Int J Theor Phys (2008) 47: 1441–1454
DOI 10.1007/s10773-007-9586-9

Quantum Information in the Frame of Coherent States Representation

Dušan Popov · Ioan Zaharie · Vjekoslav Sajfert · Ioan Luminosu · Deian Popov

Received: 27 July 2007 / Accepted: 26 September 2007 / Published online: 18 October 2007

Abstract In this paper we have showed that the qubit can be expressed through the coherent states. Consequently, a message, i.e. a sequence of qubits, is expressed as a tensor product of coherent states. In the quantum information theory and practice, only the code and key message are expressed as a sequence of qubits, i.e. through a quantum channel, the properly information will be transmitted by using a classical channel. Even if the most used coherent states in the quantum information theory are the coherent states of the harmonic oscillator (particularly, expressing by them the Schrödinger "cat states" and the Bell states), several authors have been demonstrated that other kind of coherent states may be used in quantum information theory. For the ensembles of qubits, we must use the density operator, in order to describe the informational content of the ensemble. The diagonal representation of the density operator, in the coherent state representation, is also useful to examine the entanglement of the states.

Keywords Quantum information · Qubit · Coherent state · Density operator

1 Introduction

Even if the quantum mechanics (QM) is a theoretical framework with which physicists are able to describe the world of subatomic particles, the usual formulation of the QM principles becomes in contradiction with our intuition. This formulation is based on some negative

D. Popov (✉) · I. Zaharie · I. Luminosu
Departamentul de Fizică Tehnică, Universitatea "Politehnica" din Timişoara, B-dul V. Pârvan No. 2, 300223 Timişoara, Romania
e-mail: dusan_popov@yahoo.co.uk

V. Sajfert
Technical Faculty "Mihajlo Pupin", University of Novi Sad, Zrenjanin, Serbia

D. Popov
Alcatel-Lucent University, B-dul. Circumvalaţiunii No. 8-12, 300012 Timişoara, Romania

points of view which evinced some facts that *we are not able to realize*: (a) we cannot perform a quantum measurement without perturbing the state of the measured system; (b) we cannot determine the particle position and momentum simultaneously and with any precision; (c) we cannot perform a simultaneously measurement of polarization of the photon in a vertical and diagonal basis; (d) we cannot represent the image (or “photography”) of individual quantum processes; (e) we cannot be able to reproduce an unknown quantum state.

As a consequence of the tasks to transform these nonintuitive points of view into positive results, it was created the quantum information theory (QIT). The QIT is a new field that attempts to quantify and describe resources of QM and processes that act on them. In other words, QIT addresses how fundamental QM laws can be used in order to improve the acquisition, transmission, and procession of information. One of the fundamental challenges of QIT is to identify and quantify the basic resources that can be used for communication in quantum theory, i.e.: classical communication (in bits), quantum communication (in qubits) and entanglement (in ebits).

The primary motivation for exploring the quantum mechanical impact on the QIT is very practical: quantum computing and quantum communication devices are on the horizon and in the near future these devices become useful and very usual.

2 Qubits and Coherent States

As well as the fundamental unit of information in classical information theory (CIT) is a bit, in QIT the fundamental unit of information is a *qubit* (i.e. quantum bit) or, generally, a *multi-qubit* (or N-*qubit*).

A qubit is a state (vector) in a two-dimensional Hilbert space of the form:

$$|\Psi\rangle = a_0|0\rangle + a_1|1\rangle = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}, \tag{1}$$

where a_0 and a_1 are the complex numbers and $|0\rangle \equiv \binom{1}{0}$, respectively $|1\rangle \equiv \binom{0}{1}$ are arbitrary base vectors from the state space. The normalization condition require that $\sum_{n=0}^{1} |a_n|^2 = 1$.

Analogously, a state with many qubits, called a multi-qubit or N-*qubit* can be written as:

$$|\Psi\rangle = \sum_{n_1,n_2,\dots,n_N=0,1} a_{n_1 n_2 \dots n_N} \, |n_1 n_2 \dots n_N\rangle, \tag{2}$$

where the 2^N basis vectors are:

$$|n_1 n_2 \dots n_N\rangle \equiv \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ \vdots \\ n_N \end{pmatrix}, \quad n_1, n_2, \dots, n_N = 0, 1, \tag{3}$$

with the normalization relation: $\sum_{n_1,n_2,\dots,n_N=0,1} |a_{n_1 n_2 \dots n_N}|^2 = 1$.

The basis vectors of an N-qubit can be assimilated with $2J+1$ eigenvectors of a spin J particle, i.e. $|J; J-m\rangle$, where $m = -J, -J+1, \dots, J$. Particularly, for $J = \frac{1}{2}$, the basis states are: $|\frac{1}{2}; -\frac{1}{2}\rangle \equiv |0\rangle$ and $|\frac{1}{2}; +\frac{1}{2}\rangle \equiv |1\rangle$.

Generally, a qubit can be represented, through a phase factor, as well [1]:

$$|\theta;\varphi\rangle = \exp\left[-\frac{\theta}{2}(\sigma_+ e^{-i\varphi} - \sigma_- e^{i\varphi})\right]|1\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle, \tag{4}$$

where $\sigma_\pm = \sigma_x \pm i\sigma_y$, and σ_x and σ_x are the Pauli matrices, while θ and φ are real parameters.

