QUANTUM MARGINAL PROBLEM AND REPRESENTATIONS OF THE SYMMETRIC GROUP

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To 60 anniversary of Alain Lascoux

Abstract. We discuss existence of mixed state ρ_{AB} of two (or multy-) component system $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ with reduced density matrices ρ_A, ρ_B and given spectra $\lambda_{AB}, \lambda_A, \lambda_B$. We give a complete solution of the problem in terms of linear inequalities on the spectra, accompanied with extensive tables of marginal inequalities, including arrays up to 4 qubits. In the second part of the paper we pursue another approach based on reduction of the problem to representation theory of the symmetric group.

Contents

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1. Introduction

The quantum marginal problem is about relations between spectrum of mixed state ρ_{AB} of two (or multi) component system $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and that of reduced states ρ_A and ρ_B. Relations of this type, for example, impose certain restrictions on manipulations with qubits in quantum information theory. The problem can be stated in plain language as follows. Let $M = [m_{ijk}]$ be cubic complex matrix and H_1, H_2, H_3 be Gram matrices formed by Hermitian dot products of parallel slices of M. We seek for relations between spectra of these matrices.

We pursue two different approach to this problem. The first is based on reduction to general Berenstein-Sjamaar theorem [5] applied to subgroup $\mathrm{SU}(\mathcal{H}_A) \times SU(\mathcal{H}_B) \subset \mathrm{SU}(\mathcal{H}_A \otimes \mathcal{H}_B)$. The relevant geometry and combinatorics are explained in section 4.1. Theorems 4.1.1 and 4.1.3 give a pretty explicit ansatz for producing quantum marginal inequalities. For systems of rank ≤ 4 they are given in Appendix. It covers all system with few hundreds, rather then thousands, marginal inequalities. An important case of an array of qubits considered separately in section 4.2. Modulo a "standard" conjecture the marginal inequalities can be produced in a very straightforward way, see Theorem 4.2.3.

The number of marginal inequalities increases drastically with rank of the system. This makes the above solution inefficient for systems of big rank. In section 5 we develop another approach based on reduction of the marginal problem to decomposition of tensor product of irreducible representations of the symmetric group, see Theorem 5.3.1. This approach, for example, allows answer questions about maximal eigenvalue of a mixed state with given margins, see Theorem 6.3.1, or its rank, see Theorem 6.4.1. However the main point here is not in new results, but in new vision, based on connections between apparently very remote subjects. In last section 7 we consider some applications of the above correspondence to back to representation theory of the symmetric group.

It is my pleasure to dedicate the paper to Alain Lascoux, who was one of the inventors and main contributor to theory of Schubert polynomials [29] and gave the first known combinatorial description of tensor product for a class of irreducible representations of the symmetric group [28]. Both topics turn out to be entangled in a surprising way with quantum marginal problem.

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2. Classical marginal problem

The classical marginal problem is about existence of a "body" or probability density $p_I(x_I) := p(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^I$, $I = \{1, 2, \ldots, n\}$ with given projections onto some coordinate subspaces $\mathbb{R}^J \subset \mathbb{R}^I$, $J \subset I$

$$p_J(x_J) = \int_{\mathbb{R}^{\bar{J}}} p_I(x_I) dx_{\bar{J}}, \quad \bar{J} = I \backslash J$$

called *margins* of p_I. The problem has a long history, see [23] and references therein. Here we give only few relevant examples.

2.0.1. *Univariant margins $p_i(x_i)$ are always compatible.* Indeed consider x_i as independent variables and define joint distribution by equation

$$p_I(x_I) = \prod_{i\in I} p_i(x_i).$$

2.0.2. The following inequality is necessary for compatibility of bivariant margins

$$\langle x_i|x_i\rangle + \langle x_j|x_j\rangle + \langle x_k|x_k\rangle + 2\langle x_i|x_j\rangle + 2\langle x_j|x_k\rangle + 2\langle x_k|x_i\rangle \geq 0.$$

Indeed LHS is equal to variance $\langle x_i + x_j + x_k \mid x_i + x_j + x_k\rangle \geq 0$ *provided* joint distribution p_{ijk} exists.

2.0.3. So called *Bell's ineqalities* in quantum mechanics are just compatibility conditions for marginal distributions corresponding to *commuting* observables [23].

2.0.4. *Discrete version of the marginal problem* is about existence of, say, cubic matrix $p_{ijk} \geq 0$ with given projections onto its facets

$$a_{ij} = \sum_k p_{ijk}, \quad b_{jk} = \sum_i p_{ijk}, \quad c_{ki} = \sum_j p_{ijk}.$$

Compatibility conditions for such projections are still unknown (so called *Planar Transport Polytope Problem*, see [49]).

2.0.5. *Restricted marginal problem* is about existence of a matrix with prescribed content, e.g. $0-1$, and given margins. In this case even univariant marginal problem becomes nontrivial. As an example recall the following classical result.

Theorem (Gale [13, 14], Ryser [42, 14])**.** *Partitions λ, μ are margins of a rectangular $0-1$ matrix iff $\lambda \prec \mu^t$.*

Here marginal values $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0$ arranged in decreasing order are treated as *Young diagram* λ with i-th row of length λ_i, μ^t stands for transpose diagram, and the *majorization* or *dominance* order $\lambda \prec \mu^t$ is defined by inequalities

$$\begin{aligned} \lambda_1 &\leq \mu_1^t \\ \lambda_1 + \lambda_2 &\leq \mu_1^t + \mu_2^t \\ \lambda_1 + \lambda_2 + \lambda_3 &\leq \mu_1^t + \mu_2^t + \mu_3^t \\ &\cdots\cdots\cdots \end{aligned}$$

The number of $0-1$ matrices with margins λ, μ is equal to the number of pairs of tableaux of conjugate shape and contents λ, μ, see [35].

We'll deal with quantum version of this problem below.

3. Quantum marginal problem

3.1. **Quantum margins.** A background of a quantum system A is Hilbert space $\mathcal{H}_A$ called *state space*. We'll consider only *finite systems*, for which $\dim \mathcal{H}_A < \infty$. Actual state of the system is described by unit vector $\psi \in \mathcal{H}_A$ for *pure state* or by non negative Hermitian operator $\rho : \mathcal{H}_A \to \mathcal{H}_A, \operatorname{Tr} \rho = 1$, called *density matrix*, for *mixed state*. Pure state ψ corresponds to projection operator $|\psi\rangle\langle\psi|$ onto ψ. Hence

$$\rho = \text{pure} \Longleftrightarrow \operatorname{rk} \rho = 1 \Longleftrightarrow \operatorname{Spec} \rho = (1, 0, 0 \cdots, 0). \tag{3.1}$$

An observable is Hermitian operator $X_A : \mathcal{H}_A \to \mathcal{H}_A$. Taking measurement of X_A while system is in state ρ produces random quantity $x_A \in \operatorname{Spec} X_A$ implicitly determined by expectations

$$\langle f(x_A)\rangle_\rho = \operatorname{Tr}(\rho f(X_A)) = \langle\psi|f(X_A)|\psi\rangle$$

of arbitrary function $f(x)$ on $\operatorname{Spec} X_A$. The second equation holds for pure state ψ.

Superposition principle of quantum mechanics implies that state space of composite system AB splits into tensor product

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$$

of state spaces of the components, as opposed to direct product $P_{AB} = P_A \times P_B$ of configuration spaces in classical mechanics. Density matrix of composite system can be written as linear combination

$$\rho_{AB} = \sum_\alpha a_\alpha L_A^\alpha \otimes L_B^\alpha \tag{3.2}$$

where L_A^α, L_B^α are linear operators in $\mathcal{H}_A, \mathcal{H}_B$ respectively. Its *reduced matrices* or *marginal states* are defined by equations

$$\rho_A = \sum_\alpha a_\alpha \operatorname{Tr}(L_B^\alpha) L_A^\alpha := \operatorname{Tr}_B(\rho_{AB}), \tag{3.3}$$

$$\rho_B = \sum_\alpha a_\alpha \operatorname{Tr}(L_A^\alpha) L_B^\alpha := \operatorname{Tr}_A(\rho_{AB}). \tag{3.4}$$

They can be characterized by the following property

$$\langle X_A \rangle_{\rho_{AB}} = \operatorname{Tr}(\rho_{AB} X_A) = \operatorname{Tr}(\rho_A X_A) = \langle X_A \rangle_{\rho_A}, \quad X_A : \mathcal{H}_A \to \mathcal{H}_A, \tag{3.5}$$

which tells that *observation of subsystem A gives the same results as if A would be in reduced state $\rho_A = \operatorname{Tr}_B(\rho_{AB})$*. This justifies the terminology.

Margins of state ρ_I of multicomponent system

$$\mathcal{H}_I = \bigotimes_{i \in I} \mathcal{H}_i = \mathcal{H}_J \otimes \mathcal{H}_{\bar{J}}, \quad J \subset I, \bar{J} = I \backslash J$$

are defined in a similar way: $\rho_J = \operatorname{Tr}_{\bar{J}}(\rho_I)$.

Example 3.1.1. In tensor algebra the above reduction $\rho_I \mapsto \rho_J, J \subset I$ is known as *contraction*. Most mathematicians are familiar with this procedure from differential geometry, where, say, Ricci curvature $\operatorname{Ric} : \mathcal{T} \to \mathcal{T}$ is defined as contraction of Riemann curvature $R : \mathcal{T} \otimes \mathcal{T} \to \mathcal{T} \otimes \mathcal{T}$ (here $\mathcal{T}$ stands for tangent bundle).

Example 3.1.2. Let's identify pure state of two component system

$$\psi = \sum_{ij} \psi_{ij} \alpha_i \otimes \beta_j \in \mathcal{H}_A \otimes \mathcal{H}_B$$

with its matrix $[\psi_{ij}]$ in orthonormal bases α_i, β_j of $\mathcal{H}_A, \mathcal{H}_B$. Then margins of ψ in respective bases are given by matrices

$$\rho_A = \psi^\dagger \psi, \quad \rho_B = \psi \psi^\dagger. \tag{3.6}$$

In striking difference with classical case margins of a pure quantum state are mixed ones (provided $\psi \neq \psi_A \otimes \psi_B$).

Example 3.1.3. A similar description holds for multicomponent systems. For example, write orthonormal components of tensor $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ into a cubic matrix $[\psi_{ijk}]$. Then univariant margins of ψ are given by *Gram matrices* formed by Hermitian dot products of parallel slices of $[\psi_{ijk}]$. It follows that $\operatorname{rk} \rho_C \leq \operatorname{rk} \rho_A \cdot \operatorname{rk} \rho_B$, because ρ_C can be written as Gram matrix of the slices of dimension $\operatorname{rk} \rho_A \cdot \operatorname{rk} \rho_B$. This inequality is a simplest manifestation of general problem about relations between margins of a pure state, which we address below.

3.2. **Marginal problem.** *General quantum marginal problem* is about existence of mixed state ρ_I of composite system

$$\mathcal{H}_I = \bigotimes_{i\in I} \mathcal{H}_i$$

with given margins ρ_J for some $J \subset I$ (cf. with classical settings $n^\circ 2$).

Additional restrictions on state ρ_I may be relevant. Here we consider only two variations:

- *Pure marginal problem*

corresponding to pure state ρ_I, and more general

- *Spectral marginal problem*

corresponding to a state with given spectrum $\lambda_I = \operatorname{Spec} \rho_I$.

Both versions are nontrivial even for univariant margins (cf. with Gale–Ryser theorm $n^\circ 2.0.5$). In this case margins ρ_i can be diagonalized by local unitary transformations and their compatibility depends only on spectra $\lambda_i = \operatorname{Spec} \rho_i$.

Pure quantum marginal problem has no classical analogue, since projection of a point (="pure state") is a point. In simplest univariant case it can be stated in plain language as follows.

Problem 3.2.1. *Let $M = [m_{ijk}]$ be complex cubic matrix and M_1, M_2, M_3 be Gram matrices formed by Hermitian dot products of parallel slices of M (see Example 3.1.3). What are possible spectra of matrices M_1, M_2, and M_3?*

3.3. **Some known results.** In contrast with classical marginal problem its quantum counterpart attracts attention only quite recently. Until now there were very few known results in this direction, which are listed below.

3.3.1. *Pure two component problem.* Equation 3.6 implies that margins of pure state $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$ are *isospectral* $\operatorname{Spec} \rho_A = \operatorname{Spec} \rho_B$ if we discard zero eigenvalues. Vice versa, this condition is sufficient for compatibility. Indeed let α_i, β_i be orthonormal eigenbases of ρ_A, ρ_B with common eigenvalues λ_i. Then state $\psi = \sum_i \lambda_i \alpha_i \otimes \beta_i$ has margins ρ_A, ρ_B. This representation of ψ is known as *Schmidt decomposition.*

3.3.2. *Scalar margins.* Completely entangled state $\psi \in \mathcal{H}_I = \bigotimes_i \mathcal{H}_i$ can be characterized by its univariant margins ρ_i [47, 23]:

$$\psi \text{ is completely entangled} \Longleftrightarrow \rho_i = \text{scalar}.$$

Theorem (Klyachko [23])**.** *Pure state $\psi \in \mathcal{H}_I$ with scalar univariant margins exists iff informational capacities of the components $\delta_i = \log \dim \mathcal{H}_i$ satisfy polygonal inequalities*

$$\delta_i \le \sum_{j(\neq i)} \delta_j.$$

3.3.3. *Pure N-qubit problem.* In this case there is a simple criterion for compatibility univariant margins.

Theorem (Higuchi et al. [19], Bravyi [7])**.** *Pure N-qubit state $\psi \in \mathcal{H}^N, \dim \mathcal{H} = 2$ with univariant margins ρ_i exists iff their minimal eigenvalues λ_i satisfy polygonal inequalities*

$$\lambda_i \le \sum_{j(\neq i)} \lambda_j. \tag{3.7}$$

3.3.4. *Mixed 2-qubit problem.* This problem was solved by Sergey Bravyi.

Theorem (Bravyi [7])**.** *Mixed two-qubit state ρ_{AB} with spectrum $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ and margins ρ_A, ρ_B exists iff minimal eigenvalues λ_A, λ_B of the margins satisfy inequalities*

$$\begin{aligned} &\lambda_A \geq \lambda_3 + \lambda_4, \quad \lambda_B \geq \lambda_3 + \lambda_4, \\ &\lambda_A + \lambda_B \geq \lambda_2 + \lambda_3 + 2\lambda_4, \\ &|\lambda_A - \lambda_B| \leq \min(\lambda_1 - \lambda_3, \lambda_2 - \lambda_4). \end{aligned} \tag{3.8}$$

3.3.5. *Pure 3-qutrit problem.* Astuchi Higuchi found a criterion for compatibility of univarint margins in 3-qutrit system $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, $\dim \mathcal{H}_* = 3$ in terms of marginal spectra $\lambda_1^* \leq \lambda_2^* \leq \lambda_3^*$, $* = A, B, C$.

Theorem (Higuchi [17])**.** *Pure state $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ with margins ρ_A, ρ_B, ρ_C exists iff the following inequalities holds*

$$\begin{aligned} \lambda_2^a + \lambda_1^a &\leq \lambda_2^b + \lambda_1^b + \lambda_2^c + \lambda_1^c, \\ \lambda_3^a + \lambda_1^a &\leq \lambda_2^b + \lambda_1^b + \lambda_3^c + \lambda_1^c, \\ \lambda_3^a + \lambda_2^a &\leq \lambda_2^b + \lambda_1^b + \lambda_3^c + \lambda_2^c, \\ 2\lambda_2^a + \lambda_1^a &\leq 2\lambda_2^b + \lambda_1^b + 2\lambda_2^c + \lambda_1^c, \\ 2\lambda_1^a + \lambda_2^a &\leq 2\lambda_2^b + \lambda_1^b + 2\lambda_1^c + \lambda_2^c, \\ 2\lambda_2^a + \lambda_3^a &\leq 2\lambda_2^b + \lambda_1^b + 2\lambda_2^c + \lambda_3^c, \\ 2\lambda_2^a + \lambda_3^a &\leq 2\lambda_1^b + \lambda_2^b + 2\lambda_3^c + \lambda_2^c, \end{aligned}$$

where a, b, c is a permutation of A, B, C.

It takes 46 pages to prove this!

3.3.6. *Remark.* All the above theorems deal with univariant margins. In quantum field theory they are known as *mean fields*, and higher rank margins as *n-point correlations*. Most physical effects are governed by two-point correlations. However complete solution of bivariant marginal problem is hardly possible (even in classical case, see $n^\circ 2.0.4$). Here is a couple of sporadic facts beyond trivial compatibility relations like $\operatorname{Tr}_A \rho_{AB} = \rho_B = \operatorname{Tr}_C \rho_{BC}$.

3.3.7. Strong subadditivity [32, 15] of quantum entropy $S(\rho) = -\operatorname{Tr}(\rho \log \rho)$

$$S(\rho_{ABC}) \leq S(\rho_{AB}) + S(\rho_{BC}) - S(\rho_B)$$

imposes a restriction on bivariant margins.

3.3.8. There is no pure 4-qubit state with scalar bivariant margins [18].

4. Marginal inequalities

In this section we give a general recipe for producing marginal inequalities for arbitrary multi-component system based on Berenstin–Sjamaar theorem [5]. For n qubits our result, modulo a "standard" conjecture, amounts to a simple combinatorics, but the number of involved inequalities increases drastically with n. The marginal inequalities up to 4 qubits, and for systems of formats 2×3, 3×3, 2×4, and $2 \times 2 \times 3$ are given in Appendix.

4.1. **Main result.** To avoid technicalities we confine ourselves to two component system $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. Generalization to multicomponent systems is straightforward. We start with some auxiliary notions and results.

4.1.1. *Filtrations.* Let α be a *nonincreasing filtration* of space $\mathcal{H}$, i.e. one parametric system of subspaces $\mathcal{H}^\alpha(t)\subset\mathcal{H}, t\in\mathbb{R}$ such that $\mathcal{H}^\alpha(s)\subset\mathcal{H}^\alpha(t)$ for $s\geq t$ and

$$\mathcal{H}^\alpha(t)=\begin{cases}\mathcal{H}, & \text{for } t\ll 0,\\ 0, & \text{for } t\gg 0.\end{cases} \tag{4.1}$$

The filtration supposed to be left continuous: $\lim_{\varepsilon\to+0}\mathcal{H}^\alpha(t-\varepsilon)=\mathcal{H}^\alpha(t)$.

Denote by

$$\mathcal{H}^{[\alpha]}(t)=\lim_{\varepsilon\to+0}\frac{\mathcal{H}^\alpha(t)}{\mathcal{H}^\alpha(t+\varepsilon)} \tag{4.2}$$

composition factors of the filtration. The dimension $m_\alpha(t)=\dim\mathcal{H}^{[\alpha]}(t)$ called *multiplicity* of t in spectrum of α. Finite set of real numbers $t\in\mathbb{R}$ with positive multiplicity $m_\alpha(t)>0$ called *spectrum* of filtration α. This are just discontinuity points of *dimension function*

$$d_\alpha(t)=\dim\mathcal{H}^\alpha(t). \tag{4.3}$$

We always arrange the spectral values (counted according the multiplicities) in nonincreasing order

$$\operatorname{Spec}(\alpha): a_1\geq a_2\geq\cdots\geq a_n,\quad n=\dim\mathcal{H}_A. \tag{4.4}$$

Example 4.1.1. Hermitian operator $\alpha:\mathcal{H}\to\mathcal{H}$ defines *spectral filtration*

$$\mathcal{H}^\alpha(t)=\begin{Bmatrix}\text{subspace spanned by eigenspaces}\\ \text{of } \alpha \text{ with eigenvalues } \geq t\end{Bmatrix}.$$

Spectrum of this filtration is equal to spectrum of the operator α. Every filtration of Hilbert space $\mathcal{H}$ is a spectral one. Hence filtration is a metric independent substitution for Hermitian operator.

We often refer to spectrum of a filtration as its *type*. Filtrations α of fixed type $a=\operatorname{Spec}(\alpha)$ form *flag variety* $\mathcal{F}\ell_a(\mathcal{H})$ consisting of all chains of subspaces

$$\mathcal{H}=F^1\supset F^2\supset\cdots\supset F^\ell\supset F^{\ell+1}=0, \tag{4.5}$$

with composition factors F^i/F^{i+1} of dimension equal to multiplicity of i-th spectral value in a.

