Acta Math., 183 (1999), 71–99

Quantum hyperplane section theorem for homogeneous spaces

by

BUMSIG KIM

Pohang University of Science and Technology
Pohang, Republic of Korea

1. Introduction

Quantum cohomology of a symplectic manifold is a certain deformed ring of the ordinary cohomology ring with parameter space given by the second cohomology group. It encodes enumerative geometry of rational curves on the manifold. In general it is difficult to compute the quantum cohomology structure. On the other hand, mirror symmetry predicts an answer to a question of counting the virtual numbers of rational curves of given degrees on a three-dimensional Calabi–Yau manifold, which amounts to knowing the structure of the quantum cohomology of the manifold. In the large class of Calabi–Yau manifolds, the complete intersections in toric manifolds or homogeneous spaces, this mirror symmetry prediction [7], [3], [1], [2], can be interpreted as a quantum cohomology counterpart of the weak Lefschetz hyperplane section theorem relating cohomology algebras of the ambient manifolds and their hyperplane sections. As it is mentioned in [13], the "quantum hyperplane section conjecture" can be formulated in intrinsic terms of Gromov–Witten theory on the ambient manifold and does not require a reference to its mirror partner. In this paper we formulate and prove the conjecture for homogeneous spaces. It would be one of the highly nontrivial functorial properties enjoyed by quantum cohomology algebras. One can compute the virtual numbers of rational curves on a Calabi–Yau 3-fold complete intersection, provided one knows the quantum cohomology algebra of the ambient space. In fact, one needs to know the quantum differential equations of the space, which are certain linear differential equations arising from the flat connection in the quantum cohomology algebra. The mirror symmetry prediction is that the quantum differential equations of a Calabi–Yau manifold are equivalent in a sense to the Picard–Fuchs differential equations of another Calabi–Yau manifold. In contrast, the proposed conjecture is that there is a certain relation between quantum differential equations of a manifold and those of a nonnegative smooth zero locus of a spanned decomposable vector bundle over the manifold. The formulation of the conjecture is given in [2]. When the ambient space is a symplectic toric manifold, the conjecture is a corollary of the Givental mirror theorem [14].

Let X be a compact homogeneous space of a semi-simple complex Lie group and let V be a vector bundle over X. Let $\beta \in H_2(X, \mathbf{Z})$ which is in the Mori cone Λ of X. Suppose that $V'_\beta := \pi_* e_1^* V$ becomes a vector orbi-bundle over the Kontsevich moduli space $\overline{M}_{0,0}(X, \beta)$, where e_1 is the evaluation map at the (first) marked point from $\overline{M}_{0,1}(X, \beta)$ to X, and π is the map from $\overline{M}_{0,1}(X, \beta)$ to $\overline{M}_{0,0}(X, \beta)$ associated with "forgetting the marked point" [17]. Then one might want to compute

$$\int_{\overline{M}_{0,0}(X,\beta)} \mathrm{Euler}(V'_\beta).$$

Introduce a formal parameter $\hbar$. Then it turns out that the classes

$$G_\beta^V := (e_1)_* \frac{\mathrm{Euler}(V_\beta)}{\hbar(\hbar - c)}$$

would be better considered [12], where $V_\beta = \pi^*(V'_\beta)$ and c (depending on β) are the first Chern classes of the universal cotangent line bundles. The classes are in $H^*(X)[\hbar^{-1}]$. They recover the original integrals which we want:

$$\int_X G_\beta^V = \frac{-2}{\hbar^3} \int_{\overline{M}_{0,0}(X,\beta)} \mathrm{Euler}(V'_\beta) + o(\hbar^{-3}).$$

Consider the classes

$$G_\beta^X := (e_1)_* \frac{1}{\hbar(\hbar - c)}$$

corresponding to X itself (without V). When V is a convex decomposable vector bundle $\bigoplus L_j$ of line bundles L_j, the main result of this paper proves some explicit relationship between $A := \{G_\beta^V \mid \beta \in \Lambda\}$ and $B := \{H_\beta^V \cup G_\beta^X \mid \beta \in \Lambda\}$, where

$$H_\beta^V = \prod_j \prod_{m=0}^{\langle c_1(L_j), \beta \rangle} (c_1(L_j) + m\hbar),$$

which is the key object introduced in this sequel.

We now formulate the precise result of this paper. Let $\{p_i\}_{i=1}^k$ denote the $\mathbf{Z}_+$-basis of the closed integral Kähler/ample cone of X. Let us introduce formal parameters q_i, $i = 1, \ldots, k$, and the ring $H^*(X)[\hbar^{-1}][[q_1, \ldots, q_k]]$ of formal power series of q_i. Denote by q^β

$$\prod_{i=1}^k q_i^{\langle p_i, \beta \rangle}.$$

When $\beta=0$, let $G_0^X=1$ and $G_0^V=\text{Euler}(V)$. We want to compare generating functions J^V and I^V from A and B, respectively:

$$S^V := \sum_{\beta\in\Lambda} q^\beta G_\beta^V,$$
$$\Phi^V := \sum_{\beta\in\Lambda} q^\beta H_\beta^V \cup G_\beta^X.$$

We prove that one can be transformed to another by a unique "mirror" transformation. To describe the transformation, let

$$q_i = e^{t_i}, \quad \text{for } i=1,...,k,$$

and introduce another formal variable t_0. Define the degree of q_i by

$$c_1(TX)-c_1(V) = \sum (\deg q_i) p_i.$$

Let

$$J^V := e^{(t_0+\sum_i p_i t_i)/\hbar} S^V$$

and

$$I^V := e^{(t_0+\sum_i p_i t_i)/\hbar} \Phi^V,$$

which are formal power series of $t_1,...,t_k$, $e^{t_0/\hbar}, e^{t_1},...,e^{t_k}$ over $H^*(X)[\hbar^{-1}]$.

THEOREM 1. *Assume that* $\deg q_i \geqslant 0$ *for all* i. *Then* J^V *and* I^V *coincide up to a unique weighted homogeneous change of variables*: $t_0 \mapsto t_0 + f_0\hbar + f_{-1}$ *and* $t_i \mapsto t_i + f_i$, *where* $f_{-1},...,f_k$ *are power series of* $q_1,...,q_k$ *over* $\mathbf{Q}$ *without constant terms,* $\deg f_i=0$, $i=0,...,k$, *and* $\deg f_{-1}=1$.

Remarks. (0) J^V will be a cohomological expression of solutions to quantum differential equations associated to (X,V), which is closely related to the quantum differential equations of the smooth zero locus of V. The similar theorem can be extended to the case of decomposable concavex vector bundles V (see [19], [15]).

(1) The change of variables must be understood as replacements of $e^{(t_0+\sum_i p_i t_i)/\hbar}$ by $e^{(t_0+\sum p_i t_i)/\hbar} e^{f_0} e^{f_{-1}/\hbar} e^{\sum p_i f_i/\hbar}$ and of q_i with $q_i e^{f_i}$. So,

$$S^V \quad \text{and} \quad e^{f_0} e^{f_{-1}/\hbar} e^{\sum p_i f_i/\hbar} \Phi^V(q_1 e^{f_1},...,q_k e^{f_k})$$

are equal, where $f_{-1},...,f_k$ are some power series of $q_1,...,q_k$ over $\mathbf{Q}$ without constant terms, $\deg f_i=0$, $i=0,...,k$, and $\deg f_{-1}=1$. In fact, f_i are uniquely well-determined by the coefficients of $1=(1/\hbar)^0$ and $1/\hbar$ in the expansions of S^V and Φ^V as power series of $1/\hbar$.

(2) In the case of a symplectic toric manifold X, the similar statement is a corollary of a mirror theorem in [14], where Φ^X is explicitly found.

(3) For the proof of Theorem 1 we follow the scheme of Givental's proof [12], [14] of the mirror theorem for nonnegative complete intersections in symplectic toric manifolds.

(4) The theorem verifies the prediction [1] of virtual numbers of rational curves in Calabi–Yau 3-fold complete intersections in Grassmannians.

(5) A mirror construction is established for complete intersections in partial flag manifolds [1], [2]. Because of the known quantum cohomology structure [8], in principle there is no essential difficulty in finding G^X_β for each partial flag manifold X, even though a general formula of it is unknown.

(6) In [21] the quantum hyperplane section principle is applied to a nonconvex, nontoric manifold.

Notation. X will always be a generalized flag manifold G/P, where G is a complex semi-simple Lie group and P is a parabolic subgroup. Let T be a maximal torus of G in P, and let T act on X from the left. Let a complex torus T' act on X trivially, and let V be a $T\times T'$-equivariant convex vector bundle over X. Consider E a multiplicative class and suppose that $E(V)\in H^*_{T\times T'}(X)$ is invertible in $H^*_{(T\times T')}(X):=H^*_{T\times T'}(X)\otimes H^*_{T\times T'}$, where $H^*_{(T\times T')}$ is the quotient field of $H^*_{T\times T'}(pt)$. In §2, we will not consider the T-action on X. In §6, additionally we will assume that V is decomposable. The convexity of V is by definition that $H^1(\mathbf{P}^1,f^*V)=0$ for any morphism $f:\mathbf{P}^1\to X$. Let the $T\times T'$-equivariant line bundles U_i, $i=1,\dots,k$, form an ample basis of the ordinary Picard group. We denote $\int_X ABE(V)$ by $\langle A,B\rangle^V_0$, for $A,B\in H^*_{(T\times T')}(X)$, and also we use $\int_V A:=\int_X AE(V)$ (equivariant push forwards). The Mori cone of X will be denoted by Λ, which can be identified with $\mathbf{Z}^k_+$ with respect to the coordinates $p_i:=c_1(U_i)$. On the additive group $\mathbf{Z}^k$ we will give the standard partial ordering, so that $d:=(d_1,\dots,d_k)\geqslant 0$ means $d_i\geqslant 0$ for all i. Let v be a fixed point of X with respect to the T-action, and let ϕ_v denote the equivariant pushforward of class 1 under the embedding i_v of the fixed point v to (X,V,E); this (X,V,E), by definition, has a Frobenius structure by the pairing $\langle\cdot,\cdot\rangle^V_0$, so that $A_v:=\langle A,\phi_v\rangle^V_0=i_v^*(A)$ for $A\in H^*_{(T\times T')}(X)$. For a G-manifold M, let M^G denote the set of G-fixed points of M. We will say simply degree and dimension for complex degree and complex dimension, respectively. Let $\sum_a T_a\otimes T^a$ be the equivariant diagonal class of (X,V,E) in $X\times X$. That is, $\langle T_a,T^b\rangle^V_0=\delta_{a,b}$. In the paper we will consider various rings: the $H^*_{T\times T'}[[\hbar^{-1}]][[q]]$-formal power series ring of $\hbar^{-1},q$ over $H^*_{T\times T'}$, the $H^*_{(T\times T')}[[\hbar^{-1}]][[q]]$-formal power series ring of $\hbar^{-1},q$ over $H^*_{(T\times T')}$, and the $H^*_{(T\times T')}(\hbar)[[q]]$-formal power series ring of q over the quotient field of $H^*_{T\times T'}[\hbar]$.

