Quantum M-ary Phase Shift Keying

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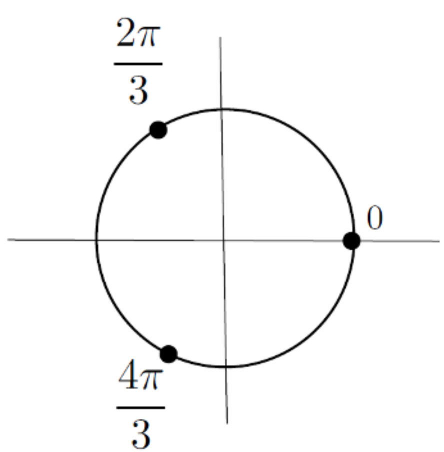
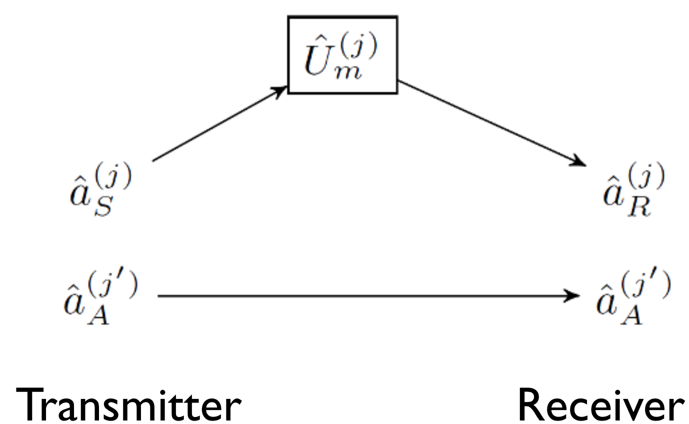
Quantum Phase Shift Keying

- Ordinary M-PSK -- Carrier signal is phase-modulated by one of M uniformly spaced phase shifts.
- Quantum-optical M-PSK
  - RF carrier → Spatiotemporal complex field mode $\mathcal{E}(\boldsymbol{\rho}, t)$ at optical frequency with associated annihilation operator $\hat{a}$; $[\hat{a}, \hat{a}^\dagger] = 1$.
  - Number operator $\hat{N} = \hat{a}^\dagger \hat{a}$; proportional to energy for quasi-monochromatic fields
  - Phase shift unitary operator $\hat{U}_\theta = e^{i\theta\hat{N}}$; M uniformly spaced phase shifts
  - Allow multiple transmitted (signal) modes $\{\hat{a}_S^{(j)}\}_{j=1}^{J}$ and ancilla modes $\{\hat{a}_A^{(j')}\}_{j'=1}^{J'}$ for pre-shared entanglement between transmitter and receiver



Phase shifts for M=3



Applications

- Communication

  [M. J. W. Hall and I. G. Fuss, Quant. Opt. **3**, 147 (1991)]

  - Appreciable loss
- Phase sensing
  - Low to moderate loss
  - Entanglement-assisted sensing feasible
- Reading a phase-encoded digital memory

  [O. Hirota, e-print arXiv:1108.4163 (2011)]

  - M=2 (bits)
  - Low to moderate loss; entanglement-assisted readout feasible

Notations & Problem Setup (I)

- J signal (S) modes, J' ancilla (A) modes
- General pure transmitter state:

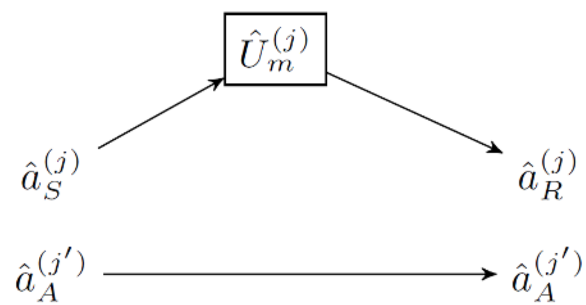
$$|\psi\rangle_{AS} = \sum_{\mathbf{k},\mathbf{n}} c_{\mathbf{k},\mathbf{n}} |\mathbf{k}\rangle_A |\mathbf{n}\rangle_S,$$

  with $|\mathbf{k}\rangle_A = |k_1\rangle \otimes \cdots \otimes |k_{J'}\rangle$ & $|\mathbf{n}\rangle_S = |n_1\rangle \otimes \cdots \otimes |n_J\rangle$ multimode ancilla and signal number states.