4.1.2. *Composition of filtrations.* Let now α, β be filtrations in spaces $\mathcal{H}_A$, $\mathcal{H}_B$ respectively. Define their *composition* $\alpha\beta$ as filtration of $\mathcal{H}_{AB}=\mathcal{H}_A\otimes\mathcal{H}_B$ given by equation

$$\mathcal{H}_{AB}^{\alpha\beta}(t)=\sum_{r+s=t}\mathcal{H}_A^\alpha(r)\otimes\mathcal{H}_B^\beta(s). \tag{4.6}$$

The type of the composition $\alpha\beta$ depends only on spectra $a=\operatorname{Spec}(\alpha)$ and $b=\operatorname{Spec}(\beta)$

$$\operatorname{Spec}(\alpha\beta)=\{a_i+b_j \text{ arranged in nonincreasing order}\}. \tag{4.7}$$

Therefore the composition of filtrations of *fixed* types a and b gives rise to morphism of flag varieties

$$\varphi_{a,b}:\mathcal{F}\ell_a(\mathcal{H}_A)\times\mathcal{F}\ell_b(\mathcal{H}_B)\to\mathcal{F}\ell_{ab}(\mathcal{H}_{AB}),\quad \alpha\times\beta\mapsto\alpha\beta, \tag{4.8}$$

where composition of spectra ab is defined by RHS of equation (4.7).

4.1.3. *Cubicles and extremal edges.* Morphism (4.8) depends only on the *order* of quantities $a_i + b_j$ in spectrum (4.7). We call pairs of spectra (a, b) and $(\tilde{a}, \tilde{b})$ to be equivalent iff they define the same morphism $\varphi_{a,b} = \varphi_{\tilde{a},\tilde{b}}$. This means that the quantities $a_i + b_j$ and $\tilde{a}_i + \tilde{b}_j$ come in the same order

$$a_i + b_j \le a_k + b_l \Leftrightarrow \tilde{a}_i + \tilde{b}_j \le \tilde{a}_k + \tilde{b}_l. \tag{4.9}$$

Note that affine transformations of the spectra

$$a_i \mapsto p \cdot a_i + q, \qquad b_j \mapsto p \cdot b_j + r, \quad p > 0; q, r \in \mathbb{R}, \tag{4.10}$$

preserve the equivalence classes. This allows reduce the spectra a, b to Weyl chambers

$$\begin{aligned} \Delta_m : a_1 \ge a_2 \ge \cdots \ge a_m, \quad \sum\nolimits_i a_i = 0, \quad m = \dim \mathcal{H}_A; \\ \Delta_n : b_1 \ge b_2 \ge \cdots \ge b_n, \quad \sum\nolimits_j b_j = 0, \quad n = \dim \mathcal{H}_B. \end{aligned} \tag{4.11}$$

The equivalence produces a decomposition of the product $\Delta_m \times \Delta_n$ into relatively open polyhedral cones. We are mostly interested in cones of maximal and minimal dimension, called *cubicles* and *extremal edges* respectively. The cubicles are just pieces into which hyperplanes

$$a_i + b_j = a_k + b_l. \tag{4.12}$$

cut $\Delta_m \times \Delta_n$. Extremal edges are given by a system of equations of this form with one dimensional space of solutions.

A cubicle consists of spectra (a, b) with pairwise distinct quantities $a_i + b_j$ coming in *fixed order*. It can be described by $m \times n$ matrix T with entries $1, 2, \dots, mn$ written in the *opposite* order to that of matrix $[a_i + b_j]$:

$$T_{ij} \ge T_{kl} \Leftrightarrow a_i + b_j \le a_k + b_l.$$

In other words, T shows a way of counting entries of matrix $[a_i + b_j]$ in *decreasing* order. The entries of T strictly increase in rows and columns, i.e. T is a *standard tableau* of rectangular shape $m \times n$. There is a famous *hook formula* [35] for the number of such tableaux, which in current settings takes form

$$\#\{\text{ standard } m \times n \text{ tableaux }\} = \frac{(mn)!}{\prod\limits_{\substack{1 \le i \le m \\ 1 \le j \le n}} (i + j - 1)} \tag{4.13}$$

and gives an *upper bound* for the number of cubicles.

Example 4.1.2. For system of format $2 \times n$ every standard tableau corresponds to a cubicle. Hence by (4.13) the number of cubicles is equal to *Catalan number*

$$\#\{\text{ cubicles }\} = \frac{1}{n+1}\binom{2n}{n}.$$

A typical extremal edge in this case comes from spectra

$$a = (1/2, -1/2), \qquad b = (b_1, b_2, \dots, b_n),$$

with $b_i - b_{i+1} = 0, 1;\quad \sum b_i = 0$. The remaining extremal edges are that of Δ_n

$$a = (0, 0), \qquad b = (\underbrace{k, k, \dots, k}_{n-k}, \underbrace{k-n, k-n, \dots, k-n}_{k}).$$

This amounts altogether to $2^{n-1} + n - 1$ extremal edges, which is of order square root of the number of cubicles. In simplest case of two qubits they are

$$\begin{aligned} a = (1, -1); \quad a = (0, \ \ 0); \quad a = (1, -1); \\ b = (1, -1); \quad b = (1, -1); \quad b = (0, \ \ 0); \end{aligned} \tag{4.14}$$

Example 4.1.3. For two qutrits (=system of format 3×3) there are 36 cubicles represented by the following tableaux and their transpose

1	4	7
2	5	8
3	6	9

1	4	6
2	5	8
3	7	9

1	4	6
2	5	7
3	8	9

1	4	5
2	6	8
3	7	9

1	4	5
2	6	7
3	8	9

1	5	6
2	3	8
4	7	9

1	3	6
2	5	7
4	8	9

1	3	7
2	5	8
4	6	9

1	3	7
2	4	8
5	6	9

1	3	6
2	4	8
5	7	9

1	3	6
2	4	7
5	8	9

1	2	6
3	4	8
5	7	9

1	2	7
3	4	8
5	6	9

1	2	5
3	6	8
4	7	9

1	2	6
3	5	8
4	7	9

1	2	7
3	5	8
4	6	9

1	3	5
2	6	7
4	8	9

1	3	5
2	6	8
4	7	9

By hook formula (4.13) there are 42 standard 3×3 tableaux, 6 of them correspond to no cubicle. There are 17 extremal edges spanned by spectra (a,b) or (b,a) below

$$\begin{array}{llll} a=(1,\ \ 0,-1); & a=(3,\ \ 0,-3); & a=(0,\ \ 0,\ \ 0); & a=(3,\ \ 0,-3);\\ b=(1,\ \ 0,-1); & b=(5,-1,-4); & b=(2,-1,-1); & b=(1,\ \ 1,-2); \end{array}$$

$$\begin{array}{llll} a=(2,-1,-1); & a=(3,\ \ 0,-3); & a=(0,\ \ 0,\ \ 0); & a=(3,\ \ 0,-3);\\ b=(2,-1,-1); & b=(4,\ \ 1,-5); & b=(1,\ \ 1,-2); & b=(2,-1,-1); \end{array}$$

$$\begin{array}{ll} a=(1,\ \ 1,-2); & a=(2,-1,-1);\\ b=(1,\ \ 1,-2); & b=(1,\ \ 1,-2). \end{array}$$

4.1.4. *Cohomology of flag varieties.* Let T be a cubicle. By definition, morphism φ_{ab} defined by equation (4.8) is independent of $(a,b)\in T$. Hence the notation $\varphi_{ab}=\varphi_T$. Besides, all three spectra a,b,ab are simple and the corresponding flag varieties $\mathcal{F}\ell_a(\mathcal{H}_A)$, $\mathcal{F}\ell_b(\mathcal{H}_B)$, $\mathcal{F}\ell_{ab}(\mathcal{H}_{AB})$ are *complete*, i.e. consist of filtrations with composition factors of dimension one. Thus every cubicle gives rise to a well defined morphism of complete flag varieties

$$\varphi_T:\mathcal{F}\ell(\mathcal{H}_A)\times\mathcal{F}\ell(\mathcal{H}_B)\to\mathcal{F}\ell(\mathcal{H}_{AB}) \tag{4.15}$$

and that of their cohomology

$$\varphi_T^*:H^*(\mathcal{F}\ell(\mathcal{H}_{AB}))\to H^*(\mathcal{F}\ell(\mathcal{H}_A))\otimes H^*(\mathcal{F}\ell(\mathcal{H}_B)). \tag{4.16}$$

Cohomology ring of complete flag variety $H^*(\mathcal{F}\ell(\mathcal{H}))$ is generated by characteristic classes $x_i=c_1(\mathcal{L}_i)$ of line bundles $\mathcal{L}_i$ with fibers equal to i-th composition factor of the flag (4.5). We call x_i *canonical generators*. Elementary symmetric functions $\sigma_i(x)$ of the canonical generators are characteristic classes of trivial bundle $\mathcal{H}$ and thus vanish. This identifies the cohomology ring with factor

$$H^*(\mathcal{F}\ell(\mathcal{H}))=\mathbb{Z}[x_1,x_2,\ldots,x_n]/(\sigma_1,\sigma_2,\ldots,\sigma_n). \tag{4.17}$$

In term of the canonical generators x_i,y_j,z_k of cohomology of flag varieties $\mathcal{F}\ell(\mathcal{H}_A)$, $\mathcal{F}\ell(\mathcal{H}_B)$, and $\mathcal{F}\ell(\mathcal{H}_{AB})$ morphism φ_T^* can be described as follows

$$\varphi_T^*:z_k\mapsto x_i+y_j \text{ for } k=T_{ij}, \tag{4.18}$$

where we identify cubicle with the corresponding tableau T and for simplicity write x_i,y_j instead of $x_i\otimes 1, 1\otimes y_j$. In other words, $z_k\mapsto x_i+y_j$ iff k-th term of the composition ab is a_i+b_j for any (a,b) in cubicle T.

The cohomology ring $H^*(\mathcal{F}\ell(\mathcal{H}))$ has a natural geometric basis consisting of so called *Schubert cocycles* σ_w, where $w\in S_n$ is a permutation of degree $n=\dim\mathcal{H}$.

They can be expressed via characteristic classes x_i in terms of *difference operators*

$$\partial_i : f(x_1, x_2, \ldots, x_n) \mapsto \frac{f(\ldots, x_i, x_{i+1}, \ldots) - f(\ldots, x_{i+1}, x_i, \ldots)}{x_i - x_{i+1}} \tag{4.19}$$

as follows. Write permutation $w \in S_n$ as product of minimal number of transpositions $s_i = (i, i+1)$

$$w = s_{i_1} s_{i_2} \cdots s_{i_\ell}. \tag{4.20}$$

The number of factors $\ell(w) = \#\{i < j \mid w(i) > w(j)\}$ called *length* of permutation w. The product

$$\partial_w := \partial_{i_1} \partial_{i_2} \cdots \partial_{i_\ell} \tag{4.21}$$

is independent of reduced decomposition (4.20) and in terms of these operators Schubert cocycle σ_w is given by equation

$$\sigma_w = \partial_{w^{-1}w_0}(x_1^{n-1} x_2^{n-2} \cdots x_{n-1}), \tag{4.22}$$

where $w_0 = (n, n-1, \ldots, 2, 1)$ is unique permutation of maximal length.

Right hand side of equation (4.22) called *Schubert polynomial* $S_w(x_1, x_2, \ldots, x_n)$, $\deg S_w = \ell(w)$. These polynomials where first introduced by Lascoux and Schützenberger [29] who studied them in a long series of papers. See [36] for further references and a concise exposition of the theory. We borrowed from [29] the following table, in which x, y, z stand for x_1, x_2, x_3.

w	S_w	w	S_w	w	S_w	w	S_w
3201	x^3y^2z	2301	x^2y^2	2031	$x^2y + x^2z$	1203	xy
2310	x^2y^2z	3021	$x^3y + x^3z$	2103	x^2y	2013	x^2
3120	x^3yz	3102	x^3y	3012	x^3	0132	$x + y + z$
3201	x^3y^2	1230	xyz	0231	$xy + yz + zx$	0213	$x + y$
1320	$x^2yz + xy^2z$	0321	$x^2y + x^2z + xy^2$	0312	$x^2 + xy + y^2$	1023	x
2130	x^2yz	1302	$x^2y + xy^2$	1032	$x^2 + xy + xz$	0123	1

Extra variables $x_{n+1}, x_{n+2}, \ldots$ being added to (4.22) leave Schubert polynomials unaltered. By that reason they are usually treated as polynomials in infinite alphabet $X = (x_1, x_2, \ldots)$. With this understanding every homogeneous polynomial can be decomposed into Schubert components as follows

$$f(X) = \sum_{\ell(w)=\deg(f)} \partial_w f \cdot S_w(X). \tag{4.23}$$

Applying this to specialization (4.18) of Schubert polynomial $S_w(Z)$ we get decomposition

$$\varphi_T(S_w(Z)) = \sum_{\ell(u)+\ell(v)=\ell(w)} c_{uv}^w(T) \cdot S_u(X) S_v(Y), \tag{4.24}$$

where

$$c_{uv}^w(T) = \partial_u \partial_v S_w(Z) \big|_{z_k = x_i + y_j}, \qquad k = T_{ij}. \tag{4.25}$$

Here operators ∂_u, ∂_v act on variables x, y respectively. Reduction of (4.24) modulo elementary symmetric functions in x and in y gives cohomology morphism (4.16) in terms of Schubert cocycles

$$\varphi_T(\sigma_w) = \sum_{\ell(u)+\ell(v)=\ell(w)} c_{uv}^w(T) \cdot \sigma_u \otimes \sigma_v. \tag{4.26}$$

4.1.5. *Marginal inequalities.* Now we are in position to state a solution of the quantum marginal problem.

Theorem 4.1.1. *Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ be two component system. Then the following conditions on spectra $\lambda^A, \lambda^B, \lambda^{AB}$ are equivalent.*

(1) *There exists mixed state ρ_{AB} with margins ρ_A, ρ_B and spectra $\lambda^{AB}, \lambda^A, \lambda^B$.*

(2) *Each time the coefficient $c_{uv}^w(T)$ of decomposition (4.26) is nonzero, the following inequality holds for all (a, b) in the cubicle T*

$$\sum_i a_i \lambda^A_{u(i)} + \sum_j b_j \lambda^B_{v(j)} \geq \sum_k (a_i + b_j)_k \lambda^{AB}_{w(k)}, \tag{4.27}$$

where $(a_i+b_j)_k$ is k-th element of the sequence a_i+b_j arranged in decreasing order.

Proof. The quantum marginal problem amounts to decomposition of projection of a coadjoint orbit of group $\mathrm{SU}(\mathcal{H}_A \otimes \mathcal{H}_B)$ into coadjoint orbits of subgroup $\mathrm{SU}(\mathcal{H}_A) \times \mathrm{SU}(\mathcal{H}_B)$.

The above discussion just puts the result into framework of Berenstein-Sjamaar theorem [5, Thm 3.2.1] applied these groups. □

4.1.2. *Remark.* Every cubicle is interior of convex hull of its extremal edges. Therefore inequality (4.27) enough to check for extremal edges only.

Example 4.1.4. Note that $c_{uv}^w(T) = 1$ for identical permutations u, v, w. Hence the following inequality holds for every extremal edge (a, b)

$$\sum_i a_i \lambda^A_i + \sum_j b_j \lambda^B_j \geq \sum_k (a_i + b_j)_k \lambda^{AB}_k. \tag{4.28}$$

We call these marginal inequalities *basic* ones.

Example 4.1.5. *"Easy" two qubit inequalities.* Extremal edges for two qubits from Example 4.1.2 give rise to basic inequalities

$$\begin{aligned}
\lambda^A_1 - \lambda^A_2 + \lambda^B_1 - \lambda^B_2 &\leq 2(\lambda^{AB}_1 - \lambda^{AB}_4),\\
\lambda^A_1 - \lambda^A_2 &\leq \lambda^{AB}_1 + \lambda^{AB}_2 - \lambda^{AB}_3 - \lambda^{AB}_4,\\
\lambda^B_1 - \lambda^B_2 &\leq \lambda^{AB}_1 + \lambda^{AB}_2 - \lambda^{AB}_3 - \lambda^{AB}_4,
\end{aligned}$$

which are equivalent to the first three inequalities in (3.8). S. Bravyi [7] calls them *easy* ones.

Extremal edge (a, b) gives rise to cohomology map

$$\varphi^*_{ab} : H^*(\mathcal{F}\ell_{ab}(\mathcal{H}_{AB})) \to H^*(\mathcal{F}\ell_a(\mathcal{H}_A)) \otimes H^*(\mathcal{F}\ell_b(\mathcal{H}_B)) \tag{4.29}$$

induced by morphism φ_{ab} from equation (4.8). In contrast with cubicle case, spectrum ab is never simple and we have to deal with varieties of *incomplete* flags.

Let α be a filtration of space $\mathcal{H}$ of dimension m and spectrum $a = \operatorname{Spec}\alpha$. Denote by $\mu(a) = (m_1, m_2, \ldots, m_\ell)$, $\sum_i m_i = m$ the multiplicities of the spectrum. Recall that flag variety $\mathcal{F}\ell_a(\mathcal{H})$ consists of the filtrations with composition factors of dimension m_i. Its cohomology known to be subring of invarinats

$$H^*(\mathcal{F}\ell_a(\mathcal{H})) = H^*(\mathcal{F}\ell(\mathcal{H}))^{S_\mu} \subset H^*(\mathcal{F}\ell(\mathcal{H})) \tag{4.30}$$

with respect to Schur subgroup $S_\mu = S_{m_1} \times S_{m_2} \times \cdots \times S_{m_\ell} \subset S_m$ acting on cohomolohy of *complete* flags by permutations of the canonical generators x_i, see [6]. An important example of such invariant comes from Schubert cocycle σ_w corresponding to a *shuffle of type* $\mu(a)$, i.e. permutation $w \in S_m$ which preserves order of the first m_1 elements, the next m_2 elements, and so on. Such Schubert cocycles form a basis of the cohomology ring (4.30) of incomplete flags.

In view of this interpretation morphism (4.29) becomes just a restriction of

$$\varphi_T^*: H^*(\mathcal{F}\ell(\mathcal{H}_{AB})) \to H^*(\mathcal{F}\ell(\mathcal{H}_A)) \otimes H^*(\mathcal{F}\ell(\mathcal{H}_B)) \tag{4.31}$$

onto the algebras of invariants for any cubicle T containing extremal edge (a, b) in its closure. Hence by (4.26)-(4.25) coefficients of the decomposition

$$\varphi_{ab}(\sigma_w) = \sum_{\ell(u)+\ell(v)=\ell(w)} c_{uv}^w(a, b) \cdot \sigma_u \otimes \sigma_v, \tag{4.32}$$

are given by equation

$$c_{uv}^w(a, b) = \partial_u \partial_v S_w(Z)\big|_{z_k = x_i + y_j}, \qquad k = T_{ij}, \tag{4.33}$$

where u, v, w are shuffles of types $\mu(a), \mu(b), \mu(ab)$ respectively. This leads to the following description of marginal inequalities in terms of extremal edges only.

Theorem 4.1.3. *All marginal inequalities for system $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ can be obtained from basic ones (4.28) by shuffles u, v, w of types $\mu(a), \mu(b), \mu(ab)$*

$$\sum_i a_i \lambda_{u(i)}^A + \sum_j b_j \lambda_{v(j)}^B \leq \sum_k (a_i + b_j)_k \lambda_{w(k)}^{AB} \tag{4.34}$$

such that coefficient $c_{uv}^w(a, b)$ of decomposition (4.32) is nonzero.

Extensive tables of marginal inequalities deduced from this theorem are given in Addendum. Details of the calculation will be published elsewhere [25].

Example 4.1.6. *"Difficult" two qubit inequality.* Let's find modifications (4.34) of the first basic inequality form Example 4.1.5

$$\lambda_1^A - \lambda_2^A + \lambda_1^B - \lambda_2^B \leq 2(\lambda_1^{AB} - \lambda_4^{AB}), \tag{4.35}$$

coming from extremal edge $a = (1, -1), b = (1, -1)$. There are two ways of counting entries of matrix

$$[a_i + b_j] = \begin{array}{|c|c|}\hline 2 & 0 \\ \hline 0 & -2 \\ \hline \end{array}$$

in decreasing order

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \qquad \text{or} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \tag{4.36}$$

which exhibit two cubicles containing (a, b) in theirs closure.