Structure of the paper. In §2, we recall a general theory of Gromov–Witten invariants and quantum cohomology. We introduce the Givental correlators S^V. In §3, we show that the equivariant correlators satisfy a certain recursion relation. In §4, we introduce the double construction and show that the correlators satisfy a polynomiality in a "double construction". In §5, we introduce a class $\mathcal{P}(X, V, \text{Euler})$ of series of $q=(q_1, ..., q_k), \hbar^{-1}$ over $H^*_{T\times T'}(X)$, where a "mirror" group acts freely and transitively. In §6, we introduce a modified correlator of S^X. It will also belong to the class $\mathcal{P}(X, V, \text{Euler})$. The modification is given by the hypergeometric correcting Euler classes H^V_β according to the decomposition type of V. In §§ 7 and 8, we analyze the torus T-action on a generalized flag manifold, its one-dimensional orbits and the representations of the global section spaces of equivariant line bundles restricted to the orbits. The analysis would be useful to find the explicit expression of Φ^X.

Acknowledgments. I am grateful to A. Givental and Y.-P. Lee for helping me to understand the paper [12], and to V. Batyrev, I. Ciocan-Fontaine, W. Fulton, B. Kreußler, E. Tjøtta, K. Wirthmüller for useful discussions on the papers [12], [14]. Also, I would like to thank the Institut Mittag-Leffler for the financial support during the yearlong program "Enumerative Geometry and its Interactions with Theoretical Physics" in 1996/97. My special thanks go to D. van Straten for numerous comments which improve the clarity of the paper.

2. Mirror symmetry

2.1. *The moduli space of stable maps.* To fix notation we recall the definition of stable maps and some elementary properties of the moduli spaces of stable maps to X [17], [10], [6]. The notion of stable maps is due to M. Kontsevich. We recommend the (survey) paper of W. Fulton and R. Pandharipande [10].

A prestable rational curve C is a connected arithmetic genus-0 projective curve with possibly nodes. The curve is not necessary irreducible. A prestable map $(f, C; x_1, ..., x_n)$ is a morphism f from C to X with fixed ordered n-many marked distinct smooth points $x_i \in C$. We will identify $(f, C; \{x_i\})$ with $(f', C', \{x'_i\})$ if there is an isomorphism h from C to C' preserving the configuration of marked points such that $f = f' \circ h$. A stable map $(f, C; \{x_i\})$ is a prestable map with only finitely many automorphisms.

Let $\bar{M}_{0,n}(X, \beta)$ be the (coarse moduli) space of all stable maps $(f, C; \{x_i\}_{i=1}^n)$ with the fixed homology type $\beta = f_*([C]) \in H_2(X, \mathbf{Z})$. Whenever it is nonempty, the moduli space is a connected([1]) compact complex *orbifold* with complex dimension $\dim X +$

([1]) For a proof of the connectedness see [20].

$\langle c_1(TX), \beta\rangle + n - 3$.

More precisely, locally near a stable map the moduli space has data of a quotient of a holomorphic domain by the (finite) group action of all automorphisms of the stable map. In the paper [10] are constructed smooth open complex domains V with finite groups Γ which act on V such that V/Γ are naturally glued together in the moduli space of stable maps. Let $X \subset_i \mathbf{P}^N$, $\beta \neq 0$, and $(X, i_*(\beta)) \neq (\mathbf{P}^1, [\text{line}])$. Here [line] denotes the line class of $H_2(\mathbf{P}^N)$. Given a stable map (f, C) (without marked points for simplicity), choose hyperplanes H_j in $\mathbf{P}^N$ satisfying that $\{H_j\}$ gives rise to a basis of $H^0(\mathbf{P}^N, \mathcal{O}(1))$, f is transversal to the hyperplanes, and their inverse images $\{x_{i,j}\}_i = f^{-1}(H_j)$ contain no nodes of C. Then the data $(C; \{x_{i,j}\})$ determines a point in the moduli space of marked stable curves. Conversely, a point in a suitable closed subvariety of an open smooth domain of the moduli space of marked stable curves naturally determines a stable map f with the extra choices of elements in $(\mathbf{C}^\times)^N$. If G is the product of the symmetric group of the elements of each group $\{x_{i,j}\}_i$, then this G has an action sending the data $(f, C; \{x_{i,j}\})$ to another by permutations of the new marked points. A $(\mathbf{C}^\times)^N$-bundle of the smooth closed subvariety is an algebraic local chart of the moduli space of stable maps at f with the induced G-action.

Example. Let $X = \mathbf{P}^2$ and f be a stable map without marked points such that f is transversal to the hyperplanes $x=0$, $y=0$ and $z=0$. Assume that no singular points of C are mapped into the hyperplanes, and $f_*[C] = 2[\text{line}]$. Consider their inverse images (Cartier divisors), a_1, a_2, b_1, b_2, c_1, c_2 in C. This information $(C; a_1, \ldots, c_2)$ as a stable curve will determine f uniquely with $(\mathbf{C}^\times)^2$-ambiguity. This $(\mathbf{C}^\times)^2$-bundle over some open subset of the smooth space $\overline{M}_{0,6}$ is the local smooth chart. Notice that for instance, $(C; a_2, a_1, b_1, b_2, c_2, c_1)$ gives rise to the same f up to isomorphism. Thus we have to take account of the quotient by the finite group permuting the elements of sets $\{a_1, a_2\}$, $\{b_1, b_2\}$ and $\{c_1, c_2\}$.

CLAIM. *The stabilizer subgroup $G_{(C;\{x_{i,j}\})}$ of G is exactly the automorphism group* $\mathrm{Aut}(f, C)$ *of* (f, C).

Proof. We shall construct a correspondence between $G_{(C;\{x_{i,j}\})}$ and $\mathrm{Aut}(f, C)$. Let $g \in G_{(C;\{x_{i,j}\})}$ which is given by one of the suitable permutations of $x_{i,j}$. So, $g(C; \{x_{i,j}\}) = (C; \{g(x_{i,j})\})$. Since the permutation does not change the stable curve $(C; \{x_{i,j}\})$, there is an isomorphism h from $(C; \{x_{i,j}\})$ to $(C; \{g(x_{i,j})\})$. The isomorphism h is unique since there is no nontrivial automorphism in the stable curve of genus 0. Of course this h gives rise to an automorphism of (f, C).

Conversely, if h is an automorphism of (f, C), then it induces an isomorphism from $(C; \{x_{i,j}\})$ to $(C; \{g(x_{i,j})\})$ for a unique permutation g which we allow. Thus we established a 1-1 correspondence, which can be easily seen to be a group homomorphism.

Remark. The action of $\mathrm{Aut}(f,C)$ may not be effective in general. For instance, see $\overline{M}_{0,0}(\mathbf{P}^1, 2[\text{line}])$.

2.2. *Gromov–Witten invariants and* $QH^*_{(T')}(V)$. There are natural morphisms on the moduli spaces, namely, evaluation maps e_i at the ith marked points and forgetting-marked-point maps π:

$$\begin{array}{ccc}\overline{M}_{0,n+1}(X,\beta) & \xrightarrow{e_{n+1}} & X \\ \pi\big\downarrow & & \\ \overline{M}_{0,n}(X,\beta). & & \end{array}$$

If s_i are the universal sections for the marked points, then $e_i = e_{n+1}\circ s_i$ (here we assume that π is the forgetful map of the last marked point). In the orbifold charts, π gives the universal family of stable maps as a fine moduli space.

Consider, for a second homology class $\beta\neq 0$ and an integer $n\geqslant 0$, the vector orbi-bundle $V_\beta=\pi_*(e^*_{n+1}(V))$. Here π is a flat morphism in the level of orbifold charts. Thus indeed, V_β is vector orbi-bundle with the fiber $H^0(C, f^*(V))$ at $(f, C; \{x_i\})$. Notice that $V_\beta = \pi^*(V_\beta)$ (it has nothing to do with marked points).

Notation. For $A_i\in H^*_{(T')}(X)$,

$$V_0 := V,$$
$$\overline{M}_{0,i}(X,0) := X \quad \text{for } i=0,1,2,$$
$$\langle A_1, ..., A_N\rangle^V_\beta := \int_{\overline{M}_{0,N}(X,\beta)} e_1^*(A_1)\cup ...\cup e_N^*(A_N)\cup E(V_\beta).$$

Then one can show that for all β

$$\sum_{\beta_1+\beta_2=\beta}\sum_a \langle A_1, A_2, T_a\rangle^V \langle T^a, A_3, A_4\rangle^V$$

are totally symmetric in A_i (see §4 in [12]). This property will be equivalent to the associativity of the quantum cohomology of $QH^*_{(T')}(V)$ which we define below.

Let us choose a basis $\{p_i\}_{i=1}^k$ of $H^2(X)$ by classes in the closed Kähler cone.

Notation.

$$q^\beta := \prod_i q_i^{\langle p_i, \beta\rangle},$$
$$\langle A_1, ..., A_N\rangle^V := \sum q^\beta \langle A_1, ..., A_N\rangle^V_\beta.$$

The quantum multiplication $\circ$ is defined by the following simple requirement: for $A, B, C \in H^*_{(T')}(X)$,

$$\langle A \circ B, C\rangle_0^V = \langle A, B, C\rangle^V$$

which is a formal power series of the parameters q_i. Thus our quantum cohomology $QH^*_{(T')}(V)$ is defined as $H^*_{(T')}(X) \otimes_{\mathbf{Q}} \mathbf{Q}[[q_1, \ldots, q_k]]$ with a product structure.

2.3. *Givental's correlators.* We review the topic after [9], [12].

2.3.1. *The flat connections and the fundamental solutions.* Now let $q_i = e^{t_i}$ with the formal parameters t_i. We have a one-parameter family of the formal $\mathcal{D}$-module structures on $QH^*_{(T')}(V)$ by giving a flat connection $\nabla_i = \hbar\partial/\partial t_i - p_i \circ$ for any nonzero $\hbar$, $i = 1, \ldots, k$. For the fundamental solutions we introduce $c_i \in H^*_{T'}(\overline{M}_{0,N}(X, \beta))$, so-called gravitational descendents. These c_i are the first Chern classes of the universal cotangent line bundles at the ith marked points. The line bundles are, by definition, the dual of the normal bundle of $s_i(\overline{M}_{0,N}(X, \beta))$ in $\overline{M}_{0,N+1}(X, \beta)$.

Notation. Let $f_i(x) \in H^*_{(T')}[y][[x]]$ for indeterminants x, y. Throughout the paper (see [12] and [22]),

$$\langle A_1 f_1(c), \ldots, A_N f_N(c); B\rangle_\beta^X := \int_{\overline{M}_{0,N}(X,\beta)} e_1^*(A_1) f_1(c_1) \ldots e_N^*(A_N) f_N(c_N) B,$$

$$\langle A_1 f_1(c), \ldots, A_N f_N(c); B\rangle_\beta^V := \langle A_1 f_1(c), \ldots, A_N f_N(c); BE(V_\beta)\rangle_\beta^X,$$

$$\langle A_1 f_1(c), \ldots, A_N f_N(c); B\rangle^X := \sum_\beta q^\beta \langle A_1 f_1(c), \ldots, A_N f_N(c); B\rangle_\beta^X,$$

$$\langle A_1 f_1(c), \ldots, A_N f_N(c); B\rangle^V := \sum_\beta q^\beta \langle A_1 f_1(c), \ldots, A_N f_N(c); B\rangle_\beta^V,$$

where $A_i \in H^*_{(T')}(X)$, $B \in H^*_{(T')}(\overline{M}_{0,N}(X, \beta))$.

The system of the first-order equations $\nabla_i \vec{s} = 0$, $i = 1, \ldots, k$, has the following complete set of $(\dim H^*(X))$-many solutions [12]:

$$\vec{s}_a := \sum_b \left\langle \frac{e^{pt/\hbar} T_a}{\hbar - c}, T_b \right\rangle^V T^b,$$

where pt denotes $\sum_{i=1}^k p_i t_i$ and $\hbar$ is a formal variable (but when $\beta = 0$, set $\hbar = 1$).