- For a (uniformly distributed) message $m \in \mathbb{Z}_M$, and $\theta_M = 2\pi/M$, the corresponding received states are

$$|\psi_m\rangle = \hat{V}^m |\psi\rangle,$$
$$\hat{V}^M = \hat{I},$$

  where $\hat{V} = \hat{I}_A \otimes \bigotimes_{j=1}^{J} e^{i\theta_M \hat{N}_S^{(j)}}$



- The received states form a symmetric set.

Notations & Problem Setup (II)

- For a given transmitter, the minimum error probability achievable at the receiver is

$$\overline{P}_e = 1 - \frac{1}{M} \max_{\{\hat{E}_m\}} \sum_{m=0}^{M-1} \mathrm{tr}\left(|\psi_m\rangle_{AR}\langle\psi_m| \hat{E}_m \right),$$

optimization over all POVM's.

- Signal energy constraint $\langle \hat{N}_S \rangle \equiv \left\langle \sum_{j=1}^{J} \hat{N}_S^{(j)} \right\rangle \leq N_S$.

$$\begin{aligned} \langle \hat{N}_S \rangle &= \sum_{\mathbf{k},\mathbf{n}} (n_1 + \cdots + n_J)\, |c_{\mathbf{k},\mathbf{n}}|^2 \\ &\equiv \sum_{\mathbf{n}} (n_1 + \cdots + n_J)\, p_{\mathbf{n}} \\ &\equiv \sum_{n=0}^{\infty} n\, p_n, \end{aligned}$$

p_n : p.m.f. of signal photon number

- Definition $\mathfrak{p} \equiv (\mathfrak{p}_0, \ldots, \mathfrak{p}_\nu, \ldots, \mathfrak{p}_{M-1})$

$$\mathfrak{p}_\nu := \sum_{n\,:\, n \equiv \nu \pmod{M}} p_n.$$

- Until further notice, we limit discussion to pure-state transmitters.

For a given N_S, we seek the transmitter state yielding minimum error probability.

Characterization Theorem (CT)

Theorem 1. *Pure transmitters with the same $\mathfrak{p}$ have the same error performance in M-ary PSK. This statement encompasses transmitters with differing J and/or J'.*

Proof sketch: Received states $\{|\psi_m\rangle_{AR}\}_{m=0}^{M-1}$:

$$|\psi_m\rangle_{AR} = \sum_{\mathbf{k},\mathbf{n}} c_{\mathbf{k},\mathbf{n}} e^{im\theta_M(n_1+\cdots+n_J)} |\mathbf{k}\rangle_A |\mathbf{n}\rangle_R.$$

Performance completely determined by the Gram matrix of the states :

$$\begin{aligned} G_{mm'} &= {}_{AR}\langle\psi_m|\psi_{m'}\rangle_{AR} \\ &= \sum_{\mathbf{n}} p_{\mathbf{n}}\, e^{-i\theta_M(m-m')(n_1+\cdots+n_J)} \\ &= \sum_{n=0}^{\infty} p_n\, e^{-i\theta_M(m-m')n} \\ &= \sum_{\nu=0}^{M-1} \mathfrak{p}_\nu\, e^{-i\theta_M(m-m')\nu}. \end{aligned}$$

Immediate Consequences of CT

- Since $\mathfrak{p}$ is a function of the *signal* photon p.m.f. alone, any given $\mathfrak{p}$ can be realized using a signal-only state, i.e., entanglement with ancillas is unnecessary.
  - Contrasts with general situation in distinguishing finite-dimensional unitaries and CP-maps:

    [G. M. D'Ariano, P. Lo Presti, and M. G. A. Paris, Phys. Rev. Lett. **87**, 270404 (2001)]
    [M. F. Sacchi, Phys. Rev. A **71**, 062340 (2005)]
- Since any given $\mathfrak{p}$ can be realized using a *single-mode* signal state, $J=1$ is sufficient.
  - Contrasts with general situation in which multiple applications of unitaries helps in their discrimination:

    [A. Acín, Phys. Rev. Lett. **87**, 177901 (2001)]

Theorem 2. *(a) For* $N_S < (M-1)/2$*, a single-mode transmitter state of the form* $|\psi\rangle_S = \sum_{\nu=0}^{M-1} \sqrt{\mathfrak{p}_\nu}\,|\nu\rangle_S$ *with* $\mathfrak{p}_\nu \geq 0$ *achieves the minimum error probability.*
(b) For $N_S \geq (M-1)/2$*, the uniform superposition state* $|\psi\rangle_S = \frac{1}{\sqrt{M}}(|0\rangle_S + \cdots + |M-1\rangle_S)$ *achieves zero error probability.*

Proof sketch:

(a) By CT, we consider only single-mode states. Then, optimum use of available energy is to concentrate probability on low photon numbers.

(b) Corresponding received states comprise the (orthonormal) Fourier basis.

Theorem 3. *(a) Among all transmitter states satisfying $\langle \hat{N}_S \rangle \leq N_S < (M-1)/2$, the minimum error probability is achieved by a single-mode state with $\mathfrak{p}$ given by*

$$\mathfrak{p}_\nu^{\mathsf{opt}} = \frac{1}{(A+\nu B)^2}, \ \nu \in \mathbb{Z}_M, \tag{21}$$

where A, B are positive constants chosen to satisfy the constraints

$$\sum_{\nu=0}^{M-1} \mathfrak{p}_\nu = 1, \quad \sum_{\nu=0}^{M-1} \nu\, \mathfrak{p}_\nu = N_S. \tag{22}$$

(b) Any transmitter state achieving zero-error discrimination must have $\mathfrak{p} = (1/M, \ldots, 1/M)$ and signal energy greater than or equal to $(M-1)/2$.

Proof sketch:

(a) Error probability of optimal (Square-root) measurement known to be:

$$\overline{P}_e = 1 - \frac{1}{M^2}\left(\sum_{m=0}^{M-1} \sqrt{\lambda_m}\right)^2,$$

Proof sketch (Contd)

where $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_{M-1})$ is an ordered eigenvalue vector of the Gram matrix, given by the Fourier transform of the first row $\mathbf{G}_0 \equiv \{G_{0m}\}$ of the Gram matrix:

$$\boldsymbol{\lambda} = \mathcal{F}[\mathbf{G}_0]$$

Recall that:

$$G_{mm'} = \sum_{\nu=0}^{M-1} \mathfrak{p}_\nu \, e^{-i\theta_M (m-m')\nu}$$

so that

$$\mathbf{G}_0 = M \cdot \mathcal{F}^{-1}[\mathfrak{p}]$$

Therefore, $\boldsymbol{\lambda} = M\,\mathfrak{p}$ and $\overline{P}_e = 1 - \frac{1}{M}\left(\sum_{m=0}^{M-1} \sqrt{\mathfrak{p}_m}\right)^2.$

Result follows from constrained optimization over $\mathfrak{p}$.

(b) Calculation.

Mixed-state transmitters

Theorem 4. *Let ρ_{AS} be a mixed state with ensemble decomposition $\rho_{AS} = \sum_j \pi_j |\psi_j\rangle_{AS\,AS}\langle\psi_j|$ and with signal energy $\mathrm{tr}(\rho_{AS}\hat{N}_S) \leq N_S$. A transmitter preparing the ensemble $\{|\psi_j\rangle_{AS}\}$ with probabilities $\{\pi_j\}$ and a receiver making optimal measurements conditioned on knowledge of j cannot beat the performance of the state of Theorem 3.*