One can demonstrate that the above expression is just a coherent state (CS) belonging to the $SU(2)$ quantum group. This may be realized by beginning from the definition of the Klauder–Perelomov CSs [2] (for a finite dimensional quantum system), e.g. for a particle with spin J:

$$|z;J\rangle = \exp(\eta(z)J_+ - \eta^*(z)J_-)|0;J\rangle = e^{zJ_+}e^{\ln(1+|z|^2)J_3}e^{-z^*J_-}|0;J\rangle, \tag{5}$$

where $\eta(z) = \frac{z}{|z|}\tan(|z|)$. We obtain the following expansion of CSs in the Fock vectors basis:

$$|z;J\rangle = \frac{1}{(1+|z|^2)^J}\sum_{m=0}^{2J}\frac{z^m}{\sqrt{\rho(J;m)}}|m;J\rangle, \tag{6}$$

where:

$$\rho(J;m) = \frac{\Gamma(m+1)\Gamma(2J-m+1)}{\Gamma(2J+1)} \tag{7}$$

are the structure (real) functions of the CSs.

Particularly, for the electron spin $J = \frac{1}{2}$ one gets:

$$\begin{aligned}|z;1/2\rangle &= \frac{1}{\sqrt{1+|z|^2}}\sum_{m=0}^{1}\frac{z^m}{\sqrt{\frac{\Gamma(m+1)\Gamma(2-m)}{\Gamma(2)}}}|m;1/2\rangle \\ &= \frac{1}{\sqrt{1+|z|^2}}(|0;1/2\rangle + z|1;1/2\rangle). \end{aligned}\tag{8}$$

If we choose the complex parameter z so that $z = \frac{\theta}{|\theta|}\tan(\frac{\theta}{2}) = \tan(\frac{\theta}{2})e^{i\varphi}$, finally we obtain:

$$|z\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle. \tag{9}$$

This result is identical with those from [1] and shows that the storage of information can be performed by using CSs. So, the CSs are useful elements for the storage and transmission of information.

If we consider that, generally, a state of a quantum system is dependent on a real or complex parameter λ, i.e. their Fock vector is $|\lambda;n\rangle$, then the corresponding CS may be written as:

$$\begin{aligned}|z;\lambda\rangle &= \sum_{n=0}^{N-1}\langle\lambda;n|z;\lambda\rangle|\lambda;n\rangle \equiv \sum_{n=0}^{N-1}c_n(z;\lambda)|\lambda;n\rangle \\ &= c_0(z;\lambda)\begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix} + c_1(z;\lambda)\begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix} + \cdots + c_{N-1}(z;\lambda)\begin{pmatrix}0\\0\\0\\\vdots\\1\end{pmatrix}\end{aligned}$$

$$= \begin{pmatrix} c_0(z;\lambda) \\ c_1(z;\lambda) \\ c_2(z;\lambda) \\ \vdots \\ c_{N-1}(z;\lambda) \end{pmatrix}. \tag{10}$$

In this manner the CSs may be represented as the column matrices whose elements are the complex functions. This manner to express the CSs will be useful in QIT.

The completion relation or the unity decomposition is, then:

$$\int d\mu(z;\lambda)|z;\lambda\rangle\langle z;\lambda| = I_N = \sum_{n=0}^{N-1} |\lambda;n\rangle\langle\lambda;n|, \tag{11}$$

where I_N is the projector on the subspace $\mathcal{H}_N$. This relation allows the expansion of a state vector from the subspace $\mathcal{H}_N$ (usually, there are identical with the eigenvectors of the Hamilton operator H of the quantum system) as a superposition of CSs over the complex z-space:

$$|\lambda;n\rangle = \int d\mu(z;\lambda)\langle z;\lambda|\lambda;n\rangle|z;\lambda\rangle = \int d\mu(z;\lambda)c_n^*(z;\lambda)|z;\lambda\rangle, \tag{12}$$

$$\langle\lambda;n| = \int d\mu(z;\lambda)\langle\lambda;n|z;\lambda\rangle\langle z;\lambda| = \int d\mu(z;\lambda)c_n(z;\lambda)\langle z;\lambda|, \tag{13}$$

or, more explicitly:

$$\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \int d\mu(z;\lambda)c_n^*(z;\lambda) \begin{pmatrix} c_0(z;\lambda) \\ \vdots \\ c_n(z;\lambda) \\ \vdots \\ c_{N-1}(z;\lambda) \end{pmatrix} = \int d\mu(z;\lambda) \begin{pmatrix} 0 \\ \vdots \\ |c_n(z;\lambda)|^2 \\ \vdots \\ 0 \end{pmatrix} \tag{14}$$

and similarly for the corresponding bra-vectors.

