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Quantum kinematics of bosonic vortex loops

Gerald A. Goldin* Robert Owczarek, † David H. Sharp,‡

Abstract

Poisson structure for vortex filaments (loops and arcs) in 2D ideal incompressible fluid is analyzed in detail. Canonical coordinates and momenta on coadjoint orbits of the area-preserving diffeomorphism group, associated with such vortices, are found. The quantum space of states in the simplest case of "bosonic" vortex loops is built within a geometric quantization approach to the description of a quantum fluid. Fock-like structure and non-local creation and annihilation operators of quantum vortex filaments are introduced.

1 Introduction

In the papers [1, 2], Goldin, Menikoff, and Sharp analyzed from the point of view of geometric quantization the problem of quantizability of particular vortex structures that exist in superfluid helium. The geometric quantization approach is based on consideration of the symmetry group of the system. Unitary irreducible representations of this group provide us with the appropriate quantum space of states. An idealized picture of superfluid helium we adopt also in this paper assumes that the superfluid consists of an incompressible and nonviscous fluid, which could be treated as a classical fluid, and a number of vortices for which classical description is not sufficient. The vortices are collective excitations of the fluid. Their quantum description should respect the classical symmetry of the fluid. Since the configuration space and simultaneously the symmetry group for an ideal incompressible fluid is $G = SDiff(\mathbb{R}^n)$, ($n = 2$, or 3), the group of area-, or volume- preserving diffeomorphisms, for $n = 2$ or $n = 3$, respectively, (the diffeomorphisms are additionally assumed to become trivial at infinity), the problem of quantizability is equivalent to the problem of construction of appropriate irreducible unitary representations of G, connected with coadjoint orbits describing particular vortex structures.

Irreducibility of the representations is mathematically described in the language of polarizations of the orbits. The polarization means roughly division of the coordinates of the coadjoint orbits, which are the reduced phase spaces of the system, into canonical coordinates and momenta. When the polarization is established, the canonical coordinates give rise to the quantum configuration space. Quantum physical states of the system depend then only on the "coordinates," which constitute a quantum configuration space of the system, or on the conjugated "momenta." The surprising result

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of the papers [1, 2] was elimination of point vortices in $2D$, and vortex filaments in $3D$ from the list of admissible quantum objects. Instead, it was found there that the polarization exists for coadjoint orbits describing vortex loops, dipoles and arcs in $2D$, and for vortex ribbons and tubes in $3D$. Similar results for the $3D$ case obtained Owczarek in [3, 4], within a different approach based on field theoretic interpretation of knot theory [5], [6], and inspired by Peradzyński [7]. Owczarek used also the topological degrees of freedom of the knotted and linked vortices to discuss their role in thermodynamics of critical superfluid helium [8], anticipated by Goldin, Menikoff, and Sharp in [2].

Let us mention that the fact that vortex ribbons are better suited for quantization than the vortex filaments was also observed by Brylinski [9] who was studying the symplectic structure on the orbits associated with vortex filaments in $3D$. He proved a theorem which states that vortex ribbons describe manifolds, which are Lagrange manifolds with respect to the standard symplectic structure. He stated it could be a starting point for geometric quantization of such vortex structures but he did not follow this way, concentrating rather on further discussion of vortex filaments, which are not feasible in this context.

Unitarity of the representations can be established only when there is known appropriate quasi-invariant measure on the quantum configuration space. Showing existence of such a measure and its construction are difficult mathematical problems. We believe such measures exist in the physically interesting situations, and we assume their existence in the cases of vortex structures allowing quantization. Knowledge of the measures is not of vital importance for describing kinematics of the system, as in the given paper. However, one can expect they are very important in considerations involving quantum dynamics of the system.

The facts about quantizability of the vortex structures reminded above lead to possibility of appearance of interesting effects on the quantum level, associated with complicated topology of the quantum configuration spaces, like "internal statistics" of the individual $2D$ vortex loops, anyonic statistics for systems of the vortex loops, their combinations, and topological effects connected with knottedness and linkedness of vortex loops in $3D$. These problems will be discussed systematically in our future papers.

We should mention here a recent interesting paper by Speliotopoulos [13], in which the author constructs creation and annihilation operators for vortices in superfluid helium within a heuristic approach, starting from the rule of quantization of vorticity. In his approach he was not using the geometric quantization framework. He treated the vortices as point-like excitations. In the first approximation he postulated the one-vortex Hamiltonian to be harmonic, $H = \epsilon c^* c$, where c^*, c are one-vortex creation and annihilation operators. Then, to avoid some singularities in the corresponding wave function, he modified the creation and annihilation operators. This modification he interpreted as transition from the point vortices to the vortex patches. However, the vortex patches are homogeneous and not feasible for geometric quantization [1].

In this paper quantum kinematics of vortex filaments is discussed. Similar questions were addressed and answered for systems of nonrelativistic point particles, which were classified successfully long ago in terms of associated representations of the group of diffeomorphisms [10, 11]. In particular, anyons were for the first time described at rigorous mathematical level within this approach [11]. Recently, Goldin and Sharp proposed in [12] a general construction of the field creation and annihilation operators as intertwiners of the representations of diffeomorphism groups. They constructed explicitly field operators in the case of anyons, showing appearance, in a natural way, of q-commutation rules for the operators. The rules follow from the type of the representations used and are not assumed from the beginning.

In this paper we analyze first in detail the Poisson structure on the coadjoint orbits associated with the vortex loops and introduce canonical coordinates and momenta on the orbits. Next, we construct the Fock-like space of states and field operators of creation and destruction in the simplest case of "bosonic" vortex loops. The field operators depend on non-local arguments which are unparametrized loops. This formal construction is not completely explicit due to the lack of knowledge of the quasi-invariant measure on the space of unparametrized loops. In this case we showed the measure should satisfy a multiplicative property with respect to addition of new loops in order to get a hierarchy of representations.

The plan of the paper is as follows. In the second section we remind basic material on canonical symplectic structure on coadjoint orbits of the diffeomorphism groups and establish notation. In the third section the Poisson structure of vortex loops is discussed in detail. In the fourth section a formal construction of the Fock space, and of the creation and annihilation operators is presented. In the Appendix we prove some formulas given in the second section, and present for convenience of the reader some material on application of differential forms in hydrodynamics of ideal incompressible fluids.

2 Kirillov-Kostant-Souriau symplectic structure on coadjoint orbits

In this section we remind basic mathematical structures in the classical theory of ideal incompressible fluids. Our purpose is convenience of the reader (many conventions are present in the existing literature) and of the authors (who also are victims of the numerous conventions). As the guide through this material the authors chose the newest textbook in the area by Arnold and Khesin [14]. Our notation is a compromise between the one used in our previous publications on the subject and the one used in this book. We derived a number of formulas once more and present the derivations for the same purpose of convenience. Some transformations are presented in the Appendix. First, we remind the general formula of the Kirillov-Kostant-Souriau (KKS) symplectic structure on coadjoint orbits of a Lie group G. Then we specify the groups to be $SDiff(I\!R^n)$ ($n = 2$, or 3), which are the proper groups for discussing incompressible hydrodynamics.

Let for a general Lie group G, $\mathcal{G}$ will be its Lie algebra, and $\mathcal{G}^*$ its dual with respect to some pairing $< \cdot, \cdot >$: $\mathcal{G}^* \times \mathcal{G} \to I\!R$. Let $[\cdot, \cdot]$ denotes the commutator (Lie bracket) in $\mathcal{G}$. One should keep in mind that for finite dimensional groups G the dual of the Lie algebra $\mathcal{G}$, $\mathcal{G}^*$, is isomorphic as a vector space to $\mathcal{G}$, but for infinite dimensional groups $\mathcal{G}^*$ is much bigger than $\mathcal{G}$ and they are by no means isomorphic.

The left translation of the group G acting on itself is defined for every element g of G as $L_g : G \to G$ such that $L_g(h) = gh$. Since $L_{g_1g_2}(h) = (g_1g_2)h = g_1(g_2h) = (L_{g_1} \circ L_{g_2})(h)$, then $L_{g_1g_2} = L_{g_1} \circ L_{g_2}$ and the left translation is a left action on G. The right translation of a group G acting on itself is defined for every element g of G as $R_g : G \to G$ such that $R_g(h) = hg$. Since $R_{g_1g_2}(h) = h(g_1g_2) = (hg_1)g_2 = (R_{g_2} \circ R_{g_1})(h)$, then $R_{g_1g_2} = R_{g_2} \circ R_{g_1}$ and the right translation is a right action. The adjoint action of a group G on itself, or the inner automorphism of the group G is the composition $Ad_g = R_{g^{-1}} \circ L_g$, $Ad_g : G \to G$. The name automorphism is justified by the formula $Ad_g(h_1h_2) = g(h_1h_2)g^{-1} = g(h_1g^{-1}gh_2)g^{-1} = (gh_1g^{-1})(gh_2g^{-1}) = Ad_g(h_1)Ad_g(h_2)$ showing that indeed Ad_g is an automorphism acting in the group G. Of course $Ad_{g_1g_2}(h) = (g_1g_2)h(g_1g_2)^{-1} = g_1(g_2hg_2^{-1})g_1^{-1} = (Ad_{g_1} \circ Ad_{g_2})(h)$, so that Ad_g is a left action on G.

Let us remind that when there is defined a map $F : M \to M$ of a manifold M into itself, its derivative $F_*|_x$ at the point $x \in M$ is a linear operator from T_xM to $T_{F(x)}M$, $F_*|_x : T_xM \to T_{F(x)}M$. The Lie algebra $\mathcal{G}$ of a Lie group G is usually identified as a vector space with the tangent space to G at its unity. Since the action Ad_g does not move the unity of G, the derivative of this action at the unity maps the tangent space, and because of that also the vector space of the Lie algebra, $\mathcal{G}$, into itself. This action is usually also denoted Ad_g for each $g \in G$, but it acts now from $\mathcal{G}$ to $\mathcal{G}$. The definition reads for each $g \in G$: $Ad_g : \mathcal{G} \to \mathcal{G}$, $Ad_g\xi = (Ad_{g*}|_e)\xi$, $\xi \in \mathcal{G} := T_eG$. Since Ad_g acting on G satisfies $Ad_{g_1g_2} = Ad_{g_1} \circ Ad_{g_2}$, the same is true for the Ad_g acting on $\mathcal{G}$ and therefore Ad_g defines a representation of G in the vector space $\mathcal{G}$.

Let us consider then a curve $g(t)$ in G, such that $g(0) = e \in G$, $\frac{d}{dt}\Big|_{t=0} g(t) = \xi \in \mathcal{G}$. Let us introduce the adjoint representation of the Lie algebra in itself: $ad = Ad_{*e} : \mathcal{G} \to End\mathcal{G}$, $ad_\xi = \frac{d}{dt}\Big|_{t=0} Ad_{g(t)}$, $ad_\xi : \mathcal{G} \to \mathcal{G}$.

