Quantum logarithmic Sobolev inequalities and rapid mixing

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QIP 2013 Beijing

January 21, 2013

Outline

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- If $\Gamma_\sigma \mathcal{L} = \mathcal{L}^* \Gamma_\sigma$, where σ is the stationary state of $\mathcal{L}$ and $\Gamma_\sigma(X) = \sqrt{\sigma} X \sqrt{\sigma}$, the $\mathcal{L}$ is *reversible*.

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Note: we do not yet make any assumptions about locality or geometry at this point.

The question

Let $\mathcal{L}$ be the generator of a primitive reversible quantum dynamical semigroup. Given $\epsilon > 0$, for what $\tau \geq t > 0$ do we have

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What are good choices for A and b? We will argue that the Log Sobolev machinery is the finest available to answer this question.

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4. Dissipative algorithms?
5. Concentration of measure?

A few definitions to start with...

non-commutative $\mathbb{L}_p$ spaces

- The $\mathbb{L}_p$ inner product. For two hermitian operators f, g:

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- The $\mathbb{L}_p$ norm. For any hermitian operator f:

$$||f||_{p,\sigma} = \mathrm{tr}[\ |\Gamma_\sigma^{1/p}(f)|^p]^{1/p} \tag{5}$$

A few more definitions...

Variance and Entropy functionals

- The variance

$$\mathrm{Var}_\sigma(\mathrm{g}) = \mathrm{tr}[\Gamma_\sigma(\mathrm{g})\mathrm{g}] - \mathrm{tr}[\Gamma_\sigma(\mathrm{g})]^2. \tag{6}$$

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- The $\mathbb{L}_p$ relative entropies. For any hermitian operator f:

$$\mathrm{Ent}_1(\mathrm{f}) = \mathrm{tr}[\Gamma_\sigma(f)(\log(\Gamma_\sigma(f)) - \log(\sigma))] \tag{7}$$

$$-\mathrm{tr}[\Gamma_\sigma(f)]\log(\mathrm{tr}[\Gamma_\sigma(f)]) \tag{8}$$

$$\mathrm{Ent}_2(\mathrm{f}) = \mathrm{tr}[\left(\Gamma_\sigma^{1/2}(f)\right)^2 \log\left(\Gamma_\sigma^{1/2}(f)\right)] \tag{9}$$

$$-\frac{1}{2}\mathrm{tr}[\left(\Gamma_\sigma^{1/2}(f)\right)^2 \log(\sigma)]$$

$$-\frac{1}{2}\|f\|_{2,\sigma}^2 \log\left(\|f\|_{2,\sigma}^2\right).$$

Yet more... (sorry!)

Dirichlet Forms

$$\mathcal{E}_1(f) = -\frac{1}{2}\mathrm{tr}[\Gamma_\sigma(\mathcal{L}(f))(\log(\Gamma_\sigma(f)) - \log(\sigma))] \quad (10)$$

$$\mathcal{E}_2(f) = -\langle f, \mathcal{L}(f)\rangle_\sigma \, . \quad (11)$$

Useful identities:

$$\mathrm{Var}(\Gamma_\sigma^{-1}(\rho)) = \chi^2(\rho,\sigma), \qquad \mathrm{Ent}_2(\Gamma_\sigma^{-1}(\rho)) = \mathrm{S}(\rho||\sigma) \quad (12)$$

Spectral Gap and Log-Sobolev constant

- The spectral gap of $\mathcal{L}$:

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Note: one can in fact define a whole family of Log Sobolev constants α_p, with $p \geq 0$.

Theorem

Let $\mathcal{L}$ denote the generator of a primitive reversible semigroup with fixed point σ. Then,

1. χ^2 **bound**:

$$\begin{aligned} ||\rho_t - \sigma||_1 &\leq \sqrt{\chi^2(\rho_t, \sigma)} \\ &\leq \sqrt{\chi^2(\rho, \sigma)} e^{-\lambda t} \leq \sqrt{1/\sigma_{\min}} e^{-\lambda t}. \end{aligned} \tag{15}$$

Where $\sigma_{\min}$ denotes the smallest eigenvalue of the fixed point σ.

