^2} - \frac{(b_2 - b_1)^2}{3(q+1)} \right] \right\},
\end{aligned}$$

$$\begin{aligned}
& \left| \psi''(b_1) \right| \left[\frac{(qb_1 + b_2)^2}{2(1+q)^2} - b_1 \frac{qb_1 + b_2}{1+q} + \frac{q(b_2 - b_1)^2}{(1+q)^2} - \frac{(b_2 - b_1)^2}{3(q+1)} \right] \\
& + \max \left\{ -\left| \psi''(b_1) \right| \left[\frac{2b_1^3 + b_2^3 - 3b_1b_2^2}{6(b_2 - b_1)} \right], \right. \\
& \left. -\left| \psi''(b_2) \right| \left[\frac{2b_1^3 + b_2^3 - 3b_1b_2^2}{6(b_2 - b_1)} \right] \right\} \\
& = \max \left\{ \left| \psi''(b_2) \right|, \left| \psi''(b_1) \right| \right\} \left[\frac{(qb_1 + b_2)^2}{2(1+q)^2} - b_1 \frac{qb_1 + b_2}{1+q} + \frac{q(b_2 - b_1)^2}{(1+q)^2} - \frac{(b_2 - b_1)^2}{3(q+1)} \right] \\
& - \min \left\{ \left| \psi''(b_1) \right|, \left| \psi''(b_2) \right| \right\} \left[\frac{2b_1^3 + b_2^3 - 3b_1b_2^2}{6(b_2 - b_1)} \right],
\end{aligned}$$

which gives the inequality in item (iii). \square

Theorem 3.6 Let $\psi \in C^2([b_1, b_2])$, and let $|\psi''|$ be a convex function. Then for $q \in (0, 1)$, the following inequality holds:

$$\begin{aligned}
& \left| \frac{q\psi(b_1) + \psi(b_2)}{1+q} - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \psi(x) {}_{b_1}d_q x \right| \\
& \leq \left| \psi''(b_1) \right| \left[\frac{4b_1b_2 + q(b_1 + b_2)^2}{12(q+1)} \right] + \left| \psi''(b_2) \right| \left[\frac{q(b_2 - b_1)^2 - 2(q+1)b_1b_2}{12(q+1)} \right].
\end{aligned}$$

Proof Employing (3.8), we get:

$$\begin{aligned}
& \left| \frac{q\psi(b_1) + \psi(b_2)}{1+q} - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \psi(x) {}_{b_1}d_q x \right| \\
& = \left| \int_{b_1}^{b_2} \left[\frac{G(b_2, u)}{q+1} + \frac{1}{b_2 - b_1} \left\{ \frac{(u - b_1)^2}{q+1} + (b_2 - u)(u - b_1) \right\} \right] \psi''(u) du \right| \\
& = \left| \int_{b_1}^{b_2} \left[\frac{b_1 - u}{q+1} + \frac{1}{b_2 - b_1} \left\{ \frac{(u - b_1)^2}{q+1} + (b_2 - u)(u - b_1) \right\} \right] \psi''(u) du \right|. \\
& \leq \int_{b_1}^{b_2} \left[\frac{b_1 - u}{q+1} + \frac{1}{b_2 - b_1} \left\{ \frac{(u - b_1)^2}{q+1} - u^2 + (b_1 + b_2)u - b_1b_2 \right\} \right] |\psi''(u)| du.
\end{aligned}$$

Putting $u = (1-t)b_1 + tb_2$ with $t \in [0, 1]$, we get

$$\begin{aligned}
& = \int_0^1 \left[\frac{b_1 - (1-t)b_1 - tb_2}{q+1} + \frac{1}{b_2 - b_1} \left\{ \frac{((1-t)b_1 + tb_2 - b_1)^2}{q+1} - ((1-t)b_1 + tb_2)^2 \right. \right. \\
& \quad \left. \left. + (b_1 + b_2)((1-t)b_1 + tb_2) - b_1b_2 \right\} \right] |\psi''((1-t)b_1 + tb_2)| (b_2 - b_1) dt \\
& \leq \int_0^1 \left[\frac{t(b_1 - b_2)}{q+1} + \frac{1}{b_2 - b_1} \left\{ \frac{t^2(b_2 - b_1)^2}{q+1} - (1-t)^2b_1^2 - t^2b_2^2 - 2(1-t)tb_1b_2 \right. \right. \\
& \quad \left. \left. + (b_1 + b_2)b_1(1-t) + (b_1 + b_2)b_2t - b_1b_2 \right\} \right] [(1-t)|\psi''(b_1)| + t|\psi''(b_2)|] (b_2 - b_1) dt \\
& = (b_2 - b_1) |\psi''(b_1)| \int_0^1 \left[\frac{t(1-t)(b_1 - b_2)}{q+1} + \frac{1}{b_2 - b_1} \left\{ \frac{t^2(1-t)(b_2 - b_1)^2}{q+1} \right. \right. \\
& \quad \left. \left. - (1-t)^3b_1^2 - t^2(1-t)b_2^2 - 2(1-t)^2tb_1b_2 + (b_1 + b_2)b_1(1-t)^2 \right\} \right] dt
\end{aligned}$$

