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Quantum Logic, State Space Geometry and Operator Algebras

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Abstract. The problem of characterising those quantum logics which can be identified with the lattice of projections in a JBW-algebra or a von Neumann algebra is considered. For quantum logics which satisfy the countable chain condition and which have no Type I_2 part, a characterisation in terms of geometric properties of the quantum state space is given.

Introduction

Quantum logics, as defined below, are σ-complete orthomodular lattices. They have been vigorously investigated in recent years. In most mathematical formulations of the foundations of quantum mechanics the lattice of "questions" associated with a physical system is a quantum logic.

Important examples of quantum logics are, in order of successive generalisation:

(a) The lattice of all closed subspaces of a separable Hilbert space.
(b) The lattice of all projections in a von Neumann algebra.
(c) The lattice of all projections in certain Jordan operator algebras known as JBW-algebras.

Characterisation of those quantum logics isomorphic to (a) have been obtained by Piron, in 1964, (see [8]), and by Wilbur [9], in 1977. Can one characterise those quantum logics isomorphic to the lattice of all projections in a von Neumann algebra, or in a JBW-algebra, by geometric properties of the quantum state space of a quantum logic?

We obtain a partial solution to this problem by restricting our attention to quantum logics which satisfy the countable chain condition and which have no Type I_2 part (see below for definitions). We show that, when Q is such a quantum logic, there are three geometric properties which will be satisfied by the quantum state space of Q if, and only if, Q is isomorphic to the lattice of all projections in a JBW-algebra.

We also, as a corollary, give a geometric characterisation of those orthomodular lattices which are isomorphic to the projection lattice of a countably decomposable von Neumann algebra with no Type I_2 direct summand.

Let L be an orthomodular lattice with orthocomplementation $x \mapsto x^\perp$. A *probability measure*, ϕ, on L is a non-negative real valued function, $\phi: L \to \mathbb{R}_+$, such that $\phi(0)=0$, $\phi(1)=1$, and if (x_n) is a sequence, in L, of mutually disjoint elements for which $\vee x_n$ exists, then $\phi(\vee x_n) = \Sigma \phi(x_n)$. (The sequence (x_n) is said to be *mutually disjoint* if $x_n \leqq x_m^\perp$, for every m, n, with $m \neq n$.) The set of all probability measures on L is a convex set which we shall denote by K_L.

The convex set K_L is said to be *strongly full* if the following three properties are satisfied.

(1) For $x, y \in L$ we have $x \leqq y$ if

$$\{\phi \in K_L : \phi(x) = 1\} \subseteq \{\phi \in K_L : \phi(y) = 1\}.$$

(2) Whenever $x, y \in L$ and $\phi \in K_L$ with $\phi(x) = \phi(y) = 1$, then $\phi(x \wedge y) = 1$.
(3) Whenever ϕ lies in a proper norm-exposed face of K_L, then $\phi(x) = 0$ for some non-zero element x of L. (A face F of K_L is said to be norm-exposed if there exists a bounded affine function, b, on K_L such that $b > 0$ on $K_L \backslash F$ and $b = 0$ on F.)

The orthomodular lattice, L, is said to satisfy the *countable chain conditions* (abbreviated c.c.c.) if every family of mutually disjoint elements in L is at most countable. It is said that L is a *quantum logic* if $\vee x_n$ exists in L whenever the sequence (x_n), of elements of L, is mutually disjoint. It is easy to see that a quantum logic which satisfies the c.c.c. is a complete orthomodular lattice.

Consider the orthomodular lattice L and let x be an element of L. The order interval, $L[0, x] = \{y \in L; y \leqq x\}$, is an orthomodular sublattice of L with the complementation $y \to x \wedge y^\perp$. The element x of L is said to be *abelian* if $L[0, x]$ is distributive. The elements y and z of L are said to *commute* if y and z generate a distributive sublattice of L. The set of all those elements of L which commute with every other element of L is said to be the *centre*, $Z(L)$, of L. It is said that L is *factor* if $Z(L) = \{0, 1\}$.