We observe that the following relation holds:

$$\int d\mu(z;\lambda)c_n^*(z;\lambda)c_j(z;\lambda) = \delta_{nj}. \tag{15}$$

Consequently, a state of N-qubits may be written as follows:

$$|\Psi\rangle = \sum_{n=0}^{2^N-1} a_n \int d\mu(z;\lambda)c_n^*(z;\lambda)|z;\lambda\rangle \equiv \int d\mu(z;\lambda) f^*(z;\lambda)|z;\lambda\rangle, \tag{16}$$

$$\langle\Psi| = \sum_{n=0}^{2^N-1} a_n \int d\mu(z;\lambda)c_n(z;\lambda)\langle z;\lambda| \equiv \int d\mu(z;\lambda) f(z;\lambda)\langle z;\lambda|, \tag{17}$$

where we have used the following notations:

$$f^*(z;\lambda) = \sum_{n=0}^{2^N-1} a_n c_n^*(z;\lambda), \tag{18}$$

$$f(z;\lambda) = \sum_{n=0}^{2^N-1} a_n c_n(z;\lambda), \tag{19}$$

and where, on the one hand, for the simplicity of notation, we have used the index n that signify in fact the number sequence $n_1 n_2 \dots n_N \dots$, each of them may bring values 0 or 1 and, on the other hand, generally, $f(z;\lambda)$ and $f^*(z;\lambda)$ are known complex functions [3].

So, in fact we have passed from the discrete-variable (dv) to continuous variable (cv) QIT. The main motivation to deal with continuous variables in QIT is of practical reasons: the essential steps in quantum communication protocols (preparing, unitary manipulating, and measuring entangled quantum states) is achievable in quantum optics by using continuous amplitudes of the quantized electromagnetic field [4].

The above relation is in fact the quintessence of the synergetic connection between the quantum information (represented here by N-qubit $|\Psi\rangle$) and the quantum mechanics (represented by coherent state $|z;\lambda\rangle$).

As it is well-known, the properly information is transmitted through a message through a classical channel. In QIT it is used also a quantum channel, but only for transmission of the lock-message. Evidently, the key-message, used by the receptor, is also of the quantum nature, identical with the lock-message.

The message is in fact a sequence of n qubits each with a probability p_i to be in a pure state $|\Psi_i\rangle$, i.e.:

$$|\mathcal{M}_n\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle \otimes \cdots \otimes |\Psi_n\rangle = \bigotimes_{i=1}^{n} |\Psi_i\rangle. \tag{20}$$

This message is in a Hilbert space $\mathcal{H}^{\otimes}$, that is, it is obtained as the tensor product of the individual Hilbert spaces of the qubits that compose it: $\mathcal{H}^{\otimes} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$.

Due to the connection between the Fock vectors $|\lambda;n\rangle$ and the CSs $|z;\lambda\rangle$, it is evident that a message finally may be expressed through the tensor product of CSs:

$$|\mathcal{M}_n\rangle \sim |z_1;\lambda\rangle \otimes |z_2;\lambda\rangle \otimes \cdots \otimes |z_n;\lambda\rangle \equiv \bigotimes_{i=1}^{n} |z_i;\lambda\rangle. \tag{21}$$

In quantum theory of information (QIT) this kind of tensorial products, composed only of the CSs $|z_i;\lambda\rangle$ can constitute a quantum key $|\mathcal{M}_n\rangle_{Q\text{-key}}$, which can be experimentally realized (see, e.g. [5]). Of course, each quantum key must be associated with a unique lock state $|\mathcal{M}_n\rangle_{Q\text{-lock}}$ which is composed by an identical tensorial product of CSs, i.e. we must have $|\mathcal{M}_n\rangle_{Q\text{-lock}} = |\mathcal{M}_n\rangle_{Q\text{-key}}$. So, the first system (Alice) posses $|\mathcal{M}_n\rangle_{Q\text{-lock}}$ and the second system (Bob) $|\mathcal{M}_n\rangle_{Q\text{-key}}$ and the last must compare their key string of CSs with the lock string.

The fundamental advantage of the CSs use in QIT is their non-demolition character. For example, if Bob compare they key-state with the lock-state of Alice, then the later state will not be destroyed [5].

3 Density Operators

In order to represent ensembles of quantum states or mixed quantum states instead of state vectors we must use the density operator ρ. Given n qubits (or N-qubits) each in a pure state

$|\Psi_j\rangle$, $j = 1, 2, \ldots, n$ and each with the probability w_j of being selected from the ensemble, we can define the associated density operator ρ.

Generally, a quantum information source X is defined as a set or sequence $\{w_j; |\Psi_j\rangle\}$, where any use of the source has as a result the production of a quantum state $|\Psi_j\rangle$ with the probability w_j.

Then, the density operator ρ associated with the source $X = \{w_{nj}; |\Psi_j\rangle\}$ is:

$$\rho = \sum_j w_j |\Psi_j\rangle\langle\Psi_j|. \tag{22}$$

It may be showed that the density operator can be represented as the diagonal expansion over the CSs projectors:

$$\rho = \int d\mu(z;\lambda)|z;\lambda\rangle P(|z|^2;\lambda)\langle z;\lambda|, \tag{23}$$

where $P(|z|^2;\lambda)$ is a quasi distribution function or P-quasi distribution function, which is independent on the phase of CSs variable z, but dependent on the parameter λ.

In addition, a message has a density operator as the tensor product of the density operators of the individual qubits:

$$\rho = \bigotimes_i \rho_i = \bigotimes_i \left[\int d\mu(z_i;\lambda) P(|z_i|^2;\lambda)|z_i;\lambda\rangle\langle z_i;\lambda|\right]. \tag{24}$$

As well as in CIT we use the formalism of the informational sources and the distribution functions of probability, in QIT one uses the formalism of the informational sources and the density operator, for the pure states, as well as for the mixed quantum states.