Proof sketch:

$$\overline{P}_e[\rho_{AS}] \geq \sum_j \pi_j \overline{P}_e[|\psi_j\rangle_{AS}]$$ (Knowledge of *j* cannot hurt)

$$= \sum_j \pi_j \overline{P}_e[|\psi_j^*\rangle_S]$$ (States have the same $\mathfrak{p}$)

$$\geq \overline{P}_e[|\overline{\psi}\rangle_S]$$ (Concavity of $\overline{P}_e$ in $\mathfrak{p}$)

$$\geq \overline{P}_e[|\psi^{\text{opt}}\rangle_S] \,.$$ (Definition of optimal state)

$|\psi_j^*\rangle_S$:Theorem 2 state with the same $\mathfrak{p}$ as $|\psi_j\rangle_{AS}$

$|\overline{\psi}\rangle_S$:Theorem 2 state with $\mathfrak{p} = \sum_j \pi_j \mathfrak{p}_j$

$|\psi^{\text{opt}}\rangle_S$: Optimum Theorem 3 state of energy N_S

Measurement operators of SRM

- The optimal square-root measurement consists of rank-one POVM elements $\hat{\Pi}_m = |\chi_m\rangle_{AR}\langle\chi_m|$, with

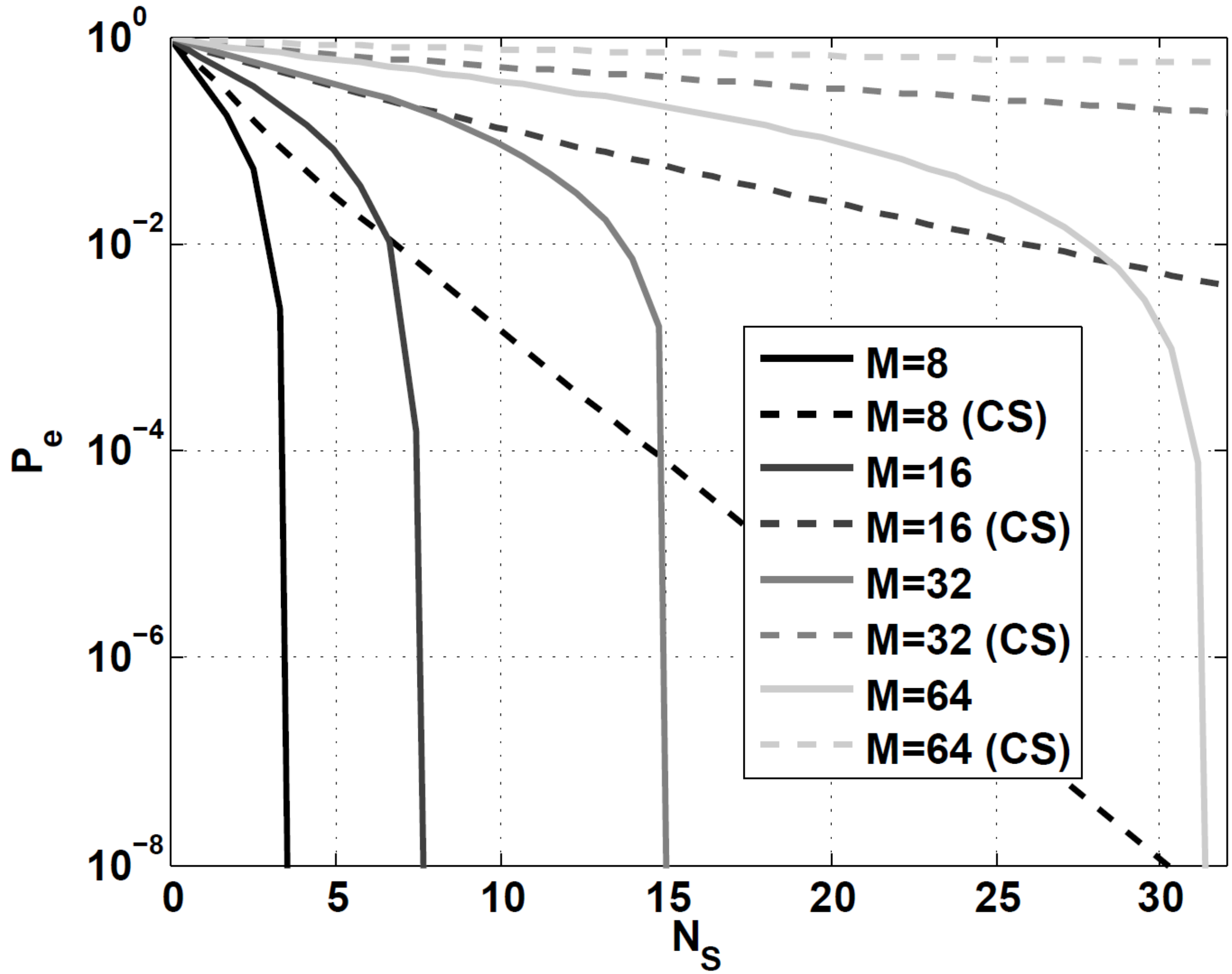
$$|\chi_m\rangle_{AR} = \left(\sum_{n=0}^{M-1} |\psi_n\rangle_{AR}\langle\psi_n| \right)^{-1/2} |\psi_m\rangle_{AR}$$