$$\begin{aligned}
& + (b_1 + b_2)b_2t(1-t) - b_1b_2(1-t) \Big\} \Big] dt + (b_2 - b_1) |\psi''(b_2)| \int_0^1 \left[\frac{t^2(b_1 - b_2)}{q+1} \right. \\
& + \frac{1}{b_2 - b_1} \left\{ \frac{t^3(b_2 - b_1)^2}{q+1} - t(1-t)^2b_1^2 + t^3b_2^2 + 2(1-t)t^2b_1b_2 \right. \\
& + (b_1 + b_2)b_1t(1-t) + (b_1 + b_2)b_2t^2 - b_1b_2t \Big\} \Big] dt \\
& = |\psi''(b_1)| \int_0^1 \left[\frac{(t^2 - t)(b_2 - b_1)^2}{q+1} + \frac{(t^2 - t^3)(b_2 - b_1)^2}{q+1} - (1-t)^3b_1^2 - (t^2 - t^3)b_2^2 \right. \\
& - 2(1-t)^2tb_1b_2 + (b_1 + b_2)b_1(1-t)^2 + (b_1 + b_2)b_2(t - t^2) - b_1b_2(1-t) \Big] dt \\
& + |\psi''(b_2)| \int_0^1 \left[\frac{-t^2(b_2 - b_1)^2}{q+1} + \frac{t^3(b_2 - b_1)^2}{q+1} - t(1-t)^2b_1^2 - t^3b_2^2 - 2(t^2 - t^3)b_1b_2 \right. \\
& + (b_1 + b_2)b_1(t - t^2) + (b_1 + b_2)b_2t^2 - b_1b_2t \Big] dt \\
& = |\psi''(b_1)| \left[\frac{(\frac{t^3}{3} - \frac{t^2}{2})(b_2 - b_1)^2}{q+1} + \frac{(\frac{t^3}{3} - \frac{t^4}{4})(b_2 - b_1)^2}{q+1} + \frac{(1-t)^4}{4}b_1^2 - \left(\frac{t^3}{3} - \frac{t^4}{4}\right)b_2^2 \right. \\
& + b_1b_2 \left(t^2 + \frac{t^4}{2} - \frac{4t^3}{3} \right) - (b_1 + b_2)b_1 \frac{(1-t)^3}{3} + (b_1 + b_2)b_2 \left(\frac{t^2}{2} - \frac{t^3}{3} \right) \\
& - b_1b_2 \left(t - \frac{t^2}{2} \right) \Big]_0^1 + |\psi''(b_2)| \left[\frac{-t^3(b_2 - b_1)^2}{3(q+1)} + \frac{t^4(b_2 - b_1)^2}{4(q+1)} - \left(\frac{t^2}{2} + \frac{t^4}{4} - \frac{2t^3}{3} \right) b_1^2 \right. \\
& - \frac{t^4}{4}b_2^2 - 2 \left(\frac{t^3}{3} - \frac{t^4}{4} \right) b_1b_2 + (b_1 + b_2)b_1 \left(\frac{t^2}{2} - \frac{t^3}{3} \right) + (b_1 + b_2)b_2 \frac{t^3}{3} - b_1b_2 \frac{t^2}{2} \Big]_0^1 \\
& = |\psi''(b_1)| \left[\frac{-(b_2 - b_1)^2}{6(q+1)} + \frac{(b_2 - b_1)^2}{12(q+1)} - \frac{b_1^2}{4} - \frac{b_2^2}{12} + b_1b_2 \left(1 + \frac{1}{2} - \frac{4}{3} \right) \right. \\
& + \frac{(b_1 + b_2)b_1}{3} + \frac{(b_1 + b_2)b_2}{6} - \frac{b_1b_2}{2} \Big] + |\psi''(b_2)| \left[\frac{-(b_2 - b_1)^2}{3(q+1)} + \frac{(b_2 - b_1)^2}{4(q+1)} \right. \\
& - \left(\frac{1}{2} + \frac{1}{4} - \frac{2}{3} \right) b_1^2 - \frac{b_2^2}{4} - \frac{b_1b_2}{3} + \frac{(b_1 + b_2)b_1}{6} + \frac{(b_1 + b_2)b_2}{3} - \frac{b_1b_2}{2} \Big] \\
& = |\psi''(b_1)| \left[\frac{4b_1b_2 + q(b_1 + b_2)^2}{12(q+1)} \right] + |\psi''(b_2)| \left[\frac{q(b_2 - b_1)^2 - 2(q+1)b_1b_2}{12(q+1)} \right]. \quad \square
\end{aligned}$$

We now present our last result.

Theorem 3.7 Let $\psi \in ([b_1, b_2])$ be such that $|\psi''|$ is a convex function. Then for any $q \in (0, 1)$, the following inequality holds:

$$\begin{aligned}
& \left| \psi'' \left(\frac{qb_1 + b_2}{1+q} \right) - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \psi(x) {}_{b_1}d_q x \right| \\
& \leq |\psi''(b_1)| \left[\frac{(b_2 - b_1)^2(5q^2 + 4q + 1) - 4(b_1 + b_2)b_1}{12(1+q)^3} + \frac{3b_1^2 - b_2^2 + 6b_1b_2}{12} \right] \\
& + |\psi''(b_2)| \left[\frac{(b_2 - b_1)^2(3(1+q)^2 + 10)}{12(1+q)^3} - \frac{(b_2 - b_1)^2}{12} \right].
\end{aligned}$$

Proof Using (3.5), we get:

$$\begin{aligned}
& \left| \psi'' \left(\frac{qb_1 + b_2}{1+q} \right) - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \psi(x) {}_{b_1}d_q x \right| \\
&= \left| \int_{b_1}^{b_2} \left[G \left(\frac{qb_1 + b_2}{q+1}, u \right) + \frac{1}{b_2 - b_1} \left\{ \frac{(u - b_1)^2}{q+1} + (b_2 - u)(u - b_1) \right\} \right] \psi''(u) du \right| \\
&\leq \int_{b_1}^{\frac{qb_1+b_2}{1+q}} \left| b_1 - u + \frac{1}{b_2 - b_1} \left\{ \frac{(u - b_1)^2}{q+1} + (b_2 - u)(u - b_1) \right\} \right| |\psi''(u)| du \\
&\quad + \int_{\frac{qb_1+b_2}{1+q}}^{b_2} \left| \frac{b_1 - b_2}{1+q} + \frac{1}{b_2 - b_1} \left\{ \frac{(u - b_1)^2}{q+1} + (b_2 - u)(u - b_1) \right\} \right| |\psi''(u)| du \\
&= \int_{b_1}^{\frac{qb_1+b_2}{1+q}} \left[u - b_1 - \frac{1}{b_2 - b_1} \left\{ \frac{(u - b_1)^2}{q+1} - u^2 + (b_1 + b_2)u - b_1 b_2 \right\} \right] |\psi''(u)| du \\
&\quad + \int_{\frac{qb_1+b_2}{1+q}}^{b_2} \left[\frac{b_2 - b_1}{1+q} - \frac{1}{b_2 - b_1} \left\{ \frac{(u - b_1)^2}{q+1} - u^2 + (b_1 + b_2)u - b_1 b_2 \right\} \right] |\psi''(u)| du \\
&= (b_2 - b_1) \int_0^{\frac{1}{1+q}} \left[(1-t)b_1 + tb_2 - b_1 - \frac{1}{b_2 - b_1} \left\{ \frac{((1-t)b_1 + tb_2 - b_1)^2}{q+1} \right. \right. \\
&\quad \left. \left. - ((1-t)b_1 + tb_2)^2 + (b_1 + b_2)((1-t)b_1 + tb_2) - b_1 b_2 \right\} \right] |\psi''((1-t)b_1 + tb_2)| dt \\
&\quad + \int_{\frac{1}{1+q}}^1 \left[\frac{b_2 - b_1}{1+q} - \frac{1}{b_2 - b_1} \left\{ \frac{((1-t)b_1 + tb_2 - b_1)^2}{q+1} - ((1-t)b_1 + tb_2)^2 \right. \right. \\
&\quad \left. \left. + (b_1 + b_2)((1-t)b_1 + tb_2) - b_1 b_2 \right\} \right] |\psi''((1-t)b_1 + tb_2)|(b_2 - b_1) dt \\
&\leq (b_2 - b_1) \int_0^{\frac{1}{1+q}} \left[t(b_2 - b_1) - \frac{1}{b_2 - b_1} \left\{ \frac{t^2(b_2 - b_1)^2}{q+1} - (1-t)^2 b_1^2 - t^2 b_2^2 \right. \right. \\
&\quad \left. \left. - 2t(1-t)b_1 b_2 + (b_1 + b_2)b_1(1-t) + t(b_1 + b_2)b_2 - b_1 b_2 \right\} \right] [(1-t)|\psi''(b_1)| \\
&\quad + t|\psi''(b_2)|] dt + \int_{\frac{1}{1+q}}^1 \left[\frac{b_2 - b_1}{1+q} - \frac{1}{b_2 - b_1} \left\{ \frac{t^2(b_2 - b_1)^2}{q+1} - (1-t)^2 b_1^2 \right. \right. \\
&\quad \left. \left. - t^2 b_2^2 - 2t(1-t)b_1 b_2 + (b_1 + b_2)b_1(1-t) + t(b_1 + b_2)b_2 - b_1 b_2 \right\} \right] \\
&\quad \times [(1-t)|\psi''(b_1)| + t|\psi''(b_2)|](b_2 - b_1) dt \\
&= (b_2 - b_1) |\psi''(b_1)| \int_0^{\frac{1}{1+q}} \left[t(1-t)(b_2 - b_1) - \frac{1}{b_2 - b_1} \left\{ \frac{t^2(1-t)(b_2 - b_1)^2}{q+1} \right. \right. \\
&\quad \left. \left. - (1-t)^3 b_1^2 - t^2(1-t)b_2^2 - 2t(1-t)^2 b_1 b_2 + (b_1 + b_2)b_1(1-t)^2 \right. \right. \\
&\quad \left. \left. + t(1-t)(b_1 + b_2)b_2 - b_1 b_2(1-t) \right\} \right] + (b_2 - b_1) |\psi''(b_2)| \int_0^{\frac{1}{1+q}} [t^2(b_2 - b_1) \\
&\quad - \frac{1}{b_2 - b_1} \left\{ \frac{t^3(b_2 - b_1)^2}{q+1} - t(1-t)^2 b_1^2 - t^3 b_2^2 - 2t^2(1-t)b_1 b_2 \right. \\
&\quad \left. \left. + (b_1 + b_2)b_1 t(1-t) \right\}] dt + \int_{\frac{1}{1+q}}^1 [t^2(b_2 - b_1) - \frac{1}{b_2 - b_1} \left\{ \frac{t^3(b_2 - b_1)^2}{q+1} \right. \\
&\quad \left. \left. - t(1-t)^2 b_1^2 - t^3 b_2^2 - 2t^2(1-t)b_1 b_2 \right. \right. \\
&\quad \left. \left. + (b_1 + b_2)b_1 t(1-t) \right\}] | \psi''(b_2) | dt + \int_{\frac{1}{1+q}}^1 [t^2(b_2 - b_1) - \frac{1}{b_2 - b_1} \left\{ \frac{t^3(b_2 - b_1)^2}{q+1} \right. \\
&\quad \left. \left. - t(1-t)^2 b_1^2 - t^3 b_2^2 - 2t^2(1-t)b_1 b_2 \right. \right. \\
&\quad \left. \left. + (b_1 + b_2)b_1 t(1-t) \right\}] | \psi''(b_2) | dt + \int_{\frac{1}{1+q}}^1 [t^2(b_2 - b_1) - \frac{1}{b_2 - b_1} \left\{ \frac{t^3(b_2 - b_1)^2}{q+1} \right. \\
&\quad \left. \left. - t(1-t)^2 b_1^2 - t^3 b_2^2 - 2t^2(1-t)b_1 b_2 \right. \right. \\
&\quad \left. \left. + (b_1 + b_2)b_1 t(1-t) \right\}] | \psi''(b_2) | dt + \int_{\frac{1}{1+q}}^1 [t^2(b_2 - b_1) - \frac{1}{b_2 - b_1} \left\{ \frac{t^3(b_2 - b_1)^2}{q+1} \right. \\
&\quad \left. \left. - t(1-t)^2 b_1^2 - t^3 b_2^2 - 2t^2(1-t)b_1 b_2 \right. \right. \\
&\quad \left. \left. + (b_1 + b_2)b_1 t(1-t) \right\}] | \psi''(b_2) | dt + \int_{\frac{1}{1+q}}^1 [t^2(b_2 - b_1) - \frac{1}{b_2 - b_1} \left\{ \frac{t^3(b_2 - b_1)^2}{q+1} \right. \\
&\quad \left. \left. - t(1-t)^2 b_1^2 - t^3 b_2^2 - 2t^2(1-t)b_1 b_2 \right. \right. \\
&\quad \left. \left. + (b_1 + b_2)b_1 t(1-t) \right\}] | \psi''(b_2) | dt + \int_{\frac{1}{1+q}}^1 [t^2(b_2 - b_1) - \frac{1}{b_2 - b_1} \left\{ \frac{t^3(b_2 - b_1)^2}{q+1} \right. \\
&\quad \left. \left. - t(1-t)^2 b_1^2 - t^3 b_2^2 - 2t^2(1-t)b_1 b_2 \right. \right. \\
&\quad \left. \left. + (b_1 + b_2)b_1 t(1-t) \right\}] | \psi''(b_2) | dt + \dots
\end{aligned}$$