The following two formulas show that $\vec{s}_a$ are indeed solutions to the quantum differential system $\nabla_i \vec{s} = 0$.

Using that $c_i - \pi^*(c_i)$ is the fundamental class Δ_i represented by the section

$$s_i \colon \overline{M}_{0,n}(X, \beta) \to \overline{M}_{0,n+1}(X, \beta)$$

and $c_i \cup \Delta_i = 0$ (the image of s_i is isomorphic to $\overline{M}_{0,3}(X,0)\times_X \overline{M}_{0,n}(X,\beta)$), it is easy to derive the so-called fundamental class axiom and the divisor axiom [22], [12]. Let $f_i(x)$ be a polynomial with coefficients in $\pi^*(H^*_{(T')}(\overline{M}_{0,n}(X,d)))$. Let D be a divisor class in $H^*_{(T')}(X)$. Then (for $n>0$)

$$\langle f_1(c),...,f_n(c),1\rangle_\beta^V = \sum_i \left\langle f_1(c),...,\frac{f_i(c)-f_i(0)}{c},...,f_n(c)\right\rangle_\beta^V$$

(where we abuse the notation "$f_i(\pi^*(c))=f_i(c)$") and

$$\begin{aligned}\langle f_1(c),...,f_n(c),D\rangle_\beta^V &= \langle D,\beta\rangle\langle f_1(c),...,f_n(c)\rangle_\beta^V \\ &+\sum_i \left\langle f_1(c),...,f_{i-1}(c), D\cup\frac{f_i(c)-f_i(0)}{c}, f_{i+1}(c),...,f_n(c)\right\rangle_\beta^V.\end{aligned}$$

Consider

$$e^{pt/\hbar}S^V := \sum_a \langle \vec{s}_a,1\rangle_0^V T^a = e^{pt/\hbar}\sum_a \left\langle \frac{T_a}{\hbar(\hbar-c)}\right\rangle^V T^a = e^{pt/\hbar}(1+o(1/\hbar)),$$

which is the main object in this paper. This S^V will be called Givental's correlator for (X,V,E). It is an element in $H^*_{(T')}(X)[\hbar^{-1}][[q_1,...,q_k]]$. Notice that S^V for (X,V,Euler) is homogeneous of degree 0 if we let $\sum(\deg q_i)p_i=c_1(TX)-c_1(V)$, $\deg\hbar=1$ and $\deg A=b$ if $A\in H^{2b}_{T'}(X)$.

The quantum $\mathcal{D}$-module of $QH^*_{(T')}(V)$ is defined by the $\mathcal{D}$-module generated by $\langle\vec{s},1\rangle_0^V$ for all flat sections $\vec{s}$. When there is no V considered, we denote by $QH^*(X)$ the quantum cohomology. That is, using $\langle\, ...\,\rangle^X$, we define $QH^*(X)$.

Remark. Suppose that a differential operator $P(\hbar\partial/\partial t_i, e^{t_i},\hbar)$ with coefficients in $H^*_{(T')}$ annihilates $\langle\vec{s},1\rangle_0^V$ for all flat sections $\vec{s}$. Then $P(p_1,...,p_k,q_1,...,q_k,0)$ holds in $QH^*_{(T')}(V)$ [12].

2.3.2. *Examples.* The projective space $\mathbf{P}^n$: It is well known that in the quantum cohomology ring $QH^*(\mathbf{P}^n)$, $(p\circ)^{n+1}=q$, where $p=c_1(\mathcal{O}(1))$ and q is given with respect to the line class dual to p. The corresponding operator is $(\hbar d/dt)^{n+1}-e^t$. The solutions are explicitly known in [11]. S^X is

$$1+\sum_{d>0} e^{dt}\frac{1}{((p+\hbar)(p+2\hbar)\dots(p+d\hbar))^{n+1}}.$$

The complete flag manifolds $F(n)$: Let $F(n)$ be the set of all complete flags $(\mathbf{C}^1\subset \dots\subset\mathbf{C}^n)$ in $\mathbf{C}^n$. The usual cohomology ring is $\mathbf{Q}[x_1,x_2,...,x_n]/(I_1,...,I_n)$ where x_i are the Chern classes of $(S_i/S_{i-1})^*$, S_i are the universal subbundles with fibers $\mathbf{C}^i$, and I_i are the ith elementary symmetric polynomials of $x_1, ..., x_n$. Let us use as a basis of $H_2(F(n), \mathbf{Z})$ duals of the first Chern classes of $(S_i)^*$, $i=1,...,n-1$. They are in the edges of the closed Kähler cone.

Let $A(x_i)$ be a matrix

$$\begin{pmatrix} x_1 & q_1 & 0 & 0 & \dots & 0 \\ -1 & x_2 & q_2 & 0 & \dots & \\ & \ddots & & & \ddots & \\ 0 & \dots & & -1 & x_{n-1} & q_{n-1} \\ 0 & \dots & & 0 & -1 & x_n \end{pmatrix}.$$

Then the quantum relations are generated by the coefficients of the characteristic polynomial of the matrix $A(x_i)$.

The corresponding differential operators turn out to be obtained by the same method using $A(x_i)$ with arguments $\hbar\partial/\partial t_1$ instead of x_1, $\hbar\partial/\partial t_i - \hbar\partial/\partial t_{i-1}$ instead of x_i, and $-\hbar\partial/\partial t_{n-1}$ instead of x_n [13], [16]. These differential operators are the integrals of the quantized Toda lattices. The quadratic differential operator of them can be easily derived. In fact, given a quantum relation of $F(n)$ between the divisors x_i, there is a unique operator satisfying that its symbol becomes the relation, and that it annihilates $\langle \vec{s}, 1\rangle_0$ for all flat sections $\vec{s}$. In general, the explicit cohomological expression S^X of solutions to the quantum differential operators are not known.

2.3.3. *The general quintic hypersurface in* $\mathbf{P}^4$. Let Y be a smooth degree-5 hypersurface in $\mathbf{P}^4$. Y is not a homogeneous space. However, using the virtual fundamental class $[\overline{M}_{0,n}(Y,\beta)]^{\mathrm{virt}}$ [4], [5], [18], one can define also the quantum cohomology $QH^*(Y)$ of Y. It is expected that

$$\langle A_1 f_1(c), ..., A_N f_N(c)\rangle^Y = \langle A_1 f_1(c), ..., A_N f_N(c)\rangle^{\mathcal{O}(5)}.$$

Let p be the induced class of the hyperplane divisor in $\mathbf{P}^4$. The quantum relation is $(p\circ)^4=0$. The corresponding operator is, however, *not* $(\hbar d/dt)^4$, but

$$\left(\hbar\frac{d}{dt}\right)^2\left(\frac{(\hbar d/dt)^2}{5+f(q)}\right),$$

where $\langle p\circ p, p\rangle_0^Y = 5+f(q)$. Notice that in this *Calabi–Yau* 3-fold case, we loose the whole information of quantum cohomology when one considers only the quantum relation, $(p\circ)^4=0$. The unknown $f(q)$ was conjectured by physicists [7]. The general idea of the prediction is the following. Roughly speaking, in theoretical physics, there are quantum field theories associated to Calabi–Yau 3-folds by the A-model and the B-model. What we have constructed so far are A-model objects for Calabi–Yau 3-folds. On the other hand, using a family of the so-called mirror manifolds which are also Calabi–Yau 3-folds, conjecturally one may construct the equivalent quantum field theory by the B-model. The corresponding mirror partner of a quantum differential equation/quantum $\mathcal{D}$-module is the Picard–Fuchs differential equation/Gauss–Manin connection of the mirror family. It was predicted that they are equivalent by a certain transformation. In [7] are obtained the conjectural mirror family of quintics, the Picard–Fuchs differential equation and the transformation. That is how the prediction is made. The prediction is now proven to be correct by Givental [12].

2.4. *The idea of the proof of Theorem* 1. To describe the idea, let us notice that Givental's proof [14] of the mirror conjecture for the nonnegative toric complete intersections can be divided into three parts. (He shows in the paper that the mirror phenomenon occurs also in non-Calabi–Yau manifolds.) Let X be a symplectic toric manifold with a big torus T, and V be a $T\times T'$-equivariant decomposable convex vector bundle over X, where T' acts on X trivially. Consider the $T\times T'$-equivariant correlator corresponding to S^V, which will be denoted also by S^V.

(1) In the A-part, it is proven that

(a) the $T\times T'$-equivariant solution vector $S^V\in H^*_{(T\times T')}(X)[[q,\hbar^{-1}]]$ has an "almost recursion relation",

(b) it satisfies the polynomiality in the so-called "double construction", and

(c) it is uniquely determined by the above two properties with the asymptotical behavior $S^V=1+o(1/\hbar)$.

(2) In the B-part, another ($T\times T'$-equivariant hypergeometric) vector Φ^V, presumably given by the mirror symmetry conjecture, is constructed. It is verified that it also satisfies (a) and (b) using a toric (naive) compactification of holomorphic maps from $\mathbf{P}^1$ to X.

(3) When $c_1(X)-c_1(V)$ is nonnegative and E=Euler, there is a suitable equivalence transformation between Φ^V and S^V.

In this paper, for a $T\times T'$-equivariant decomposable convex vector bundle V over any compact homogeneous X of a semi-simple complex Lie group G, we will show that S^V satisfies property (1) above. In this case, T is a maximal torus of G.

We define Φ^V which corresponds to Φ^V of the toric case in property (2): Let $\Phi^X=S^X=\sum_d \Phi^X_d q^d$. For Φ^V, we will find a modification $H'_d\in H^*_{T\times T'}(X)[\hbar]$ (depending on V and d) such that if $\Phi^V:=\sum_d \Phi^X_d H'_d q^d$, then

(A) Φ^V (after the restriction to the fixed points) has the almost recursion relation exactly like S^V, and

(B) Φ^V has the polynomial property in the double construction.

In fact, we design H'_d to satisfy (A) and (B).

Finally, when E=Euler and $c_1(TX)-c_1(V)$ is nonnegative, we will prove that a certain operation will transform S^V to Φ^V, since they satisfy the same almost recursion relation and the polynomiality of the double construction.

3. The almost recursion relations

As in §2 let X be a homogeneous manifold G/P where G is a complex semi-simple Lie group and P is a parabolic subgroup. Let T be a maximal torus. The T-action has only isolated fixed points $\{v, w, ...\}$. The one-dimensional invariant orbit of T is analyzed in detail in §§7 and 8. For a moment we need the fact that the closures of orbits form a finite number of projective lines $\mathbf{P}^1$ connecting a fixed point v to another fixed point w. For a given equivariant vector bundle W over a $T\times T'$-space M, we use $[W]$ which denote the element in the K-group $K^0_{T\times T'}(M)$ corresponding to the $T\times T'$-vector bundle W.

The torus action on X induces the natural action on the moduli space of stable maps by the functorial property. Since the evaluation maps are $T\times T'$-equivariant, the pullbacks of $T\times T'$-bundles have natural actions in the orbifold sense. In turn, V_β has the induced $T\times T'$-action. *All ingredients in* §2 *are from now on the equivariant ones.* We would like to evaluate S^V as a specialization of the equivariant one corresponding to S^V. We use the same notation $S^V \in H^*_{(T\times T')}(X)[[\hbar^{-1}]][[q_1, ..., q_k]]$ for the equivariant one. Notice that S^V might have power series of $\hbar^{-1}$ in each coefficient of q^d, since c are not anymore nilpotent. Using the localization theorem, we shall find an "almost recursion relation" on the equivariant Givental correlator. To begin with, we summarize the fixed points of the induced action on the moduli space of stable maps. If a stable map represents a fixed point in the moduli space, the image of the map should lie in the closure of the one-dimensional orbits. The special points are mapped to isolated fixed points. Let us denote by $\alpha_{v,w}$ the character of the tangent space of a one-dimensional orbit connecting an isolated fixed point v to another w. Then, $-\alpha_{v,w}$ is the character of the tangent line of the one-dimensional orbit $o(v,w)$ at w. We use $\beta_{v,w}$ to stand for the second homology class represented by the ray. Denote by $o(v)$ the set of all fixed points $w\neq v$ which can be connected by a one-dimensional orbit with v.