By Theorem 4.1.3 the modifications come from components of $\varphi_{ab}^*(\sigma_w)$ for shuffles w of type $(1, 2, 1)$ and length $\ell(w) \leq 2$. Using first cubicle in (4.36) the calculation amounts to specialization (4.18)

$$z_1 = x_1 + y_1, \quad z_2 = x_2 + y_1, \quad z_3 = x_1 + y_2, \quad z_4 = x_2 + y_2$$

of Schubert polynomial $S_w(Z)$ taken modulo relations

$$x_1 + x_2 = y_1 + y_2 = 0, \qquad x_1 x_2 = y_1 y_2 = 0.$$

From the table on page 10 it follows

$$\varphi_{ab}^*(\sigma_{2134}) = \varphi_{ab}^*(\sigma_{1243}) = x_1 + y_1 = \sigma_{21} \otimes \sigma_{12} + \sigma_{12} \otimes \sigma_{21},$$
$$\varphi_{ab}^*(\sigma_{2143}) = \varphi_{ab}^*(\sigma_{3124}) = \varphi_{ab}^*(\sigma_{1342}) = 2x_1 y_1 = 2\sigma_{21} \otimes \sigma_{21}.$$

The first line gives rise to 4 modified inequalities

$$\lambda_2^A - \lambda_1^A + \lambda_1^B - \lambda_2^B \leq 2(\lambda_2^{AB} - \lambda_4^{AB}),$$
$$\lambda_1^A - \lambda_2^A + \lambda_2^B - \lambda_1^B \leq 2(\lambda_2^{AB} - \lambda_4^{AB}),$$
$$\lambda_2^A - \lambda_1^A + \lambda_1^B - \lambda_2^B \leq 2(\lambda_1^{AB} - \lambda_3^{AB}),$$
$$\lambda_1^A - \lambda_2^A + \lambda_2^B - \lambda_1^B \leq 2(\lambda_1^{AB} - \lambda_3^{AB}),$$

which can be squeezed into the following one

$$|\,(\lambda_1^A-\lambda_2^A)-(\lambda_1^B-\lambda_2^B)\,|\le 2\min(\lambda_1^{AB}-\lambda_3^{AB},\lambda_2^{AB}-\lambda_4^{AB}).$$

This is the last inequality in (3.8) designated by S. Bravyi [7] as a *difficult* one. All the other modified inequalities are redundant. We'll discuss the underlying reason for this in the next section.

4.1.6. *Redundancy.* Redundancy of marginal inequalities is an important issue. For example, for two qutrits there are 17 extremal edges (a,b), 2298 permutations $w\in S_9$ of length ≤ 12, in addition $\varphi_{ab}^*(\sigma_w)$ may have many components, and each gives a marginal inequality. Among those tens of thousands inequalities only 397 are independent.

Conjectur 4.1.4. *All marginal constraints are given by inequalities* (*4.34*) *of theorem 4.1.3 with* $c_{uv}^w=1$.

This "standard" conjecture may be true in framework of general Berenstein-Sjamaar theorem [5]. It is valid for multicomponent quantum systems of rank ≤ 4, see Appendix. Apparently the conjecture should follow from rigidity argument of Belkale [2], who proved it for Hermitian spectral problem, see section 5.2. In this case the inequalities listed in the conjecture are also independent [27]. However for quantum marginal problem this is not the case. For example, for system of format $2\times 2\times 3$ even some basic inequalities are redundant. Redundancy also occurs in a counterpart of the Hermitian spectral problem for groups other then $\mathrm{SU}(n)$. In this case Belkale and Kumar [3] have a refined procedure which gives independent inequalities, at least in some examples.

4.2. **Arrays of qubits.** In this section we study in details the system of n qubit, which is of fundamental importance for quantum information. In this case, modulo Conjecture 4.1.4, the marginal inequalities can be written down in a straightforward manner from the list of extremal edges.

4.2.1. *Cubicles and extremal edges.* Let's consider array of qubits

$$\mathcal{H}^{\otimes n}=\mathcal{H}_1\otimes\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_n,\quad \dim\mathcal{H}_i=2$$

and denote by α_i a filtration of $\mathcal{H}_i$, normalized to Weyl chamber (4.11), so that

$$\operatorname{Spec}\alpha_i=\pm a_i,\quad a_i\ge 0.$$

Henceforth we'll identify the spectrum with nonnegative number $a_i\ge 0$. As in (4.8) composition of filtrations gives rise to morphism

$$\varphi_a:\mathcal{F}\ell_{a_1}(\mathcal{H}_1)\times\mathcal{F}\ell_{a_2}(\mathcal{H}_2)\times\cdots\times\mathcal{F}\ell_{a_n}(\mathcal{H}_n)\to\mathcal{F}\ell_A(\mathcal{H}_1\otimes\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_n),\tag{4.37}$$

where on the left hand side a stands for vector of spectra $(a_1,a_2,\dots,a_n)$, while on the right hand side A refers to their composition

$$A=a_1a_2\cdots a_n=\{\pm a_1\pm a_2\pm\cdots\pm a_n \text{ arranged in nonincreasing order}\}.$$

Note that flag variety $\mathcal{F}\ell_{a_i}(\mathcal{H}_i)$ amounts to projective line $\mathbb{P}^1$ (=Riemann sphere) for $a_i>0$, or to a point for $a_i=0$. Similar to two component system, φ_a depends only on the order of 2^n quantities

$$\pm a_1\pm a_2\pm\cdots\pm a_n\tag{4.38}$$

which are just dot products of vector a and a vertex $(\pm1,\pm1,\dots,\pm1)$ of n-cube. Hence n-qubit cubicle is determined by the order of pairwise distinct projections of vertices of the n-cube onto positive direction a. Equivalently, cubicles are pieces into which hyperplanes

$$\varepsilon_1a_1+\varepsilon_1a_1+\cdots+\varepsilon_na_n=\delta_1a_1+\delta_1a_1+\cdots+\delta_na_n,\quad \varepsilon,\delta=\pm1,\tag{4.39}$$

cut positive octant. Equation of the wall (4.39) separating two cubicles can be recast in the form

$$H_\eta : \eta_1 a_1 + \eta_2 a_2 + \cdots + \eta_n a_n = 0, \quad \eta_i = 0, \pm 1, \tag{4.40}$$

which tells that H_η is orthogonal to vector η pointing to center of a face of the cube.

Every extremal edge is intersection of $n-1$ walls, hence given by an independent system of homogeneous equations with $(n-1)\times n$ matrix M filled by $0, \pm 1$. The solution is given by a vector of maximal minors taken with alternating signs $(-1)^i M_i$. Their absolute values

$$(|M_1|, |M_2|, \ldots, |M_n|)$$

gives a positive solution for another such matrix M' and spans an extremal edge. Computer implementation of this brute force approach allows to find all extremal edges up to 6 qubits. The table below shows the number of extremal edges counted up to a permutation of qubits, say normalized by condition $a_1 \le a_2 \le \ldots \le a_n$.

# qubits	2	3	4	5	6
# edges	2	4	12	125	11344

Example 4.2.1. *Three qubits.* Let $0 \le a \le b \le c$ be three spectra in increasing order. Then among 8 quantities $\pm a \pm b \pm c$ only one inequality is indefinite

$$a{+}b{+}c \ge -a{+}b{+}c \ge a{-}b{+}c \ge a{+}b{-}c \gtreqless -a{-}b{+}c \ge -a{+}b{-}c \ge a{-}b{-}c \ge -a{-}b{-}c$$

and depends on whether a, b, c satisfy triangle inequality or not. In the former case the order is

$$a{+}b{+}c \ge -a{+}b{+}c \ge a{-}b{+}c \ge a{+}b{-}c \ge -a{-}b{+}c \ge -a{+}b{-}c \ge a{-}b{-}c \ge -a{-}b{-}c,$$

and in the later

$$a{+}b{+}c \ge -a{+}b{+}c \ge a{-}b{+}c \ge -a{-}b{+}c \ge a{+}b{-}c \ge -a{+}b{-}c \ge a{-}b{-}c \ge -a{-}b{-}c.$$

Thus the system has two cubicles up to permutations of qubits and 12 altogether. The two cubicles have 4 extremal edges

$$(0,0,1); \quad (0,1,1); \quad (1,1,1); \quad (1,1,2). \tag{4.41}$$

Example 4.2.2. The 12 extremal edges $a_1 \le a_2 \le a_3 \le a_4$ for 4 qubits are as follows

$$\begin{array}{llll} (0,0,0,1), & (0,0,1,1), & (0,1,1,1), & (0,1,1,2),\\ (1,1,1,1), & (1,1,1,2), & (1,1,1,3), & (1,1,2,2),\\ (1,1,2,3), & (1,1,2,4), & (1,2,2,3), & (1,2,3,4). \end{array}$$

4.2.1. *Remark.* It would be nice to understand better geometry and combinatorics of the system of hyperplanes H_η. Note that in a configuration of N central hyperplanes in general position there are $\binom{N}{n-1}$ lines of intersections. This gives an *upper bound* for the number of lines of intersection in the system of $(3^n-1)/2$ walls (4.40) and leads to estimate

$$\sum_{\substack{\text{extremal edges}\\ a_1\le a_2\le\cdots\le a_n}} \frac{1}{|\operatorname{Stab}(a)|} \le \frac{1}{2^{n-1}n!}\binom{(3^n-1)/2}{n-1},$$

where $\operatorname{Stab}(a)$ is stabilizer of edge a in group of monomial permutations $a_i \mapsto \pm a_j$. In worst case scenario it may happens that the system H_η becomes stochastic for $n \gg 1$. Then there may be no reasonable description of its edges and cubicles, except statistical one.

4.2.2. *Marginal inequalities.* Every vector a in cubicle T gives rise to morphism (4.37) of *complete* flag varieties depending only on T

$$\varphi_T : \mathbb{P}(\mathcal{H}_1) \times \mathbb{P}(\mathcal{H}_2) \times \cdots \times \mathbb{P}(\mathcal{H}_n) \to \mathcal{F}\ell(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n), \tag{4.42}$$

where we identify flag variety of i-th qubit with projective line $\mathbb{P}(\mathcal{H}_i)$. By theorem 4.1.1 marginal inequalities come from the induced cohomology morphism

$$\varphi_T^* : H^*(\mathcal{F}\ell(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n)) \to \mathbb{Z}[x_1, x_2, \ldots, x_n]/(x_i^2 = 0). \tag{4.43}$$

Here we use isomorphism $H^*(\mathbb{P}(\mathcal{H}_i)) \cong \mathbb{Z}[x_i]/(x_i^2 = 0)$, where $x_i = \sigma_{21}$ is Schubert cocycle. In terms of canonical generators (4.17) the morphism amounts to specialization

$$\varphi_T^* : z_k \mapsto \pm x_1 \pm x_2 \pm \cdots \pm x_n, \tag{4.44}$$

where the signs are taken from k-th term of the sequence $\pm a_1 \pm a_2 \pm \cdots \pm a_n$ arranged in decreasing order. Applying this to Schubert cocycle σ_w we get decomposition

$$\varphi_T^*(\sigma_w) = \sum c^w_{b_1 b_2 \ldots b_n}(T) \cdot x_1^{b_1} x_2^{b_2} \ldots x_n^{b_n} \tag{4.45}$$

into sum of square free monomials with exponents $b_i = 0, 1$. The theorem 4.1.1 tells that each time $c^w_{b_1 b_2 \ldots b_n}(T)$ is nonzero the following inequality holds for all $a \in T$

$$\sum_i (-1)^{b_i} a_i(\rho_1^{(i)} - \rho_2^{(i)}) \le \sum_k (\pm a_1 \pm a_2 \pm \cdots \pm a_n)_k \rho_{w(k)}, \tag{4.46}$$

where ρ is n qubit state with spectrum $\rho_1 \ge \rho_2 \ge \cdots \ge \rho_{2^n}$ and margins $\rho^{(i)}$, $(\pm a_1 \pm a_2 \pm \cdots \pm a_n)_k$ is k-th term of the sequence $\pm a_1 \pm a_2 \pm \cdots \pm a_n$ arranged in decreasing order, and sign $(-1)^{b_i}$ reflects the effect of transposition of eigenvalues $\rho_1^{(i)} \ge \rho_2^{(i)}$ of reduced state $\rho^{(i)}$.

Recall that according to Conjecture 4.1.4 all marginal constraints come from inequalities (4.46) with $c^w_{b_1 b_2 \ldots b_n}(T) = 1$.

Claim 4.2.2. *For n qubit system components of multiplicity one in $\varphi_T^*(\sigma_w)$ can appear only for identical substitution $w = 1$ or for odd transposition $w = (2j-1, 2j)$.*

Proof. Indeed $\varphi_T^*(\sigma_w)$ is obtained from Schubert polynomial $S_w(Z)$ by specialization (4.44). Since $\pm x_1 \pm x_2 \pm \cdots \pm x_n \equiv x_1 + x_2 + \cdots + x_n \mod 2$, then

$$\varphi_T^*(\sigma_w) \equiv S_w(1, 1, \ldots, 1)(x_1 + x_2 + \cdots + x_n)^{\ell(w)} \mod 2. \tag{4.47}$$

Let $\ell(w) = \sum_\alpha 2^\alpha$ be binary decomposition of $\ell(w)$. Then

$$(x_1 + x_2 + \cdots + x_n)^{\ell(w)} \equiv \prod_\alpha (x_1^{2^\alpha} + x_2^{2^\alpha} + \cdots + x_n^{2^\alpha}) \mod 2. \tag{4.48}$$

Hence multiplicity free monomials with *odd* coefficient can appear only for permutation of length $\ell(w) \le 1$, i.e. for $w = 1$ or for transposition $w = (i, i+1)$. Note that

$$S_{(i,i+1)}(Z) = z_1 + z_2 + \cdots + z_i.$$

Therefore for even transposition $w = (i, i+1)$ coefficient $S_w(1, 1, \ldots, 1) = i$ in (4.47) is even. So we proved that $\varphi_T^*(\sigma_w)$ contains a monomial with odd coefficient iff $w = (2j-1, 2j)$ or $w = 1$. □

Recall that identical substitution gives rise to basic marginal inequality

$$\sum_i a_i(\rho_1^{(i)} - \rho_2^{(i)}) \le \sum_k (\pm a_1 \pm a_2 \pm \cdots \pm a_n)_k \rho_k \tag{4.49}$$

for every extremal edge a. From Claim 4.2.2 it follows

Theorem 4.2.3 (modulo Conjecture 4.1.4)**.** *All marginal inequaliteis for n qubits can be obtained from basic ones (4.49) by odd transposition $\rho_{2j-1} \leftrightarrow \rho_{2j}$ in the right hand side, accompanied with sign change $a_i(\rho_1^{(i)} - \rho_2^{(i)}) \mapsto -a_i(\rho_1^{(i)} - \rho_2^{(i)})$ of one term in the left hand side.*

4.2.4. *Remark.* Right hand side of basic inequality (4.49) is invariant under permutation of a_i. To get a strongest inequality we have to arrange them in the same order as quantities $\delta_i = \rho_1^{(i)} - \rho_2^{(i)}$ to make LHS maximal possible. Let's assume for certainty

$$\delta_1 \le \delta_2 \le \cdots \le \delta_n, \qquad a_1 \le a_2 \le \cdots \le a_n. \tag{4.50}$$

To get the strongest modified inequality one have to revert the sign of the minimal term $a_1\delta_1 \mapsto -a_1\delta_1$ in the LHS.

Example 4.2.3. *Three qubits.* The above procedure being applied to extremal edges (4.41) for three qubits returns the following list of marginal inequalities grouped by their extremal edges. The first inequality in each group is the basic one. The transposed eigenvalues in modified inequalities typeset in bold face. We expect $\delta_i = \rho_1^{(i)} - \rho_2^{(i)}$ to be arranded in increasing order $\delta_1 \le \delta_2 \le \delta_3$.

$$\delta_3 \le \rho_1 + \rho_2 + \rho_3 + \rho_4 - \rho_5 - \rho_6 - \rho_7 - \rho_8.$$

$$\delta_2 + \delta_3 \le 2\rho_1 + 2\rho_2 - 2\rho_7 - 2\rho_8.$$

$$\begin{aligned}
\delta_1 + \delta_2 + \delta_3 &\le 3\rho_1 + \rho_2 + \rho_3 + \rho_4 - \rho_5 - \rho_6 - \rho_7 - 3\rho_8,\\
-\delta_1 + \delta_2 + \delta_3 &\le 3\boldsymbol{\rho_2} + \boldsymbol{\rho_1} + \rho_3 + \rho_4 - \rho_5 - \rho_6 - \rho_7 - 3\rho_8,\\
-\delta_1 + \delta_2 + \delta_3 &\le 3\rho_1 + \rho_2 + \rho_3 + \rho_4 - \rho_5 - \rho_6 - \boldsymbol{\rho_8} - 3\boldsymbol{\rho_7}.
\end{aligned}$$

$$\begin{aligned}
\delta_1 + \delta_2 + 2\delta_3 &\le 4\rho_1 + 2\rho_2 + 2\rho_3 - 2\rho_6 - 2\rho_7 - 4\rho_8,\\
-\delta_1 + \delta_2 + 2\delta_3 &\le 4\boldsymbol{\rho_2} + 2\boldsymbol{\rho_1} + 2\rho_3 - 2\rho_6 - 2\rho_7 - 4\rho_8,\\
-\delta_1 + \delta_2 + 2\delta_3 &\le 4\rho_1 + 2\rho_2 + 2\boldsymbol{\rho_4} - 2\rho_6 - 2\rho_7 - 4\rho_8,\\
-\delta_1 + \delta_2 + 2\delta_3 &\le 4\rho_1 + 2\rho_2 + 2\rho_3 - 2\boldsymbol{\rho_5} - 2\rho_7 - 4\rho_8,\\
-\delta_1 + \delta_2 + 2\delta_3 &\le 4\rho_1 + 2\rho_2 + 2\rho_3 - 2\rho_6 - 2\boldsymbol{\rho_8} - 4\boldsymbol{\rho_7}.
\end{aligned}$$

One can check[1] that the above inequalities are independent, and other inequalities (4.46) follows from these ones, in conformity with Conjecture 4.1.4. The later is still valid for four qubits, however in this case two modified inequalities are redundant, see Appendix. Actually there are many "trivial" redundant inequalities coming from transposition of eigenvalues ρ_{2j-1}, ρ_{2j} entering in RHS of basic inequality (4.49) with the same coefficient. In settings of theorem 4.1.3, which deals with extremal edges rather then cubicles, such transpositions are forbidden and the "trivial" redundancy never occurs. 4.2.4.

5. Representation theoretical interpretation

Apparently marginal inequalities give a complete solution of the quantum marginal problem. This however may be an illusion, since the number of inequalities increases drastically with rank of the system. In this section we pursue another approach, based on reduction of the univariant marginal problem to *representation theory* of the symmetric group. We recall first the basic facts of the latter [20, 35, 43].

[1]Thanks to Maple simplex package.

5.1. **Digest of representation theory.** Let's start with ensemble of N identical systems with state space

$$\mathcal{H}^{\otimes N} = \underbrace{\mathcal{H} \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H}}_{N}. \tag{5.1}$$

Unitary $\mathrm{SU}(\mathcal{H})$ and symmetric S_N groups act on this space from the left and from the right by formulae

$$g : \psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_N \mapsto g\psi_1 \otimes g\psi_2 \otimes \cdots \otimes g\psi_N, \qquad g \in \mathrm{SU}(\mathcal{H}),$$
$$\sigma : \psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_N \mapsto \psi_{\sigma(1)} \otimes \psi_{\sigma(2)} \otimes \cdots \otimes \psi_{\sigma(N)}, \qquad \sigma \in \mathrm{S}_N.$$

Issai Schur in his celebrated thesis [44] of 1901 found decomposition of tensor space (5.1) into irreducible components with respect to these actions

$$\mathcal{H}^{\otimes N} = \bigoplus_{\lambda} \mathcal{H}_\lambda \otimes \mathcal{S}_\lambda, \tag{5.2}$$

where $\mathcal{H}_\lambda$ and $\mathcal{S}_\lambda$ are irreducible representations of the unitary and the symmetric groups respectively. The components describe tensors of different types of symmetry. They are parameterized by *Young diagrams*

$$\lambda : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq 0, \qquad \lambda_1 + \lambda_2 + \cdots + \lambda_d = N$$

of N cells and no more then $d = \dim \mathcal{H}$ rows of length λ_i. Representations $\mathcal{H}_\lambda$ and $\mathcal{S}_\lambda$ can be intrinsically characterized as follows.

- Let V be irreducible representation of $\mathrm{SU}(\mathcal{H})$ and λ be maximal diagram in *lexicographic order* such that diagonal subgroup has eigenvector ψ of weight λ, that is
$$\mathrm{diag}(x_1, x_2, \ldots, x_d)\psi = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_d^{\lambda_d} \psi.$$
Then $V \simeq \mathcal{H}_\lambda$.
- Let now M be irreducible representation of S_N, and λ be maximal diagram in the *majorization order* (see $n^\circ 2.0.5$) such that M contains a nonzero invariant with respect to permutations in *rows* of diagram λ (filled in arbitrary way by numbers $1, 2, \ldots, N$). Then $M \simeq \mathcal{S}_\lambda$.

Example 5.1.1. One row diagram λ corresponds to trivial representation of the symmetric group. Representation $\mathcal{H}_\lambda$ in this case consists of *symmetric tensors* in $\mathcal{H}^{\otimes N}$ (Bose–Einstein statistics).

