Max-Planck-Institut
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Quantum logic is undecidable

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by

Tobias Fritz

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Abstract. We investigate the first-order theory of closed subspaces of complex Hilbert spaces in the signature $(\vee, \perp, 0, 1)$, where '$\perp$' is the orthogonality relation. Our main result is that already its purely implicational fragment is undecidable: there is no algorithm to decide whether an implication between equations in the language of orthomodular lattices is valid in all complex Hilbert spaces. This is a corollary of a recent result of Slofstra in combinatorial group theory. It follows upon reinterpreting that result in terms of the hypergraph approach to contextuality, for which it constitutes a proof of the *inverse sandwich conjecture*. It can also be interpreted as stating that a certain quantum satisfiability problem is undecidable.

Introduction

Quantum logic starts with the idea that quantum theory can be understood as a theory of physics in which standard Boolean logic gets replaced by a different form of logic, where various rules, such as the distributivity of logical and over logical or, are relaxed [1, 2]. This builds on the observation that $\{0,1\}$-valued observables behave like logical propositions: such an observable is a projection operator on Hilbert space, and it can be identified with the closed subspace that it projects onto. The conjunction (logical *and*) translates into the intersection of subspaces, while disjunction (logical *or*) is interpreted as forming the closed subspace spanned by two subspaces. In this way, the closed subspaces of a complex Hilbert space $\mathcal{H}$ form the *complex Hilbert lattice* $\mathcal{C}(\mathcal{H})$, which is interpreted as the lattice of 'quantum propositions' and forms a particular kind of orthomodular lattice [3–5].

However, the theory of orthomodular lattices is quite rich and contains many objects other than complex Hilbert lattices. So in order to understand the laws of quantum logic, one has to find additional properties which characterize the latter kind of objects. Much effort has been devoted to this question, resulting in partial characterizations such as Piron's theorem [6, 7], Wilbur's theorem [8] and Solèr's theorem [9][1]. However, the axioms for complex Hilbert lattices that these results suggest are quite sophisticated: atomicity, completeness or the existence of an infinite orthonormal sequence. These are conditions that cannot be expressed algebraically, i.e. as *first-order* properties using just a finite number of variables, algebraic operations, and quantifiers. Fortunately, there has also been a substantial amount of work on first-order properties enjoyed by complex Hilbert

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[1] See also the survey [10] for a more recent exposition from a geometrical perspective.

lattices, and in particular on equational laws that hold in all complex Hilbert lattices $\mathcal{C}(\mathcal{H})$, such as the algorithmic approach advocated by Megill and Pavičić [11, 12]. Such algorithmic approaches are what our present contribution is about: we prove that there cannot exist any algorithm to decide whether an *implication* between equations in complex Hilbert lattices is valid.

To make this statement precise, we keep the lattice operations notationally separate from the external logical connectives and denote the latter in plain English:

Theorem 1. *There is no algorithm to decide whether an implication of the form*

$$(E_1 \text{ and } E_2 \text{ and } \dots \text{ and } E_k) \;\text{ implies }\; (0 = 1) \tag{1}$$

holds in every complex Hilbert lattice $\mathcal{C}(\mathcal{H})$, where each E_i has one of the following two forms:

- *an equation of the form $P_1 \vee \dots \vee P_m = 1$, where the P_j are free variables denoting projections and 1 is the identity projection;*
- *an orthogonality relation $P_1 \perp P_2$ between two free variables, or equivalently the equation $P_1^\perp \vee P_2^\perp = 1$.*

Here, the consequent $0 = 1$ states that the zero projection is equal to the identity projection, or equivalently that $\mathcal{H}$ is the trivial zero-dimensional Hilbert space. Thus the proposition (1) states that the antecedents $E_1, \dots, E_k$ are jointly contradictory, meaning that the only way to satisfy them jointly is by taking the Hilbert space to be zero-dimensional, $\mathcal{H} = \{0\}$.

Remark 2. By replacing each P_i by $P_i^\perp$, it follows that Theorem 1 remains true if one replaces $\vee$ by $\wedge$ and 1 by 0, so that each E_i is either of the form $P_1 \wedge \dots \wedge P_m = 0$ or $P_1^\perp \wedge P_2^\perp = 0$.