LEMMA 1 (Recursion Lemma [12]). *Denote by ϕ_v the equivariant classes $i_*(1)$ at v, where i_v denotes the $T\times T'$-equivariant inclusion of the point v into (X, V). Then $S^V \in H^*_{T\times T'}(X)[[\hbar^{-1}]][[q_1, ..., q_k]]$ has an "almost" recursion relation, namely, for any $v\in X^T$:*

(0) *$S^V_v(q,\hbar) := \langle S^V, \phi_v\rangle^V_0 \in H^*_{(T\times T')}(\hbar)[[q]]$ and the substitution $S^V_w(q, -\alpha_{v,w}/m)$ of $\hbar$ with $-\alpha_{v,w}$ in $S_w(q,\hbar)$ is well-defined;*

(1) *the difference* R_v *of* $S_v^V(q,\hbar)$ *and the "recursion part" is a power series of* q *over the* polynomial *ring of* $1/\hbar$, *that is,*

$$R_v := S_v^V(q,\hbar) - \sum_{\substack{w\in o(v)\\ m>0}} q^{m\beta_{v,w}} \frac{(-\alpha_{v,w})/m}{\hbar(\alpha_{v,w}+m\hbar)} \cdot \frac{E(V_{v,w,m})\, i_v^*(\phi_v)}{\mathrm{Euler}(N_{v,w,m})} S_w^V(q, -\alpha_{v,w}/m)$$

is in $H^*_{(T\times T')}[\hbar^{-1}][[q]]$, *where* $V_{v,w,m}$ *is the* $T\times T'$*-representation space* $H^0(\mathbf{P}^1, f^*V)$ *with* f *the totally ramified* m*-fold map onto* $o(v,w)$ *over* v *and* w, *and* $N_{v,w,m}$ *is the* $T\times T'$*-representation space* $[H^0(\mathbf{P}^1, f^*TX)]-[0]$; *and*

(2) *furthermore, for* S^X *itself, the first term* R_v *is* 1.

We will say that the statement (1) reveals the almost recursion relation of S^V. The statement (2) shows that S_v^X have recursion relations in the ordinary sense.

Proof. First of all, using the short exact sequence

$$0 \to \mathrm{Ker} \to V_d \to e_1^*(V) \to 0$$

over $\overline{M}_{0,1}(X,d)$, we see that S^V is indeed in $H^*_{T\times T'}[[\hbar^{-1}]][[q_1,\dots,q_k]]$. (The last map in the sequence is given by the evaluation of global sections at the marked point.)

A connected component of the T-fixed loci of the moduli space $X_d := \overline{M}_{0,1}(X,d)$ is isomorphic to a product of Deline–Mumford spaces with marked points from the special points of the inverse image $f^{-1}(v)$ of the generic f in the component for all $v\in X^T$. Now fix a v and consider S_v^V. It is enough to count the fixed locus $F^{d,v}$ where the marked point x should be mapped to the fixed point v since ϕ_v can be supported only near the point. For a stable map $(f,C;x)$ denote by C_1 the irreducible component of C containing the marked point x. Then $F^{d,v}$ is the disjoint union of

$$F_1^{d,v} := \{(f,C;x)\in F^{d,v} \mid f(C_1)=v\}$$

and

$$F_2^{d,v} := \bigcup_{\substack{w\in o(v)\\ m=1,\dots,m\beta_{v,w}\leqslant d}} F^{d,v,w,m},$$

where $F^{d,v,w,m}$ is

$$\{(f,C;x)\in F^{d,v} \mid w\in f(C_1), \deg f|_{C_1} = m\in\mathbf{N}\}.$$

S_v^V is an integral over $F^{d,v}$'s by a localization theorem for orbifolds. We claim that the integral of

$$\frac{E(V_d)\, e_1^*(\phi_v)}{\hbar(\hbar-c)}$$

over $F_1^{d,v}$ is in $H^*_{(T\times T')}[\hbar^{-1}]$. The reason is that the universal cotangent line bundle over $F_1^{d,v}$ in the moduli space has the trivial action. It implies that the equivariant class c restricted to $F_1^{d,v}$ is nilpotent.

Now we shall obtain the "almost recursion relation" from the contribution of the fixed loci $F_2^{d,v}$. Denote $d-m\beta_{v,w}$ by d'. Since C_1 is always one end of C for any $(f,C;x)\in F_2^{d,v}$, we can have a natural isomorphism from $F^{d',w}$ to $F^{d,v,w,m}$, where $F^{d',w}$ are fixed loci in $X_{d'}:=\overline{M}_{0,1}(X,d')$, consisting of the stable maps sending the marked points to w. We obtain the morphism, joining the m-covering of $o(v,w)$ to stable maps in $F^{d',w}$. By the m-covering of $o(v,w)$ we mean a totally m-ramified map from $\mathbf{P}^1\cong C_1$ to $o(v,w)$ over v and w. Let $x'=f^{-1}(w)\cap C_1$.

We claim that the normal bundles as in a K-group $K^0(F^{d,v,w,m}\cong F^{d',w})$ satisfy the equality

$$[N_{X_d/F^{d,v,w,m}}]-[N_{X_{d'}/F^{d',w}}]=[N_{v,w,m}]-[T_wX]+[T_{x'}C_1\otimes L|_{F^{d',w}}], \qquad (1)$$

where L is the universal tangent line bundle over $X_{d'}$. The reason of the claim is as follows: Recall that each fixed component is isomorphic to the product of moduli spaces of stable curves (see §3 in [17] for detail). Hence, we conclude that $[N_{X_d/F^{d,v,w,m}}]-[N_{X_{d'}/F^{d',w}}]$ (over each fixed component) is equal to a trivial bundle with nontrivial actions. The twister by action can be computed by study of action on normal spaces at $(f,C_1\cup C_2;x)\in F^{d,v,w,m}$. Let N_1 be the normal space of $F^{d,v,w,m}$ at $(f,C_1\cup C_2;x)$ and N_2 be the normal space of $F^{d',w}$ at $(f|_{C_2},C_2;x':=C_1\cap C_2)$. Then as representation spaces,

$$\begin{aligned}[N_1]=[N_2]+([H^0(C_1,f|^*_{C_1}TX)]-[H^0(C_1,TC_1)])-[T_wX]\\ +[T_{x'}C_1\otimes T_{x'}C_2]+[T_{x'}C_1]+[T_xC_1].\end{aligned}$$

Hence we conclude the claim (1) after canceling of $[H^0(C_1,TC_1)]=[0]+[T_{x'}C_1]+[T_xC_1]$.

On the other hand, the direct sum of the fiber of V_d at $(f,C_1\cup C_2;x)\in F^{d,v,w,m}$ and $V|_w$ is equal to the direct sum of the fiber of $V_{d'}$ at $(f|_{C_2},C_2;x')$ and $H^0(C_1,(f|_{C_1})^*V)$.

Thus, applying the localization theorem we obtain

$$\int_{X_d}\frac{E(V_d)e_1^*(\phi_v)}{\hbar(\hbar-c)}=I+\sum_{\substack{w\in o(m)\\ 0<m\\ m\beta_{v,w}\leqslant d}}\frac{E(V_{v,w,m})i_v^*(\phi_v)(-\alpha_{v,w}/m)}{m\hbar(\alpha_{v,w}/m+\hbar)\,\mathrm{Euler}(N_{v,w,m})}\times\int_{X_{d-m\beta_{v,w}}}\frac{E(V_{d-m\beta_{v,w}})e_1^*(\phi_w)}{(-\alpha_{v,w}/m)(-\alpha_{v,w}/m-c)},$$

where I is the integral over $F_1^{d,v}$. The factor m in $m\hbar(\alpha_{v,w}/m+\hbar)$ comes from the nature of the orbifold localization theorem. (There are m automorphisms of $f|_{C_1}$.)

Using induction on $|d|=\sum d_i$, we may assume that the integral factors in the second term are well-defined and belong to $H^*_{(T\times T')}$. (The localization theorem itself also explains them.) So, statements (0) and (1) in the lemma are proven.

Now let us prove statement (2). Since $\langle c_1(TX),\beta\rangle\geqslant 2$ for all β, by degree counting we see that there are no contributions from the integral over $F_1^{d,v}$. The reason is that $\dim \overline{M}_{0,\sum d_i+1}=(\sum d_i)-2$ is less than $2(\sum d_i)-2$ if $(d_1,...,d_k)\neq 0$ and $\dim \overline{M}_{0,1}(X,d)\geqslant 2\sum d_i+\dim X-2$. So, in the case of S^X, $R_v=1$.

4. The double construction

LEMMA 2 (Double Construction Lemma). *The double construction*

$$W(S^V):=\int_V S^V(qe^{\hbar z},\hbar)e^{\sum p_i z_i}S^V(q,-\hbar)$$

is a power series of $q_1,...,q_k$ *and* $z_1,...,z_k$ *with coefficients in* $H^*_{T\times T'}[\hbar]$.

A priori, $W(S^V)$ has coefficients in the Laurent power series ring of $\hbar^{-1}$ over $H^*_{T\times T'}$. For the proof we will make use of graph spaces and universal classes defined below.

4.1. *The main lemma.* Let L_d be the projective space of the collection of all $(f_0,...,f_N)$ such that $f_i(z_0,z_1)$ are homogeneous polynomials of degree d. L_d is isomorphic to $\mathbf{P}^{(d+1)(N+1)-1}$. Given a stable map of degree $(d,1)$ from a prestable curve C to $\mathbf{P}^N\times\mathbf{P}^1$, there is a special irreducible component C_0 of C such that C_0 has degree $(d_0,1)$ under the stable map. This special component C_0 is parametrized by $\mathbf{P}^1$ in the target space. Thus we can identify C_0 with $\mathbf{P}^1$ and keep track where the other components intersect. Suppose that the other connected components $C_1,...,C_l$ of $C-C_0$ intersect with $C_0=\mathbf{P}^1$ at $[x_1:y_1],...,[x_l:y_l]$. If the degrees of C_i are d_i under the stable map, we now associate the stable map to

$$\prod_{i=1}^{l}(y_iz_0-x_iz_1)^{d_i}(f_0^0,...,f_N^0),$$

where $(f_0^0,...,f_N^0)$ are the polynomials coming from the data of the restriction of f to C_0.

MAIN LEMMA (Givental [12]). *The above "polynomial" mapping from* $G_d(\mathbf{P}^N):=\overline{M}_{0,0}(\mathbf{P}^N\times\mathbf{P}^1,(d,1))$ *to* L_d *is a* $(\mathbf{C}^\times)^N\times\mathbf{C}^\times$*-equivariant morphism, where* $\mathbf{P}^N$ *has the diagonal* $(\mathbf{C}^\times)^N$*-action and* $\mathbf{P}^1$ *has the* $\mathbf{C}^\times$*-action by* $[z_0:z_1]\mapsto[tz_0:z_1]$ *for* $t\in\mathbf{C}^\times$.