Column diagram λ defines sign representation $\sigma \mapsto \mathrm{sgn}(\sigma)$ of S_N, while $\mathcal{H}_\lambda$ is the space of *antisymmetric tensors* in $\mathcal{H}^{\otimes N}$ (Fermi–Dirak statistics).

Other diagrams correspond to more complicate symmetry types of tensors which some physicists associate with *parastatistics*.

5.2. **Degression: Hermitian spectral problem.** Here we consider a *model example* which illustrates relation between representation theory and spectral problems. As we have seen at the end of $n^\circ 3.2$ univariant marginal problem falls into this category.

Let's start with decomposition of tensor product of irreducible representations of the unitary group $\mathrm{SU}(\mathcal{H})$

$$\mathcal{H}_\lambda \otimes \mathcal{H}_\mu = \sum_\nu C^\nu_{\lambda\mu} \mathcal{H}_\nu. \tag{5.3}$$

Representation $\mathcal{H}_\nu$ enters into the decomposition with multiplicity $C^\nu_{\lambda\mu}$ which physicists call *Clebsch–Gordon* and mathematiciens *Littlewood–Richardson* coefficient. The last authors invented in 1934 an efficient algorithm for calculation $C^\nu_{\lambda,\mu}$, known as *Littlewood–Richardson rule* [2], see [34, 35] for details. Littlewood–Richardson calculator is available at [8].

Theorem (Klyachko [24]). *The following conditions on Young dyagrams λ, μ, ν are equivalent*

(1) $C^{\nu}_{\lambda\mu} \neq 0$, *i.e.* $\mathcal{H}_\nu \subset \mathcal{H}_\lambda \otimes \mathcal{H}_\mu$.
(2) *There exist Hermitian operators $A, B, C = A + B$ in $\mathcal{H}$ with spectra λ, μ, ν.*

Here we identify Young diagram λ with integeral spectrum $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$ formed by lengths of its rows.

5.2.1. *Remark.* (i) The conditions of the theorem are actually equivalent to a system of linear inequalities on λ, μ, ν, known as *Horn–Klyachko inequalities* [24, 12].

(ii) The theorem as it stated can be applied to integral spectra only. However by scaling one can extend it to rational spectra, and by limit arguments to real spectra as well. In the last case we lose connection with representation theory, and compatibility conditions for spectra of matrices $A, B, C = A + B$ are given by H-K inequalities.

5.3. **Back to quantum marginal problem.** Let now consider tensor product of irreducible representations of the symmetric group S_N

$$\mathcal{S}_\lambda \otimes \mathcal{S}_\mu = \sum_\nu g(\lambda, \mu, \nu) \mathcal{S}_\nu. \tag{5.4}$$

The multiplicities $g(\lambda, \mu, \nu)$ in this case called *Kronecker coefficients*. In contrast with Littlewood–Richardson ones $C^{\nu}_{\lambda\mu}$ they are symmetric in λ, μ, ν and

$$g(\lambda, \mu, \nu) = \dim(\mathcal{S}_\lambda \otimes \mathcal{S}_\mu \otimes \mathcal{S}_\nu)^{S_N}. \tag{5.5}$$

The superscript refers to space of invariants $V^{S_N} = \{\, x \in V \mid x^\sigma = x \quad \forall \sigma \in S_N \,\}$.

Now we can state our main result for univariant marginal problem, which is similar in form to the previous theorem on spectra of Hermitian matrices A, B, and $A + B$.

Theorem 5.3.1. *The following conditions are equivalent*

(1) $g(m\lambda, m\nu, m\mu) \neq 0$ *for some* $m > 0$, *i.e.* $\mathcal{S}_{m\nu} \subset \mathcal{S}_{m\lambda} \otimes \mathcal{S}_{m\mu}$.
(2) *There exists mixed state of ρ_{AB} of two component system $\mathcal{H}_A \otimes \mathcal{H}_B$ with spectrum ν and margins of spectra λ, μ.*
(3) *There exists pure state $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ with univariant margins of spectra λ, μ, ν.*

Proof. The proof is a combination of well known results and runs as follows.

• As we've yet mentioned in the proof of Theorem 4.1.1 the quantum marginal problem amounts to decomposition of projection of a coadjoint orbit of group $\mathrm{SU}(\mathcal{H}_A \otimes \mathcal{H}_B)$ into coadjoint orbits of subgroup $\mathrm{SU}(\mathcal{H}_A) \times \mathrm{SU}(\mathcal{H}_B)$.

• Combining this with Heckman's theorem [16], see also in [5, Thm 3.4.2], we arrive at characterization of quantum margins by inclusion

$$\mathcal{H}^\lambda_A \otimes \mathcal{H}^\mu_B \subset \mathcal{H}^\nu_{AB} \mid_{\mathrm{SU}(\mathcal{H}_A)\times \mathrm{SU}(\mathcal{H}_B)} \tag{5.6}$$

for some spectra λ, μ, ν proportional to $\mathrm{Spec}\,\rho_A, \mathrm{Spec}\,\rho_B, \mathrm{Spec}\,\rho_{AB}$. The later, without loss of generality, expected to be rational.

• Finally, the following equation for multiplicities

$$\text{Mult. } \mathcal{H}^\mu_A \otimes \mathcal{H}^\nu_B \text{ in } \mathcal{H}^\lambda_{AB} \mid_{\mathrm{SU}(\mathcal{H}_A)\times \mathrm{SU}(\mathcal{H}_B)} = \text{Mult. } \mathcal{S}^\lambda \text{ in } \mathcal{S}^\mu \otimes \mathcal{S}^\nu \tag{5.7}$$

reduces the problem to Kronecker coefficients of the symmetric group.

[2] *"The Littlewood–Richardson rule helped to get men on the moon, but it was not proved until after they had got there."*, see [21].

• Equivalence (2) $\Leftrightarrow$ (3) is strightforward and independent of representation theory. □

In the rest of this paper we'll use the theorem in both directions to obtain new results in quantum marginal problem and in representation theory.

6. Examples and applications

6.1. **Polygonal inequalities revisited.** Admitting an abuse of language we'll use the same notation for Young diagram λ and for the corresponding irreducible representation of the symmetric group S_n. It is convenient to single out the first row of diagram $\lambda = (n - |\bar{\lambda}|, \bar{\lambda})$. The remaining part $\bar{\lambda}$ is called *reduced diagram*, and the number of its cells $d(\lambda) = |\bar{\lambda}|$ is said to be *depth* of λ. We treat n as a parameter and use parentheses to return the original diagram $(\bar{\lambda}) = (n - |\bar{\lambda}|, \bar{\lambda}) = \lambda$.

For small values of n entries of $(\mu) = (n - |\mu|, \mu)$ may be not in decreasing order. Then the following rule is applied

$$(\ldots, p, q, \ldots) = -(\ldots, q-1, p+1, \ldots) \tag{6.1}$$

to transform (μ) into a Young diagram with sign $\pm$ or into zero (for $q = p+1$). By the above convention (μ) is also understood as a *virtual representation* of S_n.

Theorem (Murnaghan [37, 38], Littlewood [33])**.** (1) *Coefficients of decomposition*

$$(\bar{\lambda}) \otimes (\bar{\mu}) = \sum_{\bar{\nu}} \bar{g}(\bar{\lambda}, \bar{\mu}, \bar{\nu})(\bar{\nu}). \tag{6.2}$$

are independent of n, and $g(\lambda, \mu, \nu) = \bar{g}(\bar{\lambda}, \bar{\mu}, \bar{\nu})$ for $n \gg 1$.

(2) *The coefficient $g(\lambda, \mu, \nu)$ vanishes except depth of the diagrams satisfies triangle inequalities*

$$d(\lambda) \le d(\mu) + d(\nu), \qquad d(\mu) \le d(\nu) + d(\lambda), \qquad d(\nu) \le d(\lambda) + d(\mu). \tag{6.3}$$

(3) *In the case of equality in (6.3) the Kronecker coefficient coincides with Littlewood-Richardson one for reduced dyagrams*

$$g(\lambda, \mu, \nu) = C^{\bar{\lambda}}_{\bar{\mu}\bar{\nu}}, \qquad d(\lambda) = d(\mu) + d(\nu). \tag{6.4}$$

6.1.1. *Remark.* Murnaghan [37] gave dozens examples of decomposition (6.2) like the following one

$$\begin{aligned}(n-2,2) \otimes (n-2,1^2) = (n-1,1) + (n-2,2) + 2(n-2,1^2) + (n-3,3)\\ + 2(n-3,2,1) + (n-3,1^3) + (n-4,3,1) + (n-4,2,1^2).\end{aligned}$$

This equation literally gives Kronecker coefficients for $n \ge 7$. Otherwise rule (6.1) should be invoked which may result in cancelation of some virtual components. See [45] for a general stabilization bound for n.

6.1.2. *Remark.* Murnaghan theorem allows to defined new product of Young diagrams

$$\bar{\lambda} \circ \bar{\mu} = \sum_{\bar{\nu}} q^{\bar{\lambda} + \bar{\mu} - \bar{\nu}} g(\bar{\lambda}, \bar{\mu}, \bar{\nu}) \bar{\nu} \tag{6.5}$$

depending on parameter q. By (6.4) the multiplication is a *deformation* of cohomology ring of (infinite) Grassmannian. The deformation is different from *quantum cohomology* of Grassmannian which plays central role in unitary spectral problem [1]. It would be very interesting to find a geometric interpretation of this deformation.

Theorem 5.3.1 allows recast the second claim of Murnaghan's theorem into the following result.

Corollary 6.1.3. *Let $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ be pure state with univariant margins of spectra λ, μ, ν. Then their depth satisfies triangle inequalities*

$$d(\lambda) \le d(\mu) + d(\nu), \qquad d(\mu) \le d(\nu) + d(\lambda), \qquad d(\nu) \le d(\lambda) + d(\mu), \tag{6.6}$$

where the depth of spectrum $\lambda_1 \ge \lambda_2 \ge \lambda_3 \ge \cdots$ is $\lambda_2 + \lambda_3 + \cdots := d(\lambda)$.

6.1.4. *Remark.* Extension of the corollary to multicomponent systems is straightforward. For N-qubit it returns polygonal inequalities (3.7). The proof given by S. Bravyi [7] actually works for depth as well.

6.2. **Quasiclassical limit.** Here we analyze the boundary case (3) of Murnaghan's theorem in terms of the marginal problem.

Let ρ_{AB} be mixed state of composite system $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ with margins ρ_A, ρ_B and spectra $\lambda_{AB}, \lambda_A, \lambda_B$. Suppose triangle inequalities for depth (6.6) degenerate into equality

$$d(\lambda_{AB}) = d(\lambda_A) + d(\lambda_B). \tag{6.7}$$

Using notations

$$\psi_{AB}, \qquad \psi_A, \qquad \psi_B$$

for eigenstates of $\rho_{AB}, \rho_A, \rho_B$ with maximal eigenvalues,

$$\overline{\mathcal{H}}_{AB}, \qquad \overline{\mathcal{H}}_A, \qquad \overline{\mathcal{H}}_B$$

for their orthogonal complements, and

$$\overline{\rho}_{AB}, \qquad \overline{\rho}_A, \qquad \overline{\rho}_B$$

for restrictions of $\rho_{AB}, \rho_A, \rho_B$ onto subspaces $\overline{\mathcal{H}}_{AB}, \overline{\mathcal{H}}_A, \overline{\mathcal{H}}_B$, one can show that equation (6.7) is equivalent to the following conditions

- $\psi_{AB} = \psi_A \otimes \psi_B$;
- $\overline{\rho}_{AB}$ has support in subspace $\overline{\mathcal{H}}_A \otimes \psi_B + \psi_A \otimes \overline{\mathcal{H}}_B \simeq \overline{\mathcal{H}}_A \oplus \overline{\mathcal{H}}_B$;
- $\overline{\rho}_A$ and $\overline{\rho}_B$ are just restrictions of $\overline{\rho}_{AB}$ onto $\overline{\mathcal{H}}_A \otimes \psi_B$ and $\psi_A \otimes \overline{\mathcal{H}}_B$,

$$\overline{\rho}_{AB} = \left[\begin{array}{c|c} \overline{\rho}_A & * \\ \hline * & \overline{\rho}_B \end{array}\right] \tag{6.8}$$

We call this *quasiclassical limit* because all the events happen in classical direct sum $\overline{\mathcal{H}}_A \oplus \overline{\mathcal{H}}_B$ rather then in quantum tensor product. In this case Theorem 5.3.1 combined with equation (6.4) and Remark 5.2.1 gives

Theorem 6.2.1. *The following conditions are equivalent*

(1) $C^{\nu}_{\lambda\mu} \neq 0$, *that is* $\mathcal{H}_\nu \subset \mathcal{H}_\lambda \otimes \mathcal{H}_\mu$.

(2) *There exist Hermitian matrices H_A, H_B, and*

$$H_{AB} = \left[\begin{array}{c|c} H_A & * \\ \hline * & H_B \end{array}\right]$$

with spectra λ, μ, and ν respectively.

(3) *Horn–Klyachko inequalities hold for λ, μ, ν.*

One have to use multiple L-R coefficients

$$\mathcal{H}_\lambda \otimes \mathcal{H}_\mu \otimes \mathcal{H}_\nu \otimes \cdots = \sum_\sigma C^\sigma_{\lambda\mu\nu\cdots} \mathcal{H}_\sigma$$

to deal with arbitrary number of diagonal blocks $\mathcal{H}_A, \mathcal{H}_B, \mathcal{H}_C, \ldots$ of spectra $\lambda, \mu, \nu, \ldots$. For blocks of size one the theorem amounts to *Horn majorization inequality*

$$\operatorname{Spec} H \succ \operatorname{diag} H$$

since for one row diagrams $a, b, c, \ldots \quad C^\sigma_{abc\cdots} \neq 0 \Leftrightarrow \sigma \succ (a, b, c, \ldots)$.

6.2.2. *Remark.* It is more natural and easy deduce Theorem 6.2.1 directly from Berensein-Sjamaar theorem [5], then deduce it from Murnaghan results. See [30] for another approach. Note also that one can control spectrum of Hermitian matrix

$$H = \left[\begin{array}{c|c} * & X \\ \hline X^* & * \end{array}\right]$$

by linear inequalities in $\operatorname{Spec} H$ and *singular* spectrum of X [31, 11]. It would be very interesting to merge these two results to gain control over spectrum of Hamiltonian

$$H_{AB} = \left[\begin{array}{c|c} H_A & H_{\text{int}} \\ \hline H_{\text{int}}^* & H_B \end{array}\right]$$

of composite classical system in terms of spectra of the components A, B, and singular spectrum of H_{int}, which measures the *strength* of their interaction. This, however, can't be done by *linear* inequalities.

6.2.3. *Remark.* Note an important *saturation property* of Littlewood–Richardson coefficients [26]

$$C^{m\nu}_{m\lambda,m\mu} \neq 0 \Longleftrightarrow C^{\nu}_{\lambda\mu} \neq 0,$$

which simplifies the statement of Theorem 6.2.1 (cf. Theorem 5.3.1). It has no analogue for Kronecker coefficients, e.g. $2^2 \subset 2^2 \otimes 2^2$ but $1^2 \not\subset 1^2 \otimes 1^2$.

Conjectur 6.2.4. *Saturation property still holds for reduced Kronecker coefficients (6.2).*

For example, the saturation conjecture is valid for *leading* reduced coefficient $\bar{g}(\bar{\lambda}, \bar{\mu}, \bar{\nu}) = C^{\bar{\lambda}}_{\bar{\mu}\bar{\nu}}$, $\bar{\mu} + \bar{\nu} = \bar{\lambda}$, see Murnaghan theorem $n^\circ 6.1$.

6.3. **Maximal eigenvalue of a state with given margins.** Here and in the next $n^\circ 6.4$ we consider the following

Problem 6.3.1. *How close to a pure state can be mixed state ρ_{AB} with given margins ρ_A, ρ_B?*

As we have seen in $n^\circ 3.3.1$ margins of a pure state are isospectral. Hence for $\operatorname{Spec} \rho_A \neq \operatorname{Spec} \rho_B$ state ρ_{AB} can't be pure, and we want to get it as close to a pure state as possible. Recall that state ρ is pure iff its maximal eigenvalue is equal to one. Hence the maximal eigenvalue, known as *spectral norm* $\|\rho\|_s$ of operator ρ, may be considered as a measure of purity.

Theorem 6.3.1. *Let ρ_A, ρ_B be marginal states of spectra*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n, \qquad \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$$

extend by zeros to make them of the same length. Then maximal spectral norm of states ρ_{AB} with margins ρ_A, ρ_B is given by equation

$$\max \|\rho_{AB}\|_s = \sum_i \min(\lambda_i, \mu_i) = 1 - \frac{1}{2} \sum_i |\lambda_i - \mu_i| \,. \tag{6.9}$$

The proof amounts to application of Theorem 5.3.1 to the following representation theoretical counterpart.

Theorem (Klemm [22], Dvir [10], Clausen & Meier [9]). *Maximal length of the first row of Young diagrams $\nu \subset \lambda \otimes \mu$ is equal to $|\lambda \cap \mu| = \sum_i \min(\lambda_i, \mu_i)$.*

It is an interesting problem to describe states ρ_{AB} with maximal spectral norm. There is a simple formula for Kronecker coefficient of representation $\nu \subset \lambda \otimes \mu$ with maximal first row [10, 9]

$$g(\lambda, \mu, \nu) = \sum_{\alpha,\beta} C^{\lambda}_{\lambda\cap\mu,\alpha} C^{\mu}_{\lambda\cap\mu,\beta}\, g(\alpha, \beta, \overline{\nu}), \qquad \nu_1 = |\lambda \cap \mu| \,, \tag{6.10}$$

which allows, in principle, recursively find maximal component $\nu \subset \lambda \otimes \mu$ in lexicographic order. For example, ν_2 is equal to maximum of $|\alpha \cap \beta|$ s.t. both Littlewood–Richardson coefficients $C^{\lambda}_{\lambda\cap\mu,\alpha}, C^{\mu}_{\lambda\cap\mu,\beta}$ are nonzero. This combinatorial problem amounts to construction of a couple of skew standard tableaux of shapes $\lambda\backslash(\lambda\cap\mu)$ and $\mu\backslash(\lambda\cap\mu)$ with as close content as possible. I skip transposition of this into the marginal problem key.

See [46] for construction of other components in $\lambda \otimes \mu$.

6.4. **Rank of a state with given margins.** Pure state can be also characterized by its rank

$$\rho \text{ pure} \Longleftrightarrow \operatorname{rk} \rho = 1.$$

Hence $\operatorname{rk}\rho$ or better $\delta(\rho) = \log \operatorname{rk}\rho$ is another measure of purity (or rather impurity). The later is just *informational capacity* of support of ρ. It follows from Example 3.1.3 that margins of a pure state satisfy triangle inequalities

$$\delta(\rho_A) \le \delta(\rho_B) + \delta(\rho_C), \quad \delta(\rho_B) \le \delta(\rho_C) + \delta(\rho_A), \quad \delta(\rho_C) \le \delta(\rho_A) + \delta(\rho_B),$$

and the same is true for $\delta(\rho_A)$, $\delta(\rho_B)$, and $\delta(\rho_{AB})$. Representation theoretical counterpart of this

$$\operatorname{ht}(\nu) \le \operatorname{ht}(\lambda)\cdot\operatorname{ht}(\mu) \quad \text{for} \quad \nu \subset \lambda \otimes \mu \tag{6.11}$$

comes back to Schur [44], see also [39]. Here the *height* $\operatorname{ht}(\nu)$ of diagram ν is the number of its rows. Note that decomposable state $\rho_{AB} = \rho_A \otimes \rho_B$ has margins ρ_A, ρ_B, and *maximal* possible rank $\operatorname{rk}\rho_{AB} = \operatorname{rk}\rho_A \operatorname{rk}\rho_B$.

Theorem 6.4.1. *There exists state ρ_{AB} of two component system with given margins ρ_A, ρ_B and*

$$\operatorname{rk}\rho_{AB} \le \max(\operatorname{rk}\rho_A, \operatorname{rk}\rho_B). \tag{6.12}$$

In addition, such a state can be taken with maximal possible spectral norm, given by Theorem 6.3.1.

The proof of the theorem once again is just a translation into marginal problem language of the following representation theoretical counterpart.

Theorem (Berele & Imbo [4])**.** *Let λ and μ be Young diagram of height $\le k$. Then tensor product $\lambda \otimes \mu$ contains a component ν of height $\le k$.*

Vallejo [46] gave an explicit construction of such a component $\nu \subset \lambda \otimes \mu$ with maximal possible first row $\nu_1 = |\lambda \cap \mu|$.