Remark 3. Instead of asking whether (1) is valid in all $\mathcal{H}$, one can alternatively negate the question and ask whether there exists a Hilbert space $\mathcal{H}$ with $\dim(\mathcal{H}) > 0$ together with an assignment of projections in $\mathcal{H}$ to the free variables such that the formula

$$E_1 \text{ and } E_2 \text{ and } \dots \text{ and } E_k$$

holds. This formulation makes it clear that we are dealing with a quantum version of the Boolean satisfiability problem—distinct from the QSAT problem introduced by Bravyi [13]—which is undecidable as per Theorem 1. As we will see in the proof of Lemma 16, if an instance of our quantum satisfiability problem is solvable, then it is also solvable with $\mathcal{H}$ infinite-dimensional separable. Thus it is sufficient to consider e.g. $\mathcal{H} = \ell^2(\mathbb{N})$ only.

The reason that we prefer the statement of Theorem 1 over the satisfiability formulation is that we are interested in the laws of quantum logic, i.e. in those statements that are valid on *all* Hilbert spaces $\mathcal{H}$.

Example 4. The implication

$$\begin{aligned}(P \vee Q = 1) \text{ and } (Q \vee R = 1) \text{ and } (R \vee P = 1)&\\ \text{and } (P \perp Q) \text{ and } (Q \perp R) \text{ and } (R \perp P) \;\text{ implies }\; (0 = 1)&\end{aligned}$$

is valid: in any nonzero Hilbert space, it is impossible to have three projections that are pairwise orthogonal and such that any two of them sum to the identity [14].

The key ingredient that leads to Theorem 1 is an undecidability result of Slofstra [15], who builds on earlier work of Cleve, Liu and Slofstra [16] and Cleve and Mittal [17]. Our contribution merely consists of having seen the connection to quantum logic via the hypergraph approach to contextuality [18]. The mathematical depth necessary for deriving such an undecidability result is to be found in Slofstra's arguments.

The statement that we actually prove first is Corollary 11, which is the *inverse sandwich conjecture* from [18]. The undecidability of quantum logic in the form of Theorem 15 is then merely a reformulation—on an even smaller set of sentences than our formulation above. As we will see, the statement remains true if one replaces 'holds in every $\mathcal{C}(\mathcal{H})$' by 'holds in $\mathcal{C}(\mathcal{H})$ for some infinite-dimensional separable Hilbert space $\mathcal{H}$'.

Corollary 11 also implies that infinitely many of the hypergraph C*-algebras $C^*(H)$ of [18] fail to be residually finite-dimensional, as per Corollary 13.

Related work. Lipshitz [19] has shown, among other things, that the purely implicational fragment of the theory of all $\mathcal{C}(\mathbb{C}^n)$ is undecidable, already in the signature $(\vee, \wedge, 0, 1)$. While this result is similar to ours, it uses techniques specific to a finite-dimensional setting (coordinatization). A result of Sherif [20] is that any first-order theory between orthomodular lattices and finite orthomodular lattices is undecidable. Herrmann [21] has proven that the equational theory of the orthomodular lattice of projections of a finite von Neumann algebra factor is decidable; this includes both the $\mathcal{C}(\mathbb{C}^n)$ and the projection ortholattices of factors of type II_1. Other work of Herrmann and Ziegler is also concerned with related decidability and complexity problems [22].

Solution groups and their group C*-algebras

Before getting to the proof of our Theorem 1, we review the essential ingredient: Slofstra's recent work in combinatorial group theory [15]. Subsequently, we will modify his intended interpretation using nonlocal games to one in terms of contextuality. From there, it is only a small step to quantum logic.

Following [17], Slofstra considers *linear systems* over $\mathbb{Z}_2$, which are linear equations $Mx = b$ with $M \in \mathbb{Z}_2^{m\times n}$ and $b \in \mathbb{Z}_2^m$. While conventional solutions have $x \in \mathbb{Z}_2^n$, a *quantum solution* [16] consists of self-adjoint operators $A_1, \ldots, A_n \in \mathcal{B}(\mathcal{H})$ such that:

- $A_i^2 = \mathbb{1}$ for all i;
- If x_i and x_j appear in the same equation, then A_i commutes with A_j;
- For each equation of the form $x_{k_1} + \ldots + x_{k_r} = b_r$, the operators satisfy
$$A_{k_1} \cdots A_{k_r} = (-1)^{b_r}\mathbb{1},$$
where the order of the factors is irrelevant due to the previous commutativity requirement.