If L is a complete orthomodular lattice, then so is $Z(L)$ ([5], [8]) and, consequently, for each x in L we can define the *central support* of x in L:

$$c(x) = \wedge \{y \in Z(L); x \leqq y\} \in Z(L).$$

We say that the complete orthomodular lattice, L, has *Type I_2 part* if there exist, in L, disjoint non-zero abelian elements x, y such that $x \vee y = c(x) = c(y)$; is disjoint abelian elements x, y can be chosen so that $x \vee y = c(x) = c(y) = 1$, then L is said to be of *Type I_2*.

Recall that the convex set, F, is said to be *spectral* if it is the base of a base-norm space, (V, F), and (V, F) is in spectral duality (see [1, Sects. 6, 7]) with $A^b(F) \simeq V^*$, where $A^b(F)$ represents the bounded affine functions on F. The spectral convex set F is *elliptic* if $P(Q - Q')P' = 0$ for all P-projections P, Q of $A^b(F)$ (Q' represents the quasi-complement of the P-projection Q, [1]).

Iochum and Schultz [6] have shown that a convex set is (affinely isomorphic to) the normal state space of a JBW-algebra if and only if it is spectral and elliptic.

Theorem. *Let L be a quantum logic satisfying the countable chain condition with no Type I_2 part. Then L is isomorphic to the lattice of all projections in a* JBW*-algebra if and only if K_L is strongly full, spectral and elliptic.*

Proof. Suppose that $K = K_L$ is strongly full, spectral and elliptic. Given an element x in L, define the element $\hat{x}$ of $A^b(K)^+$, by $\hat{x}(\lambda\phi + (1-\lambda)\psi) = \lambda\phi(x) + (1-\lambda)\psi(x)$, $\lambda \in [0,1]$, ϕ, $\psi \in K$. Notice then that condition (1) in the definition of strongly full implies that the map $L \to \hat{L} = \{\hat{x} : x \in L\}$ is an order isomorphism and that $\hat{L}$ is an orthomodular lattice, isomorphic to L, with the lattice operations defined by $\hat{x} \vee \hat{y} = (x \vee y)\hat{}$, $\hat{x} \wedge \hat{y} = (x \wedge y)\hat{}$, $(\hat{x})^\perp = (x^\perp)\hat{}$. We may therefore suppose that L is contained in $A^b(K)^+$ and that $x = \hat{x}$, for each x in L. Observe that (with the above identification) 1 is the order unit of $A^b(K)$, and that for x, y in L,

$$x^\perp = 1 - x; x \vee y = x + y, \quad \text{if } x \leqq y^\perp; x \wedge y^\perp = x - y, \quad \text{if } y \leqq x.$$

Furthermore, since K is spectral and elliptic, we can identify $A^b(K)$ with a JBW-algebra, M, which has normal state space K, by the result of Iochum and Schultz, [6, Theorem 1.5], mentioned above.

Let $\phi \in K$. The condition (2), in the definition of strongly full, implies that the set $\{x \in L; \phi(x) = 1\}$ is downward directed. Since L is a complete lattice, this means that $\phi(s_L(\phi)) = 1$, where $s_L(\phi) = \wedge \{x \in L; \phi(x) = 1\}$, the support of ϕ in L. Let $(\phi_i)_{i \in I}$ be a maximal family in K for which the $s_L(\phi_i)$ are mutually disjoint. If I is infinite, then we can take $I = \mathbb{N}$, since L satisfies the c.c.c.. It is easy to check that condition (1) implies that for each non-zero y in L there exists ψ in K such that $\psi(y) = 1$. It follows from this that $\Sigma s_L(\phi_n) = 1$. Now with $\phi = \Sigma(1/2^n)\phi_n$, we see that $s_L(\phi) = 1$ (when I is finite, the proof of the existence of such ϕ is similar). Observe now that condition (3) implies that ϕ is a faithful normal state on M. In addition, since a JBW-algebra has a faithful normal state if and only if its lattice of projections satisfies the c.c.c. (the proof is similar to the usual W^*-proof, see [7, II.3.19]), it follows that every projection in M is the support projection of some normal state on M.