Conjectur 6.4.2. *State ρ_{AB} with maximal lexicographical spectrum has minimal rank among all states with given margins ρ_A, ρ_B.*

6.5. **Relation with classical marginal problem.** Triplet of spectra λ, μ, ν is said to be *quasiclassical* iff there exist $p_{ij} \ge 0$ such that

$$\lambda = \sum_i p_{ij}, \qquad \mu = \sum_j p_{ij}, \qquad p \prec \nu, \tag{6.13}$$

where the last condition means that content of matrix $p = [p_{ij}]$ being arranged in decreasing order is majorized by ν. If the content of matrix p coincides with ν the triplet is said to be *classical*, cf. $n^\circ 2.0.5$. We borrow this definition, with a minor modification, from Sergei Bravyi [7] who proved part (1) $\Leftrightarrow$ (2) of the next theorem.

Theorem 6.5.1. *The following conditions on spectra λ, μ, ν are equivalent*

(1) *Triplet λ, μ, ν is quasiclassical.*

(2) *There exists mixed state* ρ_{AB} *such that*

$$\operatorname{Spec}\rho_{AB}\prec\nu,\quad \operatorname{Spec}\rho_A=\lambda,\quad \operatorname{Spec}\rho_B=\mu.$$

(3) *There exists mixed state* $\widetilde{\rho}_{AB}$ *such that*

$$\operatorname{Spec}\widetilde{\rho}_{AB}=\nu,\quad \operatorname{Spec}\widetilde{\rho}_A\succ\lambda,\quad \operatorname{Spec}\widetilde{\rho}_B\succ\mu.$$

The second part (1) $\Leftrightarrow$ (3) comes from Theorem 5.3.1 and standard facts from representation theory of the symmetric group:

- Let $[\lambda]$ be *permutation representation* induced from Schur subgroup $\mathrm{S}_\lambda=\mathrm{S}_{\lambda_1}\times\mathrm{S}_{\lambda_2}\times\cdots\times\mathrm{S}_{\lambda_m}\subset\mathrm{S}_n$. Then $\widetilde{\lambda}\subset[\lambda]\Longleftrightarrow\widetilde{\lambda}\succ\lambda$.
- Hence $\nu\subset\widetilde{\lambda}\otimes\widetilde{\mu}$ for some $\widetilde{\lambda}\succ\lambda$ and $\widetilde{\nu}\succ\nu$ iff $\nu\subset[\lambda]\otimes[\mu]$.
- $\nu\subset[\lambda]\otimes[\mu]\Longleftrightarrow\lambda,\mu$ are margins of integer matrix $p_{ij}\geq 0$ s.t. $p\prec\nu$.

The first property is actually a source of the definition of irreducible representation $\mathcal{S}_\lambda$ (currently shortened to λ) in $n^\circ 5.1$. The last one can be seen from decomposition of tensor product $[\lambda]\otimes[\mu]$ into direct sum of permutation modules corresponding to intersection $Z_{ij}=X_i\cap Y_j$ of partitions

$$\{1,2,\dots,n\}=\bigsqcup_i X_i,\qquad |X_i|=\lambda_i,$$
$$\{1,2,\dots,n\}=\bigsqcup_j Y_j,\qquad |Y_j|=\mu_j,$$

of types λ and μ. It follows that every component $\nu\subset[\lambda]\otimes[\mu]$ satisfies (6.13) with $p_{ij}=|Z_{ij}|$, and vice versa.

Corollary 6.5.2. *Let* ρ_{AB} *be mixed state with given margins* ρ_A,ρ_B *of spectra* λ_A,λ_B. *Suppose that* $\lambda_{AB}=\operatorname{Spec}\rho_{AB}$ *is minimal possible in majorization order. Then triplet* $\lambda_A,\lambda_B,\lambda_{AB}$ *is classical.*

6.5.3. *Remark.* Implication (3) $\Rightarrow$ (2) has a simple analytical proof. Indeed, inequality $\operatorname{Spec}\widetilde{\rho}_A\succ\lambda$ means that $\rho_A=\sum_i p_iU_i\widetilde{\rho}_AU_i^\dagger$ has spectrum λ for some unitary operators U_i and probabilities $p_i\geq 0$ [48]. In a similar way there exists $\rho_B=\sum_j q_jV_j\widetilde{\rho}_BV_j^\dagger$ with spectrum μ. Then

$$\rho_{AB}=\sum_{ij}p_iq_j(U_i\otimes V_j)\widetilde{\rho}_{AB}(U_i^\dagger\otimes V_j^\dagger)$$

has margins ρ_A,ρ_B and $\operatorname{Spec}\rho_{AB}\prec\operatorname{Spec}\widetilde{\rho}_{AB}=\nu$.

Neither *analytical* proof of (2) $\Rightarrow$ (3), nor *representation theoretical* proof of (3) $\Rightarrow$ (2) are known, see n°**??** below.

6.6. **Two qubit revisited.** Here we return back to two qubit marginal problem $n^\circ 3.3.4$ from representation theoretical perspective. By Theorem 5.3.1 it amounts to decomposition of tensor product

$$\lambda\otimes\mu=\sum_\nu g(\lambda,\mu,\nu)\nu$$

of representations defined by *two row* diagrams $\lambda=(\lambda_1,\lambda_2)$ and $\mu=(\mu_1,\mu_2)$. By Schur inequality (6.11) the components $\nu=(\nu_1,\nu_2,\nu_3,\nu_4)$ have at most four rows.

Currently neither combinatorial description nor efficient algorithm are available for Kronecker coefficients $g(\lambda,\mu,\nu)$, in contrast with Littlewood–Richardson ones. One of the first results in this direction belongs to A. Lascoux [28] who resolved the case of hook diagrams λ,μ. For two row diagrams the problem was first addressed by Remmel and Whitehead in long paper [40], and later by Rosas [41]. Unfortunately the results are too complicated to be reproduced here. Mersedes Rosas write down $g(\lambda,\mu,\nu)$ as a *difference* of the number of lattice points in two polygons. A close look at her formula shows that for symmetric group S_n of *odd degree* n one polygon can be moved into another by an affine transformation respecting the lattice. As result we get

Claim 6.6.1. *For odd n Kronecker coefficient $g(\lambda, \mu, \nu)$ for two row diagrams λ, μ is equal to the number of integers a, b satisfying the following conditions*

$$\nu_3 + \nu_4 \leq a \leq \nu_2 + \nu_4, \tag{6.14a}$$

$$\nu_2 + \nu_4 \leq b \leq \min(\nu_2 + \nu_3, \nu_1 + \nu_4), \tag{6.14b}$$

$$\lambda_2 - \mu_2 \leq b - a \leq \lambda_1 - \mu_2, \tag{6.14c}$$

$$a + b \leq \lambda_2 + \mu_2, \tag{6.14d}$$

$$a + b \equiv \lambda_2 + \mu_2 \pmod 2, \tag{6.14e}$$

where we expect, without loss of generality, $\lambda_2 \geq \mu_2$.

It follows that

$$\begin{aligned}
\lambda_2 + \mu_2 \geq a + b \geq \nu_2 + \nu_3 + 2\nu_4,\\
|\lambda_2 - \mu_2| \leq b - a \leq \min(\nu_2 - \nu_4, \nu_1 - \nu_3),\\
\lambda_2 \geq a \geq \nu_3 + \nu_4,\\
\mu_2 \geq a \geq \nu_3 + \nu_4,
\end{aligned}$$

which in turn implies Bravyi inequalities (3.8). One can easily check that the later are sufficient for $g(\lambda, \mu, \nu) \neq 0$. This gives another proof of Bravyi theorem $n°3.3.4$.

6.6.2. *Remark.* For even n a precise description of $g(\lambda, \mu, \nu)$ is more complicate. Admitting an absolute error ≤ 1 the Kronecker coefficient is equal to *weighted* number of lattice points in heptagon (6.14), assigning weight 1/2 to boundary points on the line $b - a = \lambda_1 - \mu_2$.

Available information on Kronecker coefficients of three row diagrams seems to be insufficient to deduce Higichi theorem $n°3.3.5$ or 397 independent marginal inequalities for two qutrits in section A.2.

7. Applications to representation theory

Unfortunately currently available information on Kronecker coefficients is sparse and sporadic.[3] Therefore direct applications of Theorem 5.3.1 to marginal problem is limited. We believe however that interplay between representation theory of S_n and quantum margins eventually may lead to solution of both problems simultaneously, as it happens, for example, for Hermitian spectral problem and representations of $SU(\mathcal{H})$. We leave this topic for a future study, and bound ourselves to few examples of using Theorem 5.3.1 in backward direction. One obstruction to this is lack of the saturation property (see Remark 6.2.3), which bounds rigorous results to *stable regime* of very long diagrams $m\lambda$, $m \gg 1$, although some of them may still hold in general. We'll use special notation for *stable inclusion*

$$\nu \underset{stbl}{\subset} \lambda \otimes \mu \Longleftrightarrow m\nu \subset m\lambda \otimes m\mu \text{ for some } m > 0, \tag{7.1}$$

where the diagrams are multiplied by m row-wise.

[3] Arguably this is *"... the last major problem in ordinary representation theory of S_n"* [10].

7.1. **Stable support of Kronecker coefficients.** Here we are interested in triplets of Young diagrams λ, μ, ν such that $\lambda \subset \mu \otimes \nu$, i.e. in *support* of Kronecker coefficients. We have seen, for example, that for such triplets $\mathrm{ht}(\lambda) \le \mathrm{ht}(\mu)\,\mathrm{ht}(\nu)$ (n°6.4), and $|\overline{\lambda}| \le |\overline{\mu}| + |\overline{\nu}|$ (n°6.1). It is much easier to describe *stable support*, defined by stable inclusion (7.1).

Theorem 7.1.1. *Stable inclusion* $\nu \underset{stab}{\subset} \lambda \otimes \mu$ *for diagrams* λ, μ *of bounded heights*

$$\mathrm{ht}\,\lambda \le m, \qquad \mathrm{ht}\,\mu \le n$$

is given by marginal inequalities for system of format $m \times n$ *with* $\lambda = \mathrm{Spec}\,\rho_A$, $\mu = \mathrm{Spec}\,\rho_B$, $\nu = \mathrm{Spec}\,\rho_{AB}$, .

Proof. This follows from Theorem 5.3.1 combined with marginal inequalities of section 4.1. □

For example, for two row diagrams the stable inclusion amounts to Bravyi inequalities, cf. section 6.6.

Corollary 7.1.2. *Suppose* $\nu^{(i)} \subset \lambda^{(i)} \otimes \mu^{(i)}$. *Then*

$$(\nu^{(1)} + \nu^{(2)}) \underset{stab}{\subset} (\lambda^{(1)} + \lambda^{(2)}) \otimes (\mu^{(1)} + \mu^{(2)}) \tag{7.2}$$

where addition of the diagrams is defined row-wise.

7.1.3. *Remark.* Numerical experiments suggest that inclusion (7.2) actually holds in usual sense. This leads to the following

Conjectur 7.1.4. *Triplets* λ, μ, ν *s.t.* $g(\lambda, \mu, \nu) \ne 0$ *form a semigroup with respect to row-wise addition.*

One can expect that the semigroup is finitely generated if heights of the diagrams are bounded.

7.2. **Rectangular diagrams.** In the case of *rectanguler* diagrams the previous problem has very simple answer.

Theorem 7.2.1. *Let* $\lambda, \mu\nu$ *be rectangular diagrams of heights* l, m, n. *Then stable inclusion* $\nu \underset{stab}{\subset} \lambda \otimes \mu$ *is equivalent to inequalities* $l \le mn$, $m \le nl$, $n \le lm$.

Proof. The theorem is about stable inclusion of trivial representation in $\lambda \otimes \mu \otimes \nu$. By Theorem 5.3.1 trivial representation corresponds to a *pure* state ρ_{ABC} of three component system $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ of formal $l \times m \times n$, while rectangular diagrams $\lambda\mu\nu$ correspond to *scalar* margins ρ_A, ρ_B, ρ_C. The result now follows from criterion of existence of a pure state with scalar margins. in section 3.3.2. □

7.2.2. *Remark.* The simplest proof of the above criterion comes from marginal inequalities of Theorem 4.1.1.

Appendix A. Marginal inequalities

The appendix contains marginal inequalities for systems of rank ≤ 4, i.e. for arrays up to four qubits and for systems of formats 2×3, 2×4, 3×3, and $2\times2\times3$. It covers all systems with few hundreds (rather then thousands) marginal inequalities. Other statistical data are collected in the following table.

System	Rank	Inequalities	Edges	Permutations
2×2	2	7 (4)	3	9
$2\times 2\times 2$	3	40 (10)	10	111
2×3	3	41	6	98
2×4	4	234	11	2191
3×3	4	387 (197)	17	2298
$2\times 2\times 3$	4	442 (232)	39	3914
$2\times 2\times 2\times 2$	4	805 (50)	101	3723

For system of format $p\times q\times\cdots$ the last two columns show the number of extremal edges and the number of permutations $w\in S_{p\cdot q\cdots}$ of length $\ell(w)\leq \frac{p(p-1)}{2}+\frac{q(q-1)}{2}+\cdots$. The number of marginal inequalities produced by permutation w and extremal edge E is equal to the number of components in $\varphi_E^*(\sigma_w)$. For system of rank 4 this amounts all together to hundreds of thousands inequalities. compared with seconds for rank ≤ 3). Note that the number of inequalities in most cases can be essentially reduced using symmetry w.r. to permutations of equidimensional components. These numbers are shown in parenthesis. Further reduction is possible using duality

$$(\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n)\mapsto(-\lambda_n\geq-\lambda_{n-1}\geq\cdots\geq-\lambda_1)$$

applied to every spectrum involved in a marginal inequality.

Because of huge number of marginal inequalities for systems of rank more then three, they provide only an illusion of a complete solution of the problem and have limited practical value. I reproduce the inequalities here primary as experimental data for those brave people who may try to understand them better.

Details of the calculation [25] will be published elsewhere.

A.1. **Arrays of qubits.** In this case all marginal inequalities can be produced automatically from the list of extremal edges, see n^o 4 for details. All the spectra are arranged in decreasing order $\rho_1\geq\rho_2\geq\cdots$. By technical reason we normalize mixed states to trace zero. Hence univariant margins have spectra $(\lambda,-\lambda),(\mu,-\mu)$, etc. To get marginal inequalities in standard normalization $\operatorname{Tr}\rho=1$ one have to change $2\lambda,2\mu,\ldots$ in LHS of the inequalities by $\lambda_1-\lambda_2,\mu_1-\mu_2,\ldots$. To save space we skip all the inequalities obtained from another one by a permutation of qubits. The reduced systems are complete if marginal spectra arranged in increasing order $\lambda\leq\mu\leq\cdots$.

A.1.1. *Two qubits.* For completeness we reproduce here Bravyi inequalities (3.8) for two qubits in current notations.

$$\begin{aligned}
2\lambda &\leq \nu_1+\nu_2-\nu_3-\nu_4,\\
\lambda+\mu &\leq \nu_1-\nu_4,\\
\lambda-\mu &\leq \nu_2-\nu_4,\\
\lambda-\mu &\leq \nu_1-\nu_3.
\end{aligned}$$

A.1.2. *Three qubits.* Below are 10 marginal inequalities for three qubits. By permutations of qubits they give a complete system of 40 independent constraints.

$$2\lambda \le \rho_1+\rho_2+\rho_3+\rho_4-\rho_5-\rho_6-\rho_7-\rho_8,$$
$$\lambda+\mu \le \rho_1+\rho_2-\rho_7-\rho_8,$$

$$2\lambda+\mu+\nu \le 2\rho_1+\rho_2+\rho_3-\rho_6-\rho_7-2\rho_8,$$
$$2\lambda+\mu-\nu \le 2\rho_2+\rho_1+\rho_3-\rho_6-\rho_7-2\rho_8,$$
$$2\lambda+\mu-\nu \le 2\rho_1+\rho_2+\rho_4-\rho_6-\rho_7-2\rho_8,$$
$$2\lambda+\mu-\nu \le 2\rho_1+\rho_2+\rho_3-\rho_5-\rho_7-2\rho_8,$$
$$2\lambda+\mu-\nu \le 2\rho_1+\rho_2+\rho_3-\rho_6-\rho_8-2\rho_7,$$

$$2\lambda+2\mu+2\nu \le 3\rho_1+\rho_2+\rho_3+\rho_4-\rho_5-\rho_6-\rho_7-3\rho_8,$$
$$2\lambda+2\mu-2\nu \le 3\rho_2+\rho_1+\rho_3+\rho_4-\rho_5-\rho_6-\rho_7-3\rho_8,$$
$$2\lambda+2\mu-2\nu \le 3\rho_1+\rho_2+\rho_3+\rho_4-\rho_5-\rho_6-\rho_8-3\rho_7.$$

A.1.3. *Four qubits.* In this case there are 805 independent marginal inequalities. To save space we skip inequalities obtained from a preceding one by a permutation of qubits. The resulting 50 inequalities are grouped by the extremal edges. The basic inequality stands first, and the remaining ones in the group are obtained by transposition of eigenvalues τ_{2i-1},τ_{2i} typeset in bold face and change sign of the first coefficient in LHS, see Theorem 4.2.3 for details. Note that in the fifth group odd transpositions τ_5,τ_6 and τ_{11},τ_{12} give redundant inequalities. Arguably this is the last system of marginal constraints which can be published. For 5 qubits there are more then thousand independent inequalities counted up to a permutation of qubits.

$$2\rho \le \tau_1+\tau_2+\tau_3+\tau_4+\tau_5+\tau_6+\tau_7+\tau_8-\tau_9-\tau_{10}-\tau_{11}-\tau_{12}-\tau_{13}-\tau_{14}-\tau_{15}-\tau_{16},$$

$$2\nu+2\rho \le 2\tau_1+2\tau_2+2\tau_3+2\tau_4-2\tau_{13}-2\tau_{14}-2\tau_{15}-2\tau_{16},$$

$$2\mu+2\nu+2\rho \le 3\tau_1+3\tau_2+\tau_3+\tau_4+\tau_5+\tau_6+\tau_7+\tau_8-\tau_9-\tau_{10}-\tau_{11}-\tau_{12}-\tau_{13}-\tau_{14}-3\tau_{15}-3\tau_{16},$$

$$2\mu+2\nu+4\rho \le 4\tau_1+4\tau_2+2\tau_3+2\tau_4+2\tau_5+2\tau_6-2\tau_{11}-2\tau_{12}-2\tau_{13}-2\tau_{14}-4\tau_{15}-4\tau_{16},$$

$$2\lambda+2\mu+2\nu+2\rho \le 4\tau_1+2\tau_2+2\tau_3+2\tau_4+2\tau_5-2\tau_{12}-2\tau_{13}-2\tau_{14}-2\tau_{15}-4\tau_{16},$$
$$2\lambda+2\mu+2\nu-2\rho \le 4\boldsymbol{\tau_2}+2\boldsymbol{\tau_1}+2\tau_3+2\tau_4+2\tau_5-2\tau_{12}-2\tau_{13}-2\tau_{14}-2\tau_{15}-4\tau_{16},$$
$$2\lambda+2\mu+2\nu-2\rho \le 4\tau_1+2\tau_2+2\tau_3+2\tau_4+2\tau_5-2\tau_{12}-2\tau_{13}-2\tau_{14}-2\boldsymbol{\tau_{16}}-4\boldsymbol{\tau_{15}},$$

$$2\lambda+2\mu+2\nu+4\rho \le 5\tau_1+3\tau_2+3\tau_3+3\tau_4+\tau_5+\tau_6+\tau_7+\tau_8-\tau_9-\tau_{10}-\tau_{11}-\tau_{12}-3\tau_{13}-3\tau_{14}-3\tau_{15}-5\tau_{16},$$
$$2\lambda+2\mu-2\nu+4\rho \le 5\boldsymbol{\tau_2}+3\boldsymbol{\tau_1}+3\tau_3+3\tau_4+\tau_5+\tau_6+\tau_7+\tau_8-\tau_9-\tau_{10}-\tau_{11}-\tau_{12}-3\tau_{13}-3\tau_{14}-3\tau_{15}-5\tau_{16},$$
$$2\lambda+2\mu-2\nu+4\rho \le 5\tau_1+3\tau_2+3\tau_3+3\tau_4+\tau_5+\tau_6+\tau_7+\tau_8-\tau_9-\tau_{10}-\tau_{11}-\tau_{12}-3\tau_{13}-3\tau_{14}-3\boldsymbol{\tau_{16}}-5\boldsymbol{\tau_{15}},$$