In the classical information theory (CIT) the Shannon entropy plays an important role, while in quantum information theory (QIT) this role is assumed by the von Neumann entropy. For a pair quantum source–density operator $\{X;\rho\}$, the von Neumann entropy is defined as:

$$S(\rho) = -\langle \ln \rho\rangle = -\operatorname{Tr}\rho(\ln\rho). \tag{25}$$

In the representation of CSs this expression becomes:

$$S(\rho) = -\int d\mu(z;\lambda)P(|z|,\lambda)\langle z;\lambda|\ln\rho|z;\lambda\rangle. \tag{26}$$

For a quantum system which is characterized by the Hamiltonian H, with a quantum canonical distribution, for which the density operator is

$$\rho = \frac{1}{Z(\beta)}e^{-\beta H}, \tag{27}$$

where $Z(\beta)$ is the partition function, this expression becomes:

$$S(\rho) = \ln Z(\beta) - \beta\int d\mu(z;\lambda)P(|z|,\lambda)\langle z;\lambda|H|z;\lambda\rangle. \tag{28}$$

We evince here that the canonical distribution may be useful e.g. in the transmission of the qubits through long optical fiber. In this case the P-quasi distribution function is a

Gaussian which depend on absolute temperature T through the mean occupation number of photons

$$\overline{n} = (\exp(\beta\hbar\omega) - 1)^{-1}. \tag{29}$$

The von Neumann entropy is always nonnegative, $S(\rho) \geq 0$. It is equal to zero only if ρ corresponds to a pure state, $\rho = |\Psi\rangle\langle\Psi|$.

If we consider a system of two parts A and B, with the whole density operator $\rho^{(AB)}$, the entropy of the joint system is:

$$S_{AB} = -\operatorname{Tr}[\rho^{(AB)} \ln \rho^{(AB}], \tag{30}$$

while the entropies of their individual parts (subsystems) is determined via the reduced density operators:

$$S_{A;B} = -\operatorname{Tr}_{A;B}[\rho^{(AB)} \ln \rho^{(AB}]. \tag{31}$$

It is always available the Araki and Lieb inequality [6]:

$$|S_A - S_B| \leq S_{AB} \leq S_A + S_B, \tag{32}$$

which shows the limits of the joint entropy and, particularly, that if the total system is in a pure state, then $S_A = S_B$. From the QIT point of view, the entropy can be regarded as the amount of uncertainty contained within the density operator.

4 Different Kinds of CSs

The most used CSs in QIT are the harmonic oscillator (HO) CSs, which are defined as eigenstates of the annihilation operator a:

$$a|z; \lambda\rangle = z|z; \lambda\rangle. \tag{33}$$

Instead of HO-CSs, one may define some other kinds of CSs, so called generalized CSs, in order to use them in QIT [1, 7], for finite, as well as for infinite dimensional representations of the quantum systems. We have also constructed and examined the properties of CSs for different oscillators: Morse oscillator (MO) [8–10] and pseudoharmonic oscillator (PHO) [11–13].

We reproduce here only the main results obtained for PHO CSs, i.e. the definition of CSs, the expansion of CSs in the Fock vectors basis and the integration measure.

(a) *Barut–Girardello CSs* (BG-CSs)—as the eigenvectors of the lowering operator K_-—only for the infinite dimensional representations of the $SU(1,1)$ quantum group [14]:

$$K_-|z; k\rangle = z|z; k\rangle. \tag{34}$$

The solution of this equation is [11, 14]:

$$|z; k\rangle = \sqrt{\frac{|z|^2}{I_{2k-1}(2|z|)}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!\Gamma(n+2k)}} |n; k\rangle, \tag{35}$$

and the integration measure:

$$d\mu(z; k) = \frac{d^2 z}{\pi} K_{2k-1}(2|z|) I_{2k-1}(2|z|), \tag{36}$$

where $\Gamma(x)$ is gamma Euler function, $I_{2k-1}(2|z|)$, $K_{2k-1}(2|z|)$—the Bessel functions. The BG-CSs are definite on the whole complex plane.

(b) *Klauder–Perelomov CSs* (KP-CSs)—by the action of the generalized displacement operator on the vacuum state—for the infinite and finite dimensional representations of the $SU(1,1)$ and $SU(2)$ quantum groups [2].

For the infinite dimensional systems (CSs based on the $SU(1,1)$):

$$|z;k\rangle = \exp(\eta(z)K_+ - \eta^*(z)K_-)|0;k\rangle w = e^{zK_+}e^{\ln(1-|z|^2)K_3}e^{-z^*K_-}|0;k\rangle, \tag{37}$$

where $\eta(z) = \frac{z}{|z|}\tan(|z|)$. The solution of this equation is [2, 13]:

$$|z;k\rangle = (1-|z|^2)^k \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\frac{\Gamma(n+1)\Gamma(2k)}{\Gamma(2k+n)}}}|n;k\rangle, \tag{38}$$

and the integration measure:

$$d\mu(z;k) = (2k-1)\frac{d^2z}{\pi}\frac{1}{(1-|z|^2)^2}. \tag{39}$$

These CSs are definite only on a unit ray disc of the complex plane.