The fact that quantum solutions solve the given linear system multiplicatively instead of additively is purely conventional, and allows for simpler notation. The most famous example of a quantum solution of a linear system that is unsolvable over $\mathbb{Z}_2$ is the Mermin-Peres magic square [23]. The quantum solution for $\mathcal{H} = \mathbb{C}$ are precisely the conventional solutions over $\mathbb{Z}$, written multiplicatively as $A_i = (-1)^{x_i}$.

The quantum solutions of a linear system are controlled by representations of a certain group associated to the system:

Definition 5 (Cleve, Liu, Slofstra [16]). *Let $Mx = b$ be a linear system over $\mathbb{Z}_2$. Its* solution group *is the finitely presented group Γ with generators $g_1, \ldots, g_n$ and J subject to the relations:*

- *$g_i^2 = 1$ for all i;*
- *If x_i and x_j appear in the same equation, then $g_i g_j = g_j g_i$;*
- *For each equation of the form $x_{k_1} + \ldots + x_{k_r} = b_r$, the generators satisfy*
$$g_{k_1} \cdots g_{k_r} = J^{b_r}.$$

These relations are precisely such that quantum solutions of $Mx = b$ on a Hilbert space $\mathcal{H}$ are in bijective correspondence with those unitary representations $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ for which $\pi(J) = -\mathbb{1}$.

Slofstra [**15**, Theorem 3.1] has shown that every finitely presented group embeds into a solution group in a particular way. Concerning undecidability, the following essential result was derived in the proof of [**15**, Corollary 3.3].

Theorem 6 (Slofstra). *Given a linear system $Mx = b$, it is undecidable to determine whether $J = 1$ in the associated solution group. Equivalently, it is undecidable to determine whether the linear system has a quantum solution.*

We now move on to considering the ramifications of this result, first for the hypergraph approach to contextuality [**18**] and then for quantum logic, including the proof of Theorem 1.

Consequences for the hypergraph approach to contextuality

For us, a *hypergraph* is a pair $H = (V, E)$ consisting of a finite set of vertices V and a subset $E \subseteq 2^V$ with $\cup E = V$. Their relevance lies in the observation that hypergraphs provide a convenient and powerful language to analyze quantum contextuality [**18**]:

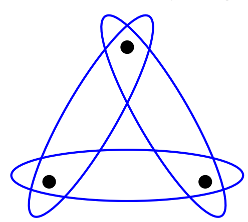
Definition 7. *A* quantum representation *of a hypergraph $H = (V, E)$ consists of:*

- *A Hilbert space $\mathcal{H}$ with $\dim(\mathcal{H}) > 0$,*
- *a family of projections $(P_v)_{v \in V}$ in $\mathcal{H}$ assigned to the vertices of H such that for each edge $e \in E$, the associated projections form a partition of unity,*

$$\sum_{v \in e} P_v = \mathbb{1}. \qquad (2)$$

In the case where all projections have rank 1, this is related to the notion of Kochen-Specker configuration: a finite collection of vectors in a Hilbert space such that certain particular subsets of these vectors form orthonormal bases. The concept of *quantum model* [**18**, Definition 5.1.1] on a hypergraph—considered as a contextuality scenario—is implicitly based on our notion of quantum representation. In general, the idea is that the hyperedges $e \in E$ label measurements with outcomes $v \in V$, and some outcomes may be shared between several measurements, corresponding to vertices being incident to several hyperedges.

Example 8. A quantum representation of the hypergraph



would yield a nontrivial solution to the antecedents of Example 4. Thus this hypergraph does not have any quantum representation.

The quantum representations of a hypergraph are equivalently given by the representations of the *hypergraph C*-algebra*, which is the finitely presented C*-algebra

$$C^*(H) := \left\langle P_v : v \in V \;\middle|\; P_v = P_v^* = P_v^2,\ \sum_{v \in e} P_v = \mathbb{1}\ \forall e \in E \right\rangle,$$

as already introduced in [18, Section 8.3]. We quickly record a standard observation for future reference:

FACT 9. *Projections that form a partition of unity* (2) *are mutually orthogonal,* $P_v P_w = \delta_{v,w} P_v$.