$$2\lambda+2\mu+2\nu+6\rho \le 6\tau_1+4\tau_2+4\tau_3+4\tau_4+2\tau_5+2\tau_6+2\tau_7-2\tau_{10}-2\tau_{11}-2\tau_{12}-4\tau_{13}-4\tau_{14}-4\tau_{15}-6\tau_{16},$$
$$2\lambda+2\mu-2\nu+6\rho \le 6\boldsymbol{\tau_2}+4\boldsymbol{\tau_1}+4\tau_3+4\tau_4+2\tau_5+2\tau_6+2\tau_7-2\tau_{10}-2\tau_{11}-2\tau_{12}-4\tau_{13}-4\tau_{14}-4\tau_{15}-6\tau_{16},$$
$$2\lambda+2\mu-2\nu+6\rho \le 6\tau_1+4\tau_2+4\tau_3+4\tau_4+2\tau_5+2\tau_6+2\boldsymbol{\tau_8}-2\tau_{10}-2\tau_{11}-2\tau_{12}-4\tau_{13}-4\tau_{14}-4\tau_{15}-6\tau_{16},$$
$$2\lambda+2\mu-2\nu+6\rho \le 6\tau_1+4\tau_2+4\tau_3+4\tau_4+2\tau_5+2\tau_6+2\tau_7-2\boldsymbol{\tau_9}-2\tau_{11}-2\tau_{12}-4\tau_{13}-4\tau_{14}-4\tau_{15}-6\tau_{16},$$
$$2\lambda+2\mu-2\nu+6\rho \le 6\tau_1+4\tau_2+4\tau_3+4\tau_4+2\tau_5+2\tau_6+2\tau_7-2\tau_{10}-2\tau_{11}-2\tau_{12}-4\tau_{13}-4\tau_{14}-4\boldsymbol{\tau_{16}}-6\boldsymbol{\tau_{15}},$$

$$2\lambda+2\mu+4\nu+4\rho \le 6\tau_1+4\tau_2+4\tau_3+2\tau_4+2\tau_5+2\tau_6-2\tau_{11}-2\tau_{12}-2\tau_{13}-4\tau_{14}-4\tau_{15}-6\tau_{16},$$
$$2\lambda-2\mu+4\nu+4\rho \le 6\boldsymbol{\tau_2}+4\boldsymbol{\tau_1}+4\tau_3+2\tau_4+2\tau_5+2\tau_6-2\tau_{11}-2\tau_{12}-2\tau_{13}-4\tau_{14}-4\tau_{15}-6\tau_{16},$$
$$2\lambda-2\mu+4\nu+4\rho \le 6\tau_1+4\tau_2+4\boldsymbol{\tau_4}+2\boldsymbol{\tau_3}+2\tau_5+2\tau_6-2\tau_{11}-2\tau_{12}-2\tau_{13}-4\tau_{14}-4\tau_{15}-6\tau_{16},$$
$$2\lambda-2\mu+4\nu+4\rho \le 6\tau_1+4\tau_2+4\tau_3+2\tau_4+2\tau_5+2\tau_6-2\tau_{11}-2\tau_{12}-2\boldsymbol{\tau_{14}}-4\boldsymbol{\tau_{13}}-4\tau_{15}-6\tau_{16},$$
$$2\lambda-2\mu+4\nu+4\rho \le 6\tau_1+4\tau_2+4\tau_3+2\tau_4+2\tau_5+2\tau_6-2\tau_{11}-2\tau_{12}-2\tau_{13}-4\tau_{14}-4\boldsymbol{\tau_{16}}-6\boldsymbol{\tau_{15}},$$

$$\begin{aligned}
2\lambda+2\mu+4\nu+6\rho &\le 7\tau_1+5\tau_2+5\tau_3+3\tau_4+3\tau_5+\tau_6+\tau_7+\tau_8-\tau_9-\tau_{10}-\tau_{11}-3\tau_{12}-3\tau_{13}-5\tau_{14}-5\tau_{15}-7\tau_{16},\\
2\lambda-2\mu+4\nu+6\rho &\le 7\boldsymbol{\tau_2}+5\boldsymbol{\tau_1}+5\tau_3+3\tau_4+3\tau_5+\tau_6+\tau_7+\tau_8-\tau_9-\tau_{10}-\tau_{11}-3\tau_{12}-3\tau_{13}-5\tau_{14}-5\tau_{15}-7\tau_{16},\\
2\lambda-2\mu+4\nu+6\rho &\le 7\tau_1+5\tau_2+5\boldsymbol{\tau_4}+3\boldsymbol{\tau_3}+3\tau_5+\tau_6+\tau_7+\tau_8-\tau_9-\tau_{10}-\tau_{11}-3\tau_{12}-3\tau_{13}-5\tau_{14}-5\tau_{15}-7\tau_{16},\\
2\lambda-2\mu+4\nu+6\rho &\le 7\tau_1+5\tau_2+5\tau_3+3\tau_4+3\boldsymbol{\tau_6}+\boldsymbol{\tau_5}+\tau_7+\tau_8-\tau_9-\tau_{10}-\tau_{11}-3\tau_{12}-3\tau_{13}-5\tau_{14}-5\tau_{15}-7\tau_{16},\\
2\lambda-2\mu+4\nu+6\rho &\le 7\tau_1+5\tau_2+5\tau_3+3\tau_4+3\tau_5+\tau_6+\tau_7+\tau_8-\tau_9-\tau_{10}-\boldsymbol{\tau_{12}}-3\boldsymbol{\tau_{11}}-3\tau_{13}-5\tau_{14}-5\tau_{15}-7\tau_{16},\\
2\lambda-2\mu+4\nu+6\rho &\le 7\tau_1+5\tau_2+5\tau_3+3\tau_4+3\tau_5+\tau_6+\tau_7+\tau_8-\tau_9-\tau_{10}-\tau_{11}-3\tau_{12}-3\boldsymbol{\tau_{14}}-5\boldsymbol{\tau_{13}}-5\tau_{15}-7\tau_{16},\\
2\lambda-2\mu+4\nu+6\rho &\le 7\tau_1+5\tau_2+5\tau_3+3\tau_4+3\tau_5+\tau_6+\tau_7+\tau_8-\tau_9-\tau_{10}-\tau_{11}-3\tau_{12}-3\tau_{13}-5\tau_{14}-5\boldsymbol{\tau_{16}}-7\boldsymbol{\tau_{15}},
\end{aligned}$$

$$\begin{aligned}
2\lambda+4\mu+4\nu+6\rho &\le 8\tau_1+6\tau_2+4\tau_3+4\tau_4+2\tau_5+2\tau_6+2\tau_7-2\tau_{10}-2\tau_{11}-2\tau_{12}-4\tau_{13}-4\tau_{14}-6\tau_{15}-8\tau_{16},\\
-2\lambda+4\mu+4\nu+6\rho &\le 8\boldsymbol{\tau_2}+6\boldsymbol{\tau_1}+4\tau_3+4\tau_4+2\tau_5+2\tau_6+2\tau_7-2\tau_{10}-2\tau_{11}-2\tau_{12}-4\tau_{13}-4\tau_{14}-6\tau_{15}-8\tau_{16},\\
-2\lambda+4\mu+4\nu+6\rho &\le 8\tau_1+6\tau_2+4\tau_3+4\tau_4+2\tau_5+2\tau_6+2\boldsymbol{\tau_8}-2\tau_{10}-2\tau_{11}-2\tau_{12}-4\tau_{13}-4\tau_{14}-6\tau_{15}-8\tau_{16},\\
-2\lambda+4\mu+4\nu+6\rho &\le 8\tau_1+6\tau_2+4\tau_3+4\tau_4+2\tau_5+2\tau_6+2\tau_7-2\boldsymbol{\tau_9}-2\tau_{11}-2\tau_{12}-4\tau_{13}-4\tau_{14}-6\tau_{15}-8\tau_{16},\\
-2\lambda+4\mu+4\nu+6\rho &\le 8\tau_1+6\tau_2+4\tau_3+4\tau_4+2\tau_5+2\tau_6+2\tau_7-2\tau_{10}-2\tau_{11}-2\tau_{12}-4\tau_{13}-4\tau_{14}-6\boldsymbol{\tau_{16}}-8\boldsymbol{\tau_{15}},
\end{aligned}$$

$$\begin{aligned}
2\lambda+2\mu+4\nu+8\rho &\le 8\tau_1+6\tau_2+6\tau_3+4\tau_4+4\tau_5+2\tau_6+2\tau_7-2\tau_{10}-2\tau_{11}-4\tau_{12}-4\tau_{13}-6\tau_{14}-6\tau_{15}-8\tau_{16},\\
2\lambda-2\mu+4\nu+8\rho &\le 8\boldsymbol{\tau_2}+6\boldsymbol{\tau_1}+6\tau_3+4\tau_4+4\tau_5+2\tau_6+2\tau_7-2\tau_{10}-2\tau_{11}-4\tau_{12}-4\tau_{13}-6\tau_{14}-6\tau_{15}-8\tau_{16},\\
2\lambda-2\mu+4\nu+8\rho &\le 8\tau_1+6\tau_2+6\boldsymbol{\tau_4}+4\boldsymbol{\tau_3}+4\tau_5+2\tau_6+2\tau_7-2\tau_{10}-2\tau_{11}-4\tau_{12}-4\tau_{13}-6\tau_{14}-6\tau_{15}-8\tau_{16},\\
2\lambda-2\mu+4\nu+8\rho &\le 8\tau_1+6\tau_2+6\tau_3+4\tau_4+4\boldsymbol{\tau_6}+2\boldsymbol{\tau_5}+2\tau_7-2\tau_{10}-2\tau_{11}-4\tau_{12}-4\tau_{13}-6\tau_{14}-6\tau_{15}-8\tau_{16},\\
2\lambda-2\mu+4\nu+8\rho &\le 8\tau_1+6\tau_2+6\tau_3+4\tau_4+4\tau_5+2\tau_6+2\boldsymbol{\tau_8}-2\tau_{10}-2\tau_{11}-4\tau_{12}-4\tau_{13}-6\tau_{14}-6\tau_{15}-8\tau_{16},\\
2\lambda-2\mu+4\nu+8\rho &\le 8\tau_1+6\tau_2+6\tau_3+4\tau_4+4\tau_5+2\tau_6+2\tau_7-2\boldsymbol{\tau_9}-2\tau_{11}-4\tau_{12}-4\tau_{13}-6\tau_{14}-6\tau_{15}-8\tau_{16},\\
2\lambda-2\mu+4\nu+8\rho &\le 8\tau_1+6\tau_2+6\tau_3+4\tau_4+4\tau_5+2\tau_6+2\tau_7-2\tau_{10}-2\boldsymbol{\tau_{12}}-4\boldsymbol{\tau_{11}}-4\tau_{13}-6\tau_{14}-6\tau_{15}-8\tau_{16},\\
2\lambda-2\mu+4\nu+8\rho &\le 8\tau_1+6\tau_2+6\tau_3+4\tau_4+4\tau_5+2\tau_6+2\tau_7-2\tau_{10}-2\tau_{11}-4\tau_{12}-4\boldsymbol{\tau_{14}}-6\boldsymbol{\tau_{13}}-6\tau_{15}-8\tau_{16},\\
2\lambda-2\mu+4\nu+8\rho &\le 8\tau_1+6\tau_2+6\tau_3+4\tau_4+4\tau_5+2\tau_6+2\tau_7-2\tau_{10}-2\tau_{11}-4\tau_{12}-4\tau_{13}-6\tau_{14}-6\boldsymbol{\tau_{16}}-8\boldsymbol{\tau_{15}},
\end{aligned}$$

$$\begin{aligned}
2\lambda+4\mu+6\nu+8\rho &\le 10\tau_1+8\tau_2+6\tau_3+4\tau_4+4\tau_5+2\tau_6+2\tau_7-2\tau_{10}-2\tau_{11}-4\tau_{12}-4\tau_{13}-6\tau_{14}-8\tau_{15}-10\tau_{16},\\
-2\lambda+4\mu+6\nu+8\rho &\le 10\boldsymbol{\tau_2}+8\boldsymbol{\tau_1}+6\tau_3+4\tau_4+4\tau_5+2\tau_6+2\tau_7-2\tau_{10}-2\tau_{11}-4\tau_{12}-4\tau_{13}-6\tau_{14}-8\tau_{15}-10\tau_{16},\\
-2\lambda+4\mu+6\nu+8\rho &\le 10\tau_1+8\tau_2+6\boldsymbol{\tau_4}+4\boldsymbol{\tau_3}+4\tau_5+2\tau_6+2\tau_7-2\tau_{10}-2\tau_{11}-4\tau_{12}-4\tau_{13}-6\tau_{14}-8\tau_{15}-10\tau_{16},\\
-2\lambda+4\mu+6\nu+8\rho &\le 10\tau_1+8\tau_2+6\tau_3+4\tau_4+4\boldsymbol{\tau_6}+2\boldsymbol{\tau_5}+2\tau_7-2\tau_{10}-2\tau_{11}-4\tau_{12}-4\tau_{13}-6\tau_{14}-8\tau_{15}-10\tau_{16},\\
-2\lambda+4\mu+6\nu+8\rho &\le 10\tau_1+8\tau_2+6\tau_3+4\tau_4+4\tau_5+2\tau_6+2\boldsymbol{\tau_8}-2\tau_{10}-2\tau_{11}-4\tau_{12}-4\tau_{13}-6\tau_{14}-8\tau_{15}-10\tau_{16},\\
-2\lambda+4\mu+6\nu+8\rho &\le 10\tau_1+8\tau_2+6\tau_3+4\tau_4+4\tau_5+2\tau_6+2\tau_7-2\boldsymbol{\tau_9}-2\tau_{11}-4\tau_{12}-4\tau_{13}-6\tau_{14}-8\tau_{15}-10\tau_{16},\\
-2\lambda+4\mu+6\nu+8\rho &\le 10\tau_1+8\tau_2+6\tau_3+4\tau_4+4\tau_5+2\tau_6+2\tau_7-2\tau_{10}-2\boldsymbol{\tau_{12}}-4\boldsymbol{\tau_{11}}-4\tau_{13}-6\tau_{14}-8\tau_{15}-10\tau_{16},\\
-2\lambda+4\mu+6\nu+8\rho &\le 10\tau_1+8\tau_2+6\tau_3+4\tau_4+4\tau_5+2\tau_6+2\tau_7-2\tau_{10}-2\tau_{11}-4\tau_{12}-4\boldsymbol{\tau_{14}}-6\boldsymbol{\tau_{13}}-8\tau_{15}-10\tau_{16},\\
-2\lambda+4\mu+6\nu+8\rho &\le 10\tau_1+8\tau_2+6\tau_3+4\tau_4+4\tau_5+2\tau_6+2\tau_7-2\tau_{10}-2\tau_{11}-4\tau_{12}-4\tau_{13}-6\tau_{14}-8\boldsymbol{\tau_{16}}-10\boldsymbol{\tau_{15}}.
\end{aligned}$$

A.2. **Two qutrits.** Below are 197 marginal inequalities for system of two qutrits $\mathcal{H}\otimes\mathcal{H}$, $\dim\mathcal{H}=3$. Together with inequalities obtained by transposition of qutrits they form a complete and independent system of 387 marginal constraints. One can check that for mixed state of rank three (i.e. for $\nu_i=0, i>3$) the system amounts to Higuchi inequalities n^o 3.3.5.

$$\begin{aligned}
2\lambda_1-\lambda_2-\lambda_3 &\le 2\nu_1+2\nu_2+2\nu_3-\nu_4-\nu_5-\nu_6-\nu_7-\nu_8-\nu_9,\\
\lambda_1+\lambda_2-2\lambda_3 &\le \nu_1+\nu_2+\nu_3+\nu_4+\nu_5+\nu_6-2\nu_7-2\nu_8-2\nu_9,
\end{aligned}$$

$$\begin{aligned}
\lambda_1+\lambda_2-2\lambda_3+\mu_1+\mu_2-2\mu_3 &\le 2\nu_1+2\nu_2+2\nu_3+2\nu_4-\nu_5-\nu_6-\nu_7-\nu_8-4\nu_9,\\
\lambda_1+\lambda_2-2\lambda_3+\mu_1+\mu_3-2\mu_2 &\le 2\nu_1+2\nu_2+2\nu_3+2\nu_4-\nu_5-\nu_6-\nu_7-\nu_9-4\nu_8,\\
\lambda_1+\lambda_2-2\lambda_3+\mu_2+\mu_3-2\mu_1 &\le 2\nu_1+2\nu_2+2\nu_3+2\nu_6-\nu_4-\nu_5-\nu_7-\nu_8-4\nu_9,\\
\lambda_1+\lambda_2-2\lambda_3+\mu_2+\mu_3-2\mu_1 &\le 2\nu_1+2\nu_2+2\nu_3+2\nu_4-\nu_5-\nu_6-\nu_8-\nu_9-4\nu_7,
\end{aligned}$$

$$\begin{aligned}
2\lambda_1-\lambda_2-\lambda_3+2\mu_1-\mu_2-\mu_3 &\le 4\nu_1+\nu_2+\nu_3+\nu_4+\nu_5-2\nu_6-2\nu_7-2\nu_8-2\nu_9,\\
2\lambda_1-\lambda_2-\lambda_3+2\mu_2-\mu_1-\mu_3 &\le 4\nu_2+\nu_1+\nu_3+\nu_4+\nu_5-2\nu_6-2\nu_7-2\nu_8-2\nu_9,\\
2\lambda_1-\lambda_2-\lambda_3+2\mu_3-\mu_1-\mu_2 &\le 4\nu_3+\nu_1+\nu_2+\nu_4+\nu_5-2\nu_6-2\nu_7-2\nu_8-2\nu_9,\\
2\lambda_1-\lambda_2-\lambda_3+2\mu_3-\mu_1-\mu_2 &\le 4\nu_1+\nu_2+\nu_3+\nu_5+\nu_6-2\nu_4-2\nu_7-2\nu_8-2\nu_9.
\end{aligned}$$

$$\lambda_1+\lambda_2-2\lambda_3+2\mu_1-\mu_2-\mu_3\le 3\nu_1+3\nu_2-3\nu_8-3\nu_9,$$
$$\lambda_1+\lambda_3-2\lambda_2+2\mu_1-\mu_2-\mu_3\le 3\nu_1+3\nu_3-3\nu_8-3\nu_9,$$
$$\lambda_1+\lambda_2-2\lambda_3+2\mu_2-\mu_1-\mu_3\le 3\nu_1+3\nu_2-3\nu_7-3\nu_9,$$
$$\lambda_2+\lambda_3-2\lambda_1+2\mu_1-\mu_2-\mu_3\le 3\nu_2+3\nu_3-3\nu_8-3\nu_9,$$
$$\lambda_1+\lambda_3-2\lambda_2+2\mu_2-\mu_1-\mu_3\le 3\nu_2+3\nu_3-3\nu_8-3\nu_9,$$
$$\lambda_1+\lambda_2-2\lambda_3+2\mu_3-\mu_1-\mu_2\le 3\nu_2+3\nu_3-3\nu_8-3\nu_9,$$
$$\lambda_2+\lambda_3-2\lambda_1+2\mu_1-\mu_2-\mu_3\le 3\nu_1+3\nu_2-3\nu_7-3\nu_8,$$
$$\lambda_1+\lambda_3-2\lambda_2+2\mu_2-\mu_1-\mu_3\le 3\nu_1+3\nu_2-3\nu_7-3\nu_8,$$
$$\lambda_1+\lambda_2-2\lambda_3+2\mu_3-\mu_1-\mu_2\le 3\nu_1+3\nu_2-3\nu_7-3\nu_8,$$