For the finite dimensional systems (CSs based on the $SU(2)$):

$$|z;J\rangle = \exp(\eta(z)J_+ - \eta^*(z)J_-)|0;J\rangle = e^{zJ_+}e^{\ln(1+|z|^2)J_3}e^{-z^*J_-}|0;J\rangle, \tag{40}$$

where $\eta(z) = \frac{z}{|z|}\tan(|z|)$. The solution of this equation is [7]:

$$|z;J\rangle = \frac{1}{(1+|z|^2)^J}\sum_{m=0}^{2J} \frac{z^m}{\sqrt{\frac{\Gamma(m+1)\Gamma(2J-m+1)}{\Gamma(2J+1)}}}|m;J\rangle, \tag{41}$$

and the integration measure:

$$d\mu(z;J) = (2J+1)\frac{d^2z}{\pi}\frac{1}{(1+|z|^2)^2}. \tag{42}$$

These CSs are definite on the whole complex plane.

(c) *Gazeau–Klauder CSs* (GK-CSs)—are a set of normalized vectors $\{|\zeta;\gamma;k\rangle, \zeta \geq 0, \gamma \in (-\infty, +\infty)\}$, defined as follows [15, 16]:

$$|\zeta;\gamma;k\rangle = c_0(\zeta;k)\sum_{n=0}^{\infty}\frac{\zeta^{\frac{n}{2}}}{\sqrt{\rho(n;k)}}e^{-i\gamma e_n(k)}|n;k\rangle, \tag{43}$$

where $e_n(k)$ are the eigenvalues of the dimensionless Hamiltonian $\widetilde{H}$:

$$\widetilde{H}|n;k\rangle = e_n(k)|n;k\rangle, \qquad \rho(n;k) = \prod_{j=0}^{n} e_j(k), \qquad \rho(0;k) = 1. \tag{44}$$

Besides this definition, the GK-CSs can be obtained also algebrically, by acting with the Hamiltonian exponential operator on the BG-CSs [13]:

$$|\zeta;\gamma;k\rangle = \exp(\eta\widetilde{H})|z;k\rangle \tag{45}$$

and, by the variable change $z=\sqrt{\zeta}$, $\eta=-\mathrm{i}\gamma$, we pass to the above expression.

5 Entanglement in the Frame of CSs Representation

Quantum entanglement is a quantum mechanical phenomenon, without any classical counterpart, in which the quantum states of two or more systems have to be described with reference to each other, even though the individual systems may be spatially separated. Quantum entanglement make it possible to realize quantum-information processing including quantum teleportation, cryptography and quantum computation [17–19].

Let's be a quantum bipartite system containing two disjoint parts A (Alice) and B (Bob), with the Fock basis $|n_i^{(A)}\rangle$ and $|n_j^{(B)}\rangle$ respectively. A typical correlated state of the joint system may be expanded using these basis:

$$|\Psi_k^{(AB)}\rangle = \sum_{i,j} C_{ij}^{(k)}|n_i^{(A)}\rangle\otimes|n_j^{(B)}\rangle. \tag{46}$$

For the joint system, in the mixed state, the density operator may be defined as:

$$\rho^{(AB)} = \sum_k w_k^{(AB)}|\Psi_k^{(AB)}\rangle\langle\Psi_k^{(AB)}|, \tag{47}$$

where $w_k^{(AB)}$ are the weights or the probabilities to find the joint system in the state $|\Psi_k^{(AB)}\rangle$. The weights satisfy $0\le w_k^{(AB)}\le 1$ and $\sum_k w_k^{(AB)}=1$. This operator contains complete information about the joint system.

One of individual parts is characterized through the reduced density operator, by calculating the trace over the variables of the other part, e.g.:

$$\begin{aligned}\rho^{(A)} = \mathrm{Tr}_B\,\rho^{(AB)} &= \sum_l \langle n_l^{(B)}|\rho^{(AB)}|n_l^{(B)}\rangle \\ &= \sum_k w_k^{(AB)}\sum_{i,j,l} C_{il}^{(k)}C_{jl}^{(k)*}|n_i^{(A)}\rangle\langle n_j^{(A)}|.\end{aligned} \tag{48}$$

So, the reduced density operator $\rho^{(A)}$ contains information about the part A, when the result of measurement on part B is discarded.

A bipartite (or, generally, N-partite) mixed state acting on the joint Hilbert space $\mathcal{H}=\bigotimes_{i=1}^{N\ge 2}\mathcal{H}_i$ is separable (or fully separable) if it can be written as a mixture or convex sum of tensor products of subsystem states [20]:

$$\rho^{(AB\ldots Z)} = \sum_k w_k\rho_k^{(A)}\otimes\rho_k^{(B)}\otimes\cdots\otimes\rho_k^{(Z)} = \sum_k w_k\bigotimes_{s=1}^{N}\rho_k^{(S)}, \tag{49}$$

where $S=A,B,\ldots,Z$ are N disjoint subsets of the joint set.

Any state that cannot be cast into the above mentioned form (or appropriately approximated by such a state for infinite dimensional joint systems) will be called *entangled* state [21].