Our central new observation is this:

LEMMA 10. *There is an algorithm to compute, for every linear system* $Mx = b$*, a hypergraph* H *such that quantum solutions of the linear system are in bijective correspondence with quantum representations of the hypergraph.*

This observation should not be surprising, since the original considerations around linear systems and solution groups [16,17] were inspired by the Mermin-Peres *magic square*, one of the most startling examples of quantum contextuality [24]. The hypergraph construction in the following proof is along the lines of the *measurement protocols* of [18, Appendix D], combined with forming an *induced subscenario* [18, Definition 2.5.1].

PROOF. Let a linear system $Mx = b$ be given, with $M \in \mathbb{Z}_2^{m\times n}$ and $b \in \mathbb{Z}_2^m$. We write $[n] := \{1, \ldots, n\}$, and

$$N(r) = \{ \, i \in [n] \mid M_{r,i} = 1 \, \}$$

for the set of variables that are contained in the r-th equation, with $r \in [m]$. We think of $i \in [n]$ as indexing an observable A_i with values in $\mathbf{2} := \{-1, +1\}$, and each $N(r)$ as indexing a set of measurements that commute and is therefore jointly implementable. Correspondingly, our hypergraph contains two kinds of outcome-representing vertices,

$$V := \{ \, v_i^\alpha \; : \; i \in [n], \, \alpha \in \mathbf{2} \, \} \, \cup \, \{ \, w_r^\beta \; : \; r \in [m], \, \beta \in \mathbf{2}_\pm^{N(r)} \, \},$$

where the set $\mathbf{2}_\pm^{N(r)}$ consists of all those functions $\beta : N(r) \to \mathbf{2}$ which have the correct parity in the sense that $\prod_{i\in N(r)} \beta(i) = (-1)^{b_r}$.

The hyperedges will also be of three kinds: first, $\{v_i^{-1}, v_i^{+1}\}$ for every $i \in [n]$, which is intended to correspond to a measurement of the $\mathbf{2}$-valued observable A_i; second, $\{w_r^\beta \; : \; \beta \in \mathbf{2}_\pm^{N(r)}\}$ for every $r \in [m]$, which corresponds to the possible outcomes of a joint measurement of the A_i with $i \in N(r)$; and third, the sets of the form

$$\{ \, v_i^\alpha, \, w_r^\beta \; : \; \beta(i) = -\alpha \, \} \tag{3}$$

for every fixed $r \in [m]$, $i \in N(r)$ and $\alpha \in \mathbf{2}$. Intuitively, this is the set of outcomes of the measurement protocol which first measures A_i, and if the outcome is $-\alpha$, then also conducts a joint measurement of all other observables A_j with $j \in N(r)$, resulting in a joint outcome $\beta \in \mathbf{2}_\pm^{N(r)}$ with $\beta(i) = -\alpha$. This ends the definition of the relevant hypergraph H.

We now show how quantum representations of this hypergraph correspond to quantum solutions of the linear system. First, given a quantum representation of the hypergraph, we obtain a quantum solution of the linear system by taking

$$A_i := P_{v_i^{+1}} - P_{v_i^{-1}} = 2P_{v_i^{+1}} - \mathbb{1} \tag{4}$$

for all $i \in [n]$. We need to show that this indeed results in a quantum solution by checking that the $P_{v_i^{+1}} - P_{v_i^{-1}}$ are unitary and satisfy the required relations. Unitarity is clear since $2P_{v_i^{+1}} - 1$ is a symmetry, which thereby also shows that the relation $A_i^2 = 1$ is respected. For the commutativity

$A_iA_j = A_jA_i$ with $i, j \in N(r)$, we use Fact 9 together with the partition of unity relation associated to the second kind of edge (3): this relation implies that

$$P_{v_i^{+1}} - P_{v_i^{-1}} = \sum_{\beta\,:\,\beta(i)=+1} P_{w_r^\beta} - \sum_{\beta\,:\,\beta(i)=-1} P_{w_r^\beta},$$

and similarly for j, so that we can apply Fact 9. This expression is also useful for checking that the relation $\prod_{i\in N(r)} \left(P_{v_i^{+1}} - P_{v_i^{-1}}\right) = (-1)^{b_r}\mathbb{1}$ holds as well, which then follows from $\sum_\beta w_r^\beta = \mathbb{1}$ upon using that every β has even parity.