The commutator in the vector space of the Lie algebra $\mathcal{G} = T_eG$ is the operation $[\cdot,\cdot] : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ defined using the adjoint action of $\mathcal{G}$ on itself, by $[\xi, \eta] = ad_\xi\eta$

The commutator defined this way satisfies the standard axioms of the Lie algebra: it is linear in both arguments, it is antisymmetric, and it satisfies the Jacobi identity.

Now, let us work out the formula for the KKS form on coadjoint orbits of G. Given duality $< \cdot, \cdot >$ between $\mathcal{G}^*$— the dual of $\mathcal{G}$, and $\mathcal{G}$, one can define the coadjoint action of $\mathcal{G}$ on $\mathcal{G}^*$ as the dual of the adjoint action:

$$\begin{array}{ccc} < ad^*_\xi\mu, \eta > & = & < \mu, ad_\xi\eta > = < \mu, [\xi, \eta] > \\ \| & & \\ < coad_\xi\mu, \eta >, & & \xi, \eta \in \mathcal{G},\ \mu \in \mathcal{G}^* \end{array} \tag{1}$$

This action is the infinitesimal version of the coadjoint action of the group G on $\mathcal{G}^*$:

$$< Ad^*_g\mu, \xi > = < \mu, Ad_g\xi >, \ \ g \in G,\ \xi \in \mathcal{G},\ \mu \in \mathcal{G}^* \tag{2}$$

which is a right action:

$$\begin{aligned} < Ad^*_{g_1g_2}\mu, \xi > &= < \mu, Ad_{g_1g_2}\xi > = < \mu, (Ad_{g_1} \circ Ad_{g_2})(\xi) > = \\ &= < \mu, Ad_{g_1}(Ad_{g_2}(\xi)) > = < Ad^*_{g_1}(\mu), Ad_{g_2}(\xi) > = \\ &= < Ad^*_{g_2}(Ad^*_{g_1}(\mu)), \xi > = < (Ad^*_{g_2} \circ Ad^*_{g_1})(\mu), \xi > \end{aligned} \tag{3}$$

The definition of the (KKS) symplectic form reads:

$$\Omega_\mu(\xi_{\mathcal{G}^*}(\mu), \eta_{\mathcal{G}^*}(\mu)) := < \mu, [\xi, \eta] > \tag{4}$$

where:
$\mu \in \mathcal{G}^*$
, $\xi_{\mathcal{G}^*}(\mu)$ (resp. $\eta_{\mathcal{G}^*}$) is the value of a vector field $\xi_{\mathcal{G}^*}$(resp. $\eta_{\mathcal{G}^*}$) associated to an element $\xi \in \mathcal{G}$ (resp. $\eta \in \mathcal{G}$), at the point $\mu \in \mathcal{G}^*$ (μ determines the coadjoint orbit).

With every $\mu \in \mathcal{G}^*$ is associated a coadjoint orbit of G, the one which is passing through μ:

$$\mathcal{O}_\mu := \{ad^*_g(\mu) : g \in G\} \tag{5}$$

In the context of hydrodynamics one considers the groups of the type $G = SDiff(M)$, where M is a manifold equipped with a volume element, usually $M = I\!R^2$ or $I\!R^3$ with the standard Euclidean volume element and $SDiff(M)$ consists of volume preserving smooth (C^∞) diffeomorphisms of M. For M not compact the diffeomorphisms are usually additionally assumed to become trivial sufficiently quickly at infinity. The group action is defined as $\phi_1\phi_2 = \phi_1 \circ \phi_2$ for any $\phi_1, \phi_2 \in G$. With this convention we also follow the book [14], instead of our previous papers, where the order of composition was the opposite one. Our convention was leading to the commutator in the Lie algebra $\mathcal{G}$ being equal to the standard commutator of vector fields. The convention of [14], and used also in some other important publications on geometric approach to classical hydrodynamics, e.g. in [15], leads to the commutator in the Lie algebra, which adds the minus sign to the commutator of vector fields. The Lie algebra of G, $\mathcal{G}$, consists of vectors tangent to G at identity e of G. These vectors can be identified with divergenceless vector fields on M, the space of which we denote $SVect(M)$. The vector fields vanish quickly at infinity, as a result of diffeomorphisms becoming trivial at infinity. Let us establish what are the general Ad and ad actions in this case. The left group action in G is $L_{\phi_1}(\phi_2) = \phi_1\phi_2 = \phi_1 \circ \phi_2$. Correspondingly, $R_{\phi_1}(\phi_2) = \phi_2 \circ \phi_1$, Then $(Ad\phi)(\psi) = \phi\psi\phi^{-1} = \phi \circ \psi \circ \phi^{-1}$. The result of this action on ψ is a new diffeomorphism of M. Let us establish the induced Ad action of ϕ in $\mathcal{G} = T_eG = SVect(M)$. Let $\psi_t^{\bar{v}}$ be a one-parameter subgroup in G connected with a vector field $\bar{v}(x)$, $x \in M$, on the manifold M. The $Ad\phi$ acts on $\psi_t^{\bar{v}}$ by $Ad\phi\,(\psi_t^{\bar{v}}) = \phi \circ \psi_t^{\bar{v}} \circ \phi^{-1}$. Its differential, taken at e, is, accordingly to the chain rule, equal to

$$Ad\phi_*\big|_e \bar{v} = (D_\phi \bar{v}) \circ \phi^{-1} \tag{6}$$

, or, in coordinates:

$$(Ad(\phi)v)^j(x) = \left[\left(\frac{\partial \phi(y)^j}{\partial y^k} v^k\right) \circ \phi^{-1}\right](x) = \frac{\partial \phi(y)^j}{\partial y^k}\bigg|_{y=\phi^{-1}(x)} v^k\left(\phi^{-1}(x)\right) = \frac{\partial x^j}{\partial\left[\phi^{-1}(x)\right]^k} v^k\left(\phi^{-1}(x)\right) \tag{7}$$

It means the matrix D_ϕ is given at x by:

$$[D_\phi]^j_k(x) = \frac{\partial \phi(x)^j}{\partial x^k} \tag{8}$$

and at $\phi^{-1}(x)$ by:

$$[D_\phi]^j_k\left(\phi^{-1}(x)\right) = \frac{\partial x^j}{\partial\left[\phi^{-1}(x)\right]^k} \tag{9}$$

Since the vector fields on M are identified as physical velocity fields of the fluid, this is the rule of action of diffeomorphisms in the space of velocity fields. This transformation rule has its counterpart as adjoint action of the diffeomorphisms on the stream functions. Since we do not want to introduce here the full apparatus of differential forms which is very convenient in discussion of hydrodynamics on general manifolds (they can be curved and of any dimension), we discuss this issue only for the case of $M = I\!R^2$ or $I\!R^3$, as it was done in our previous publications [1, 2]. However, some elements of the general case we discuss in the Appendix. Stream functions are introduced due to divergencelessness of the velocity fields, $div\bar{v} = 0$. In $I\!R^3$ the stream function is introduced by $\bar{v} = -curl\bar{\chi}_{\bar{v}}$, and in $I\!R^2$ by $\bar{v} = -curl\chi_{\bar{v}}$, $v^i = -\epsilon^{ij}(\chi_{\bar{v}})_{,j}$, ϵ_{ij} is the usual antisymmetric symbol, $\epsilon_{12} = 1$.

Let us turn the reader's attention to the difference in sign in comparison with the previous publications. The change is made in order to follow the conventions from the book [14]. One can see that the stream function is not uniquely defined. One can add to it a gradient of a function ($I\!R^3$)

or a constant ($\mathbb{R}^2$). In the $2D$ case, which is of special interest in this paper, one can integrate $\chi_{\bar{v}}$ out of $\bar{v}$, by introducing $\bar{w} = \nabla^{(2)}\chi_{\bar{v}} = (v_2, -v_1) = \bar{v}^{\perp}$, and then integrating $\bar{w}$ along any trajectory from infinity to $\bar{x}$, $\chi_{\bar{v}} = \int_{\infty}^{\bar{x}} \bar{w}d\bar{l}$. It is easy to see that the integral does not depend on the trajectory and that $\chi_{\bar{v}}$ satisfies the required conditions of the stream function. An interesting and convenient relation that satisfies such a χ is

$$\chi_{\{\bar{v}_1,\bar{v}_2\}} = \bar{v}_2 \times \bar{v}_1 \tag{10}$$

where$\{\cdot,\cdot\}$ denotes the standard commutator of vector fields $\{\bar{v}_1, \bar{v}_2\} = (\bar{v}_1\nabla)\bar{v}_2 - (\bar{v}_2\nabla)\bar{v}_1$. This relation one can easily obtain from the identity:

$$curl(\bar{a} \times \bar{b}) = (\bar{b}\nabla)\bar{a} - (\bar{a}\nabla)\bar{b} + \bar{a}\, div\, \bar{b} - \bar{b}\, div\, \bar{a} \tag{11}$$

applied to $\bar{a} = \bar{v}_2$, $\bar{b} = \bar{v}_1$, $div\, \bar{v}_1 = div\, \bar{v}_2 = 0$.

The nonuniqueness in the definition of the stream function leads to some complication in establishing the rule for the adjoint action of the diffeomorphism group. One can prove only that the proposed transformation leads to the proper action on the velocity fields. The proper transformation rules for $\bar{\chi}$ in $\mathbb{R}^3$ and χ in $\mathbb{R}^2$ are:

$$[Ad\phi(\chi)]_i = \chi_i' = \left(\frac{\partial y^j}{\partial[\phi(y)]^i}\chi_j\right) \circ \phi^{-1} \tag{12}$$

, so

$$\chi_i'(x) = \frac{\partial[\phi^{-1}(x)]^j}{\partial x^i}\chi_j\left(\phi^{-1}(x)\right) \tag{13}$$

, and in $2D$

$$\chi_i'(x) = \chi_i\left(\phi^{-1}(x)\right) \tag{14}$$

The calculations showing that such transformation rules for the stream functions under adjoint action of the diffeomorphism group lead to the proper adjoint action on the velocity fields are given in the Appendix. We also present there some elements of the approach through differential forms to the description of hydrodynamics, in particular to the stream function.