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2. **Log-Sobolev bound**:

$$\begin{aligned} ||\rho_t - \sigma||_1 &\leq \sqrt{2S(\rho_t||\sigma)} \\ &\leq \sqrt{2S(\rho||\sigma)} e^{-\alpha_1 t} \leq \sqrt{2\log(1/\sigma_{\min})} e^{-\alpha_1 t}. \end{aligned} \tag{16}$$

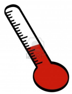
Where $\sigma_{\min}$ denotes the smallest eigenvalue of the fixed point σ.

Interpretation

Davies generators describe the dissipative dynamics resulting as the weak (or singular) coupling limit of a system coupled to a large heat bath. For these *thermal maps*, the Log-Sobolev constant is the minimal normalized rate of change of the free energy of the system:

$$\alpha_1 = \inf_{\rho} \partial_t \log \left[F(\rho_t) - F(\rho_\beta) \right] \Big|_{t=0} , \qquad (17)$$

where $F(\rho) = \mathrm{tr}[\rho H] - \frac{1}{\beta} S(\rho)$ is the free energy of the system, and ρ_β is the Gibbs state.



Mathematical results

Theorem (Partial ordering)

Let $\mathcal{L}$ be a primitive reversible Liouvillian with stationary state σ. The Log-Sobolev constants α_1, α_2 and the spectral gap λ of $\mathcal{L}$ are related as:

$$\alpha_2 \leq \alpha_1 \leq \lambda. \tag{18}$$

Theorem (Hypercontractivity)

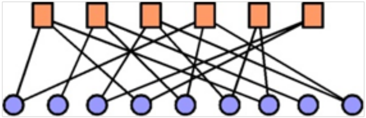
Let $\mathcal{L}$ be a primitive Liouvillian with stationary state σ, and let $T_t = e^{t\mathcal{L}}$ be its associated semigroup. Then

1. *If there exists a $\alpha > 0$ such that $||T_t||_{(2,\sigma)\to(p(t),\sigma)} \leq 1$ for all $t > 0$ and $2 \leq p(t) \leq 1 + e^{2\alpha t}$. Then $\mathcal{L}$ satisfies LS_2 with $\alpha_2 \geq \alpha$.*
2. *If $\mathcal{L}$ is weakly $\mathbb{L}_p$-regular, and has an LS_2 constant α_2, then $||T_t||_{(2,\sigma)\to(p(t),\sigma)} \leq 1$ for all $t > 0$ when $2 \leq p(t) \leq 1 + e^{2\alpha_2 t}$. If, furthermore, $\mathcal{L}$ is strongly $\mathbb{L}_p$ regular, then the above holds for all $t > 0$ when $2 \leq p(t) \leq 1 + e^{4\alpha_2 t}$.*

Quantum expanders

Quantum Expander: (sequence of) quantum channel with i) a fixed number of Kraus operators (D), and ii) the spectral gap λ of the channel is asymptotically independent of dimension d. Then,

$$\frac{(1-2/d)\lambda}{\log(d-1)} \leq \alpha_2 \leq \log D \frac{4+\log\log d}{2\log 3d/4} \tag{19}$$



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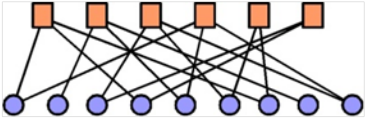

The mixing time is of order $\log d$

Suppose that $\mathcal{L}$ describes the open system dynamics on a lattice of qudits. Assume furthermore that $\mathcal{L}$ is: i) **primitive and reversible**, ii) **local**, and iii) has a Log Sobolev constant α_1 which is **system size independent**. Then we get

(strong) clustering of correlations

$$\langle O_A O_B \rangle_\sigma - \langle O_A \rangle_\sigma \langle O_B \rangle_\sigma \leq K \log\left(\frac{1}{\sigma_{\min}}\right) e^{-\alpha_1 d(A,B)/v} \tag{20}$$

where K is volume like.

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Stability of Liouvillians

Let $\mathcal{Q}$ be a local perturbation of $\mathcal{L}$, and $\mathcal{L}' = \mathcal{L} + \mathcal{Q}$ with stationary state σ', then

$$||\sigma - \sigma'||_1 \leq \frac{||\mathcal{Q}||_{1-1}}{\alpha_1} \left(\log \left(\log \left(\frac{1}{\sigma_{\min}} \right) \right) + 1 \right) \tag{21}$$

Thank you for your attention!

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