$$\begin{aligned}
& + t^2(b_1 + b_2)b_2 - tb_1b_2 \Bigg] dt + (b_2 - b_1)|\psi''(b_1)| \int_{\frac{1}{1+q}}^1 \left[\frac{b_2 - b_1}{1+q}(1-t) \right. \\
& - \frac{1}{b_2 - b_1} \left\{ \frac{t^2(1-t)(b_2 - b_1)^2}{q+1} - (1-t)^3b_1^2 - t^2(1-t)b_2^2 - 2t(1-t)^2b_1b_2 \right. \\
& + (b_1 + b_2)b_1(1-t)^2 + t(1-t)(b_1 + b_2)b_2 - b_1b_2(1-t) \Bigg\} \Bigg] dt + (b_2 - b_1)|\psi''(b_2)| \\
& \times \int_{\frac{1}{1+q}}^1 \left[\frac{b_2 - b_1}{1+q}t - \frac{1}{b_2 - b_1} \left\{ \frac{t^3(b_2 - b_1)^2}{q+1} - t(1-t)^2b_1^2 - t^3b_2^2 - 2t^2(1-t)b_1b_2 \right. \right. \\
& + (b_1 + b_2)b_1t(1-t) + t^2(b_1 + b_2)b_2 - tb_1b_2 \Bigg\} \Bigg] dt \\
& = (b_2 - b_1)|\psi''(b_1)| \int_0^{\frac{1}{1+q}} \left[(t - t^2)(b_2 - b_1) - \frac{1}{b_2 - b_1} \left\{ \frac{(t^2 - t^3)(b_2 - b_1)^2}{q+1} \right. \right. \\
& - (1-t)^3b_1^2 - (t^2 - t^3)b_2^2 - 2(t + t^3 - 2t^2)b_1b_2 + (b_1 + b_2)b_1(1-t)^2 \\
& + (t - t^2)(b_1 + b_2)b_2 - b_1b_2(1-t) \Bigg\} \Bigg] + (b_2 - b_1)|\psi''(b_2)| \int_0^{\frac{1}{1+q}} \left[t^2(b_2 - b_1) \right. \\
& - \frac{1}{b_2 - b_1} \left\{ \frac{t^3(b_2 - b_1)^2}{q+1} - (t + t^3 - 2t^2)b_1^2 - t^3b_2^2 - 2(t^2 - t^3)b_1b_2 \right. \\
& + (b_1 + b_2)b_1(t - t^2) \\
& + t^2(b_1 + b_2)b_2 - tb_1b_2 \Bigg\} \Bigg] dt + (b_2 - b_1)|\psi''(b_1)| \int_{\frac{1}{1+q}}^1 \left[\frac{b_2 - b_1}{1+q}(1-t) \right. \\
& - \frac{1}{b_2 - b_1} \left\{ \frac{(t^2 - t^3)(b_2 - b_1)^2}{q+1} - (1-t)^3b_1^2 - (t^2 - t^3)b_2^2 - 2(t + t^3 - t^2)b_1b_2 \right. \\
& + (b_1 + b_2)b_1(1-t)^2 + (t - t^2)(b_1 + b_2)b_2 - b_1b_2(1-t) \Bigg\} \Bigg] dt + (b_2 - b_1)|\psi''(b_2)| \\
& \times \int_{\frac{1}{1+q}}^1 \left[\frac{b_2 - b_1}{1+q}t - \frac{1}{b_2 - b_1} \left\{ \frac{t^3(b_2 - b_1)^2}{q+1} - (t + t^3 - 2t^2)b_1^2 - t^3b_2^2 - 2(t^2 - t^3)b_1b_2 \right. \right. \\
& + (b_1 + b_2)b_1(t - t^2) + t^2(b_1 + b_2)b_2 - tb_1b_2 \Bigg\} \Bigg] dt \\
& = (b_2 - b_1)|\psi''(b_1)| \left[\left(\frac{t^2}{2} - \frac{t^3}{3} \right) (b_2 - b_1) - \frac{1}{b_2 - b_1} \left\{ \frac{(\frac{t^3}{3} - \frac{t^4}{4})(b_2 - b_1)^2}{q+1} \right. \right. \\
& + \frac{(1-t)^4}{4}b_1^2 - \left(\frac{t^3}{3} - \frac{t^4}{4} \right) b_2^2 - 2 \left(\frac{t^2}{2} + \frac{t^4}{4} - 2\frac{t^3}{3} \right) b_1b_2 - (b_1 + b_2)b_1 \frac{(1-t)^3}{3} \\
& + \left(\frac{t^2}{2} - \frac{t^3}{3} \right) (b_1 + b_2)b_2 - b_1b_2 \left(t - \frac{t^2}{2} \right) \Bigg]_0^{\frac{1}{1+q}} + (b_2 - b_1)|\psi''(b_2)| \\
& \times \left[\frac{t^3}{3}(b_2 - b_1) - \frac{1}{b_2 - b_1} \left\{ \frac{t^4(b_2 - b_1)^2}{4(q+1)} - \left(\frac{t^2}{2} + \frac{t^4}{4} - \frac{2t^3}{3} \right) b_1^2 - 2 \left(\frac{t^3}{3} - \frac{t^4}{4} \right) b_1b_2 \right. \right. \\
& - \frac{t^4}{4}b_2^2 + (b_1 + b_2)b_1 \left(\frac{t^2}{2} - \frac{t^3}{3} \right) + \frac{t^3}{3}(b_1 + b_2)b_2 - \frac{t^2}{2}b_1b_2 \Bigg\} \Bigg]_0^{\frac{1}{1+q}} \\
& + (b_2 - b_1)|\psi''(b_1)|
\end{aligned}$$