Notice that the $\mathbf{C}^\times$-action on L_d is given by

$$[f_0(z_0,z_1):...:f_N(z_0,z_1)]\mapsto[f_0(t^{-1}z_0,z_1):...:f_N(t^{-1}z_0,z_1)]$$

for $t\in\mathbf{C}^\times$.

4.2. *The universal class.* The $T\times T'$-equivariant spanned line bundle U_i over X gives rise to the $T\times T'$-equivariant morphism $\mu_0^i\colon X\to\mathbf{P}^N$, and so we obtain

$$\begin{array}{ccccc} (\mu_d^i)^*(\mathcal{O}(1)) & & & & \mathcal{O}(1) \\ \downarrow & & & & \downarrow \\ G_d(X) & \longrightarrow & G_{d_i}(\mathbf{P}^N) & \longrightarrow & L_{d_i} \end{array}$$

where $G_d(X)$ is the graph space $\overline{M}_{0,0}(X\times\mathbf{P}^1,(d,1))$, and μ_d^i is the $T\times T'\times\mathbf{C}^\times$-equivariant map from $G_d(X)$ to L_{d_i}. On $\mathcal{O}(1)$ we choose the lifted $\mathbf{C}^\times$-action coming from an action on the vector space of all $(N+1)$-multiples $(f_0,\dots,f_N)$ of degree-d_i homogeneous polynomials f_i, where the action is given by

$$(f_0(z_0,z_1),\dots,f_N(z_0,z_1))\mapsto(f_0(z_0,tz_1),\dots,f_N(z_0,tz_1))$$

for $t\in\mathbf{C}^\times$.

Denote by $P_i=c_1((\mu_d^i)^*\mathcal{O}(1))$ the $T\times T'\times\mathbf{C}^\times$-equivariant Chern class. It is said to be a universal class in the paper [14].

Denote by W_d the vector orbi-bundle over $G_d(X)$ with the fiber $H^0(C,\psi^*\pi_1^*V)$ at (C,ψ): Consider

$$\begin{array}{ccc} G_{d,1}(X) & \xrightarrow{e_1} & X\times\mathbf{P}^1 \\ {\scriptstyle\pi}\downarrow & & \uparrow \\ G_d(X) & & \pi_1^*V \end{array}$$

where $G_{d,1}(X)$ denotes the graph space with one marked point, and π_1 is the projection of $X\times\mathbf{P}^1$ to the first factor X. Then $W_d:=\pi_*e_1^*\pi_1^*V$.

4.3. *Proof of Lemma* 2. It is enough to show the equality

$$\sum_d q^d\int_{G_d(X)}e^{Pz}E(W_d)=\int_V S^V(q,\hbar)e^{pz}S^V(qe^{-\hbar z},-\hbar).$$

The left integral is a $T\times T'\times\mathbf{C}^\times$-equivariant push forward with $\hbar$ as $c_1(\mathcal{O}(1))$ over $\mathbf{P}^\infty$, and the right one is a $T\times T'$-equivariant push forward with a formal variable $\hbar$.

We will apply the localization theorem. Let us analyze the $\mathbf{C}^\times$-action fixed loci $G_d(X)^{\mathbf{C}^\times}$ of $G_d(X)$. $G_d(X)^{\mathbf{C}^\times}$ is isomorphic to

$$\sum_{d^{(1)}+d^{(2)}=d}\overline{M}_{0,1}(X,d^{(1)})\times_X\overline{M}_{0,1}(X,d^{(2)}).$$

Suppose $|d^{(1)}|+|d^{(2)}|\neq 0$. The normal bundle is as follows:

When $|d^{(1)}|\cdot|d^{(2)}|=0$: The codimension is 2 (one from the nodal condition and the other from the condition of the image of the nodal point). Then the Euler class of the normal bundle is $\hbar(\hbar-c_0)$, or $-\hbar(-\hbar-c_\infty)$, where c_0 and c_∞ are the Chern classes of universal cotangent line bundles of the first marked point over $\overline{M}_{0,1}(X,d^{(1)})$ and $\overline{M}_{0,1}(X,d^{(2)})$, respectively. Here we assume the following convention: $0=[0:1]$, $\infty=[1:0]$, the associated equivariant line bundle to the character 1 of the group $\mathbf{C}^\times$ has $\hbar$ as its equivariant Chern class.

When $|d^{(1)}|\cdot|d^{(2)}|\neq 0$: The codimension is 4 and the Euler class is

$$\hbar(\hbar-c_0)(-\hbar)(-\hbar-c_\infty).$$

Here, for instance, $c_0\in H^2(\overline{M}_{0,1}(X,d^{(1)})\times_X\overline{M}_{0,1}(X,d^{(2)}))$ is the pullback of the Chern class of the universal cotangent line bundle of the first factor of $\overline{M}_{0,1}(X,d^{(1)})\times_X\overline{M}_{0,1}(X,d^{(2)})$.

Let us analyze P_i restricted to $G_d(X)^{\mathbf{C}^\times}$. Consider the commutative diagram

$$\begin{array}{ccc} G_{d^{(1)},d^{(2)}}(X):=\overline{M}_{0,1}(X,d^{(1)})\times_X\overline{M}_{0,1}(X,d^{(2)}) & \xrightarrow{\mu_d^i} & L_{d_i}\ni z_0^{d_i^{(1)}}z_1^{d_i^{(2)}}[x_0:\dots:x_N] \\ \downarrow{\scriptstyle\pi_2} & & \uparrow \qquad\quad \uparrow \\ \overline{M}_{0,1}(X,d^{(2)}) & \xrightarrow{\mu_0^i\circ e_1} & \mathbf{P}^N\ni[x_0:\dots:x_N], \end{array}$$

where the first vertical map π_2 is the projection and under the second vertical map $\mathbf{P}^N$ is embedded into L_{d_i} as the $\mathbf{C}^\times$-action fixed locus of the part $\{z_0^{d^{(1)}}z_1^{d^{(2)}}[x_0:\dots:x_N]\mid[x_0:\dots:x_N]\in\mathbf{P}^N\}$. One concludes that $e_1^*\circ(\mu_0^i)^*(c_1(\mathcal{O}(1)|_{\mathbf{P}^N}))=e_1^*(p_i)-d_i^{(2)}\hbar$, and so

$$\sum P_iz_i|_{G_{d^{(1)},d^{(2)}}}=\sum(\pi_2^*e_1^*(p_i)-d_i^{(2)}\hbar)z_i.$$

Since

$$\begin{aligned} &S^V(q,\hbar)\,e^{pz}S^V(qe^{-\hbar z},-\hbar) \\ &\quad=\sum_{a,b,d^{(1)},d^{(2)}}\left\langle\frac{T_a}{\hbar(\hbar-c)}\right\rangle_{d^{(1)}}^V T^aq^{d^{(1)}}e^{pz}\left\langle\frac{T^b}{-\hbar(-\hbar-c)}\right\rangle_{d^{(2)}}^V T_bq^{d^{(2)}}e^{-d^{(2)}\hbar z}, \end{aligned}$$

we see that

$$\begin{aligned} \int_V S^V(q,\hbar)\,e^{pz}S^V(qe^{-\hbar z},-\hbar)&=\sum_{a,d^{(1)},d^{(2)}}\left\langle\frac{T_a}{\hbar(\hbar-c)}\right\rangle_{d^{(1)}}^V\left\langle\frac{T^ae^{pz-d^{(2)}\hbar z}}{-\hbar(-\hbar-c)}\right\rangle_{d^{(2)}}^V q^{d^{(1)}+d^{(2)}} \\ &=\sum_d q^d\int_{G_{d^{(1)},d^{(2)}}(X)}\frac{e^{(\pi_2^*e_1^*p-d^{(2)}\hbar)z}E(W_d)}{[N_{G_d(X)/G_{d^{(1)},d^{(2)}}(X)}]} \\ &=\sum_d q^d\int_{G_d(X)}e^{Pz}E(W_d), \end{aligned}$$

after applying the localization theorem only for the $\mathbf{C}^\times$-action on $G_d(X)$.

5. The class $\mathcal{P}(\mathcal{C})$ and mirror transformations

5.1. *The class* $\mathcal{P}(\mathcal{C})$. Let $\mathcal{C}$ be the collection of given data of $C_{v,w,m}\in H^*_{(T\times T')}$, $\alpha_{v,w}\in H^*_{T\times T'}$ and $\beta_{v,w}\in\Lambda-0$, for all $(v,w,m)\in X^T\times X^T\times\mathbf{N}$ with $v\in o(w)$. Here $\mathbf{N}$ is the set of positive integers. Assume that $(p_i)_w-(p_i)_v=-\langle p_i,\beta_{v,w}\rangle\alpha_{v,w}$ for all $i=1,...,k$. Define the degree of $\hbar$ as 1. Let $q_1,...,q_k$ be formal parameters with some given nonnegative degrees. Define the degree of a homogeneous class of $H^b_{T\times T'}(X)$ as $\frac{1}{2}b$. Let $\mathcal{P}(\mathcal{C})$ be the class of all $Z(q,\hbar)\in H^*_{T\times T'}(X)[[\hbar^{-1},q]]$ of homogeneous degree 0 such that

(a) $Z(0,\hbar)=1$, $Z_v(q,\hbar):=\langle Z,\phi_v\rangle_0^V$ is in $H^*_{(T\times T')}(\hbar)[[q]]$ for any fixed point v, and $Z_w(q,-\alpha_{v,w}/m)$ are well-defined for all $v\in o(w)$, $m>0$ (m are positive integers);

(b) the almost recursion relation for each fixed point v holds, that is, by definition,

$$R_v:=Z_v(q,\hbar)-\sum_{\substack{m>0\\ w\in o(v)}}q^{m\beta_{v,w}}\frac{C_{v,w,m}}{\hbar(\alpha_{v,w}+m\hbar)}Z_w(q,-\alpha_{v,w}/m)$$

is in $H^*_{(T\times T')}[\hbar^{-1}][[q]]$, where

$$q^{m\beta_{v,w}}:=\prod_i q_i^{m\langle p_i,\beta_{v,w}\rangle};$$

and

(c) in the double construction,

$$W(Z)(q,z):=\int_V Z(qe^{\hbar z},\hbar)e^{\sum p_iz_i}Z(q,-\hbar)$$

is in $H^*_{T\times T'}[\hbar][[z,q]]$. (Here we use the multi-index notation for $z=(z_1,...,z_k)$ and $q=(q_1,...,q_k)$.)

Whenever the data $\mathcal{C}$ come from (X,V,E) as in Lemma 1, we denote the class by $\mathcal{P}(X,V,E)$. So, in the case

$$C_{v,w,m}:=C^V_{v,w,m}:=\frac{(-\alpha_{v,w})/mE(V_{v,w,m})i_v^*(\phi_v)}{\mathrm{Euler}(N_{v,w,m})},$$

$\alpha_{v,w}$ is the character of $T_vo(v,w)$, $\beta_{v,w}=[o(v,w)]\in H_2(X,\mathbf{Z})$, and

$$c_1(TX)-c_1(V)=\sum_{i=1,...,k}(\deg q_i)p_i.$$

So far, we have proved that S^V for E=Euler is in class $\mathcal{P}(X,V,\mathrm{Euler})$.

Below we introduce on $\mathcal{P}(\mathcal{C})$ a transformation group generated by the following three types of operations.

(1) Multiplication by $f(q)$: Let $f(q)=\sum_{d\geqslant 0}f_dq^d$, where $f_d\in\mathbf{Q}$, $f(q)$ is homogeneous of degree 0, and $f(0)=1$. Then $f(q)Z\in\mathcal{P}(\mathcal{C})$.