The dual of $\mathcal{G}$, $\mathcal{G}^*$, is not isomorphic to $\mathcal{G}$, which is a standard feature of infinite-dimensional vector spaces, and consists of generalized (co-)vector fields (in general, components of the generalized vector fields are distributional). The duality between $\mathcal{G}$ and $\mathcal{G}^*$ is given by the pairing:

$$< \bar{A}(\bar{x}), \bar{v}(\bar{x}) >:= \int_{\mathbb{R}^n} \bar{A}(\bar{x})\bar{v}(\bar{x})d^n x = \int_{\mathbb{R}^n} A_i(\bar{x})v^i(\bar{x})d^n x \tag{15}$$

Moreover, after introducing the stream function $\bar{\chi}_{\bar{v}}$ the formula (15) can be written in $3D$ in the form:

$$\int_{\mathbb{R}^3} \bar{A}(\bar{x})\bar{v}(\bar{x})d^3x = -\int_{\mathbb{R}^3} \bar{A}(\bar{x})\, curl\, \bar{\chi}_{\bar{v}} d^3x = -\int_{\mathbb{R}^3} curl\, \bar{A}(\bar{x})\bar{\chi}_{\bar{v}}(\bar{x})d^3x = -\int_{\mathbb{R}^3} \bar{\omega}(\bar{x})\bar{\chi}_{\bar{v}}(\bar{x})d^3x \tag{16}$$

where $\bar{\omega}(\bar{x}) = curl\, \bar{A}(\bar{x})$ is a generalized vorticity field, and where we used the identity:

$$\int_{\mathbb{R}^3} \bar{a}(\bar{x}) \cdot curl\bar{b}(\bar{x})d^3x = \int_{\mathbb{R}^3} \bar{b}(\bar{x}) \cdot curl\bar{a}(\bar{x})d^3x \tag{17}$$

valid for any vector fields in $\mathbb{R}^3$ vanishing at infinity.

The question is: what are the concrete forms for the actions in the case of $G = SDiff(I\!R^n)$, $\mathcal{G}^* = SVect^*(I\!R^n)$, $n = 2, 3$?

The proper definition of the adjoint action of G on $\mathcal{G}$ should be such that infinitesimalized it should give the adjoint action of $\mathcal{G}$ on $\mathcal{G}$. These requirements satisfies the action given by (6), (7).

With such definition of the adjoint action, let us consider the one-parameter subgroup $\phi_t^{\bar{u}}$ of G which is the solution of the ODE:

$$\begin{aligned} \frac{d\phi_t^{\bar{u}}(\bar{x})}{dt} &= \bar{u}\left(\phi_t^{\bar{u}}(\bar{x})\right) \\ \phi_{t=0}^{\bar{u}}(\bar{x}) &= \bar{x} \end{aligned}$$

describing the vector field $\bar{u}(\bar{x}) \in SVect(I\!R^n)$. Then, the adjoint action of the 1-parameter subgroup reads:

$$(Ad(\phi_t^{\bar{u}})\bar{v})^j = \frac{\partial[\phi_t^{\bar{u}}(x)]^j}{\partial x^k} v^k \circ (\phi_t^{\bar{u}})^{-1} \tag{18}$$

Expanding $\phi_t^{\bar{u}}$ in the vicinity of $t = 0$, $\phi_t^{\bar{u}} = id + tu^j \frac{\partial}{\partial x^j} + ...$, where the dots mean terms of higher degree in t, and its inverse $(\phi_t^{\bar{u}})^{-1} = id - tu^j \frac{\partial}{\partial x^j} + ...$ one gets after some manipulations :

$$(Ad(\phi_t^{\bar{u}})v)^j = v^j - t\,[(\bar{u}\nabla)\bar{v} - (\bar{v}\nabla)\bar{u}]^j + ... \tag{19}$$

, where dots denote terms of higher order in t.

As a result:

$$[\bar{u}, \bar{v}] = ad(\bar{u})\bar{v} = \left.\frac{d}{dt}\right|_{t=0} [Ad(\phi_t^{\bar{u}})\bar{v}] = \left.\frac{d}{dt}\right|_{t=0} [v - t\{u, v\} + \ldots] = -\{u, v\} \tag{20}$$

so that the commutator in the Lie algebra $\mathcal{G}$ is equal to minus the commutator of vector fields, as was explained in the above discussion.

Now, let us establish the explicit form of the coadjoint action and of the symplectic form. Let us concentrate on the $3D$ case, because the $2D$ vector fields can be embedded in the $3D$ space so that the formulas for the $3D$ case will be applicable also in the $2D$ situation. Let us take as $\mu \in \mathcal{G}^*$ an element $\bar{A}(\bar{x}) \in SVect(I\!R^n)^*$ (a generalized velocity field) and as $\xi, \eta \in \mathcal{G}$ two divergenceless vector fields $\bar{u}_1(\bar{x})$, $\bar{u}_2(\bar{x})$. Then the formula for the KKS form on the coadjoint orbit through $\bar{A}(\bar{x})$, taken at this point, reads:

$$\Omega_{\bar{A}(\bar{x})}(u_{1,\mathcal{G}^*}(\bar{A}(\bar{x})), u_{2,\mathcal{G}^*}(\bar{A}(\bar{x}))) = < \bar{A}(\bar{x}), [\bar{u}_1(\bar{x}), \bar{u}_2(\bar{x})] > = -\int_{I\!R^3} \bar{A}(\bar{x}) \cdot \{\bar{u}_1(\bar{x}), \bar{u}_2(\bar{x})\} d^3x \tag{21}$$

Next, using the identity:

$$\{\bar{u}_1(\bar{x}), \bar{u}_2(\bar{x})\} = -curl[\bar{u}_1(\bar{x}) \times \bar{u}_2(\bar{x})] \tag{22}$$

which is valid for divergenceless vector fields, and using (17) , we obtain the formula:

$$\Omega_{\bar{A}(\bar{x})}(u_{1,\mathcal{G}^*}(\bar{A}(\bar{x})), u_{2,\mathcal{G}^*}(\bar{A}(\bar{x}))) = \int_{I\!R^3} curl\bar{A}(\bar{x}) \cdot [\bar{u}_1(\bar{x}) \times \bar{u}_2(\bar{x})] d^3x \tag{23}$$

Defining the generalized vorticity field $\bar{\omega}(\bar{x}) \in \mathcal{G}^*$ as $\bar{\omega}(\bar{x}) := curl\bar{A}(\bar{x})$, one gets the formula:

$$\Omega_{\bar{A}(\bar{x})}(u_{1,\mathcal{G}^*}(\bar{A}(\bar{x})), u_{2,\mathcal{G}^*}(\bar{A}(\bar{x}))) = \int_{I\!R^3} \bar{\omega}(\bar{x}) \cdot [\bar{u}_1(\bar{x}) \times \bar{u}_2(\bar{x})] d^3x \tag{24}$$

3 Poisson structure for vortex loops and arcs in $I\!R^2$ from KKS symplectic form

Let us turn now to the case of a coadjoint orbit associated with a vortex loop or arc in $I\!R^2$. Let us use further the common name filament for either a loop or an arc.

The vorticity field can be written for the case of a vortex filament in $I\!R^2$ as:

$$\bar{\omega}(\bar{x}) = \bar{e}_z \kappa \int_0^{2\pi} d\alpha \; \delta(x - C_x(\alpha))\delta(y - C_y(\alpha)) \tag{25}$$

where $\bar{C}(\alpha)$ is a parametrized filament in $I\!R^2$, and κ is proportional to the total vorticity for this filament. The parametrized filament is a map $\bar{C} : [0, 2\pi] \ni \alpha \rightarrow \bar{C}(\alpha) \in I\!R^2$, $\alpha \in [0, 2\pi]$. Therefore, the KKS symplectic form reads:

$$\Omega_{\bar{A}(\bar{x})}(u_{1,\mathcal{G}^*}(\bar{A}(\bar{x})), u_{2,\mathcal{G}^*}(\bar{A}(\bar{x}))) \tag{26}$$

$$= \int_{I\!R^2} dx\, dy \; \bar{e}_z \kappa \int_0^{2\pi} d\alpha \; \delta(x - C_x(\alpha))\delta(y - C_y(\alpha))(\bar{u}_1(\bar{x}) \times \bar{u}_2(\bar{x})) \tag{27}$$

$$= \kappa \int_0^{2\pi} d\alpha \; [u_{1x}(C_x(\alpha), C_y(\alpha))u_{2y}(C_x(\alpha), C_y(\alpha)) - u_{2x}(C_x(\alpha), C_y(\alpha))u_{1y}(C_x(\alpha), C_y(\alpha))]$$

$$= \kappa \int_0^{2\pi} d\alpha \; (\delta C_x(\alpha) \wedge \delta C_y(\alpha)) \, (u_1(\bar{C}(\alpha)), u_2(\bar{C}(\alpha)))$$

, where $C_x(\alpha), C_y(\alpha)$ are x and y components of $\bar{C}(\alpha)$ in $I\!R^2$, respectively, $\delta C_x(\alpha)$ and $\delta C_y(\alpha)$ are infinitesimal changes in $C_x(\alpha)$, $C_y(\alpha)$.

Therefore, the KKS form on the coadjoint orbit characterized by $\bar{C}(\alpha)$ can be written as the following 2-form:

$$\Omega_{\bar{C}(\alpha)} = \kappa \int_0^{2\pi} d\alpha \; \delta C_x(\alpha) \wedge \delta C_y(\alpha) \tag{28}$$

From geometric point of view it is justified to consider instead of parametrized filaments the filaments that are unparametrized. An unparametrized filament is the image of the map $\bar{C}(\alpha)$ whereas a parametrized filament is a map $\bar{C} : [0, 2\pi] \ni \alpha \mapsto \bar{C}(\alpha) \in I\!R^2$. For the unparametrized filaments the only parametrization is the natural one, with respect to the arclength parameter s, intrinsic to them. Such vortex filaments underlie a description which applies their internal characteristics only, i.e. their shape. The remaining information about the vortex filament is coded in the vorticity distribution along it. As a result we use the equivalent description of vortex filaments, in which a parametrized vortex filament is substituted by a pair (Γ, γ) consisting of an unparametrized vortex filament Γ and vorticity distribution function γ, which can be expressed as a function of the arclength parameter s. One expects that the canonical coordinates on the coadjoint orbits will be expressed in terms of these quantities. Therefore, it is desirable to write down all the formulas in terms of these quantities. There are two reasons in favor of such a description. One is purely geometric and was mentioned above. Namely, this description is based on intrinsic characteristics of vortex filaments and not on the properties of the surrounding of the filaments. The other argument goes further into physics and is based on the remark from [1] that unparametrized filaments, $\Gamma(s)$, should be canonical coordinates on the coadjoint orbits.

Let the function $\gamma(s)$ be defined by

$$\gamma(s) := \frac{d\alpha}{ds} \tag{29}$$

Let us calculate action of the symplectic form on two vector fields in the intrinsic coordinates, using internal tangent and normal components of the vector fields. In these coordinates $\bar{u}_i(\bar{x}) = (u_{it}(\bar{x}), u_{in}(\bar{x}))$, $i = 1, 2$, where $u_{it}(\bar{x})$ is the tangent and $u_{in}(\bar{x})$ is the normal to the curve component of the vector field $\bar{u}(\bar{x})$, taken at the point $\bar{x}$ of the filament.