- For the optimal state of Theorem 3, the POVM is a von Neumann measurement of the Pegg-Barnett unitary phase operator, i.e., the QFT on $\mathrm{span}\,\{|0\rangle_R, \ldots, |M-1\rangle_R\}$ with the measurement vectors:

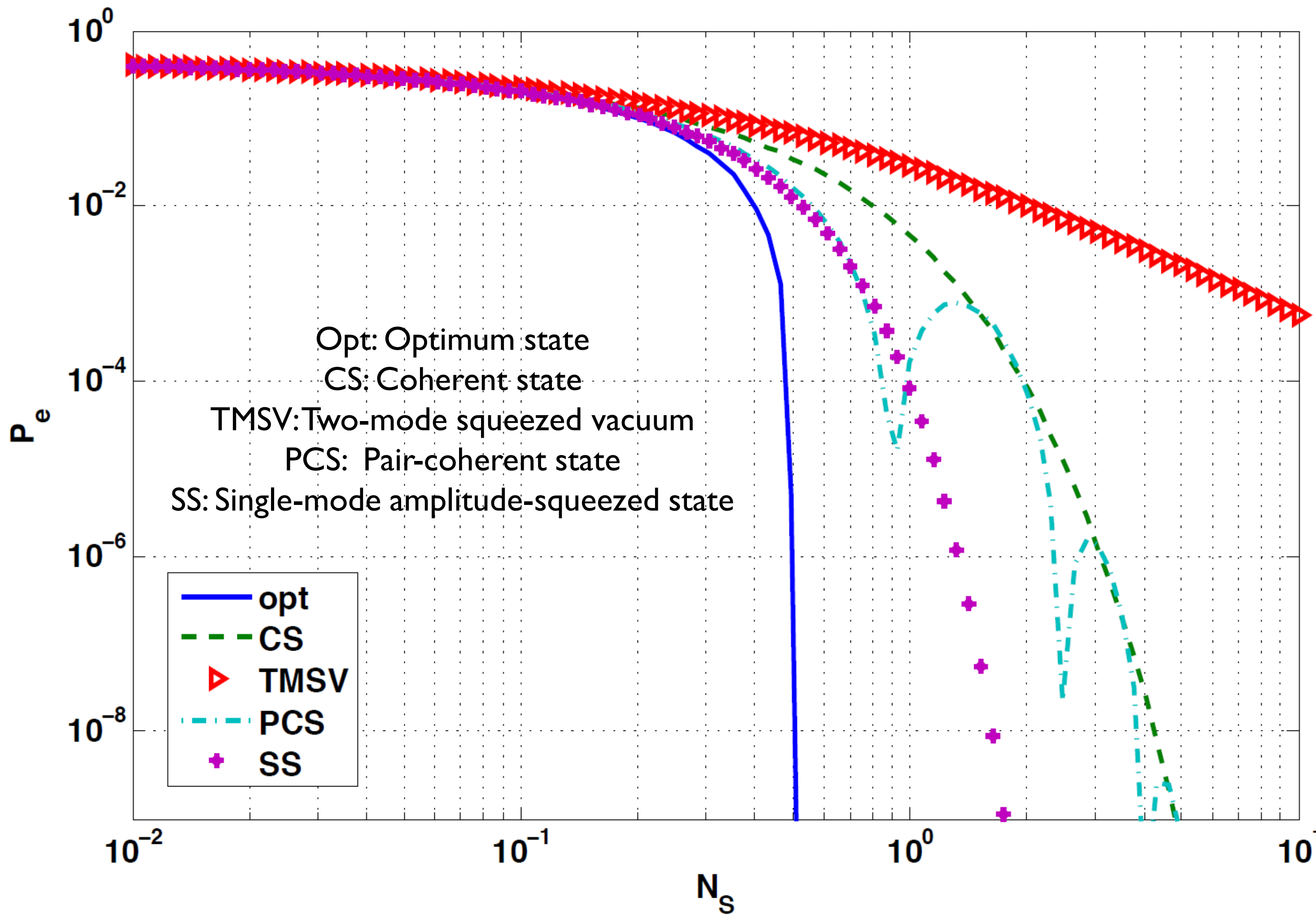
$$|\chi_m\rangle_R = \frac{1}{\sqrt{M}} \sum_{n=0}^{M-1} e^{imn\theta_M} |n\rangle_R$$