(2) Multiplication by $\exp(f(q)/\hbar)$: Let $f(q)=\sum_{d>0} f_d q^d$, where f_d are in $H^*_{T\times T'}$. Suppose $\deg(f(q))=1$. Then $Z^{\mathrm{new}}:=\exp(f(q)/\hbar)Z$ is still in $\mathcal{P}(\mathcal{C})$.

(3) Coordinate changes: Consider a transformation

$$Z \to Z^{\mathrm{new}} := \exp\left(\sum_i f_i(q)p_i/\hbar\right) Z(q\exp(f(q)),\hbar),$$

where $f_i(q)=\sum_{d>0} f_i^{(d)} q^d$ of homogeneous degree 0, $f_i^{(d)}\in\mathbf{Q}$, and

$$q\exp(f(q)) = (q_1\exp(f_1(q)),\ldots,q_k\exp(f_k(q))).$$

Then Z^{new} is still in $\mathcal{P}(\mathcal{C})$.

Let us call the transformation group the mirror group.

THEOREM 2 ([14]). *Suppose that* $\deg q_i$ *are nonnegative and that there is at least one element of the form* $1+o(\hbar^{-1})$ *in the class* $\mathcal{P}(\mathcal{C})$. *Then the mirror group action on* $\mathcal{P}(\mathcal{C})$ *is free and transitive.*

First, we will check (1), (2) and (3); and prove the so-called uniqueness lemma and then the theorem above.

Proof of (1). First, $Z^{\mathrm{new}}:=fZ$ is homogeneous of degree 0, $f(0)Z(0,\hbar)=1$, fZ_v are in $H^*_{(T\times T')}(\hbar)[[q]]$, and of course $Z^{\mathrm{new}}_w(q,-\alpha_{v,w}/m)$ are well-defined. Second,

$$Z^{\mathrm{new}}_v = f(q)R_v + \sum q^{m\beta_{v,w}} \frac{C_{v,w,m}}{\hbar(\alpha_{v,w}+m\hbar)} Z^{\mathrm{new}}_w(q,-\alpha_{v,w}/m).$$

Thus fZ has the almost recursion relation.

Finally,

$$W^{\mathrm{new}} := \int_V Z^{\mathrm{new}}(qe^{\hbar z},\hbar)e^{pz}Z^{\mathrm{new}}(q,-\hbar) = f(qe^{\hbar z})f(q)W,$$

which still has the polynomial coefficients in $H^*_{T\times T'}[\hbar]$.

Proof of (2). The new Z^{new} is homogeneous of degree 0, $Z^{\mathrm{new}}(0,\hbar)=1$, Z^{new}_v are in $H^*_{(T\times T')}(\hbar)[[q]]$, and $Z^{\mathrm{new}}_w(q,-\alpha_{v,w}/m)$ are well-defined. Since

$$\exp\left(\frac{f(q)}{\hbar}+\frac{mf(q)}{\alpha_{v,w}}\right) = 1+(\alpha_{v,w}+m\hbar)g_{\alpha_{v,w},m}$$

and $g_{\alpha_{v,w},m}$ is a q-series with polynomial coefficients in $H^*_{(T\times T')}[\hbar^{-1}]$, Z^{new} has the almost recursion relation.

Once again,

$$W^{\mathrm{new}} = \exp\left(\frac{1}{\hbar}(f(qe^{\hbar z}) - f(q))\right)W.$$

But $f(qe^{\hbar z}) - f(q) = \sum_{d>0} f_d((e^{\hbar z})^d - 1)q^d$ is a (z, q)-series with polynomial coefficients in $\hbar H^*_{T\times T'}[\hbar]$.

Proof of (3). The Z^{new} is homogeneous of degree 0, $Z^{\mathrm{new}}(0,\hbar)=1$, Z_v^{new} are in $H^*_{(T\times T')}(\hbar)[[q]]$, and $Z_w^{\mathrm{new}}(q, -\alpha_{v,w}/m)$ make sense.

Since $(p_i)_w - (p_i)_v = -\langle p_i, \beta_{v,w}\rangle \alpha_{v,w}$,

$$\sum_i f_i(q)(p_i)_v/\hbar = \sum_i f_i(q)(p_i)_w/(-\alpha_{v,w}/m) - m\sum_i \langle p_i, \beta_{v,w}\rangle f_i(q) + \sum_i \frac{f_i(q)(p_i)_v}{\alpha_{v,w}\hbar}(m\hbar + \alpha_{v,w}).$$

The exponential of the last term on the right can be denoted by $1+(\alpha_{v,w}+m\hbar)g_{\alpha_{v,w},m}$ where $g_{\alpha_{v,w},m}$ is a q-series with coefficients which are in $H^*_{(T\times T')}[\hbar^{-1}]$. Z^{new} satisfies the almost recursion relation.

Consider the double construction

$$\begin{aligned} W^{\mathrm{new}}(q,z) &= \int_V e^{f(qe^{\hbar z})p/\hbar} Z(qe^{\hbar z}e^{f(qe^{\hbar z})}, \hbar)e^{pz}e^{-f(q)p/\hbar}Z(qe^{f(q)}, -\hbar) \\ &= W\left(qe^{f(q)}, z + \frac{f(qe^{\hbar z}) - f(q)}{\hbar}\right). \end{aligned}$$

But since $f(qe^{\hbar z}) - f(q)$ is divisible by $\hbar$, W^{new} is a polynomial (q, z)-series.

By the way, the inverse transformation can be given by unique $g_i(q)\in \mathbf{Q}[[q]]$, $i = 1, \ldots, k$, satisfying $f_i(q_1e^{g_1(q)}, \ldots, q_ke^{g_k(q)}) = -g_i(q)$ and $g_i(0)=0$ for all i.

LEMMA 3 (Uniqueness Lemma). *Let $Z=\sum_{d\geqslant 0} Z_dq^d$ and $Z'=\sum_{d\geqslant 0} Z'_dq^d$ be series in $\mathcal{P}(C)$. Suppose $Z\equiv Z'$ modulo $(1/\hbar)^2$. Then $Z'=Z$.*

Proof. We may suppose that $Z'_d = Z_d$ for all $0\leqslant d < d_0$ for some $d_0 \neq (0, \ldots, 0)$. Let

$$D(\hbar) := Z'_{d_0} - Z_{d_0} = A\hbar^{-2r-1} + B\hbar^{-2r} + \ldots = \hbar^{-2r}(A/\hbar + B + O(\hbar)),$$

where $A, B\in H^*_{T\times T'}(X)$. ($A$ might be 0.) This is possible since $\langle D, \phi_v\rangle_0$ for all v are polynomials of $1/\hbar$ over $H^*_{(T\times T')}$, and so D is a polynomial of $1/\hbar$ over $H^*_{T\times T'}(X)$. Consider the coefficient of q^{d_0} in $W(Z') - W(Z)$, which can be set $\delta(D) = \int_V e^{(p+d_0\hbar)z}D(\hbar) + e^{pz}D(-\hbar)$. If $r=0$, then $D=0$ since $D\equiv 0$ modulo $(1/\hbar)^2$. Assume $r\geqslant 1$. We shall show that $A=0=B$, which implies by induction that $D=0$. Notice that, since $\delta(D)$ is a polynomial of $\hbar$,

$$O(\hbar^2)=\hbar^{2r}\delta(D)=\int_V e^{(p+d_0\hbar)z}(A/\hbar+B+O(\hbar))+e^{pz}(-A/\hbar+B+O(\hbar))$$
$$=\int_V e^{pz}Ad_0z+2Be^{pz}+O(\hbar).$$

So,

$$0=d_0z\int_V e^{pz}A+2\int_V Be^{pz}=\sum_{v\in X^T}(d_0ze^{p_vz}A_v+2e^{p_vz}B_v)\frac{1}{i_v^*(\phi_v)},$$

where A_v and B_v are the restrictions of A and B to the fixed point v, respectively. Since p_vz are different as v are different (this can be seen in §§ 7 and 8), e^{p_vz} and $d_0ze^{p_vz}$ are linearly independent over $H^*_{(T\times T')}$. So we conclude that $A_v=0=B_v$ for all v, and hence $A=0=B$.

5.2. *Proof of Theorem* 2. It suffices to show the transitivity of the action. Let Z_1 and Z_2 be in class $\mathcal{P}(\mathcal{C})$ and let $Z_1=1+o(1/\hbar)$.

Since $\deg q\geqslant 0$, we may let

$$Z_2=Z_2^{(0)}+Z_2^{(1)}\frac{1}{\hbar}+o(1/\hbar),$$

where $Z_2^{(0)}\in H^*_{T\times T'}(X)[[q]]$ is homogeneous of degree 0 and $Z_2^{(1)}$ is homogeneous of degree 1. Furthermore, $Z_2^{(0)}(q)\in H^*_{T\times T'}(X)[[q]]$ is a q-series with coefficients in $\mathbf{Q}$, $Z_2^{(0)}(0)=1$, and $Z_2^{(1)}(q)$ is a q-series with coefficients in $H^*_{T\times T'}[p]$ by degree counting.

We may let

$$\frac{Z_2^{(1)}(q)}{Z_2^{(0)}(q)}=\sum_i(f_i(q)\cdot p_i)+g(q),$$

where $f_i(q)$ are pure q-series over $\mathbf{Q}$ of degree 0 and $g(q)$ are of degree 1 in $H^*_{T\times T'}[[q]]$. In addition, $f_i(0)=0=g(0)$.

Now, consider operations on Z_1: first, coordinate changes,

$$Z_1'=\exp(f(q)p/\hbar)Z_1(q\exp(f(q)),\hbar)=1+\frac{1}{\hbar}f(q)p+o(1/\hbar),$$

second, multiplication by $\exp(g(q)/\hbar)$,

$$Z_1''=\exp(g(q)/\hbar)Z_1'=1+\frac{1}{\hbar}(f(q)p+g(q))+o(1/\hbar),$$

and finally, multiplication by $Z_2^{(0)}(q)$,

$$Z_1'''=Z_2^{(0)}(q)Z_1''=Z_2^{(0)}+\frac{1}{\hbar}Z_2^{(1)}+o(1/\hbar).$$

According to the uniqueness lemma, the last one, Z_1''', must be equal to Z_2 since $Z_1'''\cong Z_2$ modulo $(1/\hbar)^2$.

5.3. *Transformation from J^V to I^V.* We explain the transformation introduced in the introduction. Let $\widetilde{Z}$ be the nonequivariant specialization of Z. Let Z_1 and Z_2 be in class $\mathcal{P}(\mathcal{C})$ and let $Z_1=1+o(1/\hbar)$. Now let us specialize the equivariant setting to the nonequivariant one. Let $J^V=e^{(t_0+pt)/\hbar}\widetilde{Z}_1(q)$ and $I^V=e^{(t_0+pt)/\hbar}\widetilde{Z}_2(q)$. Then, they are equivalent up to the unique coordinate change $t_0\mapsto t_0+f_0(q)\hbar+f_{-1}(q)$ and $t\mapsto t_i+f_i(q)$, $i=1,\ldots,k$, where $f_j\in\mathbf{Q}[[q]]$ for all j, f_0 and f_i $(i=1,\ldots,k)$ have degree 0, f_{-1} has degree 1, and $f_j(0)=0$ for all j.

6. The modified B-series

This second part departs in perspective from Givental's papers [12], [14].