Let us calculate the action of the symplectic form on the pair of vector fields $\bar{u}_1(\bar{x})$, $\bar{u}_2(\bar{x})$:

$$\Omega_{\bar{A}(\bar{x})}(u_{1,\mathcal{G}^*}(\bar{A}(\bar{x})), u_{2,\mathcal{G}^*}(\bar{A}(\bar{x}))) = \int_{\mathbb{R}^2} d^2\bar{x}\, \bar{e}_z \kappa \int ds\, \gamma(s)\delta(\bar{x} - \Gamma(s))(\bar{u}_1(\bar{x}) \times \bar{u}_2(\bar{x})) \tag{30}$$

In these coordinates the symplectic form reads:

$$\Omega_{\gamma(s),\Gamma(s)} = \kappa \int ds \gamma(s) \delta s \wedge \delta\Gamma(s) \tag{31}$$

Let us find a function $f(s)$ such that $df = \gamma(s)ds$. Obviously, $f(s) = \int_0^s \gamma(s')ds' =: \tilde{\gamma}(s)$, where the zero in the lower limit of the integral can be taken arbitrarily.

Then:

$$\Omega_{\tilde{\gamma}(s),\Gamma(s)} = \kappa \int ds\ \delta\tilde{\gamma}(s) \wedge \delta\Gamma(s) \tag{32}$$

which is the canonical form of the symplectic structure.

Now, we will calculate the Poisson bracket for the coadjoint orbit in the case of a vortex filament, in both systems of coordinates. First, let us consider the parametrization of the coadjoint orbit by $C_x(\alpha), C_y(\alpha)$. Let us take two functions on the coadjoint orbit: $F(\bar{C}(\alpha)), G(\bar{C}(\alpha))$. The Poisson bracket for these two functions is defined by the formula

$$\{F, G\}(\bar{C}(\alpha)) = \Omega_{\bar{C}(\alpha)}(X_F, X_G) \tag{33}$$

, where X_F, X_G are the vector fields associated to the functions F, G, accordingly to the formula:

$$i_{X_F}(\Omega_{\bar{C}(\alpha)}) = -dF(\bar{C}(\alpha)) \tag{34}$$

Again, we used the convention from [14].

In general

$$dF(\bar{C}(\alpha)) = \int_0^{2\pi} d\alpha \left[\frac{\delta F}{\delta C_x(\alpha)}\delta C_x(\alpha) + \frac{\delta F}{\delta C_y(\alpha)}\delta C_y(\alpha)\right] \tag{35}$$

The expression for the vector field associated to $F(\bar{C}(\alpha))$ can be written as:

$$X_F = \int_0^{2\pi} d\alpha \left[X_{F,x}(\alpha)\frac{\delta}{\delta C_x(\alpha)} + X_{F,y}(\alpha)\frac{\delta}{\delta C_y(\alpha)}\right] \tag{36}$$

where $\frac{\delta}{\delta C_x(\alpha)}, \frac{\delta}{\delta C_y(\alpha)}$ are defined as the dual basis to $\delta C_x(\alpha), \delta C_y(\alpha)$, in the sense of the following formulas:

$$\begin{aligned} &\frac{\delta}{\delta C_x(\alpha)}\rfloor \delta C_x(\beta) = \delta(\alpha - \beta), \\ &\frac{\delta}{\delta C_y(\alpha)}\rfloor \delta C_x(\beta) = 0, \\ &\frac{\delta}{\delta C_x(\alpha)}\rfloor \delta C_y(\beta) = 0, \\ &\frac{\delta}{\delta C_y(\alpha)}\rfloor \delta C_y(\beta) = \delta(\alpha - \beta) \end{aligned} \tag{37}$$

As the next step, we would like to express the components of the vector field X_F by (derivatives of) the function $F(\bar{C}(\alpha))$.

Since

$$i_{X_F}(\Omega_{\bar{C}(\alpha)}) = \kappa \int_0^{2\pi} d\alpha \; [X_{F,x}\delta C_y(\alpha) - X_{F,y}\delta C_x(\alpha)] \tag{38}$$

then, from comparison with the expression for $dF(\bar{C}(\alpha))$ we get:

$$X_{F,x}(\alpha) = -\frac{1}{\kappa}\frac{\delta F}{\delta C_y(\alpha)} \tag{39}$$

$$X_{F,y}(\alpha) = \frac{1}{\kappa}\frac{\delta F}{\delta C_x(\alpha)} \tag{40}$$

As a result:

$$X_F = \int_0^{2\pi} d\alpha \; \frac{1}{\kappa}\left[-\frac{\delta F}{\delta C_y(\alpha)}\frac{\delta}{\delta C_x(\alpha)} + \frac{\delta F}{\delta C_x(\alpha)}\frac{\delta}{\delta C_y(\alpha)}\right] \tag{41}$$

Finally, we get the expression for the Poisson bracket of two functions on the coadjoint orbit:

$$\{F, G\}(\bar{C}(\alpha)) = \Omega_{\bar{C}(\alpha)}(X_F, X_G) = -\int_0^{2\pi} d\alpha \; \frac{1}{\kappa}\left[\frac{\delta F}{\delta C_x(\alpha)}\frac{\delta G}{\delta C_y(\alpha)} - \frac{\delta F}{\delta C_y(\alpha)}\frac{\delta G}{\delta C_x(\alpha)}\right] \tag{42}$$

Then, applying similar procedure as when we used coordinates $C_x(\alpha)$, $C_y(\alpha)$, we obtain the following formula for the Poisson bracket in coordinates $\tilde{\gamma}(s), \Gamma(s)$:

$$\begin{aligned} \{F, G\}(\tilde{\gamma}(s), \Gamma(s)) &= \Omega_{\tilde{\gamma}(s),\Gamma(s)}(X_F, X_G) = \\ &= \frac{1}{\kappa}\int ds \left[\frac{\delta F}{\delta \Gamma(s)}\frac{\delta G}{\delta \tilde{\gamma}(s)} - \frac{\delta F}{\delta \tilde{\gamma}(s)}\frac{\delta G}{\delta \Gamma(s)}\right] \end{aligned} \tag{43}$$

The coordinates $\tilde{\gamma}(s)$, $\Gamma(s)$ are then canonical coordinates and momenta of the coadjoint orbit. The formula agrees with the result of [1] concerning the polarization of the orbits. The quantum configuration space is then the space of unparametrized filaments $\Gamma(s)$. The little group is the subgroup of the group of area- preserving diffeomorphisms that consists of those diffeomorphisms which preserve the filament as a set, and which could change arbitrarily the accumulative vorticity distribution $\tilde{\gamma}(s)$, preserving only the total vorticity of the loop.

Let us consider an example of calculation of the Poisson bracket for a concrete pair of functions on the coadjoint orbit. The relation $i_{X_F}(\omega) = -dF$ defines the vector field X_F on the coadjoint orbit, associated to a function F on the orbit. The Poisson bracket on the orbit is defined by;

$$\{F, G\} := \omega(X_F, X_G) \tag{44}$$

Assuming that the vector fields X_F, X_G can be identified with vector fields $X^{\bar{u}}, X^{\bar{v}}$ associated to the vector fields $\bar{u}, \bar{v}$ in $I\!R^2$ (elements of the Lie algebra of $SDiff(I\!R^2)$ the symplectic structure for a vortex filament is given by:

$$\omega(X^{\bar{u}}, X^{\bar{v}}) = \kappa \int_0^{2\pi} d\alpha[(u^1 \circ C)(v^2 \circ C) - (u^2 \circ C)(v^1 \circ C)](\alpha) \tag{45}$$

This integral will be rewritten in a convenient $2D$ filament local system of coordinates associated with the loop and defined in its neighborhood.

This system of coordinates can be introduced as follows: At every point of the filament there are assigned the tangent and normal vectors to the curve at this point. As a a result we have two vector fields defined on the curve: one is the field of the tangent vectors to the filament, the other one is the field of normal vectors to the filament. These two fields can be parallelly transported in the plane, treated as a Euclidean space, to a neighborhood of the curve, defining two vector fields in this neighborhood. One should consider the family of integral curves of these vector fields. The coordinates for a point in this neighborhood are then defined as follows. One coordinate is the arclength coordinate along the integral curve of the "normal" vector field, with zeroth value on the filament. Let us call this coordinate $x^{\perp}$. The second coordinate is defined as the value of the arclength parameter on the filament at the point of intersection of the filament with the integral curve of normal vector field passing through this point, or, in the case of an arc, its value on the prolonged arc. This coordinate is called s, following its notation for the filament. The local system of coordinates is then $(x^{\perp}, s)$. The symplectic structure on the coadjoint orbit connected with the vortex filament can be formally rewritten in terms of the above introduced coordinates as:

$$\omega(X^{\bar{u}}, X^{\bar{v}}) = \kappa \int\int d^2x \delta^{(1)}(x^{\perp}) I_{[0,L_{tot}]}(s)\gamma(s)[u^1(x)v^2(x) - u^2(x)v^1(x)] \tag{46}$$

where $d^2x = dx^{\perp}ds$, $I_{[0,L_{tot}]}$ is the characteristic function of the sector $[0, L_{tot}]$, i.e.

$$I_{[0,L_{tot}]} = \begin{cases} 1, & s \in [0, L_{tot}] \\ 0, & \text{other } s \end{cases} \tag{47}$$

L_{tot}— is the total length of the filament. Let us establish now the equation which should be satisfied by the vector field X_F on the coadjoint orbit, associated to a function F on this orbit. Let us assume it is also of the form $X^{\bar{u}}$ for some vector field $\bar{u}$ on $I\!R^2$. We should take into account the relation (42) above. Let us contract both sides of (42) with a vector field $X^{\bar{v}}$, where $\bar{v}$ denotes an arbitrary vector field on $I\!R^2$, and take the value of the resulting function at the point of the coadjoint orbit which is the vortex filament. The right hand side of (42) then becomes:

$$dF(X^{\bar{v}})|_C = X^{\bar{v}}(F)|_C = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} F\left(\phi^{\bar{v}}_{\epsilon} \circ C\right) \tag{48}$$

where $\phi^{\bar{v}}_{\epsilon}$ is the one-parameter group of diffeomorphisms connected with the vector field $\bar{v}$:

$$\left(\frac{d\phi^{\bar{v}}_{\epsilon}(x)}{d\epsilon} = \bar{v}\left(\phi^{\bar{v}}_{\epsilon}(x)\right), \quad \epsilon \in [0,1], \quad \phi^{\bar{v}}_{\epsilon=0}(x) = x\right) \tag{49}$$

The left hand side of (42) becomes just $\omega(X^{\bar{u}}, X^{\bar{v}})|_C$ which is:

$$\omega(X^{\bar{u}}, X^{\bar{v}})|_C = \kappa \int\int d^2x \delta^{(1)}(x^{\perp}) I_{[0,L_{tot}]}(s)\gamma(s)\left[u^1(x)v^2(x) - u^2(x)v^1(x)\right] \tag{50}$$