In the other direction, we put

$$P_{v_i^\alpha} := \frac{\mathbb{1} + \alpha A_i}{2}, \qquad P_{w_r^\beta} := \prod_{i\in N(r)} \frac{\mathbb{1} + \beta(i)A_i}{2},$$

where we likewise need to check that the relations are preserved, which first requires showing that both right-hand sides are projections. This is clear in the first case and holds by the commutativity assumption $A_iA_j = A_jA_i$ for $i, j \in N(r)$ in the second case. We verify the required partition of unity relations. First, $P_{v_i^{-1}} + P_{v_i^{+1}} = 1$ holds trivially. Second, if we apply the definition of $P_{w_r^\beta}$ also for β of the wrong parity, then

$$\sum_{\beta\in\mathbf{2}^{N(r)}} P_{w_r^\beta} = \prod_{i\in N(r)} \sum_{\beta\in\mathbf{2}} \frac{\mathbb{1} + \beta A_i}{2} = \mathbb{1}.$$

Since $P_{w_r^\beta} = 0$ whenever β has the wrong parity, we can ignore these terms and arrive at the desired equation. Third, the relation associated to (3) takes a bit more work: the expression $P_{v_i^\alpha} + \sum_{\beta\,:\,\beta(i)=-\alpha} P_{w_t^\beta}$ evaluates to

$$\frac{\mathbb{1} + \alpha A_i}{2} + \sum_{\beta\,:\,\beta(i)=-\alpha} \prod_{j\in N(r)} \frac{\mathbb{1} + \beta(j)A_j}{2} = \frac{\mathbb{1} + \alpha A_i}{2} + \frac{\mathbb{1} - \alpha A_i}{2} \sum_{\beta\,:\,\beta(i)=-\alpha} \prod_{j\in N(r),\, j\neq i} \frac{\mathbb{1} + \beta(j)A_j}{2}.$$

Upon expanding the product, the sum over β makes all terms cancel except for the constant one and the $\prod_{j\neq i} x_j$ one, which survives as well due to the parity constraint on β. Therefore we arrive at the expression

$$\frac{\mathbb{1} + \alpha A_i}{2} + \frac{\mathbb{1} - \alpha A_i}{2} \cdot \frac{\mathbb{1} - \alpha(-1)^{b_r}\prod_{j\neq i} A_j}{2} = \frac{\mathbb{1} + \alpha A_i}{2} + \frac{\mathbb{1} - \alpha A_i}{2} \cdot \frac{\mathbb{1} - \alpha A_i}{2} = \mathbb{1}.$$

We finally show that these constructions are inverses of each other. Starting with a quantum solution $(A_i)_{i\in[n]}$, it is immediate to show that computing the resulting projections and using (4) results in the original A_i's. A similar statement holds for the $P_{v_i^\alpha}$ in the other direction, while a short computation is required to show the same for the $P_{w_r^\beta}$,

$$\prod_{i\in N(r)} \frac{1 + \beta(i)\cdot\left(P_{v_i^{+1}} - P_{v_i^{-1}}\right)}{2} = \prod_{i\in N(r)} \left(1 - P_{v_i^{-\beta(i)}}\right) = \prod_{i\in N(r)} \sum_{\beta'\,:\,\beta'(i)=\beta(i)} P_{w_t^{\beta'}} = P_{w_t^\beta},$$

where the last step again uses Fact 9. □

In [18, Section 8], we had considered the decision problem `ALLOWS_QUANTUM`, which asks: given a hypergraph $H = (V, E)$, does it have a quantum representation?[2] Our *inverse sandwich conjecture* hypothesized that this problem is undecidable. Thanks to Slofstra's Theorem 6, we are now in a position to prove this:

Corollary 11 (Inverse sandwich conjecture). *There is no algorithm to determine whether a given hypergraph has a quantum representation.*

Proof. If there was such an algorithm, then Lemma 10 would provide an algorithm to determine whether a given linear system has a quantum solution. This is in contradiction with Theorem 6. □

This also implies that there are hypergraphs that have quantum representations, but only in infinite Hilbert space dimension [18, Section 8]. Translating Slofstra's explicit example [15, Corollary 3.2] into a hypergraph using the prescription of Lemma 4 will provide an explicit (but large) example.