Let X be a homogeneous manifold with the torus $T\times T'$-action. From now on let $V=L_1\oplus\ldots\oplus L_l$ be an equivariant decomposable convex vector bundle over X, where L_i are line bundles. In this section, $\{T_a\}$ and $\{T^b\}$ denote bases of $H^*(X)$ dual to each other with respect to the usual Poincaré paring $\langle\cdot,\cdot\rangle_0^X$.

6.1. *The correcting Euler classes.* Let $x=(x_1,\ldots,x_l)$ be indeterminants.

Define a polynomial of x over $\mathbf{Z}[\hbar]$ for $\beta\in\Lambda$:

$$H_\beta(x,\hbar):=\prod_{i=1}^{l}\prod_{m=0}^{\langle c_1(L_i),\beta\rangle}(x_i+m\hbar).$$

Set

$$H'_\beta(x,\hbar):=\frac{H_\beta(x,\hbar)}{\prod x_i}.$$

We treat each linear factor $(x_i+m\hbar)$ of H_β as a Chern character. Define

$$\Phi^V(q,\hbar):=\sum_{d\in\Lambda}\sum_a q^d\left\langle\frac{T_a}{\hbar(\hbar-c)}\right\rangle_d^X T^a E(H'_d(x,\hbar))(c_1(L),\hbar),$$

where $c_1(L)=(c_1(L_1),\ldots,c_1(L_k))$.

CLAIM. (1) $(p_i)_w=(p_i)_v-\langle p_i,\beta_{v,w}\rangle\alpha_{v,w}$,

(2) $c_1(L)_w=c_1(L)_v-\langle c_1(L),\beta_{v,w}\rangle\alpha_{v,w}$,

(3) $E(V_{v,w,m})=E(H_{m\beta_{v,w}})(c_1(L)_v,-\alpha_{v,w}/m)$.

Proof. Let U be any equivariant convex line bundle. On the ray $o(v,w)$ $(\cong\mathbf{P}^1)$, we have a homogeneous coordinate $[z_0:z_1]$ such that the induced action on the ray is linear (because of the equivariant embedding theorem). We have also global sections $z_0^n,z_0^{n-1}z_1,\ldots,z_1^n$ of the restriction $U|_{\mathbf{P}^1}$ of U to the ray, where $[1:0]=w$, $[0:1]=v$ and

$n=\langle c_1(U),\beta_{v,w}\rangle$. We know that z_0^n, z_1^n and z_0/z_1 have the characters $c_1(U)_v$, $c_1(U)_w$ and $\alpha_{v,w}$, respectively. This concludes the proof.

The first claim shows that we have a well-defined class $\mathcal{P}(X,V,E)$. (Otherwise, the mirror group transformation may not preserve the class $\mathcal{P}(X,V,E)$.)

THEOREM 3. *Suppose that $c_1(TX)-c_1(V)$ is in the closed ample cone, and let $E=$ Euler. Then Φ^V is in the class $\mathcal{P}(X,V,\mathrm{Euler})$.*

Notice that for $\beta'\leqslant\beta$

$$H_\beta(x-\langle c_1(L),\beta'\rangle\hbar,\hbar)=H_{\beta'}(x,-h)H'_{\beta-\beta'}(x,\hbar), \tag{2}$$

$$H'_\beta(x,\hbar)=H'_{\beta'}(x,h)H'_{\beta-\beta'}(x+\langle c_1(L),\beta'\rangle\hbar,\hbar), \tag{3}$$

which will show the polynomiality of the double construction and the almost recursion relation for Φ^V, respectively.

6.2. *The proof of Theorem* 3. The homogeneity of Φ^V is clear when E=Euler, and the rest of the properties will be proven for general E.

First of all, it is a simple check to see

$$\Phi^V\in H^*_{T\times T'}[[\hbar^{-1}]][[q]].$$

For the polynomiality, consider

$$\begin{aligned}\int_V \Phi^V(q,\hbar)e^{pz}\Phi^V(qe^{-\hbar z},-\hbar)=\sum_d\sum_{d^{(1)}+d^{(2)}=d,a} q^{d^{(1)}}\left\langle\frac{T_aE(H_{d^{(1)}})(c_1(L),\hbar)}{E(V)\hbar(\hbar-c)}\right\rangle^X_{d^{(1)}}\\ \times q^{d^{(2)}}\left\langle\frac{T^ae^{(p-d^{(2)}\hbar)z}E(H_{d^{(2)}})(c_1(L),-\hbar)}{-\hbar(-\hbar-c)}\right\rangle^X_{d^{(2)}},\end{aligned} \tag{4}$$

where $\langle T_a,T^b\rangle^X_0=\delta_{a,b}$. Let us use the notation and facts in §4.3. Since

$$E(H_{d^{(1)}})(c_1(L),\hbar)E(H_{d^{(2)}})(c_1(L),-\hbar)=E(H_d(x-\langle c_1(L),d^{(2)}\rangle\hbar,\hbar))(c_1(L),\hbar)E(V)$$

from (2), the universal class $U(c_1(L))$ corresponding to $c_1(L)$ restricted to $G_{d_1,d_2}(X)$ is

$$\pi_2^*e_1^*(c_1(L))-\langle c_1(L),d^{(2)}\rangle\hbar,$$

and $e_1\circ\pi_1=e_1\circ\pi_2$, (4) is equal to

$$\sum_d q^d\int_{G_{d^{(1)},d^{(2)}}(X)}\frac{e^{(\pi_2^*e_1^*p-d^{(2)}\hbar)z}E(H_d)(U(c_1(L)),\hbar)}{[N_{G_d(X)/G_{d^{(1)},d^{(2)}}(X)}]}$$

$$=\sum_d q^d\int_{G_d(X)}e^{Pz}E(H_d)(U(c_1(L)),\hbar),$$

which shows the polynomiality.

Now let us check the almost recursion relation. Let

$$S_v^X(q,\hbar) := \langle S^X, \phi_v^X \rangle_0^X = \sum_d S_{v,d}^X(\hbar) q^d.$$

Since (if $d \neq 0$)

$$S_{v,d}^X(\hbar) = \sum_{\substack{w \in o(v) \\ 0<m \\ m\beta_{v,w} \leqslant d}} \frac{C_{v,w,m}^X}{\hbar(\alpha_{v,w}+m\hbar)} S_{w,d-m\beta_{v,w}}^X\Big(-\frac{\alpha_{v,w}}{m}\Big)$$

and

$$E(H'_\beta)\Big(c_1(L)_v, -\frac{\alpha_{v,w}}{m}\Big) = \frac{E(V_{v,w,m})}{E(V)_v} E(H'_{\beta-m\beta_{v,w}})\Big(c_1(L)_w, -\frac{\alpha_{v,w}}{m}\Big)$$

from (3) and the claim, we obtain that

$$\Phi_{v,d}^V(\hbar) := \langle \Phi_d^V(\hbar), \phi_v \rangle_0^V = R_{v,d} + \sum_{\substack{w \in o(v) \\ 0<m \\ m\beta_{v,w} \leqslant d}} \frac{C_{v,w,m}^X E(V_{v,w,m})}{E(V)_v \hbar(\alpha_{v,w}+m\hbar)}$$

$$\times \Phi_{w,d-m\beta_{v,w}}^X\Big(-\frac{\alpha_{v,w}}{m}\Big) E(H'_{d-m\beta_{v,w}})\Big(c_1(L)_w, -\frac{\alpha_{v,w}}{m}\Big),$$

where $R_{v,d}$ is indeed a polynomial of $1/\hbar$ over $H^*_{(T\times T')}$ since

$$\Phi_d^V = (e_1)_* \frac{c^{\langle c_1(X),d\rangle-2}}{\hbar^{\langle c_1(X),d\rangle-1}(\hbar-c)} \in \frac{1}{\hbar^{\langle c_1(X),d\rangle}} H^*_{T\times T'}[[\hbar^{-1}]].$$

However, since

$$C_{v,w,m}^V = \frac{C_{v,w,m}^X E(V_{v,w,m})}{E(V)_v},$$

$\Phi_v^V(q,\hbar)$ has the same almost recursion coefficients $C_{v,w,m}^V$ with S^V.

6.3. *A proof of Main Theorem* 1. Recursion Lemma 1 and Double Construction Lemma 2 show that S^V is in class $\mathcal{P}(X, V, \text{Euler})$. Certainly S^V is of the form $1+o(\hbar^{-1})$. According to Theorem 3, Φ^V also belongs to $\mathcal{P}(X, V, \text{Euler})$. Then Theorem 2 results in a proof of the main theorem. (We use the condition that E=Euler, in order to make sure that S^V and Φ^V are homogeneous of degree 0.)

7. Grassmannians

7.1. *Notation.* Let $e_1, ..., e_n$ form the standard basis of $\mathbf{C}^n$, $T{=}(\mathbf{C}^\times)^n$ be the complex torus, and $X{:=}\mathrm{Gr}(k,n)$ be the Grassmannian, the set of all k-subspaces in $\mathbf{C}^n$. As usual, let T act on $\mathrm{Gr}(k,n)$ by the diagonal action. The fixed points $v{=}(i_1, ..., i_k)$ are then the k-planes generated by the vectors $e_{i_1}, ..., e_{i_k}$. Denote by $\mathbf{C}^n \times X$ the trivial vector bundle with the standard action. Then we may consider L, the determinant of the bundle dual to the T-equivariant universal k-subbundle of $\mathbf{C}^n \times X$. Define $V{=}L^{\otimes l}$, $l{>}0$. Denote by p the equivariant class $c_1(L)$. We may identify $H^*(BT)$ with $\mathbf{Q}[\varepsilon_1, ..., \varepsilon_n]$ by the correspondence that ε_i is also denoted the equivariant Chern class of the line bundle over a point equipped with T-action as the representation of the character ε_i. With respect to the Chern class of L, we shall write $d{\in}\mathbf{Z}{=}H_2(X, \mathbf{Z})$.

7.2. *A-series.*

7.2.1. *Fixed points.* Let v be, say, $(1,2,...,k)$. Then around the point, a local chart can be described by

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \\ x_{1,1} & x_{1,2} & \dots & x_{n-k,k} \\ \vdots & \vdots & & \vdots \\ x_{n-k,1} & x_{n-k,2} & \dots & x_{n-k,k} \end{pmatrix}.$$

For each complex value $(x_{i,j})$ the column vectors in the matrix span a k-plane which stands for a point in $\mathrm{Gr}(k,n)$. Then in the chart the action by $(t_1, ..., t_n){\in}T$ is described as

$$\begin{pmatrix} x_{1,1} & x_{1,2} & \dots & x_{n-k,k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n-k,1} & x_{n-k,2} & \dots & x_{n-k,k} \end{pmatrix} \mapsto \begin{pmatrix} t_1^{-1}t_{k+1}x_{1,1} & t_2^{-1}t_{k+1}x_{1,2} & \dots & t_k^{-1}t_{k+1}x_{1,k} \\ \vdots & \vdots & \ddots & \vdots \\ t_1^{-1}t_n x_{n-k,1} & t_2^{-1}t_n x_{n-k,2} & \dots & t_k^{-1}t_n x_{n-k,k} \end{pmatrix}.$$

In each isolated fixed point of the Grassmannian there are $\dim \mathrm{Gr}(k,n)$-many one-dimensional orbits (rays) passing through the point. For instance, if $v{=}(1,2,...,k)$, then there is only one ray (v,w) from v to $w{=}(..., \hat{\imath}, ..., j)$ for any $i{\leqslant}k{<}j$. These rays have degree $1{\in}H_2(X,\mathbf{Z})$.