- No practical realization of this measurement is known.

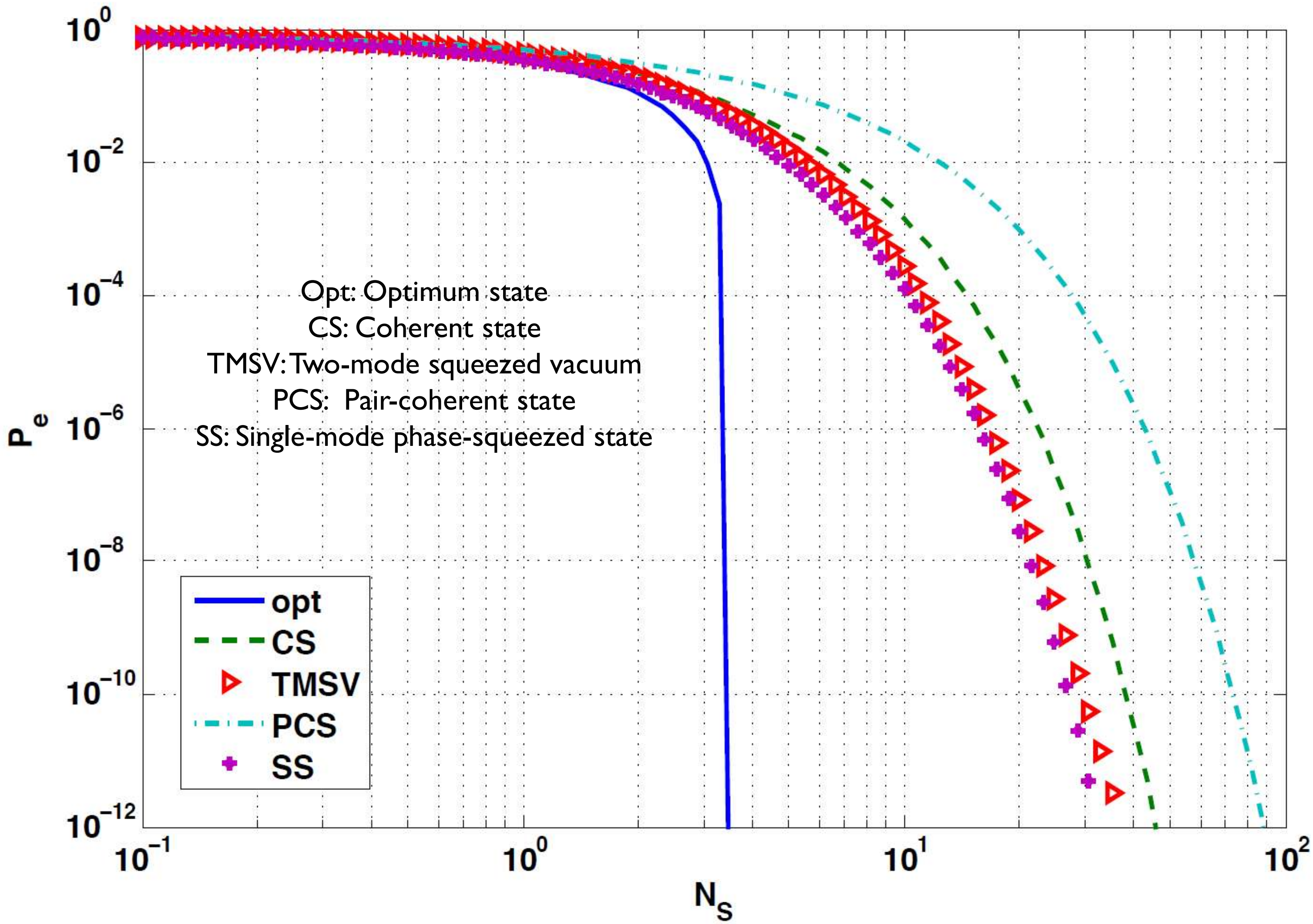
Optimum performance vs. Coherent state performance