$$\begin{aligned}
& \times \left[\frac{b_2 - b_1}{1+q} \left(t - \frac{t^2}{2} \right) - \frac{1}{b_2 - b_1} \left\{ \frac{\left(\frac{t^3}{3} - \frac{t^4}{4} \right) (b_2 - b_1)^2}{q+1} + \frac{(1-t)^4}{4} b_1^2 - \left(\frac{t^3}{3} - \frac{t^4}{4} \right) b_2^2 \right. \right. \\
& - 2 \left(\frac{t^2}{2} + \frac{t^4}{4} - \frac{2t^3}{3} \right) b_1 b_2 - (b_1 + b_2) b_1 \frac{(1-t)^3}{3} + \left(\frac{t^2}{2} - \frac{t^3}{3} \right) (b_1 + b_2) b_2 \\
& \left. \left. - b_1 b_2 \left(t - \frac{t^2}{2} \right) \right\} \right]_{\frac{1}{1+q}}^1 + (b_2 - b_1) |\psi''(b_2)| \left[\frac{b_2 - b_1}{2(1+q)} t^2 - \frac{1}{b_2 - b_1} \left\{ \frac{t^4 (b_2 - b_1)^2}{4(q+1)} \right. \right. \\
& - \left(\frac{t^2}{2} + \frac{t^4}{4} - \frac{2t^3}{3} \right) b_1^2 - \frac{t^4}{4} b_2^2 - 2 \left(\frac{t^3}{3} - \frac{t^4}{4} \right) b_1 b_2 + (b_1 + b_2) b_1 \left(\frac{t^2}{2} - \frac{t^3}{3} \right) \\
& \left. \left. + \frac{t^3}{3} (b_1 + b_2) b_2 - \frac{t^2}{2} b_1 b_2 \right\} \right]_{\frac{1}{1+q}}^1 \\
& = (b_2 - b_1) |\psi''(b_1)| \left[\left(\frac{1}{2(1+q)^2} - \frac{1}{3(1+q)^3} \right) (b_2 - b_1) - \frac{1}{b_2 - b_1} \left\{ \frac{q^4 b_1^2}{4(1+q)^4} \right. \right. \\
& - \frac{b_1^2}{4} + \frac{\left(\frac{1}{3(1+q)^3} - \frac{1}{4(1+q)^4} \right) (b_2 - b_1)^2}{q+1} - \left(\frac{1}{3(1+q)^3} - \frac{1}{4(1+q)^4} \right) b_2^2 \\
& - 2 \left(\frac{1}{2(1+q)^2} + \frac{1}{4(1+q)^4} - \frac{2}{3(1+q)^3} \right) b_1 b_2 - \frac{q^3 (b_1 + b_2) b_1}{3(1+q)^3} + \frac{(b_1 + b_2) b_1}{3(1+q)^3} \\
& \left. \left. + \left(\frac{1}{2(1+q)^2} - \frac{1}{3(1+q)^3} \right) (b_1 + b_2) b_2 - \frac{(1+2q) b_1 b_2}{2(1+q)^2} \right\} \right] \\
& + (b_2 - b_1) |\psi''(b_2)| \left[\frac{b_2 - b_1}{3(1+q)^3} - \frac{1}{b_2 - b_1} \left\{ \frac{(b_2 - b_1)^2}{4(q+1)^5} - \left(\frac{1}{2(1+q)^2} \right. \right. \right. \\
& + \frac{1}{4(1+q)^4} - \frac{2}{3(1+q)^3} \left. \right) b_1^2 - \frac{1}{4(1+q)^4} b_2^2 - 2 \left(\frac{1}{3(1+q)^3} - \frac{1}{4(1+q)^4} \right) b_1 b_2 \\
& \left. \left. + (b_1 + b_2) b_1 \left(\frac{1}{2(1+q)^2} - \frac{1}{3(1+q)^3} \right) + \frac{(b_1 + b_2) b_2}{3(1+q)^3} - \frac{b_1 b_2}{2(1+q)^2} \right\} \right] \\
& + (b_2 - b_1) |\psi''(b_1)| \left[\frac{b_2 - b_1}{2(1+q)} - \frac{(b_2 - b_1)(1+2q)}{2(1+q)^3} - \frac{1}{b_2 - b_1} \left\{ \frac{(b_2 - b_1)^2}{12(q+1)} \right. \right. \\
& - \frac{\left(\frac{1}{3(1+q)^3} - \frac{1}{4(1+q)^4} \right) (b_2 - b_1)^2}{q+1} - \frac{q^4 b_1^2}{4(1+q)^4} - \left(\frac{1}{3} - \frac{1}{4} \right) b_2^2 \\
& + \left(\frac{1}{3(1+q)^3} - \frac{1}{4(1+q)^4} \right) b_2^2 - 2 \left(\frac{1}{2} + \frac{1}{4} - \frac{2}{3} \right) b_1 b_2 + 2 \left(\frac{1}{2(1+q)^2} \right. \\
& + \frac{1}{4(1+q)^4} - \frac{2}{3(1+q)^3} \left. \right) b_1 b_2 + \frac{q^3 (b_1 + b_2) b_1}{3(1+q)^3} + \left(\frac{1}{2} - \frac{1}{3} \right) (b_1 + b_2) b_2 \\
& \left. \left. - \left(\frac{1}{2(1+q)^2} - \frac{1}{3(1+q)^3} \right) (b_1 + b_2) b_2 - \frac{b_1 b_2}{2} + \frac{b_1 b_2 (1+2q)}{2(1+q)^2} \right\} \right] \\
& + (b_2 - b_1) |\psi''(b_2)| \left[\frac{b_2 - b_1}{2(1+q)} - \frac{b_2 - b_1}{2(1+q)^3} - \frac{1}{b_2 - b_1} \left\{ \frac{(b_2 - b_1)^2}{4(q+1)} - \frac{(b_2 - b_1)^2}{4(q+1)^5} \right. \right. \\
& - \left(\frac{1}{2} + \frac{1}{4} - \frac{2}{3} \right) b_1^2 + \left(\frac{1}{2(1+q)^2} + \frac{1}{4(1+q)^4} - \frac{2}{3(1+q)^3} \right) b_1^2 - \frac{b_2^2}{4} + \frac{b_2^2}{4(1+q)^4} \\
& \left. \left. - 2 \left(\frac{1}{3} - \frac{1}{4} \right) b_1 b_2 + 2 \left(\frac{1}{3(1+q)^3} - \frac{1}{4(1+q)^4} \right) b_1 b_2 + (b_1 + b_2) b_1 \left(\frac{1}{2} - \frac{1}{3} \right) \right\} \right]
\end{aligned}$$