Remark 12. In terms of fancier language, one can phrase Lemma 10 as saying that the maximal group C*-algebra [25] of the solution group associated to the linear system is, after taking the quotient by the relation $J = -1$, computably isomorphic to a hypergraph C*-algebra. At the purely algebraic level of *$*$-algebras*, the analogous statement is still true with the same proof, but one needs to throw in the orthogonality relations of Fact 9 separately when defining the finitely presented $*$-algebra associated to a hypergraph.

Corollary 13. *There are infinitely many hypergraphs H for which $C^*(H)$ is not residually finite-dimensional.*

Proof. If every $C^*(H)$ was residually finite-dimensional, then we could use the algorithm of [26] to determine whether $\|1\| = 0$ or $\|1\| = 1$ in $C^*(H)$, which are the only two possibilities depending on whether $C^*(H) = 0$ (no quantum representation) or $C^*(H) \neq 0$ (there is a quantum representation). This contradicts Corollary 11. If there were only finitely many exceptions to this residual finite-dimensionality, then there would also have to exist an algorithm which simply treats these exceptional cases separately. □

As far as we know, this is the first time that the strategy of [26] has been successfully employed to show that some finitely presented C*-algebras are not residually finite-dimensional. We do not know *which ones* of these C*-algebras fail to be residually finite-dimensional, and in particular the decidability status of the problem "For given H, is $C^*(H)$ residually finite-dimensional?" is unclear.

In light of Remark 12, we also wonder:

Problem 14. *Is the word problem for the algebras*

$$\mathbb{Q}[H] := \left\langle P_v \,:\, v \in V \;\middle|\; P_v = P_v^2,\ \sum_{v \in e} P_v = \mathbb{1}\ \forall e \in E \right\rangle,$$

i.e. without imposing the relations of Fact 9, solvable? Uniformly solvable?

[2] The formulation of [18] asks for the existence of a quantum model on H, but this is clearly equivalent: the existence of a quantum model requires the existence of a quantum representation to begin with; conversely, one can use a quantum representation and an arbitrary state in its underlying Hilbert space to obtain a quantum model.

Consequences for quantum logic

The projection operators on a Hilbert space are in bijective correspondence with the closed subspaces. This lets us translate the observations of the previous section into statements about quantum logic.

Although we try to avoid too much jargon, it will be helpful to utilize the basic terminology of model theory [27]. We work in the signature $(\vee, \perp, 1)$, where $\perp$ is a binary relation; any orthomodular lattice can also be considered a structure of this signature due to Remark **??**. We follow [27] in using the shorthand notation $\bar{P} := P_1, \ldots, P_n$ for a list of variables, using notation which suggests that we are still thinking in terms of projections.

It is a standard fact that forming a partition of unity (2) at the level of projections is equivalent to the associated subspaces being pairwise orthogonal and spanning the entire space, which translates into the atomic formula

$$OC(\bar{P}) = OC(P_1, \ldots, P_n) := \operatorname*{\mathsf{And}}_{i \neq j} \, (\, P_i \perp P_j \,) \; \mathsf{and} \; (P_1 \vee \ldots \vee P_n = 1) \, , \tag{5}$$

where the notation 'OC' reminds us of *orthogonality* and *completeness*. All formulas that we use are built out of atomic formulas of this form. If we want to say that already a certain subset $\{P_i \,:\, i \in e\}$ of these projections indexed by $e \subseteq [n]$ forms a partition of unity, then we simply write $OC(\bar{P}_e)$.

THEOREM 15. *The theory of complex Hilbert lattices $\mathcal{C}(\mathcal{H})$ in the signature $(\vee, \perp, 0, 1)$ is undecidable: there is no algorithm to decide whether for a given hypergraph $H = (V, E)$, the formula*

$$\left(\operatorname*{\mathsf{And}}_{e \in E} OC(\bar{P}_e) \right) \; \mathsf{implies} \; \left(0 = 1\right) \tag{6}$$

holds for all V-indexed sets of projections $\{P_v\}_{v \in V}$ or not.