Performance of some standard states (M=2)

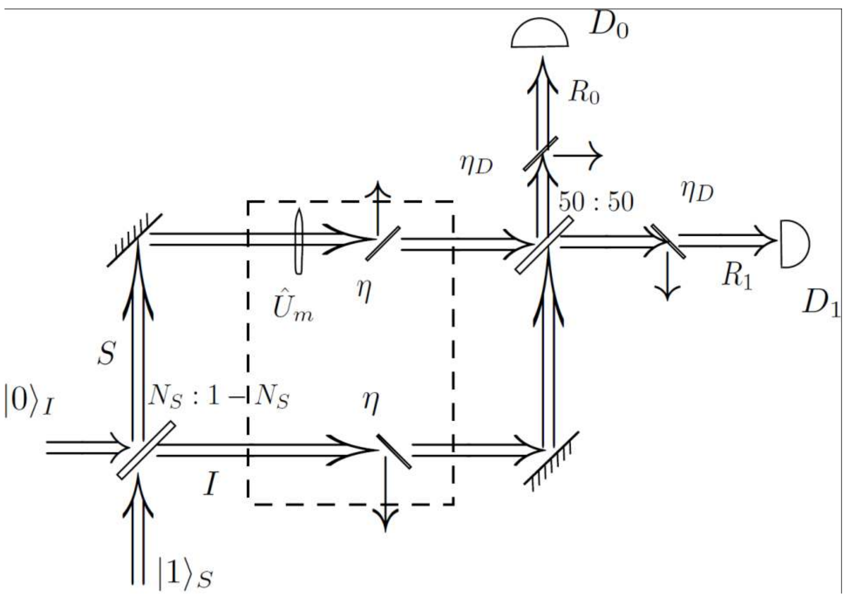


Performance of some standard states (M=8)



Optimum binary state

- For $N_S < 1/2$ the optimum transmitter state is
$$|\psi\rangle = \sqrt{1-N_S}\,|0\rangle_S + \sqrt{N_S}\,|1\rangle_S.$$
- Achievable error probability
$$\overline{P}_e^{\text{binary}} = 1/2 - \sqrt{N_S(1-N_S)}.$$
- An implementation of the optimal performance using linear optics and single-photon sources:



- Even with loss, the error probability conditioned on no erasure is optimal.

Conclusion and Outlook

- We have studied a natural generalization of phase-based communication in quantum optics.
- We have characterized and obtained the optimum transmitter states and performance under a signal energy constraint.
- We have obtained a realizable implementation of the binary case.
- For general M, both transmitter preparation and the required POVM measurement appear to be hard to implement.
- Performance bounds under realistic limitations including loss are desirable.

- Reference: eprint arxiv.org/1206.0673