The equation (42) becomes therefore:

$$\kappa \int\int d^2x \delta^{(1)}(x^{\perp}) I_{[0,L_{tot}]}(s)\gamma(s)\left[u^1(x)v^2(x) - u^2(x)v^1(x)\right] = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} F^j(\phi^{\bar{v}}_{\epsilon} \circ C) \tag{51}$$

Now, let us establish the form of the vector field $X^{\bar{u}}$ for a particular but quite enlightening choice of the functions F. Namely, let us take $F^j(C) = C^j(\beta)_{j=1,2}$, it means to a filament $\bar{C} : [0, 2\pi] \ni \alpha \mapsto \bar{C}(\alpha) \in I\!R^2$ contained in the coadjoint orbit, the functions assign the first (x) $(j = 1)$, or the second (y) $(j = 2)$ component of $\bar{C}(\alpha)$ at a particular point of the filament associated to $\alpha = \beta$. Calculation of the RHS of (51) in these two cases gives easily:

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} F^j(\phi_\epsilon^{\bar{v}} \circ C) = v^j(C(\beta)) \tag{52}$$

The RHS of (51) can be also written for the function F^j as:

$$\int\int d^2x v^j(x)\delta^{(1)}(x^\perp)\delta^{(1)}(s - s(\beta_1)) \tag{53}$$

The resulting equations:

$$\begin{aligned} &\kappa \int\int d^2x \delta^{(1)}(x^\perp) I_{[0,L_{tot}]}(s)\gamma(s) \left[u^1(x)v^2(x) - u^2(x)v^1(x)\right] \\ = &\int\int d^2x v^j(x)\delta^{(1)}(x^\perp)\delta^{(1)}(s - s(\beta_1)) \end{aligned} \tag{54}$$

are solved, in particular, if the subintegral functions are just equal

$$\begin{aligned} &\kappa\delta^{(1)}(x^\perp) I_{[0,L_{tot}]}(s)\gamma(s) \left[u^1(x)v^2(x) - u^2(x)v^1(x)\right] \\ = \; &v^j(x)\delta^{(1)}(x^\perp)\delta^{(1)}(s - s(\beta_1)) \end{aligned} \tag{55}$$

For $s \in [0, L_{tot}]$ one can solve this equation if

$$\kappa\gamma(s) \left[u^1(0,s)v^2(0,s) - u^2(0,s)v^1(0,s)\right] = v^j(0,s)\delta^{(1)}(s - s(\beta)) \tag{56}$$

is satisfied. Since $\bar{v}$ is an arbitrary vector field in $I\!R^2$, $\bar{v}^j(0,s)$, $j = 1, 2$, are arbitrary vector fields on the filament. Then we have for $j = 1$

$$\begin{aligned} u^1(0,s) &= 0 \\ u^2(0,s) &= -\frac{1}{\kappa\gamma(s)}\delta^{(1)}(s - s(\beta)) \end{aligned} \tag{57}$$

, and for $j = 2$

$$\begin{aligned} u^1(0,s) &= \frac{1}{\kappa\gamma(s)}\delta^{(1)}(s - s(\beta)) \\ u^2(0,s) &= 0 \end{aligned} \tag{58}$$

Let us make two remarks:

1. In both cases the vector field $\bar{u}$ is distributional, so that from the formal point of view it does not belong to the Lie algebra $\mathcal{G}$. Using instead of δ's some models of them, one can regularize the vector field $\bar{u}$, which we do below.

2. One should realize that the solution we get describes the vector field $\bar{u}$ on the filament and not in its neighborhood. Nevertheless, one can extend the vector field on this neighborhood in a smooth and otherwise quite arbitrary way.

Now we are ready to calculate the Poisson bracket for the functions:

$$F(C) = C^1(\beta_1), \qquad G(C) = C^2(\beta_2), \tag{59}$$

defined on the coadjoint orbit connected with the vortex filament.

We introduce two vector fields u_1 and u_2 in the plane, components of which on the filament are given by the formulas found above. Namely:

$$\begin{aligned} u_1^1(0,s) &= 0 \\ u_1^2(0,s) &= -\frac{1}{\kappa\gamma(s)}\delta^{(1)}(s - s(\beta_1)) \end{aligned} \tag{60}$$

$$\begin{aligned} u_2^1(0,s) &= \frac{1}{\kappa\gamma(s)}\delta^{(1)}(s - s(\beta_2)) \\ u_2^2(0,s) &= 0 \end{aligned} \tag{61}$$

The Poisson bracket of F and G is then given by:

$$\begin{aligned} \{F,G\}|_C &= \omega(X_F, X_G)|_C = \omega(X^{u_1}, X^{u_2})|_C \\ &= \kappa \int_0^{2\pi} d\alpha \left[u_1^1\left(C(\alpha)\right) u_2^2\left(C(\alpha)\right) - u_1^2\left(C(\alpha)\right) u_2^1\left(C(\alpha)\right)\right] \\ &= \kappa \int_0^{L_{tot}} ds\gamma(s)\frac{1}{\kappa\gamma(s)}\delta^{(1)}(s - s(\beta_1))\frac{1}{\kappa\gamma(s)}\delta^{(1)}(s - s(\beta_2)) \\ &= \frac{1}{\kappa}\int_0^{L_{tot}} \frac{ds}{\gamma(s)}\delta^{(1)}(s - s(\beta_1))\delta^{(2)}(s - s(\beta_2)) \\ &= \frac{1}{\kappa}\frac{1}{\gamma(s(\beta_1))}\delta(s(\beta_1) - s(\beta_2)) \end{aligned} \tag{62}$$

This result supports the formal expression we have got for the Poisson bracket, in which the functional derivatives $\frac{\delta F}{\delta C_{1,2}(\alpha)}$ should be treated as Frechét derivatives.

In order to make this formal derivation more exact, it means one in which we deal with the proper elements of the Lie algebra (not distributional ones), let us introduce instead of the functions F^j considered above, their smoothed versions:

$$F^j(C) = \int_0^{2\pi} d\alpha \delta_a^{(1)}(\alpha - \beta)C^j(\beta) \tag{63}$$

where $\delta_a^{(1)}(\alpha - \beta)$ ia a model of the Dirac δ, it means a family of functions such that a weak limit of $\delta_a(\alpha)$ as $a \to 0$ is $\delta(\alpha)$. Repeating the derivation of the Poisson bracket, one arrives at the result, for $j = 1$:

$$\begin{aligned} u_1^1(0,s) &= 0 \\ u_1^2(0,s) &= -\frac{1}{\kappa\gamma(s)}\delta_a^{(1)}(s - s(\beta_1)) \end{aligned} \tag{64}$$

and for $j = 2$

$$\begin{aligned} u_2^1(0,s) &= \frac{1}{\kappa\gamma(s)}\delta_a^{(1)}(s - s(\beta_2)) \\ u_2^2(0,s) &= 0 \end{aligned} \tag{65}$$

We substitute the result to the above formula for the Poisson bracket:

$$\{F,G\}|_C = \kappa \int_0^{2\pi} d\alpha \left[u_1^1(C(\alpha))\, u_2^2(C(\alpha)) - u_1^2(C(\alpha))\, u_2^1(C(\alpha))\right] \tag{66}$$

where

$$\begin{aligned} F(C) &= \int_0^{2\pi} d\alpha \delta_a^{(1)}(\alpha - \beta_1) C^1(\alpha) \\ G(C) &= \int_0^{2\pi} d\alpha \delta_a^{(1)}(\alpha - \beta_2) C^2(\alpha) \end{aligned} \tag{67}$$

Then

$$\{F,G\}|_C = \kappa \int_0^{2\pi} ds\gamma(s) \frac{1}{\kappa\gamma(s)}\delta_a^{(1)}(s - s(\beta_1)) \frac{1}{\kappa\gamma(s)}\delta_a^{(1)}(s - s(\beta_2)) \tag{68}$$

Let us observe that both $\delta_a^{(1)}(s - s(\beta_1))$ and $\delta_a^{(1)}(s - s(\beta_2))$ have compact supports containing points $s(\beta_1)$ and $s(\beta_2)$, respectively. Whenever $\beta_1 \neq \beta_2$ there exists a small enough such that the supports of these two functions have no overlap, leading to $\{F,G\}|_C = 0$. Only when $\beta_1 = \beta_2$ we can not make this Poisson bracket to be zero by taking small enough a. One can easily see that the limit $a \to 0$ gives the previously obtained result, for Poisson bracket for $F^1(C) = C^1(\beta_1)$, and $F^2(C) = C^2(\beta_2)$.

4 Fock space for bosonic vortex loops

In this section we construct the Fock space of states for vortex loops. We do it in the simplest case of bosonic loops. Despite its relative simplicity, this case is quite rich and general. It will serve as the fundamental construction in our further studies concerning vortex structures in a quantum ideal fluid, which will be discussed in our future publications. The consideration is rather formal due to the lack of knowledge of the quasi-invariant measure on the configuration space necessary to introduce unitary representations. Nevertheless, as we discussed in the introduction, till we consider only kinematics and not the dynamics of the theory, this lack of measure should not have important influence.

In the approach to quantum systems based on the diffeomorphism groups, initiated and developed by Dashen, Sharp, Goldin, and Menikoff [10, 16, 17], the states of a system are described by representations of the diffeomorphism group. The states are functionals defined on the configuration space, with values in a complex vector space. The general form of representations of the diffeomorphism group can be written as follows:

$$(V(\phi)\Psi)(\Gamma) = \chi_\phi(\Gamma)\Psi(\phi\Gamma)\sqrt{\frac{d\mu_\phi}{d\mu}(\Gamma)} \tag{69}$$

In this formula, Γ is an element of the configuration space (which can be very well a set of N objects). Next, ϕ is any diffeomorphism (in the case of superfluid helium it belongs to the subgroup

$SDiff(I\!R^n)$, $n = 2$ or 3, of the full diffeomorphism group). The $\chi_\phi(\Gamma)$ denotes a unitary cocycle, which should satisfy the relation

$$\chi_{\phi_1 \circ \phi_2}(\Gamma) = \chi_{\phi_2}(\Gamma)\chi_{\phi_1}(\phi_2\Gamma) \tag{70}$$

in order that $V(\phi)$ be a representation ($V(\phi_1 \circ \phi_2) = V(\phi_2) \circ V(\phi_1)$). As a source of cocycles serve unitary representations of the subgroup of the full diffeomorphism group that consists of the diffeomorphisms preserving the configuration Γ as a set. For this subgroup the cocycle condition becomes the representation condition, as one can easily see. The cocycles describe statistics in nonrelativistic quantum theory as was shown in [10]. The square-root expression $\sqrt{\frac{d\mu_\phi}{d\mu}(\Gamma)}$ in the formula (69) should be included in order that $V(\phi)$ be unitary with respect to the measure μ. The derivative $\frac{d\mu_\phi}{d\mu}(\Gamma)$ is the Radon-Nikodym derivative of the measure transformed by ϕ with respect to the original one, taken at Γ. It is worth to mention that the Radon-Nikodym derivative should satisfy rather obvious condition $\frac{d\mu_{\phi_1 \circ \phi_2}}{d\mu}(\Gamma) = \frac{d\mu_{\phi_1 \circ \phi_2}}{d\mu_{\phi_2}}(\Gamma)\frac{d\mu_{\phi_2}}{d\mu}(\Gamma)$ in order that $V(\phi)$ be a representation. This general formulation of a quantum theory works very well in the case of nonrelativistic point particles, including bosons, fermions, and anyons [10, 11, 18].