Generally, on the one hand, a pure state of a bipartite system $|\Psi^{(AB)}\rangle$ is said to be *unentangled* or *separable* if and only if it exists two pure states $|\Phi^{(A)}\rangle$ and $|\Xi^{(B)}\rangle$, so

$$|\Psi^{(AB)}\rangle = |\Phi^{(A)}\rangle \otimes |\Xi^{(B)}\rangle. \tag{50}$$

Otherwise, the pure state $|\Psi^{(AB)}\rangle$ is said to be *entangled* or *inseparable*.

If we used, for a bipartite system, the previously deduced CSs representation of a state of N-qubits (16):

$$|\Phi^{(A)}\rangle = \int d\mu(z;\lambda) f^*(z;\lambda)|z;\lambda\rangle, \tag{51}$$

$$|\Xi^{(B)}\rangle = \int d\mu(\sigma;\lambda) g^*(\sigma;\lambda)|\sigma;\lambda\rangle, \tag{52}$$

where z and σ are complex variables, then the joint state become:

$$|\Psi^{(AB)}\rangle = \iint d\mu(z;\lambda) d\mu(\sigma;\lambda) F^*(z,\sigma;\lambda)|z;\lambda\rangle \otimes |\sigma;\lambda\rangle. \tag{53}$$

It is evident that, depending on the structure of function $F^*(z,\sigma;\lambda)$, it exists two situations:

(a) If the function $F^*(z,\sigma;\lambda)$ allow the separate variable factorization:

$$F^*(z,\sigma;\lambda) = f^*(z;\lambda)\, g^*(\sigma;\lambda), \tag{54}$$

then the state $|\Psi^{(AB)}\rangle$ is nonentangled or separable.

(b) Contrary, if the function $F^*(z,\sigma;\lambda)$ not allow the separate variable factorization:

$$F^*(z,\sigma;\lambda) \neq f^*(z;\lambda)\, g^*(\sigma;\lambda), \tag{55}$$

then the state $|\Psi^{(AB)}\rangle$ is entangled or inseparable.

In this manner we have transferred the entanglement problem from the eigenvectors to the complex functions $f^*(z;\lambda)$ and $g^*(\sigma;\lambda)$. So, if the function $F^*(z,\sigma;\lambda)$ admits a variable separation, the state $|\Psi^{(AB)}\rangle$ is separable or unentangled, otherwise we have to deal with an entangled or inseparable pure joint state.

On the other hand, for the mixed states, the entangled or unentangled character of the joint state must be examined by using the diagonal representation of the density operator (23).

Let's we consider that $|\alpha,\beta;\lambda\rangle$ is a pure CS of the joint system. If we can find two CSs $|\alpha;\lambda\rangle$ and $|\beta;\lambda\rangle$, for the two individual parts A, respectively B, where $\alpha,\beta \in \mathcal{C}$ are the complex variables, so that:

$$|\alpha,\beta;\lambda\rangle = |\alpha;\lambda\rangle \otimes |\beta;\lambda\rangle, \tag{56}$$

then the CS $|\alpha,\beta;\lambda\rangle$ is said to be separable or unentangled. Otherwise this CS is an entangled or inseparable CS.

For a mixed joint state of a bipartite system, the corresponding two-mode density operator can be written in terms of the diagonal CS representation as

$$\rho^{(AB)} = \int d\mu(\alpha,\beta;\lambda)|\alpha,\beta;\lambda\rangle P(|\alpha|^2,|\beta|^2;\lambda)\langle\alpha,\beta;\lambda|. \tag{57}$$

Because of the fact that the P-quasi distribution function $P(|\alpha|^2, |\beta|^2; \lambda)$ can be measured in experiments, in the next we will examine the properties of this function.

As it is well-known, the two-mode CSs $|\alpha, \beta; \lambda\rangle$ for a bipartite system may be defined in different manners, e.g. a family of CSs, called the pair CSs which are non-Gaussian in nature [22], the q-deformed CSs related to $su_q(2)$-algebra [23], or CSs for $su_q(1,1)$-algebra [24]. E.g., if the parts A and B have the $SU(1,1)$ or $SU(2)$ group symmetry, the individual CSs (i.e. the CSs corresponding only to one of the part of system) may be defined as Barut–Girardello [14] or Perelomov [2] coherent states, depending on the fact that the corresponding Hilbert spaces $\mathcal{H}_A$ or $\mathcal{H}_B$ are infinite or finite dimensional spaces [1].

We will examine here only the CSs of the Barut–Girardello kind [14]. For example, if the lowering operators for the $su(1,1)$ algebras are $K_-^{(A)}$ and $K_-^{(B)}$ respectively, then we can define the Barut–Girardello CSs as:

$$K_-^{(A)}|\alpha, \beta; \lambda\rangle = \alpha|\alpha, \beta; \lambda\rangle, \tag{58}$$

$$K_-^{(B)}|\alpha, \beta; \lambda\rangle = \beta|\alpha, \beta; \lambda\rangle, \tag{59}$$

acting only on the subspaces $\mathcal{H}_A$, respectively $\mathcal{H}_B$.