$$\lambda_1-\lambda_3+\mu_1-\mu_3\le 2\nu_1+\nu_2+\nu_3-\nu_7-\nu_8-2\nu_9,$$
$$\lambda_1-\lambda_2+\mu_1-\mu_2\le 2\nu_1+\nu_3+\nu_4-\nu_7-\nu_8-2\nu_9,$$
$$\lambda_1-\lambda_2+\mu_1-\mu_2\le 2\nu_1+\nu_2+\nu_5-\nu_7-\nu_8-2\nu_9,$$
$$\lambda_2-\lambda_3+\mu_2-\mu_3\le 2\nu_1+\nu_2+\nu_3-\nu_5-\nu_8-2\nu_9,$$
$$\lambda_2-\lambda_3+\mu_2-\mu_3\le 2\nu_1+\nu_2+\nu_3-\nu_6-\nu_7-2\nu_9,$$
$$\lambda_1-\lambda_3+\mu_2-\mu_3\le 2\nu_2+\nu_1+\nu_3-\nu_7-\nu_8-2\nu_9,$$
$$\lambda_1-\lambda_3+\mu_2-\mu_3\le 2\nu_1+\nu_2+\nu_4-\nu_7-\nu_8-2\nu_9,$$
$$\lambda_1-\lambda_3+\mu_2-\mu_3\le 2\nu_1+\nu_2+\nu_3-\nu_6-\nu_8-2\nu_9,$$
$$\lambda_1-\lambda_3+\mu_1-\mu_2\le 2\nu_1+\nu_2+\nu_4-\nu_7-\nu_8-2\nu_9,$$
$$\lambda_1-\lambda_3+\mu_1-\mu_2\le 2\nu_1+\nu_2+\nu_3-\nu_6-\nu_8-2\nu_9,$$
$$\lambda_1-\lambda_3+\mu_1-\mu_2\le 2\nu_1+\nu_2+\nu_3-\nu_7-\nu_9-2\nu_8,$$
$$\lambda_1-\lambda_2+\mu_2-\mu_3\le 2\nu_2+\nu_1+\nu_4-\nu_7-\nu_8-2\nu_9,$$
$$\lambda_1-\lambda_2+\mu_2-\mu_3\le 2\nu_2+\nu_1+\nu_3-\nu_6-\nu_8-2\nu_9,$$
$$\lambda_1-\lambda_2+\mu_2-\mu_3\le 2\nu_1+\nu_2+\nu_3-\nu_6-\nu_9-2\nu_8,$$
$$\lambda_1-\lambda_2+\mu_2-\mu_3\le 2\nu_1+\nu_2+\nu_4-\nu_7-\nu_9-2\nu_8,$$
$$\lambda_1-\lambda_2+\mu_2-\mu_3\le 2\nu_2+\nu_1+\nu_3-\nu_7-\nu_9-2\nu_8,$$
$$\lambda_1-\lambda_2+\mu_2-\mu_1\le 2\nu_2+\nu_3+\nu_4-\nu_7-\nu_8-2\nu_9,$$
$$\lambda_1-\lambda_2+\mu_2-\mu_1\le 2\nu_1+\nu_2+\nu_3-\nu_6-\nu_7-2\nu_8,$$
$$\lambda_2-\lambda_3+\mu_3-\mu_2\le 2\nu_2+\nu_3+\nu_4-\nu_7-\nu_8-2\nu_9,$$
$$\lambda_2-\lambda_3+\mu_3-\mu_2\le 2\nu_1+\nu_2+\nu_3-\nu_6-\nu_7-2\nu_8,$$
$$\lambda_1-\lambda_3+\mu_3-\mu_1\le 2\nu_3+\nu_1+\nu_4-\nu_7-\nu_8-2\nu_9,$$
$$\lambda_1-\lambda_3+\mu_3-\mu_1\le 2\nu_2+\nu_3+\nu_4-\nu_7-\nu_8-2\nu_9,$$
$$\lambda_1-\lambda_3+\mu_3-\mu_1\le 2\nu_1+\nu_2+\nu_3-\nu_6-\nu_7-2\nu_8,$$
$$\lambda_1-\lambda_3+\mu_3-\mu_1\le 2\nu_1+\nu_2+\nu_3-\nu_6-\nu_9-2\nu_7,$$
$$\lambda_1-\lambda_2+\mu_3-\mu_2\le 2\nu_3+\nu_1+\nu_4-\nu_7-\nu_8-2\nu_9,$$
$$\lambda_2-\lambda_3+\mu_2-\mu_1\le 2\nu_1+\nu_2+\nu_3-\nu_6-\nu_9-2\nu_7,$$
$$\lambda_1-\lambda_2+\mu_3-\mu_1\le 2\nu_3+\nu_2+\nu_4-\nu_7-\nu_8-2\nu_9,$$
$$\lambda_1-\lambda_2+\mu_3-\mu_1\le 2\nu_3+\nu_1+\nu_5-\nu_7-\nu_8-2\nu_9,$$
$$\lambda_1-\lambda_2+\mu_3-\mu_1\le 2\nu_2+\nu_1+\nu_3-\nu_6-\nu_7-2\nu_8,$$
$$\lambda_1-\lambda_2+\mu_3-\mu_1\le 2\nu_2+\nu_3+\nu_4-\nu_7-\nu_9-2\nu_8,$$
$$\lambda_1-\lambda_2+\mu_3-\mu_1\le 2\nu_1+\nu_2+\nu_3-\nu_6-\nu_8-2\nu_7,$$
$$\lambda_2-\lambda_3+\mu_3-\mu_1\le 2\nu_3+\nu_2+\nu_4-\nu_7-\nu_8-2\nu_9,$$
$$\lambda_2-\lambda_3+\mu_3-\mu_1\le 2\nu_2+\nu_1+\nu_3-\nu_6-\nu_7-2\nu_8,$$
$$\lambda_2-\lambda_3+\mu_3-\mu_1\le 2\nu_2+\nu_3+\nu_4-\nu_7-\nu_9-2\nu_8,$$
$$\lambda_2-\lambda_3+\mu_3-\mu_1\le 2\nu_1+\nu_2+\nu_3-\nu_5-\nu_9-2\nu_7,$$
$$\lambda_2-\lambda_3+\mu_3-\mu_1\le 2\nu_1+\nu_2+\nu_3-\nu_6-\nu_8-2\nu_7.$$

$$3\lambda_1-3\lambda_3+\mu_1+\mu_2-2\mu_3\le 4\nu_1+4\nu_2+\nu_3+\nu_4+\nu_5-2\nu_6-2\nu_7-2\nu_8-5\nu_9,$$
$$3\lambda_1-3\lambda_3+\mu_1+\mu_3-2\mu_2\le 4\nu_1+4\nu_3+\nu_2+\nu_4+\nu_5-2\nu_6-2\nu_7-2\nu_8-5\nu_9,$$
$$3\lambda_1-3\lambda_3+\mu_1+\mu_3-2\mu_2\le 4\nu_1+4\nu_2+\nu_3+\nu_4+\nu_6-2\nu_5-2\nu_7-2\nu_8-5\nu_9,$$
$$3\lambda_2-3\lambda_3+\mu_1+\mu_2-2\mu_3\le 4\nu_1+4\nu_2+\nu_3+\nu_4+\nu_6-2\nu_5-2\nu_7-2\nu_8-5\nu_9,$$
$$3\lambda_1-3\lambda_3+\mu_1+\mu_3-2\mu_2\le 4\nu_1+4\nu_2+\nu_3+\nu_4+\nu_5-2\nu_6-2\nu_7-2\nu_9-5\nu_8,$$
$$3\lambda_1-3\lambda_2+\mu_1+\mu_2-2\mu_3\le 4\nu_1+4\nu_2+\nu_3+\nu_4+\nu_5-2\nu_6-2\nu_7-2\nu_9-5\nu_8,$$
$$3\lambda_1-3\lambda_3+\mu_2+\mu_3-2\mu_1\le 4\nu_2+4\nu_3+\nu_1+\nu_4+\nu_5-2\nu_6-2\nu_7-2\nu_8-5\nu_9,$$
$$3\lambda_2-3\lambda_3+\mu_1+\mu_3-2\mu_2\le 4\nu_2+4\nu_3+\nu_1+\nu_4+\nu_5-2\nu_6-2\nu_7-2\nu_8-5\nu_9,$$
$$3\lambda_1-3\lambda_3+\mu_2+\mu_3-2\mu_1\le 4\nu_1+4\nu_3+\nu_2+\nu_4+\nu_6-2\nu_5-2\nu_7-2\nu_8-5\nu_9,$$
$$3\lambda_1-3\lambda_3+\mu_2+\mu_3-2\mu_1\le 4\nu_1+4\nu_2+\nu_3+\nu_5+\nu_6-2\nu_4-2\nu_7-2\nu_8-5\nu_9,$$
$$3\lambda_1-3\lambda_2+\mu_1+\mu_3-2\mu_2\le 4\nu_1+4\nu_2+\nu_3+\nu_5+\nu_6-2\nu_4-2\nu_7-2\nu_8-5\nu_9,$$
$$3\lambda_1-3\lambda_3+\mu_2+\mu_3-2\mu_1\le 4\nu_1+4\nu_2+\nu_3+\nu_4+\nu_6-2\nu_5-2\nu_7-2\nu_9-5\nu_8,$$
$$3\lambda_2-3\lambda_3+\mu_1+\mu_3-2\mu_2\le 4\nu_1+4\nu_2+\nu_3+\nu_4+\nu_6-2\nu_5-2\nu_7-2\nu_9-5\nu_8,$$
$$3\lambda_1-3\lambda_3+\mu_2+\mu_3-2\mu_1\le 4\nu_1+4\nu_3+\nu_2+\nu_4+\nu_5-2\nu_6-2\nu_7-2\nu_9-5\nu_8,$$
$$3\lambda_1-3\lambda_2+\mu_1+\mu_3-2\mu_2\le 4\nu_1+4\nu_3+\nu_2+\nu_4+\nu_5-2\nu_6-2\nu_7-2\nu_9-5\nu_8,$$
$$3\lambda_1-3\lambda_3+\mu_2+\mu_3-2\mu_1\le 4\nu_1+4\nu_2+\nu_3+\nu_4+\nu_5-2\nu_6-2\nu_8-2\nu_9-5\nu_7,$$
$$3\lambda_1-3\lambda_2+\mu_2+\mu_3-2\mu_1\le 4\nu_1+4\nu_3+\nu_2+\nu_5+\nu_6-2\nu_4-2\nu_7-2\nu_8-5\nu_9,$$
$$3\lambda_1-3\lambda_2+\mu_2+\mu_3-2\mu_1\le 4\nu_2+4\nu_3+\nu_1+\nu_4+\nu_5-2\nu_6-2\nu_7-2\nu_9-5\nu_8,$$
$$3\lambda_2-3\lambda_3+\mu_2+\mu_3-2\mu_1\le 4\nu_2+4\nu_3+\nu_1+\nu_4+\nu_5-2\nu_6-2\nu_7-2\nu_9-5\nu_8,$$
$$3\lambda_2-3\lambda_3+\mu_2+\mu_3-2\mu_1\le 4\nu_1+4\nu_2+\nu_3+\nu_4+\nu_6-2\nu_5-2\nu_8-2\nu_9-5\nu_7.$$

$$3\lambda_1-3\lambda_3+2\mu_3-\mu_1-\mu_2\le 5\nu_1+2\nu_2+2\nu_3+2\nu_4-\nu_5-\nu_6-\nu_9-4\nu_7-4\nu_8,$$
$$3\lambda_1-3\lambda_2+2\mu_2-\mu_1-\mu_3\le 5\nu_1+2\nu_2+2\nu_3+2\nu_4-\nu_5-\nu_6-\nu_9-4\nu_7-4\nu_8,$$
$$3\lambda_2-3\lambda_3+2\mu_3-\mu_1-\mu_2\le 5\nu_1+2\nu_2+2\nu_3+2\nu_6-\nu_4-\nu_5-\nu_8-4\nu_7-4\nu_9,$$
$$3\lambda_1-3\lambda_2+2\mu_3-\mu_1-\mu_2\le 5\nu_2+2\nu_1+2\nu_3+2\nu_4-\nu_5-\nu_6-\nu_9-4\nu_7-4\nu_8,$$
$$3\lambda_2-3\lambda_3+2\mu_3-\mu_1-\mu_2\le 5\nu_2+2\nu_1+2\nu_3+2\nu_4-\nu_5-\nu_6-\nu_9-4\nu_7-4\nu_8,$$
$$3\lambda_1-3\lambda_3+2\mu_1-\mu_2-\mu_3\le 5\nu_1+2\nu_2+2\nu_3+2\nu_4-\nu_5-\nu_6-\nu_7-4\nu_8-4\nu_9,$$
$$3\lambda_1-3\lambda_3+2\mu_2-\mu_1-\mu_3\le 5\nu_2+2\nu_1+2\nu_3+2\nu_4-\nu_5-\nu_6-\nu_7-4\nu_8-4\nu_9,$$
$$3\lambda_2-3\lambda_3+2\mu_1-\mu_2-\mu_3\le 5\nu_2+2\nu_1+2\nu_3+2\nu_4-\nu_5-\nu_6-\nu_7-4\nu_8-4\nu_9,$$
$$3\lambda_1-3\lambda_3+2\mu_2-\mu_1-\mu_3\le 5\nu_1+2\nu_2+2\nu_3+2\nu_5-\nu_4-\nu_6-\nu_7-4\nu_8-4\nu_9,$$
$$3\lambda_1-3\lambda_2+2\mu_1-\mu_2-\mu_3\le 5\nu_1+2\nu_2+2\nu_3+2\nu_5-\nu_4-\nu_6-\nu_7-4\nu_8-4\nu_9,$$
$$3\lambda_1-3\lambda_3+2\mu_2-\mu_1-\mu_3\le 5\nu_1+2\nu_2+2\nu_3+2\nu_4-\nu_5-\nu_6-\nu_8-4\nu_7-4\nu_9,$$
$$3\lambda_1-3\lambda_3+2\mu_3-\mu_1-\mu_2\le 5\nu_3+2\nu_1+2\nu_2+2\nu_4-\nu_5-\nu_6-\nu_7-4\nu_8-4\nu_9,$$
$$3\lambda_1-3\lambda_3+2\mu_3-\mu_1-\mu_2\le 5\nu_2+2\nu_1+2\nu_3+2\nu_5-\nu_4-\nu_6-\nu_7-4\nu_8-4\nu_9,$$
$$3\lambda_1-3\lambda_2+2\mu_2-\mu_1-\mu_3\le 5\nu_2+2\nu_1+2\nu_3+2\nu_5-\nu_4-\nu_6-\nu_7-4\nu_8-4\nu_9,$$
$$3\lambda_1-3\lambda_3+2\mu_3-\mu_1-\mu_2\le 5\nu_1+2\nu_2+2\nu_3+2\nu_6-\nu_4-\nu_5-\nu_7-4\nu_8-4\nu_9,$$
$$3\lambda_2-3\lambda_3+2\mu_2-\mu_1-\mu_3\le 5\nu_1+2\nu_2+2\nu_3+2\nu_6-\nu_4-\nu_5-\nu_7-4\nu_8-4\nu_9,$$
$$3\lambda_1-3\lambda_3+2\mu_3-\mu_1-\mu_2\le 5\nu_1+2\nu_2+2\nu_3+2\nu_5-\nu_4-\nu_6-\nu_8-4\nu_7-4\nu_9,$$
$$3\lambda_1-3\lambda_3+2\mu_3-\mu_1-\mu_2\le 5\nu_2+2\nu_1+2\nu_3+2\nu_4-\nu_5-\nu_6-\nu_8-4\nu_7-4\nu_9,$$
$$3\lambda_2-3\lambda_3+2\mu_2-\mu_1-\mu_3\le 5\nu_2+2\nu_1+2\nu_3+2\nu_4-\nu_5-\nu_6-\nu_8-4\nu_7-4\nu_9,$$
$$3\lambda_1-3\lambda_2+2\mu_3-\mu_1-\mu_2\le 5\nu_3+2\nu_1+2\nu_2+2\nu_5-\nu_4-\nu_6-\nu_7-4\nu_8-4\nu_9.$$