We would like to generalize the construction to include also extended objects. In particular, we would like to describe this way quantum vortices in an ideal fluid (hopefully this construction applies to vortices in superfluid helium). For such structures (as was established in [1] and confirmed by considerations from the third section of the present paper) the configuration space consists of unparametrized loops. Particular configurations are then multiloops, it means sets consisting of some specified numbers of unparametrized loops.

The configuration space can be split into a set-theoretic sum of subspaces, each of which consists of multiloops with a specified number of component loops. There is a corresponding splitting of the full Hilbert space $\mathcal{H}$ into component subspaces $\mathcal{H}_N$ characterized by fixed numbers of loops. This is similar to what we get in the case of point particles. The new element when one deals with the extended objects is further splitting of the subspace of the configuration space with fixed number of objects into pieces invariant with respect to the action of diffeomorphisms. These are topological sectors discussed in [19]. There is the corresponding splitting of the Hilbert spaces $\mathcal{H}_N$.

In the case of point particles, creation and annihilation operators of the particles are usually introduced. In the approach based on the diffeomorphism groups these operators are intertwiners of the corresponding representations of the groups. The cocycles introduce the statistics by implying corresponding commutation relations for the creation and annihilation operators. In particular, q-commutators for creation and annihilation operators result from appropriate cocycles for the case of anyons [12]. In the paper [12] was proposed also a general scheme for obtaining creation and annihilation operators as intertwiners of the representations. The scheme should work also for the case of nonrelativistic extended objects, such as the vortices in superfluid helium. Let us construct the representations and the operators in the case of bosonic vortex loops, it means in the case where all cocycles are assumed to be trivial (identically equal to one).

Since all cocycles in the case of bosonic vortex loops are identically equal to 1, the complications due to the different topological sectors in subspace of the configuration space of multiloops with fixed number of loops, do not appear. The only element which changes in the way that influences the full theory is the number of loops. For this reason this case is formally very similar to the case of bosonic point particles. Nevertheless, in this case the arguments of the creation and annihilation operators are unparametrized loops rather than points in the physical space.

The wave functions of this quantum theory are complex-valued functionals of the multiloops. The representation of the diffeomorphism group is defined as follows:

$$[V(\phi)\Psi]_N(\{\Gamma_1,...,\Gamma_N\}) = \Psi(\{\phi\Gamma_1,...,\phi\Gamma_N\})\sqrt{\frac{d\mu_\phi^{(N)}}{d\mu^{(N)}}(\{\Gamma_1,...,\Gamma_N\})} \tag{71}$$

with $\{\Gamma_1,...,\Gamma_N\}$ being a multiloop consisting of N unparametrized loops. Here $\phi\Gamma_j$ denotes the natural action of the diffeomorphism ϕ on the loop Γ_j, with j being any integer from the set $\{1,2,...,N\}$.

Let us define then the operators $\psi^*(\Gamma) : \mathcal{H}_N \longrightarrow \mathcal{H}_{N+1}$ and $\psi(\Gamma) : \mathcal{H}_{N+1} \longrightarrow \mathcal{H}_N$:

$$[\psi(\Gamma)\Psi]_N(\{\Gamma_1,...,\Gamma_N\}) := \Psi_{N+1}(\{\Gamma_1,...,\Gamma_N,\Gamma\}) \tag{72}$$

$$[\psi^*(\Gamma)\Psi]_N(\{\Gamma_1,...,\Gamma_N\}) := \sum_{j=1}^{N}\delta(\Gamma,\Gamma_j)\Psi_{N-1}(\{\Gamma_1,...,\hat{\Gamma}_j,...,\Gamma_N\}) \tag{73}$$

where $\delta(\Gamma,\Gamma')$ is formally defined for functions on the configuration space consisting of single unparametrized loops by:

$$\int d\mu(\Gamma')\delta(\Gamma,\Gamma')h(\Gamma') = h(\Gamma) \tag{74}$$

Since we do not know the quasi-invariant measure on the space of loops, this formal definition can not be written more explicitly. Straightforward calculation gives the commutation relations for the creation and annihilation operators defined above:

$$[\psi(\Gamma),\psi(\Gamma')] = 0, \quad [\psi^*(\Gamma),\psi^*(\Gamma')] = 0, \quad [\psi(\Gamma),\psi^*(\Gamma')] = \delta(\Gamma,\Gamma') \tag{75}$$

Necessary conditions for the operators $\psi(\Gamma)$, $\psi^*(\Gamma)$ and representations $V(\phi)$ to form a hierarchy are intertwining properties of the form proposed in [12]:

$$\begin{aligned} V_{N+1}(\phi)\psi^*(h) &= \psi^*(V_{N=1}(\phi)h)V_N(\phi) \qquad \text{and} \\ V_N(\phi)\psi(h) &= \psi(V_{N=1}(\phi)h)V_{N+1}(\phi) \end{aligned} \tag{76}$$

where $\psi(h) = \int d\mu(\Gamma)h(\Gamma)\psi(\Gamma)$ and $\psi^*(h) = \int d\mu(\Gamma)h(\Gamma)\psi^*(\Gamma)$ are creation and annihilation operators averaged with respect to $h \in \mathcal{H}_{N=1}$. These intertwining properties lead to some interesting conditions on the quasi-invariant measure: A condition on the quasi-invariant measure which is sufficient to satisfy the intertwining properties of $\psi(\Gamma),\psi^*(\Gamma)$ reads:

$$\forall j \in \{1,...,N\} \;\; \frac{d\mu_\phi^{(N)}}{d\mu^{(N)}}(\{\Gamma_1,...,\Gamma_j,...,\Gamma_N\}) = \frac{d\mu_\phi^{(N-1)}}{d\mu^{(N-1)}}(\{\Gamma_1,...,\hat{\Gamma}_j,...,\Gamma_N\})\frac{d\mu_\phi^{(1)}}{d\mu^{(1)}}(\{\Gamma_j\}) \tag{77}$$

Iteration of this formula leads to the following multiplicative property of the Radon-Nikodym derivative of the measure μ:

$$\frac{d\mu_\phi^{(N)}}{d\mu^{(N)}}(\{\Gamma_1,...,\Gamma_N\}) = \prod_{j=1}^{N}\frac{d\mu_\phi^{(1)}}{d\mu^{(1)}}(\{\Gamma_j\}) \tag{78}$$

This is the condition which, when satisfied by the quasi-invariant measure, ensures existence of a hierarchy of representations of the diffeomorphism group. The representations from the hierarchy are indexed by the numbers of loops. The simplest example of such a measure is the one which is a product measure. Namely for a multiloop $\{\Gamma_1, ..., \Gamma_N\}$ it is the product of the one-loop measures:

$$d\mu^{(N)}(\{\Gamma_1, ..., \Gamma_N\}) = d\mu^{(1)}(\{\Gamma_1\})...d\mu^{(1)}(\{\Gamma_N\}) \tag{79}$$

However, we are actually interested in more general measures which should distinguish among the cases with nonoverlapping and overlapping vortex loops. Such measures should depend on the areas of overlapped regions (for complicated overlaps there could be many such regions). Let the areas are $\beta_1, ..., \beta_r$. Since the diffeomorphisms are area-preserving, the quantities $\beta_1, ..., \beta_r$ are all preserved by the diffeomorphisms. One of us discussed in [19] the problem of quasi-invariant measures on the spaces of loops. He considered the general properties which should be satisfied by such measures. We can assume, going along his proposals, that the measure for a multiloop $\{\Gamma_1, ..., \Gamma_N\}$ consisting of N loops with r distinguished regions of overlap with the areas $\beta_1, ..., \beta_r$ can be expressed as:

$$d\mu^{(N)}(\{\Gamma_1, ..., \Gamma_N\}) = f(\beta_1, ...\beta_r)d\mu^{(1)}(\{\Gamma_1\})...d\mu^{(1)}(\{\Gamma_N\}) \tag{80}$$

, where f is a general function of $\beta_1, ..., \beta_r$ yet to be specified. One can assume that for $\beta_1 = ... = \beta_r = 0$, $f(\beta_1, ...\beta_r) = 1$. It is easy to prove that for a measure satisfying this condition the Radon-Nikodym derivative also satisfies the multiplicative property:

$$\begin{aligned} \frac{d\mu_\phi^{(N)}}{d\mu^{(N)}}(\{\Gamma_1, ..., \Gamma_N\}) &= \frac{d\mu^{(N)}(\{\phi\Gamma_1, ..., \phi\Gamma_N\})}{d\mu^{(N)}(\{\Gamma_1, ..., \Gamma_N\})} \\ &= \frac{f(\phi_*\beta_1, \ldots, \phi_*\beta_r)d\mu^{(1)}(\{\phi\Gamma_1\})\ldots d\mu^{(1)}(\{\phi\Gamma_N\})}{f(\beta_1, \ldots, \beta_r)d\mu^{(1)}(\{\Gamma_1\})\ldots d\mu^{(1)}(\{\Gamma_N\})} \\ &= \frac{f(\beta_1, \ldots, \beta_r)}{f(\beta_1, \ldots, \beta_r)}\frac{d\mu^{(1)}(\{\phi\Gamma_1)\}}{d\mu^{(1)}(\{\Gamma_1\})}\cdot\ldots\cdot\frac{d\mu^{(1)}(\{\phi\Gamma_N)\}}{d\mu^{(1)}(\{\Gamma_N)\}} = \prod_{i=1}^{N}\frac{d\mu_\phi^{(1)}}{d\mu^{(1)}}(\{\Gamma_i\}) \end{aligned} \tag{81}$$

It would be desirable to establish the general form of the N-loop measure in terms of 1-loop measures which would satisfy the multiplicative property for the Radon-Nikodym derivative. Nevertheless, we have shown above quite general and physically relevant examples of such relations.

5 Summary and Outlook

In this paper the Poisson structure for coadjoint orbits associated with 2D vortex filaments has been found and analyzed. In particular, the canonical coordinates and momenta have been introduced. These are: unparametrized filaments, and integrated vorticity function, respectively. Since unparametrized filaments serve as canonical coordinates for the system, this result confirms results of [1] obtained using the polarization considerations.

We showed the construction of the quantum space of states, in the general case of vortex loops with bosonic statistics. Bosonic statistics in this case occurs by our taking all unitary cocycles characterizing the representations identically equal to one. This construction is rather formal because we do not know the appropriate quasi-invariant measure.