By expressing the CSs $|\alpha, \beta; \lambda\rangle$ as a development on the joint Fock basis $|n_i^{(A)}\rangle \otimes |n_j^{(B)}\rangle$, i.e.:

$$|\alpha, \beta; \lambda\rangle = \sum_{n_i^{(A)}, n_j^{(B)}=0}^{\infty} C_{n_i^{(A)} n_j^{(B)}}(\alpha, \beta; \lambda)|n_i^{(A)}\rangle \otimes |n_j^{(B)}\rangle \tag{60}$$

and using the relations:

$$K_-^{(A)}|n_i^{(A)}\rangle = \sqrt{n_i^{(A)}(n_i^{(A)} + 2k - 1)}\,|n_i^{(A)} - 1\rangle \tag{61}$$

and similar for $K_-^{(B)}$, after the straightforward calculations, we obtain [24]:

$$|\alpha, \beta; k\rangle = \mathcal{N}(|\alpha|)\mathcal{N}(|\beta|) \sum_{n_i^{(A)}, n_j^{(B)}=0}^{\infty} \frac{\alpha^{n_i^{(A)}}}{\sqrt{\rho(n_i^{(A)}; k)}} \frac{\beta^{n_i^{(A)}}}{\sqrt{\rho(n_j^{(B)}; k)}} |n_i^{(A)}\rangle \otimes |n_j^{(B)}\rangle, \tag{62}$$

where we have used the following notations for the normalization factors $\mathcal{N}(|\alpha|)$, respectively for the structure constants $\rho(n_i^{(A)}; k)$:

$$\mathcal{N}(|\alpha|) \equiv \sqrt{\frac{|\alpha|^2}{I_{2k-1}(2|\alpha|)}}, \tag{63}$$

$$\rho(n_i^{(A)}; k) = \Gamma(n_i^{(A)} + 1)\Gamma(n_i^{(A)} + 2k) \tag{64}$$

and similar for the part B.

Here $I_{2k-1}(x)$ is the modified Bessel function of the first kind and $k \in 1/2, 1, 3/2, \ldots$ is the Bargmann index labeling the irreducible representations, while $k(k-1)$ is the value of the Casimir operator.

This equation immediately leads to the factorization form:

$$|\alpha, \beta; k\rangle = |\alpha; k\rangle \otimes |\beta; k\rangle \tag{65}$$

ensuring that the joint CSs $|\alpha,\beta;k\rangle$ is separable.

We will point out here that the same result is obtained if the joint CSs $|\alpha,\beta;k\rangle$ is defined by using the primitive coproduct structure of the classical generators of $SU(1,1)$ group, but using the q-deformed $su(1,1)$ algebra for the construction of CSs for a bipartite composite system, these state becomes entangled [24].

Therefore, for the separable CSs case, the density operator can be written as:

$$\rho^{(AB)} = \int d\mu(\alpha;\lambda)d\mu(\beta;\lambda)|\alpha;\lambda\rangle \otimes |\beta;\lambda\rangle P(|\alpha|^2,|\beta|^2;\lambda)\langle\alpha;\lambda| \otimes \langle\beta;\lambda|. \tag{66}$$

In order to fulfill the identity operator decomposition

$$\int d\mu(\alpha,\beta;\lambda)|\alpha,\beta;\lambda\rangle\langle\alpha,\beta;\lambda| = I, \tag{67}$$

here we have considered that the integration measure must be split into a measure product involving separate complex variables:

$$d\mu(\alpha,\beta;\lambda) = d\mu(\alpha;\lambda)\, d\mu(\beta;\lambda). \tag{68}$$

Consequently, in order to separate the density operators for parts A and B, the P-quasi distribution function $P(|\alpha|^2,|\beta|^2;\lambda)$ must be also separable:

$$P(|\alpha|^2,|\beta|^2;\lambda) = P(|\alpha|^2;\lambda)P(|\beta|^2;\lambda). \tag{69}$$

Finally, we obtain:

$$\rho^{(AB)} = \rho^{(A)} \otimes \rho^{(B)}, \tag{70}$$

with

$$\rho^{(A)} = \int d\mu(\alpha;\lambda)|\alpha;\lambda\rangle P(|\alpha|^2;\lambda)\langle\alpha;\lambda| \tag{71}$$

and similarly for part B.

Then, the joint density operator $\rho^{(AB)}$ in CSs representation is separable or unentangled if and only if the CSs and P-quasi distribution function of the joint system are simultaneous separable. Otherwise, the joint density operator $\rho^{(AB)}$ (and, of course, their corresponding mixed state) is inseparable or entangled.

From the point of view of the entropy of joint system (30) and their individual parts (31), the correlation entropy of subsystems A and B is given by [25]

$$I_{\mathrm{corr}} = S_A + S_B - S_{AB}. \tag{72}$$

It is showed that if the two parts A and B are uncorrelated or separable, then I_{corr} vanishes. For any bipartite pure state, the join entropy S_{AB} becomes zero.