$$\begin{aligned}
& - (b_1 + b_2)b_1 \left(\frac{1}{2(1+q)^2} - \frac{1}{3(1+q)^3} \right) + \frac{(b_1 + b_2)b_2}{3} - \frac{(b_1 + b_2)b_2}{3(1+q)^3} - \frac{b_1b_2}{2} \\
& + \frac{b_1b_2}{2(1+q)^2} \Bigg] \\
& = (b_2 - b_1) |\psi''(b_1)| \left[\left(\frac{1}{2(1+q)^2} - \frac{1}{3(1+q)^3} \right) (b_2 - b_1) - \frac{1}{b_2 - b_1} \left\{ \frac{q^4 b_1^2}{4(1+q)^4} \right. \right. \\
& - \frac{b_1^2}{4} + \frac{(\frac{1}{3(1+q)^3} - \frac{1}{4(1+q)^4})(b_2 - b_1)^2}{q+1} - \left(\frac{1}{3(1+q)^3} - \frac{1}{4(1+q)^4} \right) b_2^2 \\
& - 2 \left(\frac{1}{2(1+q)^2} + \frac{1}{4(1+q)^4} - \frac{2}{3(1+q)^3} \right) b_1b_2 - \frac{q^3(b_1 + b_2)b_1}{3(1+q)^3} + \frac{(b_1 + b_2)b_1}{3(1+q)^3} \\
& + \left(\frac{1}{2(1+q)^2} - \frac{1}{3(1+q)^3} \right) (b_1 + b_2)b_2 - \frac{(1+2q)b_1b_2}{2(1+q)^2} \Bigg\} + \frac{b_2 - b_1}{2(1+q)} \\
& - \frac{(b_2 - b_1)(1+2q)}{2(1+q)^3} - \frac{1}{b_2 - b_1} \left\{ \frac{(b_2 - b_1)^2}{12(q+1)} - \frac{(\frac{1}{3(1+q)^3} - \frac{1}{4(1+q)^4})(b_2 - b_1)^2}{q+1} \right. \\
& - \frac{q^4 b_1^2}{4(1+q)^4} - \left(\frac{1}{3} - \frac{1}{4} \right) b_2^2 + \left(\frac{1}{3(1+q)^3} - \frac{1}{4(1+q)^4} \right) b_2^2 - 2 \left(\frac{1}{2} + \frac{1}{4} - \frac{2}{3} \right) b_1b_2 \\
& + 2 \left(\frac{1}{2(1+q)^2} + \frac{1}{4(1+q)^4} - \frac{2}{3(1+q)^3} \right) b_1b_2 + \frac{q^3(b_1 + b_2)b_1}{3(1+q)^3} \\
& + \left(\frac{1}{2} - \frac{1}{3} \right) (b_1 + b_2)b_2 \\
& - \left. \left(\frac{1}{2(1+q)^2} - \frac{1}{3(1+q)^3} \right) (b_1 + b_2)b_2 - \frac{b_1b_2}{2} + \frac{b_1b_2(1+2q)}{2(1+q)^2} \right\} \Bigg] \\
& + (b_2 - b_1) |\psi''(b_2)| \left[\frac{b_2 - b_1}{3(1+q)^3} - \frac{1}{b_2 - b_1} \left\{ \frac{(b_2 - b_1)^2}{4(q+1)^5} - \left(\frac{1}{2(1+q)^2} \right. \right. \right. \\
& + \frac{1}{4(1+q)^4} - \frac{2}{3(1+q)^3} \Bigg) b_1^2 - \frac{1}{4(1+q)^4} b_2^2 - 2 \left(\frac{1}{3(1+q)^3} - \frac{1}{4(1+q)^4} \right) b_1b_2 \\
& + (b_1 + b_2)b_1 \left(\frac{1}{2(1+q)^2} - \frac{1}{3(1+q)^3} \right) + \frac{(b_1 + b_2)b_2}{3(1+q)^3} - \frac{b_1b_2}{2(1+q)^2} \Bigg\} \\
& + \frac{b_2 - b_1}{2(1+q)} - \frac{b_2 - b_1}{2(1+q)^3} - \frac{1}{b_2 - b_1} \left\{ \frac{(b_2 - b_1)^2}{4(q+1)} - \frac{(b_2 - b_1)^2}{4(q+1)^5} - \left(\frac{1}{2} + \frac{1}{4} - \frac{2}{3} \right) b_1^2 \right. \\
& + \left(\frac{1}{2(1+q)^2} + \frac{1}{4(1+q)^4} - \frac{2}{3(1+q)^3} \right) b_1^2 - \frac{b_2^2}{4} + \frac{b_2^2}{4(1+q)^4} - 2 \left(\frac{1}{3} - \frac{1}{4} \right) b_1b_2 \\
& + 2 \left(\frac{1}{3(1+q)^3} - \frac{1}{4(1+q)^4} \right) b_1b_2 + (b_1 + b_2)b_1 \left(\frac{1}{2} - \frac{1}{3} \right) - (b_1 + b_2)b_1 \left(\frac{1}{2(1+q)^2} \right. \\
& - \left. \frac{1}{3(1+q)^3} \right) + \frac{(b_1 + b_2)b_2}{3} - \frac{(b_1 + b_2)b_2}{3(1+q)^3} - \frac{b_1b_2}{2} + \frac{b_1b_2}{2(1+q)^2} \Bigg\} \Bigg] \\
& = |\psi''(b_1)| \left[\left(\frac{1}{2(1+q)^2} - \frac{1}{3(1+q)^3} \right) (b_2 - b_1)^2 - \frac{q^4 b_1^2}{4(1+q)^4} \right. \\
& + \frac{b_1^2}{4} - \frac{(\frac{1}{3(1+q)^3} - \frac{1}{4(1+q)^4})(b_2 - b_1)^2}{q+1} + \left(\frac{1}{3(1+q)^3} + \frac{1}{4(1+q)^4} \right) b_2^2 \\
& + 2 \left(\frac{1}{2(1+q)^2} + \frac{1}{4(1+q)^4} - \frac{2}{3(1+q)^3} \right) b_1b_2 + \frac{q^3(b_1 + b_2)b_1}{3(1+q)^3} - \frac{(b_1 + b_2)b_1}{3(1+q)^3}
\end{aligned}$$