It is worth to mention that similar constructions should be valid also in the case of other nonrelativistic extended objects appearing in the form of loops, e.g. bosonic nonrelativistic strings. Further steps in our studies will be: on the one side extension of the kinematical results to other vortex structures of physical interest, including vortex dipoles in 2D and vortex ribbons and tubes in 3D, on the other side introduction of dynamics to the systems of vortices: writing the Hamiltonian in terms of the canonical coordinates and momenta, then writing quantum Euler equations for vortices, and finally, using the Hamiltonian, description of thermodynamics of the systems. The role of topological degrees of freedom of the vortices should become apparent in this approach. These problems will be discussed in our future publications.

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Appendix

In this appendix we show first that the transformation rules for the stream functions in $3D$ and in $2D$ given in the text lead to the proper transformation rules for velocites, which are the elements of the algebra $SVect(I\!R^n)$, $n = 2, 3$, i.e. quickly vanishing divergenceless vector fields on $I\!R^n$, $n = 2, 3$. We do it within two approaches. In the first approach we show these facts by straightforward naive calculations. In the second one we do this in a geometric framework, using differential forms.

In $3D$ $Ad\phi$ acts in $\mathcal{G}= SVect(I\!R^3)$ by: $\mathcal{G}= SVect(I\!R^3) \ni \bar{v}(\bar{x}) \mapsto Ad\phi(\bar{v}(\bar{x})) \in SVect(I\!R^3) =\mathcal{G}$,

$$(Ad\phi)\bar{v} = (D_\phi \bar{v}) \circ \phi^{-1}, \tag{82}$$

$$[(Ad\phi)\bar{v}]^j(x) = \frac{\partial \phi^j(y)}{\partial y^k}(\phi^{-1}(x))v^k(\phi^{-1}(x)) = \frac{\partial x^j}{\partial [\phi^{-1}(x)]^k} v^k(\phi^{-1}(x)) \tag{83}$$

Our claim is that if the transformation rule for the action of $Ad\phi$ on the stream function $\chi_{\bar{v}}$ is $(Ad\phi)\bar{\chi}_{\bar{v}} = [D_{\phi^{-1}}]^T \bar{\chi}_{\bar{v}} \circ \phi^{-1}$, then the transformation of $\bar{v}$ under $Ad\phi$ will be of the above form.

Proof:

By definition $\bar{v} = -curl\bar{\chi}_{\bar{v}}$, $v^k = -\epsilon^{klm}\partial_l \chi_m$. Our transformation rule for $\bar{v}$ under the action of $Ad\phi$ is

$$[(Ad\phi)\bar{v}]^j(x) = \bar{v}'^j(x) = \frac{\partial x^j}{\partial [\phi^{-1}(x)]^l}\left(-\epsilon^{lmn}\frac{\partial}{\partial [\phi^{-1}(x)]^m}\left[\chi_n(\phi^{-1}(x))\right]\right) \tag{84}$$

Our claim is that this expression (84) should be equal to

$$-\epsilon^{jlm}\frac{\partial}{\partial x^l}(\chi'_m(x)) \tag{85}$$

Our notation means that $\partial_k \phi^j \circ \phi^{-1}$ is a function of $\bar{x}$, for which at $\bar{x}$ we have the value

$$\left[\frac{\partial \phi^j(y)}{\partial y^k}\right](\phi^{-1}(\bar{x})) = \frac{\partial x^j}{\partial [\phi^{-1}(x)]^k} \tag{86}$$

In coordinates the proposed transformation rule reads:

$$\chi'_m = \left[\partial_m(\phi^{-1})^n\right] \chi_n \circ \phi^{-1} \tag{87}$$

or

$$\chi'_m(x) = \frac{\partial[\phi^{-1}(x)]^n}{\partial x^m}\chi_n(\phi^{-1}(x)) \tag{88}$$

Let us substitute this form of $\chi'_m(x)$ to (85) and check if this is equal to (84). The equality proves our thesis.

We have:

$$\begin{aligned}
-\epsilon^{jlm}\frac{\partial}{\partial x^l}\left(\chi'_m(x)\right) &= -\epsilon^{jlm}\frac{\partial}{\partial x^l}\left[\frac{\partial[\phi^{-1}(x)]^n}{\partial x^m}\chi_n\left(\phi^{-1}(x)\right)\right] \\
&= -\epsilon^{jlm}\frac{\partial}{\partial x^m}\left[\phi^{-1}(x)\right]^n\frac{\partial}{\partial x^l}\left[\chi_n\left(\phi^{-1}(x)\right)\right] - \epsilon^{jlm}\frac{\partial^2}{\partial x^l\partial x^m}\left[\phi^{-1}(x)\right]^n\chi_n\left(\phi^{-1}(x)\right) \\
&= -\epsilon^{jlm}\frac{\partial}{\partial x^m}\left[\phi^{-1}(x)\right]^n\frac{\partial}{\partial x^l}\left[\chi_n\left(\phi^{-1}(x)\right)\right] \\
&= -\epsilon^{jlm}\frac{\partial}{\partial x^m}\left[\phi^{-1}(x)\right]^n\frac{\partial\left[\phi^{-1}(x)\right]^p}{\partial x^l}\frac{\partial}{\partial\left[\phi^{-1}(x)\right]^p}\left[\chi_n(\phi^{-1}(x))\right]
\end{aligned} \tag{89}$$

We use further the identity

$$\frac{\partial\left[\phi^{-1}(x)\right]^q}{\partial x^j}\frac{\partial x^j}{\partial\left[\phi^{-1}(x)\right]^k} = \delta_k^{\ q} \tag{90}$$

Let us then compare, without loss of generality, instead of (84) and (85) these same equations but contracted with $-\frac{\partial[\phi^{-1}(x)]^q}{\partial x^j}$. Then

$$\begin{aligned}
-\frac{\partial\left[\phi^{-1}(x)\right]^q}{\partial x^j}(84) &= \frac{\partial\left[\phi^{-1}(x)\right]^q}{\partial x^j}\frac{\partial x^j}{\partial\left[\phi^{-1}(x)\right]^l}\epsilon^{lmn}\frac{\partial}{\partial\left[\phi^{-1}(x)\right]^m}\left[\chi_n(\phi^{-1}(x))\right] \\
&= \epsilon^{qmn}\frac{\partial}{\partial\left[\phi^{-1}(x)\right]^m}\left[\chi_n(\phi^{-1}(x))\right]
\end{aligned} \tag{91}$$

and

$$\begin{aligned}
-\frac{\partial\left[\phi^{-1}(x)\right]^q}{\partial x^j}(85) &= \frac{\partial\left[\phi^{-1}(x)\right]^q}{\partial x^j}\epsilon^{jlm}\frac{\partial}{\partial x^l}\left[\chi'_m(x)\right] \\
&= \frac{\partial\left[\phi^{-1}(x)\right]^q}{\partial x^j}\epsilon^{jlm}\frac{\partial}{\partial x^l}\left[\frac{\partial[\phi^{-1}(x)]^n}{\partial x^m}\chi_n\left(\phi^{-1}(x)\right)\right] \\
&= \epsilon^{jlm}\frac{\partial\left[\phi^{-1}(x)\right]^q}{\partial x^j}\frac{\partial^2[\phi^{-1}(x)]^n}{\partial x^l\partial x^m}\chi_n\left(\phi^{-1}(x)\right) \\
&+ \epsilon^{jlm}\frac{\partial\left[\phi^{-1}(x)\right]^q}{\partial x^j}\frac{\partial[\phi^{-1}(x)]^n}{\partial x^m}\frac{\partial}{\partial x^l}\left[\chi_n\left(\phi^{-1}(x)\right)\right] \\
&= \epsilon^{jlm}\frac{\partial\left[\phi^{-1}(x)\right]^q}{\partial x^j}\frac{\partial\left[\phi^{-1}(x)\right]^n}{\partial x^m}\frac{\partial\left[\phi^{-1}(x)\right]^p}{\partial x^l}\frac{\partial}{\partial\left[\phi^{-1}(x)\right]^p}\left[\chi_n\left(\phi^{-1}(x)\right)\right]
\end{aligned}$$

$$= \epsilon^{qpn} det \left[\frac{\partial\phi^{-1}(x)}{\partial x}\right] \frac{\partial}{\partial\left[\phi^{-1}(x)\right]^p}\left[\chi_n\left(\phi^{-1}(x)\right)\right]$$
$$= \epsilon^{qmn} \frac{\partial}{\partial\left[\phi^{-1}(x)\right]^m}\left[\chi_n\left(\phi^{-1}(x)\right)\right] \tag{92}$$

,where we applied the fact that $det[\frac{\partial\phi^{-1}}{\partial x}] = 1$ for any volume-preserving diffeomorphism ϕ. We realize that changing appropriately the names of indices the equality

$$\frac{\partial\left[\phi^{-1}(x)\right]^q}{\partial x^j}(84) = \frac{\partial\left[\phi^{-1}(x)\right]^q}{\partial x^j}(85) \tag{93}$$

is satisfied, and this proves correctness of the proposed transformation rule for the stream function $\bar{\chi}$. □

Now, we prove an analogous transformation rule for the stream function in $2D$:

$$\chi' = \chi \circ \phi^{-1} \tag{94}$$

, where $\chi' = (Ad\phi)\chi$. In $2D$ $Ad\phi$ acts in $\mathcal{G} = SVect(I\!R^2)$ by:

$$\mathcal{G} \ni SVect(I\!R^2) \ni \bar{v}(\bar{x}) \mapsto Ad\phi(\bar{v}(\bar{x})) \in SVect(I\!R^2) = \mathcal{G} \tag{95}$$

$$(Ad\phi)\bar{v} = \bar{v} \circ \phi^{-1}, \qquad [(Ad\phi)\bar{v}]^j(x) = \bar{v}^j(\phi^{-1}(x)) \tag{96}$$

The fact we prove now states that if χ transforms under $Ad\phi$ as (94), then $\bar{v}$ transforms accordingly to the above given rule (96).

Proof:

The transformation rule follows when we realize that the $2D$ diffeomorphisms are expressible as special kinds of $3D$ diffeomorphisms, which are trivial in the third dimension:

$$\phi_{3D} = \begin{pmatrix} \phi_{2D} & 0 \\ 0 & id \end{pmatrix} \tag{97}$$

As a result the matrix of derivatives D_ϕ has the form:

$$D_\phi = \begin{pmatrix} D_\phi^{2D} & 0 \\ 0 & 1 \end{pmatrix} \tag{98}$$

In the $3D$ language χ is the third component of the $3D$ stream function $\bar{\chi}$, so that the appropriate block of the matrix D_ϕ acting on χ is the one with "1". The same is true for the matrix $D_{\phi^{-1}}$. Therefore, we are left with the transformation rule for χ under the action of $Ad\phi$ defined above, which ends the proof. □

Let us discuss now the notion of a stream function, and the transformation rule for it under the action of $Ad\phi$ in the differential forms framework.