6 Concluding Remarks

Even if at their beginning, the quantum mechanics (QM) was applied to the phenomena at the microscopic scale, in the last decades of XX century has been formulated idea that the information itself may be quantified [26]. In this manner was born the quantum information theory (QIT). The connection between the information and the QM is much simple. In essence, information is anything which can be stored or coded into a certain physical state in order to be reproduced and transmitted to another physical system. As in QM the fundamental "stone" is the state of a certain physical system, in QIT this role is played by the qubits (or N-qubits), i.e. the states of two (or N-) level physical systems. So, the connection between the QIT and QM is evident. One of the main entity of the QM is the coherent state (CS). The reason for choosing CSs is that they are easy to generate and convenient to use. On the other hand, CSs may be compared non-invasievly if they are identical. We have showed how the CSs can be used in the storage of information, i.e. how the qubits can be expressed through the CSs, using their main properties (normalization, non orthogonality, the overlap, the identity resolution). We present a simple "lock and key" scheme [5], by considering a set of nonorthogonal states composed only of CSs to construct a message "lock" and their counterpart—a message "key". The fundamental idea of using the CSs is their non-demolition character, i.e. when we compare the "key" string with the "lock" string, the last is not destroyed and remain unaltered.

Finally, we have showed how the entanglement (another remarkable property of the quantum states, without the classical counterpart), very useful in QIT (for teleportation, cryptography and quantum computation) can be also expressed and analyzed through the CSs.

We have found that, on the one hand, a pure CS of a bipartite quantum system, defined in the Barut–Girardello manner, can be separable or unentangled. On the other hand, a mixed quantum state on the same system, described by the density operator $\rho^{(AB)}$ is unentangled or separable if and only if, besides the separability condition referring to the corresponding CSs, their P-quasi distribution is also a function which admit a factorization into two functions of separate variables, each characterizing the individual parts of the quantum system. In this manner we have extended the condition established in [22], i.e. if the P-quasi distribution function has no classical character, then the state is entangled.

Finding the criteria of separability of density matrixes (in modern QM and especially in QIT) is an important task. For multipartite qubit systems, this problem is very complex and, even if was reported some success [27], the problem is afar to be solved. In this context, we consider that our work, using the CSs representation of the density operator, is a little step in solving this problem.

In conclusion, we consider that the CSs may play an important role in the QIT and that the use of CSs formalism is not only of theoretical, but also of some practical importance, having in mind their experimental accessibility.

Acknowledgements The present work is performed in part by the financial support through the Grant CNCSIS A647 of the Romanian National Council of Scientific Research. We kindly acknowledge for this support.

References

1. Wang, X., Sanders, B.C., Pan, S.H.: J. Phys. A: Math. Gen. **33**, 7451 (2000)
2. Perelomov, A.: Generalized Coherent States and Their Applications. Springer, Berlin (1986)
3. Janszky, J., Gábris, A., Koniorczyk, M., Vukics, A., Adam, P.: Fortschr. Phys. **49**, 993 (2001)
4. Braunstein, S.L., van Loock, P.: Rev. Mod. Phys. **77**, 513 (2005)
5. Andersson, E., Curty, M., Jex, I.: Phys. Rev. A **74**, 022304 (2006)
6. Araki, H., Lieb, E.: Commun. Math. Phys. **18**, 160 (1970)
7. Fujii, K.: Introduction to coherent states, quantum information theory. quant-ph/0112090 (2001)
8. Daoud, M., Popov, D.: Int. J. Mod. Phys. B **18**, 325 (2004)
9. Popov, D.: Phys. Lett. A **316**, 369 (2003)
10. Popov, D., Zaharie, I., Dong, S.H.: Czech. J. Phys. **56**, 157 (2006)
11. Popov, D.: J. Phys. A: Math. Gen. **34**, 5283 (2001)

12. Popov, D.: J. Phys. A: Math. Gen. **35**, 7205 (2002)
13. Popov, D.: Density Matrix—General Properties and Applications in the Physics of Many Particle Systems. Editura Politehnica, Timişoara (2004) (in Romanian)
14. Barut, A.O., Girardello, L.: Commun. Math. Phys. **21**, 41 (1971)
15. Antoine, J.-P., Gazeau, J.-P., Monceau, P., Klauder, J.R., Penson, K.A.: J. Math. Phys. **42**, 2349 (2001)
16. Zaharie, I., Popov, D.: Rom. J. Optoelectron. **13**(1), 59 (2005)
17. Bennett, C.H., Brassard, G., Crépeau, C., Jozsa, R., Peres, A., Wootters, W.K.: Phys. Rev. Lett. **70**, 1895 (1993)
18. Ekert, A.: Phys. Rev. Lett. **67**, 661 (1991)
19. Barenco, A., Dutch, D., Ekert, A., Jozsa, R.: Phys. Rev. Lett. **74**, 4085 (1995)
20. Werner, R.F.: Phys. Rev. A **40**, 4277 (1989)
21. Eisert, J., Plenio, M.B.: Int. J. Quant. Inf. **1**, 479 (2003)
22. Agarwal, G.S., Biswas, A.: J. Opt. B **7**, 350 (2005)
23. Li, Y., Jing, S.: Mod. Phys. Lett. A **18**, 1523 (2003)
24. Chakrabarti, R., Vasan, S.S.: Entanglement via Barut–Girardello coherent state for $su_q(1,1)$ quantum algebra: bipartite composite system. math.QA/0301036 (2003)
25. Huang, H., Agarwal, G.S.: Phys. Rev. A **49**, 52 (1994)
26. Preskill, J.: Quantum Information and Computation. Lecture Notes in Physics, vol. 229. Springer, Berlin (1998)
27. Zhong, Z.Z.: Int. J. Quant. Inf. **3**, 405 (2005)