PROOF. This is now merely a restatement of Corollary 11: the tuples of projections in a Hilbert space $\mathcal{H}$ with $\dim(\mathcal{H}) > 0$ such that $\mathsf{And}_{e \in E} OC(\bar{P}_e)$ holds are precisely the quantum representations of H, while $0 = 1$ is equivalent to $\dim(\mathcal{H}) = 0$. □

Hence the undecidability already holds for the purely implicational fragment: upon unfolding all the $OC(\bar{P}_e)$'s that appear on the left of (6) via the definition (5), one obtains merely a conjunction of atomic formulas on the left-hand side of the implication.

Finally, we show that the universal theory of complex Hilbert lattices does not really depend on the Hilbert space.

LEMMA 16. *Let $\phi(\bar{P})$ be an atomic formula in the signature $(\vee, \perp, 0, 1)$.*

(a) *If $f : \mathcal{N} \to \mathcal{M}$ is a normal injective $*$-homomorphism between von Neumann algebras, then $\phi(f(P_1), \ldots, f(P_n))$ if and only if $\phi(P_1, \ldots, P_n)$.*
(b) *Let Φ be the universal sentence $\forall \bar{P} \; \phi(\bar{P})$. Then $\mathcal{H} \vDash \Phi$ for all Hilbert spaces $\mathcal{H}$ if and only if $\mathcal{H} \vDash \Phi$ for separable infinite-dimensional $\mathcal{H}$.*

Since there is only one infinite-dimensional separable Hilbert space up to isomorphism, this states equivalently that the universal theory of Hilbert spaces coincides with the universal theory of *one* particular Hilbert space, e.g. $\mathcal{H} = \ell^2(\mathbb{N})$. Therefore our undecidability result already holds at the level of a single such Hilbert space.

PROOF. (a) Induction on the complexity of ϕ. Normality of f is required for showing that $f(P_1 \vee P_2) = f(P_1) \vee f(P_2)$.

(b) The nontrivial direction is this: if $\mathcal{H} \vDash \Phi$ for separable infinite-dimensional $\mathcal{H}$, then also $\mathcal{H}' \vDash \Phi$ for any other Hilbert space $\mathcal{H}'$.

Case 1: $\mathcal{H}'$ is finite-dimensional. In this case, $\mathcal{H}' \otimes \mathcal{H}$ is isomorphic to $\mathcal{H}$, and therefore $\mathcal{H}' \otimes \mathcal{H} \vDash \Phi$. By assumption, we therefore know that $\phi(P_1 \otimes \mathbb{1}, \ldots, P_n \otimes \mathbb{1})$ for any tuple of projections $\bar{P}$ on $\mathcal{H}'$, and hence also $\phi(P_1, \ldots, P_n)$ by (a).

Case 2: $\mathcal{H}'$ is infinite-dimensional. In this case, any tuple of projections $\bar{P}$ generates a separable von Neumann algebra in $\mathcal{B}(\mathcal{H}')$. By separability, we can faithfully represent this von Neumann algebra on a separable Hilbert space, on which we know Φ to hold due to either the assumption or Case 1. We then conclude $\phi(P_1, \ldots, P_n)$ from two applications of (a). □

Remark 17. One can also conclude from our results that the theory of complex Hilbert lattices cannot be axiomatized recursively, as hypothesized in [**28**, p. 69], as follows. The lemma implies that the universal theory of Hilbert lattices is complete. Therefore, if one could enumerate the axioms recursively, then one would have a decision procedure for any universal sentence Φ by simply generating all axioms together with all their consequences until either Φ or $\neg\Phi$ has been derived.

Svozil [**28**, p. 69] also asks if it is possible to develop an axiomatization of Hilbert lattices in "purely algebraic" terms. If one interprets this as asking whether the Hilbert lattices are the class of models of some theory in first-order logic, then the answer is well-known to be negative: the Löwenheim-Skolem theorem asserts that it is impossible to axiomatize *any* uncountable structure in first-order logic. This holds irrespectively of whether one attempts to axiomatize Hilbert lattices in all Hilbert space dimensions or only in one particular dimension ≥ 2. The most that one can hope for is categoricity in the relevant cardinality, meaning that every model of the same cardinality as the intended model is isomorphic to the intended models. But there will always be models in other cardinalities as well.

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E-mail address: `fritz@mis.mpg.de`

Max Planck Institute for Mathematics in the Sciences, Leipzig, Germany