7.2.2. *The Euler classes.* Notice that the tangent space at $v{=}(1,2,...,k)$ of the ray connecting v to $w{=}(..., \hat{\imath}, ..., j)$ has the character $\alpha(v,w){=}\varepsilon_j{-}\varepsilon_i$, where $j{>}k{\geqslant}i$. Similarly one can find out the characters for the other cases.

Let $f\colon \mathbf{P}^1 \to \mathrm{Gr}(k,n)$ be an m-fold morphism totally ramifying the ray over v and w. The T-representation space $H^0(f^*L^{\otimes l})$ has the orbi-characters

$$\frac{ap_v+bp_w}{m}=lp_v-\frac{\varepsilon_j-\varepsilon_i}{m}b \quad \text{for } a+b=lm,\ a\geqslant 0,\ b\geqslant 0,$$

where $p_v=-(\varepsilon_1+\ldots+\varepsilon_k)$ and $p_w=-(\varepsilon_1+\ldots+\hat{\varepsilon}_i+\ldots+\varepsilon_k+\varepsilon_j)$ are $p=c_1(L)$ restricted to the fixed points v and w, respectively.

8. The flag manifolds

We analyze fixed points of the maximal torus actions and the invariant curves connecting two fixed points. This explicit description would be useful also to find S^X explicitly.

8.1. *The complete flag manifolds.* Let X be the set of all Borel subgroups of a simply-connected semi-simple Lie group G. It is a homogeneous space with the G-action by conjugation. Then the maximal torus T-action (fix one) has isolated fixed points. They are exactly Borel subgroups containing T since the normalizer of a Borel subgroup is so itself. The fixed points are naturally one-to-one corresponding to the set of Weyl chambers. Each Borel subgroup containing T gives rise to negative roots (our convention) of B and so a chamber associated to the positive roots. Let C be the set of chambers. The tangent line subspace associated to the positive root α has the character α.

There is, if one fixes a fixed point v, a natural correspondence between the $H^2(X,\mathbf{Z})$ and the characters of T. Then the Kähler cone is exactly the positive Weyl chamber v. Notice that the fundamental roots span the Kähler cone. Consider co-roots $\alpha^\vee$. They span the Mori cone. We can identify the Mori cone Λ with the nonnegative integer span of co-roots.

8.2. *The generalized flag manifolds.* Let X be the set of all parabolic subgroups with a given conjugate type. Let T be a maximal torus of G. Then the fixed loci are isolated fixed points consisting of parabolic subgroups containing T.

8.2.1. *Rays.* Let us choose a fixed point $P\supset T$. Then the rays at the fixed point are described by the following way. (The rays are by definition the one-dimensional orbits of T passing through P.) Fix B a Borel subgroup in P containing T. First consider the T-equivariant fibration $G/B\to G/P$ and the rational map to G/B by $\exp(zX_\alpha)\in G$, $z\in\mathbf{C}$, where X_α is an eigenvector of the positive root α. Since $\exp H\exp(zX_\alpha)\exp -H = \exp(z\exp \mathrm{ad}\, H(X_\alpha))=\exp(z\exp(\alpha(H))X_\alpha)$ for $H\in \mathrm{Lie}\, T$, we conclude that it is a T-invariant stable map. By the composition of the fibration, we obtain all the rays. They are effectively labeled by the positive roots which are not roots of P. So there are exactly $\dim X$-many rays at each fixed point. The tangent line at the ray has the character α.

8.2.2. *The Kähler cone.* Here we need the Levi decomposition of P, and then consider simple roots $\{\alpha_i\}_{i\in P(\Delta)}$ which are not roots of the semi-simple part of P. Then the fundamental roots with respect to P is, by definition, $\{\lambda_i\}_{i\in P(\Delta)}$, where λ_i are dual to $\alpha_i^\vee$.

Choose a fixed point P. We may identify $H^2(X,\mathbf{Z})$ with the set of integral weights according to the Borel–Weil theorem. Then the Kähler cone is the set of all dominant integral weights *with respect to* P.

8.2.3. *Homogeneous line bundles.* One can produce all the very ample line bundles by homogeneous line bundles associated to irreducible representations of P with highest weights λ. The weights corresponding to the very ample line bundles are exactly the positive integral combinations of the fundamental weights with respect to P. We shall denote by $\mathcal{O}(\lambda)$ the homogeneous line bundle associated to the (one-dimensional) highest weight $\lambda=\sum_{i\in P(\Delta)} a_i\lambda_i$ representation of P. It is a very ample bundle if and only if $a_i>0$ for all i.

This also shows that the ray $\mathbf{P}^1$ associated to α has the homology class "$\alpha^\vee$", in the sense that $\langle \mathbf{P}^1, c_1(\mathcal{O}(\lambda))\rangle=(\alpha^\vee,\lambda)$. *We shall use* $\alpha^\vee$ *to denote the homology class.*

8.2.4. $\sum(p_i)_v z_i$ *are distinct for distinct fixed points* v. Consider a line bundle L associated to $\lambda=\sum_{i\in P(\Delta)} a_i\lambda_i$. (Here in advance, we have to fix $P\supset T$.) Let S_α denote the Weyl group element of the reflection associated to a positive root α which is not a root of P. Then the line bundle is $L=\mathcal{O}(S_\alpha(\lambda))$ if one looks at it with respect to another "origin" $P'=\exp\big(\frac{1}{2}\pi(X_\alpha-Y_\alpha)\big)P\exp\big(-\frac{1}{2}\pi(X_\alpha-Y_\alpha)\big)$, where $[X_\alpha,Y_\alpha]=H_\alpha$, $[H_\alpha,X_\alpha]=2X_\alpha$ and $[H_\alpha,Y_\alpha]=-2Y_\alpha$. This P' is the other T-fixed point which lies in the ray associated to α which passes through P. (Because of the $SL(2,\mathbf{C})$-equivariant map from $\mathbf{P}^1$ to the ray, it is enough to check it when $G=SL(2,\mathbf{C})$, which is obvious.) Now it is clear that $\sum(p_i)_v z_i$ are distinct for distinct fixed points v.

8.2.5. $V_{v,w,m}$ *and* $N_{v,w,m}$. Let $V=\mathcal{O}(\lambda)$. Let $\psi\colon\mathbf{P}^1\to X$ be a stable map totally ramifying one of the rays, passing through $P\supset T$. Suppose that the ray is associated to a positive root α with respect to P, and that f is an m-multiple branched cover representing an isolated T-fixed point of $\overline{M}_{0,0}(X,m\alpha^\vee)$. Then the T-representation space $H^0(\mathbf{P}^1,f^*(\mathcal{O}(\lambda))$ has the characters

$$\lambda-a\frac{\alpha}{m},\quad a=0,\dots,m(\lambda,\alpha^\vee).$$

To see it, use the coordinate $z\in\mathbf{C}$ around the fixed point, and the equalities

$$\exp H\exp(zE_\alpha)\exp -H=\exp(z\exp\operatorname{ad}H(E_\alpha))=\exp(z\exp(\alpha(H))E_\alpha).$$

Similarly, $N_{v,w,m}$ has the characters

$$\delta - a\frac{\alpha}{m} \quad \text{for } \alpha \neq \delta > 0,\ a = 0, ..., m(\delta, \alpha^\vee),$$

$$\alpha - a\frac{\alpha}{m} \quad \text{for } a = 0, ..., \widehat{m}, ..., 2m,$$

where $\delta > 0$ means that δ is a positive root with respect to P.

References

[1] BATYREV, V., CIOCAN-FONTAINE, I., KIM, B. & STRATEN, D. VAN, Conifold transitions and mirror symmetry for complete intersections in Grassmannians. *Nuclear Phys. B*, 514 (1998), 640–666.
[2] — Mirror symmetry and toric degenerations of partial flag manifolds. Preprint, 1998.
[3] BATYREV, V. & STRATEN, D. VAN, Generalized hypergeometric functions and rational curves on Calabi–Yau complete intersections in toric varieties. *Comm. Math. Phys.*, 168 (1995), 493–533.
[4] BEHREND, K., Gromov–Witten invariants in algebraic geometry. *Invent. Math.*, 127 (1997), 601–617.
[5] BEHREND, K. & FANTECHI, B., The intrinsic normal cone. *Invent. Math.*, 128 (1997), 45–88.
[6] BEHREND, K. & MANIN, YU., Stacks of stable maps and Gromov–Witten invariants. *Duke Math. J.*, 85 (1996), 1–60.
[7] CANDELAS, P., OSSA, X. DE LA, GREEN, P. & PARKES, L., A pair of Calabi–Yau manifolds as an exactly soluble superconformal theory. *Nuclear Phys. B*, 359 (1991), 21–74.
[8] CIOCAN-FONTAINE, I., On quantum cohomology rings of partial flag varieties. Institut Mittag-Leffler Report No. 12, 1996/1997.
[9] DUBROVIN, B., Geometry of 2D topological field theories, in *Integrable Systems and Quantum Groups* (Montecatini Terme, 1993), pp. 120–348. Lecture Notes in Math., 1620. Springer-Verlag, Berlin, 1996.
[10] FULTON, W. & PANDHARIPANDE, R., Notes on stable maps and quantum cohomology, in *Algebraic geometry* (Santa Cruz, 1995), pp. 45–96. Proc. Sympos. Pure Math., 62:2. Amer. Math. Soc., Providence, RI, 1997.
[11] GIVENTAL, A., Homological geometry, I: Projective hypersurfaces. *Selecta Math. (N.S.)*, 1 (1995), 325–345.
[12] — Equivariant Gromov–Witten invariants. *Iternat. Math. Res. Notices*, 13 (1996), 613–663.
[13] — Stationary phase integrals, quantum Toda lattices, flag manifolds and the mirror conjecture, in *Topics in Singularity Theory: V. I. Arnold's 60th Anniversary Collection*, pp. 103–115. Amer. Math. Soc. Transl. Ser. 2, 180. Amer. Math. Soc., Providence, RI, 1997.
[14] — A mirror theorem for toric complete intersections, in *Topological Field Theory, Primitive Forms and Related Topics* (Kyoto, 1996), pp. 141–175. Progr. Math., 160. Birkhäuser Boston, Boston, MA, 1998.
[15] — Elliptic Gromov–Witten invariants and the generalized mirror conjecture. Preprint, 1998.
[16] KIM, B., Quantum cohomology of flag manifolds G/B and quantum Toda lattices. *Ann. of Math. (2)*, 149 (1999), 129–148.

[17] KONTSEVICH, M., Enumeration of rational curves via torus actions, in *The Moduli Space of Curves* (Texel Island, 1994), pp. 335–368. Progr. Math., 129. Birkhäuser Boston, Boston, 1995.

[18] LI, J. & TIAN, G., Virtual moduli cycle and Gromov–Witten invariants of algebraic varieties. *J. Amer. Math. Soc.*, 11 (1998), 119–174.

[19] LIAN, B., LIU, K. & YAU, S.-T., Mirror principle, I. *Asian J. Math.*, 1 (1997), 729–763.

[20] THOMSEN, J., Irreducibility of $\overline{M}_{0,n}(G/P,\beta)$. *Internat. J. Math.*, 9 (1998), 367–376.

[21] TJØTTA, E., Rational curves on the space of determinantal nets of conics. Thesis, University of Bergen, 1997.

[22] WITTEN, E., Two-dimensional gravity and intersection theory on moduli space, in *Surveys in Differential Geometry* (Cambridge, MA, 1990), pp. 243–310. Lehigh Univ., Bethlehem, PA, 1991.

BUMSIG KIM
Department of Mathematics
Pohang University of Science and Technology
Pohang, 790-784
Republic of Korea
bumsig@euclid.postech.ac.kr

Received February 4, 1998