The range projection, $r(a)$, of an element a in M^+ is the projection in M which is the unit element of the hereditary JBW-subalgebra of M generated by a. We note that for each x in L, $1 - r(1-x) \leqq x \leqq r(x)$. In addition, since $\phi(a) = 0$ if and only if $\phi(r(a)) = 0$, $a \in M^+$, $\phi \subset K$, it follows from (1) that the following three conditions are equivalent for elements x, y of L: (i) $x \leqq y$; (ii) $r(x) \leqq r(y)$; (iii) $x \leqq r(y)$. In particular, for x in L, $r(x) = 1$ if and only if $x = 1$.

Let now p belong to $\mathbb{P}$, the projection lattice of M, such that $p \neq 0, 1$. By the above remarks, there exist ϕ in K such that $p = s(\phi)$—the support projection of ϕ in $\mathbb{P}$. Since ϕ then lies in a proper norm exposed face of K, there exists x in L, $x \neq 1$, with $\phi(x) = 1$. But then, $\phi(1 - r(1-x)) = 1$, and so $p = s(\phi) \leqq 1 - r(1-x) \leqq x$.

It follows that $L \subseteq \mathbb{P}$. Indeed, let $y \in L$. Then $r(y)\ r(1-y) \neq 1$. Suppose that $q = r(y)r(1-y) \neq 0$. Then, it follows from the preceding paragraph that there exist a non-zero x in L such that $x \leqq q \leqq r(y)$, $r(1-y)$. Consequently, $x \leqq y$ and $x \leqq 1 - y$, implying that $x = 0$, a contradiction. Hence, $r(y)r(1-y) = 0$, which implies that $y(1-y) = 0$. Therefore y is a projection.

Let $\underline{\vee}$ denote lattice suprema in $\mathbb{P}$. Then, for x, y in L, $x \,\underline{\vee}\, y \leqq x \vee y$. On the other hand, given ϕ in K, if $\phi(x \,\underline{\vee}\, y) = 0$, then $\phi(x) = \phi(y) = 0$, so $\phi(x \vee y) = 0$, by (2). Hence, by (1), $x \vee y \leqq x \,\underline{\vee}\, y$. It follows that L is a sublattice of $\mathbb{P}$.

Finally, since, as we have seen, given $p \in \mathbb{P}$, $p \neq 0$, there exist non-zero elements of L dominated by p, we can choose a maximal mutually disjoint family $\{x_\alpha\}$ in L dominated by p. Then, for each ϕ in K_L, $\phi(\vee x_\alpha) = \Sigma \phi\{x_\alpha\} = \phi(\Sigma x_\alpha)$. So, $\vee x_\alpha = \Sigma x_\alpha \leqq p$ and hence $\vee x_\alpha = p$, by maximality. Consequently, $L = \mathbb{P}$.

To obtain the converse we shall make essential use of the results of [3] which generalizes the work of Christensen [4] and Yeadon [10, 11].

Let us now suppose that L is isomorphic to the projection lattice $\mathbb{P}$, of a JBW-algebra, M. Then $K = K_L$ is affinely isomorphic to the set of probability measures (as defined here) on $\mathbb{P}$, also denoted by K. From [3, Lemma 3.5(iii)], for example, we see that M has no Type I_2 direct summand. Therefore, since $\mathbb{P}$ satisfies the c.c.c., it follows from [3, Corollary 5.5] that K can be identified with the normal state space of M. Therefore, as can be seen from the results of [1], K is strongly full for $\mathbb{P}$ and, by [6, Theorem 1.5], we know that K is spectral and elliptic. This completes the proof.

By [6, Theorem 2.9], the normal state space of a JBW-algebra M has the global 3-ball property if, and only if, M is the self-ajoint part of a von Neumann algebra.

This observation and the above theorem gives the following corollary.

Corollary. *A quantum logic L which satisfies the countable chain condition and with no Type I_2 part is isomorphic to the projection lattice of a von Neumann algebra if and only if K_L is strongly full, spectral, elliptic and has the global 3-ball property.*

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