$$\begin{aligned}
& - \left(\frac{1}{2(1+q)^2} - \frac{1}{3(1+q)^3} \right) (b_1 + b_2)b_2 + \frac{(1+2q)b_1b_2}{2(1+q)^2} + \frac{(b_2 - b_1)^2}{2(1+q)} \\
& - \frac{(b_2 - b_1)^2(1+2q)}{2(1+q)^3} - \frac{(b_2 - b_1)^2}{12(q+1)} + \frac{\left(\frac{1}{3(1+q)^3} - \frac{1}{4(1+q)^4} \right) (b_2 - b_1)^2}{q+1} \\
& + \frac{q^4 b_1^2}{4(1+q)^4} + \left(\frac{1}{3} - \frac{1}{4} \right) b_2^2 - \left(\frac{1}{3(1+q)^3} - \frac{1}{4(1+q)^4} \right) b_2^2 + 2 \left(\frac{1}{2} + \frac{1}{4} - \frac{2}{3} \right) b_1 b_2 \\
& - 2 \left(\frac{1}{2(1+q)^2} + \frac{1}{4(1+q)^4} - \frac{2}{3(1+q)^3} \right) b_1 b_2 - \frac{q^3 (b_1 + b_2)b_1}{3(1+q)^3} \\
& - \left(\frac{1}{2} - \frac{1}{3} \right) (b_1 + b_2)b_2 \\
& + \left(\frac{1}{2(1+q)^2} - \frac{1}{3(1+q)^3} \right) (b_1 + b_2)b_2 + \frac{b_1 b_2}{2} - \frac{b_1 b_2 (1+2q)}{2(1+q)^2} \Big] \\
& + |\psi''(b_2)| \left[\frac{(b_2 - b_1)^2}{3(1+q)^3} - \frac{(b_2 - b_1)^2}{4(q+1)^5} + \left(\frac{1}{2(1+q)^2} + \frac{1}{4(1+q)^4} - \frac{2}{3(1+q)^3} \right) b_1^2 \right. \\
& + \frac{1}{4(1+q)^4} b_2^2 + 2 \left(\frac{1}{3(1+q)^3} - \frac{1}{4(1+q)^4} \right) b_1 b_2 - (b_1 + b_2)b_1 \left(\frac{1}{2(1+q)^2} \right. \\
& - \left. \frac{1}{3(1+q)^3} \right) - \frac{(b_1 + b_2)b_2}{3(1+q)^3} + \frac{b_1 b_2}{2(1+q)^2} + \frac{(b_2 - b_1)^2}{2(1+q)} + \frac{(b_2 - b_1)^2}{2(1+q)^3} \\
& - \frac{(b_2 - b_1)^2}{4(q+1)} + \frac{(b_2 - b_1)^2}{4(q+1)^5} + \left(\frac{1}{2} + \frac{1}{4} - \frac{2}{3} \right) b_1^2 - \left(\frac{1}{2(1+q)^2} + \frac{1}{4(1+q)^4} \right. \\
& - \left. \frac{2}{3(1+q)^3} \right) b_1^2 + \frac{b_2^2}{4} - \frac{b_2^2}{4(1+q)^4} + 2 \left(\frac{1}{3} - \frac{1}{4} \right) b_1 b_2 - 2 \left(\frac{1}{3(1+q)^3} \right. \\
& - \left. \frac{1}{4(1+q)^4} \right) b_1 b_2 - (b_1 + b_2)b_1 \left(\frac{1}{2} - \frac{1}{3} \right) + (b_1 + b_2)b_1 \left(\frac{1}{2(1+q)^2} \right. \\
& - \left. \frac{1}{3(1+q)^3} \right) - \frac{(b_1 + b_2)b_2}{3} + \frac{(b_1 + b_2)b_2}{3(1+q)^3} + \frac{b_1 b_2}{2} - \frac{b_1 b_2}{2(1+q)^2} \Big] \\
& = |\psi''(b_1)| \left[\left(\frac{1}{2(1+q)^2} - \frac{1}{3(1+q)^3} \right) (b_2 - b_1)^2 + \frac{b_1^2}{4} - \frac{(b_1 + b_2)b_1}{3(1+q)^3} + \frac{(b_2 - b_1)^2}{2(1+q)} \right. \\
& - \frac{(b_2 - b_1)^2(1+2q)}{2(1+q)^3} - \frac{(b_2 - b_1)^2}{12(q+1)} + \left(\frac{1}{3} - \frac{1}{4} \right) b_2^2 + 2 \left(\frac{1}{2} + \frac{1}{4} - \frac{2}{3} \right) b_1 b_2 \\
& - \left(\frac{1}{2} - \frac{1}{3} \right) (b_1 + b_2)b_2 + \frac{b_1 b_2}{2} \Big] + |\psi''(b_2)| \left[\frac{(b_2 - b_1)^2}{3(1+q)^3} + \frac{(b_2 - b_1)^2}{2(1+q)} \right. \\
& + \frac{(b_2 - b_1)^2}{2(1+q)^3} - \frac{(b_2 - b_1)^2}{4(q+1)} + \left(\frac{1}{2} + \frac{1}{4} - \frac{2}{3} \right) b_1^2 + \frac{b_2^2}{4} + 2 \left(\frac{1}{3} - \frac{1}{4} \right) b_1 b_2 \\
& - (b_1 + b_2)b_1 \left(\frac{1}{2} - \frac{1}{3} \right) - \frac{(b_1 + b_2)b_2}{3} + \frac{b_1 b_2}{2} \Big] \\