Let us discuss first the general case of incompressible, ideal hydrodynamics on a manifold M, not just on a Euclidean space $I\!R^2$ or $I\!R^3$. M should be equipped with the volume form, preserved by the diffeomorphisms, vol. This is an n-form on M, where n is the dimension of M. The condition of preservation of the volume form of M by the special group of diffeomorphisms we are interested in can be written as

$$\phi^* vol = vol \tag{99}$$

where ϕ^* is the pull-back of differential forms, induced from the natural action of ϕ on M, $\phi : M \to M$. For a 1-parameter subgroup of diffeomorphisms $\phi_t^{\bar{v}}$ associated with a vector field $\bar{v}$ on M this condition reads

$$\phi_t^{\bar{v}\,*} vol = vol \tag{100}$$

Differentiation of this formula with respect to t at $t = 0$ gives

$$L_{\bar{v}} vol = 0 \tag{101}$$

, where $L_{\bar{v}}$ is the Lie derivative. Since for a general vector field $\bar{v}$ on M (it means not necessarily volume-preserving) $L_{\bar{v}} vol$ is some n-form on M, this form must be proportional at every point of M to the standard volume n-form vol, so that

$$L_{\bar{v}} vol = f vol \tag{102}$$

, where f is a function on M. This way we can define the operator $div\bar{v}$ as just this function f. For $\bar{v}$ corresponding to volume-preserving diffeomorphisms $div\bar{v} = 0$, as we expected. One can understand better this definition of the operator $div\bar{v}$, if one observes what it means in the Euclidean case. Namely, in this case, in local coordinates $x^1, x^2, ..., x^n$ the volume form reads $vol = dx^1 \wedge dx^2 \wedge ... \wedge dx^n$, where $dx^1, dx^2, ..., dx^n$ is an orthonormal local cobasis. The Lie derivative with respect to $\bar{v}(\bar{x}) = v^i \frac{\partial}{\partial x^i}$ of the volume element then reads:

$$\begin{aligned} L_{\bar{v}}(vol) &= L_{\bar{v}}(dx^1 \wedge dx^2 \wedge ... \wedge dx^n) && (103) \\ &= L_{\bar{v}}(dx^1) \wedge dx^2 \wedge ... \wedge dx^n + dx^1 \wedge L_{\bar{v}}(dx^2) \wedge ... \wedge dx^n ... \\ &\quad + dx^1 \wedge dx^2 \wedge ... \wedge L_{\bar{v}}(dx^n) && (104) \end{aligned}$$

due to the fact that $L_{\bar{v}}$ is a 0-differentiation in $\Omega(M)$ (Cartan algebra of differential forms on M). Then, due to the Cartan formula $L_{\bar{v}} = d \circ i_{\bar{v}} + i_{\bar{v}} \circ d$:

$$\begin{aligned} L_{\bar{v}}(dx^i) &= (d \circ i_{\bar{v}})(dx^i) + (i_{\bar{v}} \circ d)(dx^i) \\ &= d(i_{\bar{v}} dx^i) + i_{\bar{v}}(ddx^i) \\ &= d(dx^i(\bar{v})) + 0 \\ &= d(\bar{v}(x^i)) = d\left(v^j \frac{\partial}{\partial x^j}(x^i)\right) \\ &= d\left(v^j \delta^i{}_j\right) = d(v^i) = \frac{\partial v^i}{\partial x^j} dx^j \end{aligned} \tag{105}$$

and

$$L_{\bar{v}}(vol) = \frac{\partial v^1}{\partial x^j} dx^j \wedge dx^2 \wedge ... \wedge dx^n + dx^1 \wedge \frac{\partial v^2}{\partial x^j} dx^j \wedge ... \wedge dx^n + ... + dx^1 \wedge dx^2 \wedge ... \wedge dx^{n-1} \wedge \frac{\partial v^n}{\partial x^j} dx^j \tag{106}$$

In every summand survives only the term in which the number j is missing among the others dx^i. Therefore:

$$\begin{aligned} L_{\bar{v}}(vol) &= \frac{\partial v^1}{\partial x^j} dx^j \wedge dx^2 \wedge ... \wedge dx^n + dx^1 \wedge \frac{\partial v^2}{\partial x^j} dx^j \wedge ... \wedge dx^n + ... \\ &\quad + dx^1 \wedge dx^2 \wedge ... \wedge dx^{n-1} \wedge \frac{\partial v^n}{\partial x^j} dx^j \\ &= \left(\frac{\partial v^i}{\partial x^i}\right) dx^1 \wedge ... \wedge dx^n = \left(\frac{\partial v^i}{\partial x^i}\right) vol \end{aligned} \tag{107}$$

In the Euclidean case $\frac{\partial v^i}{\partial x^i}$ is the expression for the divergence operator. This way we have illustrated the definition of the divergence operator in the general case.

Now, let us go back to the general case, in which vol is a fixed volume form on a manifold M. For divergenceless vector fields on M, which constitute the Lie algebra of the group of volume preserving diffeomorphisms $L_{\bar{v}}(vol) = 0$. Applying to this relation the Cartan formula $L_{\bar{v}} = d \circ i_{\bar{v}} + i_{\bar{v}} \circ d$, one gets:

$$0 = (d \circ i_{\bar{v}} + i_{\bar{v}} \circ d) vol = d\left(i_{\bar{v}}(vol)\right) + i_{\bar{v}}(d(vol)) \tag{108}$$

Since vol is a differential form of maximal degree, $d(vol) = 0$. Therefore, $i_{\bar{v}}(vol)$ is a closed $(n-1)$-form. Now, to introduce the general notion of a stream function one should make an additional assumption that the manifold M is simply connected. Then there exists such an $(n-2)$- form χ on M that

$$i_{\bar{v}}(vol) = d(\chi) \tag{109}$$

and χ is defined up to a differential of an $(n-3)$-form, $\chi \mapsto \chi + d\alpha$. One can define then a stream function as a class $[\chi]$ of $(n-2)$-forms on M, representatives of which differ by a differential of an $(n-3)$-form, and such that the relation (109) is satisfied. One can observe that this definition works well for our particular cases of $n = 3$ and $n = 2$ and the Euclidian space. Invariance of vol under diffeomorphisms leads to its invariance under $Ad\phi$-action, so that the transformation rule for $\bar{v}$ under this action dictates corresponding rule for the form χ.

In particular, in $I\!R^3$ the stream function is a $3 - 2 = 1$-form, it means it is a covector, so its transformation rule should be deducted from the invariance of the scalar product between covectors and vectors:

$$\int_{I\!R^3} \chi_i(x) v^i(x) d^3x \tag{110}$$

Under the action of the $Ad\phi$, $v^i(x)$ changes to:

$$(v')^i(x) = \frac{\partial \phi^i}{\partial x^k}(\phi^{-1}(x)) v^k(\phi^{-1}(x)) = \frac{\partial x^i}{\partial \left[\phi^{-1}(x)\right]^k} v^k(\phi^{-1}(x)) \tag{111}$$

but

$$\int_{I\!R^3} \chi_i(x) v^i(x) d^3x = \int_{I\!R^3} \chi'_i(x) v'^i(x) d^3x \tag{112}$$

so that

$$\int_{I\!R^3} \chi_i(x) v^i(x) d^3x = \int_{I\!R^3} \chi'_i(x) \frac{\partial x^i}{\partial \left[\phi^{-1}(x)\right]^k} v^k(\phi^{-1}(x)) d^3x \tag{113}$$

This formula under the change of coordinates defined by:

$$y = \phi^{-1}(x), \quad x = \phi(y) \qquad d^3x = det\left[\frac{\partial \phi(y)}{\partial y}\right] d^3y \tag{114}$$

becomes:

$$\begin{aligned}
&\int_{IR^3} \chi'_i(\phi(y))\frac{\partial[\phi(y)]^i}{\partial y^k}(y)v^k(y)det\left[\frac{\partial\phi(y)}{\partial y}\right]d^3y \\
=& \int_{IR^3} \chi'_i(\phi(y))\frac{\partial[\phi(y)]^i}{\partial y^k}(y)v^k(y)d^3y \\
=& \int_{IR^3} \chi'_k(\phi(y))\frac{\partial[\phi(y)]^k}{\partial y^i}(y)v^i(y)d^3y
\end{aligned} \tag{115}$$

Changing the name of the variable x to y in the first formula of (113), we get (with exactness to differentials, which we neglect):

$$\chi_i(y) = \chi'_k(\phi(y))\frac{\partial[\phi(y)]^k}{\partial y^i}(y) \tag{116}$$

Substituting back, $x = \phi(y)$, we get

$$\chi_i(\phi^{-1}(x)) = \chi'_k(x)\frac{\partial x^k}{\partial[\phi^{-1}(x)]^i} \tag{117}$$

Then, applying to both sides $\frac{\partial\left[\phi^{-1}(x)\right]^i}{\partial x^j}$ we get:

$$\begin{aligned}
\frac{\partial\left[\phi^{-1}(x)\right]^i}{\partial x^j}\chi_i(\phi^{-1}(x) &= \chi'_k(x)\frac{\partial[\phi^{-1}(x)]^i}{\partial x^j}\frac{\partial x^k}{\partial\left[\phi^{-1}(x)\right]^i} \\
&= \chi'_k(x)\frac{\partial x^k}{\partial x^j} = \chi'_k(x)\delta^k_j = \chi'_j(x)
\end{aligned} \tag{118}$$

In the matrix form:

$$\bar{\chi}' = \left[D_\phi^{-1}\right]^T \bar{\chi}\circ\phi^{-1} \tag{119}$$

since

$$\left[D_\phi^{-1}\right]^i_j = \frac{\partial y^i}{\partial\left[\phi(y)\right]^j} \tag{120}$$

as it is easy to check. Namely, one should check that $D_\phi \cdot D_\phi^{-1} = D_\phi^{-1}\cdot D_\phi = I$, or, in other words, that

$$[D_\phi]^i_j\,[D_\phi^{-1}]^j_k = [D_\phi^{-1}]^i_j\,[D_\phi]^j_k = \delta^i_k \tag{121}$$

Since

$$\frac{\partial[\phi(y)]^i}{\partial y^j}\frac{\partial y^j}{\partial[\phi(y)]^k} = \frac{\partial[\phi(y)]^i}{\partial[\phi(y)]^k} = \delta^i_k \tag{122}$$

and

$$\frac{\partial y^i}{\partial[\phi(y)]^j}\frac{\partial[\phi(y)]^j}{\partial y^k} = \frac{\partial y^i}{\partial y^k} = \delta^i_k \tag{123}$$

this is indeed the case. This result on the transformation rule for the stream function $\bar{\chi}$ in $3D$ agrees with the one given above. In $2D$ one uses the same arguments as above to derive the transformation rule for the $2D$ stream function χ knowing the transformation rule for the stream function in $3D$.

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