& = |\psi''(b_1)| \left[\frac{(1+3q)(b_2 - b_1)^2}{6(1+q)^3} + \frac{b_1^2}{4} - \frac{(b_1 + b_2)b_1}{3(1+q)^3} + \frac{(b_2 - b_1)^2}{2(1+q)} \right. \\
& - \frac{(b_2 - b_1)^2(1+2q)}{2(1+q)^3} - \frac{(b_2 - b_1)^2}{12(q+1)} + \frac{b_2^2}{12} + \frac{b_1 b_2}{6} - \frac{(b_1 + b_2)b_2}{6} + \frac{b_1 b_2}{2} \Big] \\
& + |\psi''(b_2)| \left[\frac{(b_2 - b_1)^2}{3(1+q)^3} + \frac{(b_2 - b_1)^2}{2(1+q)} + \frac{(b_2 - b_1)^2}{2(1+q)^3} - \frac{(b_2 - b_1)^2}{4(q+1)} + \frac{b_1^2}{12} + \frac{b_2^2}{4} \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{b_1 b_2}{6} - \frac{(b_1 + b_2)b_1}{6} - \frac{(b_1 + b_2)b_2}{3} + \frac{b_1 b_2}{2} \Big] \\
& = |\psi''(b_1)| \Big[\frac{(1+3q)(b_2-b_1)^2}{6(1+q)^3} - \frac{(b_1+b_2)b_1}{3(1+q)^3} + \frac{(b_2-b_1)^2}{2(1+q)} - \frac{(b_2-b_1)^2(1+2q)}{2(1+q)^3} \\
& \quad - \frac{(b_2-b_1)^2}{12(q+1)} + \frac{b_1^2}{4} + \frac{b_2^2}{12} + \frac{b_1 b_2}{6} - \frac{(b_1+b_2)b_2}{6} + \frac{b_1 b_2}{2} \Big] \\
& \quad + |\psi''(b_2)| \Big[\frac{(b_2-b_1)^2}{3(1+q)^3} + \frac{(b_2-b_1)^2}{2(1+q)} + \frac{(b_2-b_1)^2}{2(1+q)^3} - \frac{(b_2-b_1)^2}{4(q+1)} + \frac{b_1^2}{12} + \frac{b_2^2}{4} \\
& \quad + \frac{b_1 b_2}{6} - \frac{(b_1+b_2)b_1}{6} - \frac{(b_1+b_2)b_2}{3} + \frac{b_1 b_2}{2} \Big] \\
& = |\psi''(b_1)| \Big[\frac{(b_2-b_1)^2(5q^2+4q+1)-4(b_1+b_2)b_1}{12(1+q)^3} + \frac{3b_1^2-b_2^2+6b_1 b_2}{12} \Big] \\
& \quad + |\psi''(b_2)| \Big[\frac{(b_2-b_1)^2(3(1+q)^2+10)}{12(1+q)^3} - \frac{(b_2-b_1)^2}{12} \Big]. \quad \square
\end{aligned}$$

4 Conclusion

We revisited the Hermite–Hadamard inequality in quantum calculus. We deduced some new identities in the way. Using these identities, we obtained new estimates in this regard. Employing the method outlined in this paper, we anticipate that some other inequalities may be reestablished. More results on the Hermite–Hadamard inequality in quantum calculus can be found in [22, 26, 31, 34].

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

MAK provided the main idea and carried out the proof of Theorem 3.2. NM carried out the proof of Theorems 3.4 and 3.5. ERN carried out the proof of Theorems 3.6 and 3.7. YMC carried out the proof of Theorem 3.3, drafted the first version of the manuscript, and completed the revision of the manuscript. All authors agreed to the authorship and their order in the final manuscript, and all authors read and approved the final manuscript.

Author details

¹Department of Mathematics, University of Peshawar, Peshawar, Pakistan. ²Department of Mathematics and Computer Science, Alabama State University, Montgomery, USA. ³Department of Mathematics, Huzhou University, Huzhou, China. ⁴School of Mathematics and Statistics, Changsha University of Science & Technology, Changsha, China.

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