$$3\lambda_1 - 3\lambda_3 + 4\mu_1 + \mu_2 - 5\mu_3 \le 7\nu_1 + 4\nu_2 + 4\nu_3 + \nu_4 + \nu_5 - 2\nu_6 - 2\nu_7 - 5\nu_8 - 8\nu_9,$$
$$3\lambda_1 - 3\lambda_3 + 4\mu_2 + \mu_1 - 5\mu_3 \le 7\nu_2 + 4\nu_1 + 4\nu_3 + \nu_4 + \nu_5 - 2\nu_6 - 2\nu_7 - 5\nu_8 - 8\nu_9,$$
$$3\lambda_2 - 3\lambda_3 + 4\mu_1 + \mu_2 - 5\mu_3 \le 7\nu_2 + 4\nu_1 + 4\nu_3 + \nu_4 + \nu_5 - 2\nu_6 - 2\nu_7 - 5\nu_8 - 8\nu_9,$$
$$3\lambda_1 - 3\lambda_3 + 4\mu_2 + \mu_1 - 5\mu_3 \le 7\nu_1 + 4\nu_2 + 4\nu_4 + \nu_3 + \nu_5 - 2\nu_6 - 2\nu_7 - 5\nu_8 - 8\nu_9,$$
$$3\lambda_1 - 3\lambda_2 + 4\mu_1 + \mu_2 - 5\mu_3 \le 7\nu_1 + 4\nu_2 + 4\nu_4 + \nu_3 + \nu_5 - 2\nu_6 - 2\nu_7 - 5\nu_8 - 8\nu_9,$$
$$3\lambda_2 - 3\lambda_3 + 4\mu_1 + \mu_2 - 5\mu_3 \le 7\nu_1 + 4\nu_2 + 4\nu_4 + \nu_3 + \nu_5 - 2\nu_6 - 2\nu_7 - 5\nu_8 - 8\nu_9,$$
$$3\lambda_1 - 3\lambda_3 + 4\mu_2 + \mu_1 - 5\mu_3 \le 7\nu_1 + 4\nu_2 + 4\nu_3 + \nu_4 + \nu_6 - 2\nu_5 - 2\nu_7 - 5\nu_8 - 8\nu_9,$$
$$3\lambda_1 - 3\lambda_2 + 4\mu_1 + \mu_2 - 5\mu_3 \le 7\nu_1 + 4\nu_2 + 4\nu_3 + \nu_4 + \nu_6 - 2\nu_5 - 2\nu_7 - 5\nu_8 - 8\nu_9,$$
$$3\lambda_2 - 3\lambda_3 + 4\mu_1 + \mu_2 - 5\mu_3 \le 7\nu_1 + 4\nu_2 + 4\nu_3 + \nu_4 + \nu_5 - 2\nu_6 - 2\nu_8 - 5\nu_7 - 8\nu_9,$$
$$3\lambda_1 - 3\lambda_2 + 4\mu_1 + \mu_2 - 5\mu_3 \le 7\nu_1 + 4\nu_2 + 4\nu_3 + \nu_4 + \nu_5 - 2\nu_6 - 2\nu_7 - 5\nu_9 - 8\nu_8,$$
$$3\lambda_1 - 3\lambda_2 + 4\mu_2 + \mu_1 - 5\mu_3 \le 7\nu_2 + 4\nu_1 + 4\nu_4 + \nu_3 + \nu_5 - 2\nu_6 - 2\nu_7 - 5\nu_8 - 8\nu_9,$$
$$3\lambda_2 - 3\lambda_3 + 4\mu_2 + \mu_1 - 5\mu_3 \le 7\nu_1 + 4\nu_2 + 4\nu_4 + \nu_3 + \nu_6 - 2\nu_5 - 2\nu_7 - 5\nu_8 - 8\nu_9,$$
$$3\lambda_1 - 3\lambda_2 + 4\mu_2 + \mu_1 - 5\mu_3 \le 7\nu_2 + 4\nu_1 + 4\nu_3 + \nu_4 + \nu_6 - 2\nu_5 - 2\nu_7 - 5\nu_8 - 8\nu_9,$$
$$3\lambda_2 - 3\lambda_3 + 4\mu_2 + \mu_1 - 5\mu_3 \le 7\nu_2 + 4\nu_1 + 4\nu_3 + \nu_4 + \nu_6 - 2\nu_5 - 2\nu_7 - 5\nu_8 - 8\nu_9,$$
$$3\lambda_2 - 3\lambda_3 + 4\mu_2 + \mu_1 - 5\mu_3 \le 7\nu_1 + 4\nu_2 + 4\nu_3 + \nu_4 + \nu_6 - 2\nu_5 - 2\nu_8 - 5\nu_7 - 8\nu_9,$$
$$3\lambda_2 - 3\lambda_3 + 4\mu_2 + \mu_1 - 5\mu_3 \le 7\nu_1 + 4\nu_2 + 4\nu_4 + \nu_3 + \nu_5 - 2\nu_6 - 2\nu_8 - 5\nu_7 - 8\nu_9,$$
$$3\lambda_2 - 3\lambda_3 + 4\mu_2 + \mu_1 - 5\mu_3 \le 7\nu_2 + 4\nu_1 + 4\nu_3 + \nu_4 + \nu_5 - 2\nu_6 - 2\nu_8 - 5\nu_7 - 8\nu_9,$$
$$3\lambda_2 - 3\lambda_3 + 4\mu_1 + \mu_3 - 5\mu_2 \le 7\nu_1 + 4\nu_2 + 4\nu_3 + \nu_4 + \nu_5 - 2\nu_6 - 2\nu_9 - 5\nu_7 - 8\nu_8,$$
$$3\lambda_3 - 3\lambda_2 + 4\mu_1 + \mu_2 - 5\mu_3 \le 7\nu_1 + 4\nu_2 + 4\nu_3 + \nu_4 + \nu_5 - 2\nu_6 - 2\nu_9 - 5\nu_7 - 8\nu_8,$$
$$3\lambda_1 - 3\lambda_2 + 4\mu_2 + \mu_1 - 5\mu_3 \le 7\nu_1 + 4\nu_2 + 4\nu_3 + \nu_4 + \nu_6 - 2\nu_5 - 2\nu_7 - 5\nu_9 - 8\nu_8,$$
$$3\lambda_1 - 3\lambda_2 + 4\mu_2 + \mu_1 - 5\mu_3 \le 7\nu_1 + 4\nu_2 + 4\nu_4 + \nu_3 + \nu_5 - 2\nu_6 - 2\nu_7 - 5\nu_9 - 8\nu_8,$$
$$3\lambda_1 - 3\lambda_2 + 4\mu_2 + \mu_1 - 5\mu_3 \le 7\nu_2 + 4\nu_1 + 4\nu_3 + \nu_4 + \nu_5 - 2\nu_6 - 2\nu_7 - 5\nu_9 - 8\nu_8,$$
$$3\lambda_2 - 3\lambda_1 + 4\mu_1 + \mu_2 - 5\mu_3 \le 7\nu_1 + 4\nu_2 + 4\nu_3 + \nu_4 + \nu_5 - 2\nu_6 - 2\nu_8 - 5\nu_9 - 8\nu_7,$$
$$3\lambda_3 - 3\lambda_1 + 4\mu_1 + \mu_2 - 5\mu_3 \le 7\nu_3 + 4\nu_1 + 4\nu_4 + \nu_2 + \nu_5 - 2\nu_6 - 2\nu_7 - 5\nu_8 - 8\nu_9,$$
$$3\lambda_3 - 3\lambda_2 + 4\mu_2 + \mu_1 - 5\mu_3 \le 7\nu_2 + 4\nu_3 + 4\nu_4 + \nu_1 + \nu_5 - 2\nu_6 - 2\nu_7 - 5\nu_8 - 8\nu_9,$$
$$3\lambda_3 - 3\lambda_1 + 4\mu_1 + \mu_2 - 5\mu_3 \le 7\nu_2 + 4\nu_3 + 4\nu_4 + \nu_1 + \nu_5 - 2\nu_6 - 2\nu_7 - 5\nu_8 - 8\nu_9,$$
$$3\lambda_2 - 3\lambda_1 + 4\mu_2 + \mu_1 - 5\mu_3 \le 7\nu_1 + 4\nu_2 + 4\nu_6 + \nu_3 + \nu_4 - 2\nu_5 - 2\nu_7 - 5\nu_8 - 8\nu_9,$$
$$3\lambda_3 - 3\lambda_1 + 4\mu_1 + \mu_2 - 5\mu_3 \le 7\nu_1 + 4\nu_2 + 4\nu_6 + \nu_3 + \nu_4 - 2\nu_5 - 2\nu_7 - 5\nu_8 - 8\nu_9,$$
$$3\lambda_3 - 3\lambda_1 + 4\mu_1 + \mu_2 - 5\mu_3 \le 7\nu_1 + 4\nu_2 + 4\nu_5 + \nu_3 + \nu_6 - 2\nu_4 - 2\nu_7 - 5\nu_8 - 8\nu_9,$$
$$3\lambda_3 - 3\lambda_2 + 4\mu_2 + \mu_1 - 5\mu_3 \le 7\nu_2 + 4\nu_1 + 4\nu_3 + \nu_4 + \nu_5 - 2\nu_6 - 2\nu_9 - 5\nu_7 - 8\nu_8,$$
$$3\lambda_3 - 3\lambda_2 + 4\mu_2 + \mu_1 - 5\mu_3 \le 7\nu_1 + 4\nu_2 + 4\nu_4 + \nu_3 + \nu_5 - 2\nu_6 - 2\nu_9 - 5\nu_7 - 8\nu_8,$$
$$3\lambda_3 - 3\lambda_1 + 4\mu_1 + \mu_2 - 5\mu_3 \le 7\nu_1 + 4\nu_2 + 4\nu_4 + \nu_3 + \nu_5 - 2\nu_6 - 2\nu_9 - 5\nu_7 - 8\nu_8,$$
$$3\lambda_3 - 3\lambda_2 + 4\mu_2 + \mu_1 - 5\mu_3 \le 7\nu_1 + 4\nu_2 + 4\nu_3 + \nu_4 + \nu_6 - 2\nu_5 - 2\nu_9 - 5\nu_7 - 8\nu_8,$$
$$3\lambda_3 - 3\lambda_1 + 4\mu_1 + \mu_2 - 5\mu_3 \le 7\nu_1 + 4\nu_2 + 4\nu_3 + \nu_4 + \nu_6 - 2\nu_5 - 2\nu_9 - 5\nu_7 - 8\nu_8,$$
$$3\lambda_2 - 3\lambda_1 + 4\mu_2 + \mu_1 - 5\mu_3 \le 7\nu_2 + 4\nu_1 + 4\nu_3 + \nu_4 + \nu_5 - 2\nu_6 - 2\nu_8 - 5\nu_9 - 8\nu_7,$$
$$3\lambda_3 - 3\lambda_1 + 4\mu_1 + \mu_2 - 5\mu_3 \le 7\nu_2 + 4\nu_1 + 4\nu_3 + \nu_4 + \nu_5 - 2\nu_6 - 2\nu_8 - 5\nu_9 - 8\nu_7,$$
$$3\lambda_2 - 3\lambda_1 + 4\mu_2 + \mu_1 - 5\mu_3 \le 7\nu_1 + 4\nu_2 + 4\nu_4 + \nu_3 + \nu_5 - 2\nu_6 - 2\nu_8 - 5\nu_9 - 8\nu_7,$$
$$3\lambda_3 - 3\lambda_1 + 4\mu_1 + \mu_2 - 5\mu_3 \le 7\nu_1 + 4\nu_2 + 4\nu_4 + \nu_3 + \nu_5 - 2\nu_6 - 2\nu_8 - 5\nu_9 - 8\nu_7,$$
$$3\lambda_2 - 3\lambda_1 + 4\mu_2 + \mu_1 - 5\mu_3 \le 7\nu_1 + 4\nu_2 + 4\nu_3 + \nu_4 + \nu_6 - 2\nu_5 - 2\nu_8 - 5\nu_9 - 8\nu_7,$$
$$3\lambda_3 - 3\lambda_1 + 4\mu_1 + \mu_2 - 5\mu_3 \le 7\nu_1 + 4\nu_2 + 4\nu_3 + \nu_4 + \nu_5 - 2\nu_6 - 2\nu_9 - 5\nu_8 - 8\nu_7,$$
$$3\lambda_3 - 3\lambda_1 + 4\mu_2 + \mu_1 - 5\mu_3 \le 7\nu_3 + 4\nu_2 + 4\nu_4 + \nu_1 + \nu_5 - 2\nu_6 - 2\nu_7 - 5\nu_8 - 8\nu_9,$$
$$3\lambda_3 - 3\lambda_1 + 4\mu_2 + \mu_1 - 5\mu_3 \le 7\nu_1 + 4\nu_2 + 4\nu_6 + \nu_3 + \nu_5 - 2\nu_4 - 2\nu_7 - 5\nu_8 - 8\nu_9,$$
$$3\lambda_3 - 3\lambda_1 + 4\mu_2 + \mu_1 - 5\mu_3 \le 7\nu_1 + 4\nu_2 + 4\nu_6 + \nu_3 + \nu_4 - 2\nu_5 - 2\nu_8 - 5\nu_7 - 8\nu_9,$$
$$3\lambda_3 - 3\lambda_1 + 4\mu_2 + \mu_1 - 5\mu_3 \le 7\nu_2 + 4\nu_1 + 4\nu_4 + \nu_3 + \nu_5 - 2\nu_6 - 2\nu_9 - 5\nu_7 - 8\nu_8,$$
$$3\lambda_3 - 3\lambda_1 + 4\mu_2 + \mu_1 - 5\mu_3 \le 7\nu_2 + 4\nu_1 + 4\nu_3 + \nu_4 + \nu_6 - 2\nu_5 - 2\nu_9 - 5\nu_7 - 8\nu_8,$$
$$3\lambda_3 - 3\lambda_1 + 4\mu_2 + \mu_1 - 5\mu_3 \le 7\nu_2 + 4\nu_3 + 4\nu_4 + \nu_1 + \nu_5 - 2\nu_6 - 2\nu_7 - 5\nu_9 - 8\nu_8,$$
$$3\lambda_3 - 3\lambda_1 + 4\mu_2 + \mu_1 - 5\mu_3 \le 7\nu_1 + 4\nu_2 + 4\nu_4 + \nu_3 + \nu_6 - 2\nu_5 - 2\nu_8 - 5\nu_9 - 8\nu_7,$$
$$3\lambda_3 - 3\lambda_1 + 4\mu_2 + \mu_1 - 5\mu_3 \le 7\nu_2 + 4\nu_1 + 4\nu_3 + \nu_4 + \nu_6 - 2\nu_5 - 2\nu_8 - 5\nu_9 - 8\nu_7,$$
$$3\lambda_3 - 3\lambda_1 + 4\mu_2 + \mu_1 - 5\mu_3 \le 7\nu_1 + 4\nu_2 + 4\nu_3 + \nu_4 + \nu_6 - 2\nu_5 - 2\nu_9 - 5\nu_8 - 8\nu_7,$$
$$3\lambda_3 - 3\lambda_1 + 4\mu_2 + \mu_1 - 5\mu_3 \le 7\nu_1 + 4\nu_2 + 4\nu_4 + \nu_3 + \nu_5 - 2\nu_6 - 2\nu_9 - 5\nu_8 - 8\nu_7,$$
$$3\lambda_3 - 3\lambda_1 + 4\mu_2 + \mu_1 - 5\mu_3 \le 7\nu_2 + 4\nu_1 + 4\nu_3 + \nu_4 + \nu_5 - 2\nu_6 - 2\nu_9 - 5\nu_8 - 8\nu_7.$$

$$3\lambda_1 - 3\lambda_3 + 5\mu_1 - \mu_2 - 4\mu_3 \le 8\nu_1 + 5\nu_2 + 2\nu_3 + 2\nu_4 - \nu_5 - \nu_6 - 4\nu_7 - 4\nu_8 - 7\nu_9,$$
$$3\lambda_2 - 3\lambda_3 + 5\mu_1 - \mu_2 - 4\mu_3 \le 8\nu_2 + 5\nu_1 + 2\nu_3 + 2\nu_4 - \nu_5 - \nu_6 - 4\nu_7 - 4\nu_8 - 7\nu_9,$$
$$3\lambda_1 - 3\lambda_2 + 5\mu_1 - \mu_2 - 4\mu_3 \le 8\nu_1 + 5\nu_3 + 2\nu_2 + 2\nu_4 - \nu_5 - \nu_6 - 4\nu_7 - 4\nu_8 - 7\nu_9,$$
$$3\lambda_1 - 3\lambda_3 + 5\mu_1 - \mu_3 - 4\mu_2 \le 8\nu_1 + 5\nu_2 + 2\nu_3 + 2\nu_5 - \nu_4 - \nu_6 - 4\nu_7 - 4\nu_8 - 7\nu_9,$$
$$3\lambda_2 - 3\lambda_3 + 5\mu_1 - \mu_2 - 4\mu_3 \le 8\nu_1 + 5\nu_2 + 2\nu_3 + 2\nu_5 - \nu_4 - \nu_6 - 4\nu_7 - 4\nu_8 - 7\nu_9,$$
$$3\lambda_1 - 3\lambda_3 + 5\mu_1 - \mu_3 - 4\mu_2 \le 8\nu_1 + 5\nu_2 + 2\nu_3 + 2\nu_4 - \nu_5 - \nu_7 - 4\nu_6 - 4\nu_8 - 7\nu_9,$$
$$3\lambda_1 - 3\lambda_2 + 5\mu_1 - \mu_2 - 4\mu_3 \le 8\nu_1 + 5\nu_2 + 2\nu_3 + 2\nu_4 - \nu_5 - \nu_7 - 4\nu_6 - 4\nu_8 - 7\nu_9,$$
$$3\lambda_2 - 3\lambda_3 + 5\mu_1 - \mu_2 - 4\mu_3 \le 8\nu_1 + 5\nu_2 + 2\nu_3 + 2\nu_4 - \nu_5 - \nu_7 - 4\nu_6 - 4\nu_8 - 7\nu_9,$$
$$3\lambda_1 - 3\lambda_3 + 5\mu_1 - \mu_3 - 4\mu_2 \le 8\nu_1 + 5\nu_2 + 2\nu_3 + 2\nu_4 - \nu_5 - \nu_6 - 4\nu_7 - 4\nu_9 - 7\nu_8,$$
$$3\lambda_1 - 3\lambda_2 + 5\mu_1 - \mu_2 - 4\mu_3 \le 8\nu_1 + 5\nu_2 + 2\nu_3 + 2\nu_4 - \nu_5 - \nu_6 - 4\nu_7 - 4\nu_9 - 7\nu_8,$$
$$3\lambda_3 - 3\lambda_2 + 5\mu_1 - \mu_2 - 4\mu_3 \le 8\nu_3 + 5\nu_1 + 2\nu_2 + 2\nu_4 - \nu_5 - \nu_6 - 4\nu_7 - 4\nu_8 - 7\nu_9,$$
$$3\lambda_1 - 3\lambda_2 + 5\mu_2 - \mu_1 - 4\mu_3 \le 8\nu_2 + 5\nu_3 + 2\nu_1 + 2\nu_4 - \nu_5 - \nu_6 - 4\nu_7 - 4\nu_8 - 7\nu_9,$$
$$3\lambda_2 - 3\lambda_1 + 5\mu_1 - \mu_2 - 4\mu_3 \le 8\nu_2 + 5\nu_3 + 2\nu_1 + 2\nu_4 - \nu_5 - \nu_6 - 4\nu_7 - 4\nu_8 - 7\nu_9,$$
$$3\lambda_1 - 3\lambda_2 + 5\mu_1 - \mu_3 - 4\mu_2 \le 8\nu_1 + 5\nu_3 + 2\nu_2 + 2\nu_5 - \nu_4 - \nu_6 - 4\nu_7 - 4\nu_8 - 7\nu_9,$$
$$3\lambda_2 - 3\lambda_3 + 5\mu_1 - \mu_3 - 4\mu_2 \le 8\nu_2 + 5\nu_1 + 2\nu_3 + 2\nu_5 - \nu_4 - \nu_6 - 4\nu_7 - 4\nu_8 - 7\nu_9,$$
$$3\lambda_1 - 3\lambda_2 + 5\mu_1 - \mu_3 - 4\mu_2 \le 8\nu_1 + 5\nu_2 + 2\nu_3 + 2\nu_5 - \nu_4 - \nu_7 - 4\nu_6 - 4\nu_8 - 7\nu_9,$$
$$3\lambda_1 - 3\lambda_2 + 5\mu_1 - \mu_3 - 4\mu_2 \le 8\nu_1 + 5\nu_3 + 2\nu_2 + 2\nu_4 - \nu_5 - \nu_7 - 4\nu_6 - 4\nu_8 - 7\nu_9,$$
$$3\lambda_2 - 3\lambda_3 + 5\mu_1 - \mu_3 - 4\mu_2 \le 8\nu_2 + 5\nu_1 + 2\nu_3 + 2\nu_4 - \nu_5 - \nu_7 - 4\nu_6 - 4\nu_8 - 7\nu_9,$$
$$3\lambda_2 - 3\lambda_3 + 5\mu_1 - \mu_3 - 4\mu_2 \le 8\nu_1 + 5\nu_2 + 2\nu_3 + 2\nu_4 - \nu_5 - \nu_7 - 4\nu_6 - 4\nu_9 - 7\nu_8,$$
$$3\lambda_1 - 3\lambda_2 + 5\mu_1 - \mu_3 - 4\mu_2 \le 8\nu_1 + 5\nu_2 + 2\nu_3 + 2\nu_5 - \nu_4 - \nu_6 - 4\nu_7 - 4\nu_9 - 7\nu_8,$$
$$3\lambda_2 - 3\lambda_3 + 5\mu_1 - \mu_3 - 4\mu_2 \le 8\nu_1 + 5\nu_2 + 2\nu_3 + 2\nu_5 - \nu_4 - \nu_6 - 4\nu_7 - 4\nu_9 - 7\nu_8,$$
$$3\lambda_1 - 3\lambda_2 + 5\mu_1 - \mu_3 - 4\mu_2 \le 8\nu_1 + 5\nu_3 + 2\nu_2 + 2\nu_4 - \nu_5 - \nu_6 - 4\nu_7 - 4\nu_9 - 7\nu_8,$$
$$3\lambda_2 - 3\lambda_3 + 5\mu_1 - \mu_3 - 4\mu_2 \le 8\nu_2 + 5\nu_1 + 2\nu_3 + 2\nu_4 - \nu_5 - \nu_6 - 4\nu_7 - 4\nu_9 - 7\nu_8,$$
$$3\lambda_3 - 3\lambda_1 + 5\mu_1 - \mu_2 - 4\mu_3 \le 8\nu_3 + 5\nu_2 + 2\nu_1 + 2\nu_4 - \nu_5 - \nu_6 - 4\nu_7 - 4\nu_8 - 7\nu_9,$$
$$3\lambda_2 - 3\lambda_1 + 5\mu_1 - \mu_3 - 4\mu_2 \le 8\nu_2 + 5\nu_3 + 2\nu_1 + 2\nu_5 - \nu_4 - \nu_6 - 4\nu_7 - 4\nu_8 - 7\nu_9,$$
$$3\lambda_3 - 3\lambda_1 + 5\mu_1 - \mu_2 - 4\mu_3 \le 8\nu_2 + 5\nu_3 + 2\nu_1 + 2\nu_5 - \nu_4 - \nu_6 - 4\nu_7 - 4\nu_8 - 7\nu_9,$$
$$3\lambda_3 - 3\lambda_2 + 5\mu_1 - \mu_3 - 4\mu_2 \le 8\nu_3 + 5\nu_1 + 2\nu_2 + 2\nu_5 - \nu_4 - \nu_6 - 4\nu_7 - 4\nu_8 - 7\nu_9,$$
$$3\lambda_2 - 3\lambda_1 + 5\mu_1 - \mu_3 - 4\mu_2 \le 8\nu_2 + 5\nu_3 + 2\nu_1 + 2\nu_4 - \nu_5 - \nu_7 - 4\nu_6 - 4\nu_8 - 7\nu_9,$$
$$3\lambda_3 - 3\lambda_1 + 5\mu_1 - \mu_2 - 4\mu_3 \le 8\nu_2 + 5\nu_3 + 2\nu_1 + 2\nu_4 - \nu_5 - \nu_7 - 4\nu_6 - 4\nu_8 - 7\nu_9,$$
$$3\lambda_3 - 3\lambda_2 + 5\mu_1 - \mu_3 - 4\mu_2 \le 8\nu_3 + 5\nu_1 + 2\nu_2 + 2\nu_4 - \nu_5 - \nu_7 - 4\nu_6 - 4\nu_8 - 7\nu_9,$$
$$3\lambda_3 - 3\lambda_1 + 5\mu_1 - \mu_2 - 4\mu_3 \le 8\nu_3 + 5\nu_1 + 2\nu_2 + 2\nu_4 - \nu_5 - \nu_7 - 4\nu_6 - 4\nu_8 - 7\nu_9,$$
$$3\lambda_3 - 3\lambda_1 + 5\mu_1 - \mu_2 - 4\mu_3 \le 8\nu_1 + 5\nu_2 + 2\nu_3 + 2\nu_6 - \nu_4 - \nu_7 - 4\nu_5 - 4\nu_8 - 7\nu_9,$$
$$3\lambda_3 - 3\lambda_2 + 5\mu_1 - \mu_3 - 4\mu_2 \le 8\nu_1 + 5\nu_2 + 2\nu_3 + 2\nu_5 - \nu_6 - \nu_7 - 4\nu_4 - 4\nu_8 - 7\nu_9,$$
$$3\lambda_3 - 3\lambda_1 + 5\mu_1 - \mu_2 - 4\mu_3 \le 8\nu_1 + 5\nu_2 + 2\nu_3 + 2\nu_5 - \nu_6 - \nu_7 - 4\nu_4 - 4\nu_8 - 7\nu_9,$$
$$3\lambda_2 - 3\lambda_1 + 5\mu_1 - \mu_3 - 4\mu_2 \le 8\nu_1 + 5\nu_2 + 2\nu_3 + 2\nu_4 - \nu_5 - \nu_9 - 4\nu_6 - 4\nu_7 - 7\nu_8,$$
$$3\lambda_3 - 3\lambda_1 + 5\mu_1 - \mu_2 - 4\mu_3 \le 8\nu_1 + 5\nu_2 + 2\nu_3 + 2\nu_4 - \nu_5 - \nu_9 - 4\nu_6 - 4\nu_7 - 7\nu_8,$$
$$3\lambda_2 - 3\lambda_1 + 5\mu_1 - \mu_3 - 4\mu_2 \le 8\nu_2 + 5\nu_3 + 2\nu_1 + 2\nu_4 - \nu_5 - \nu_6 - 4\nu_7 - 4\nu_9 - 7\nu_8,$$
$$3\lambda_3 - 3\lambda_2 + 5\mu_1 - \mu_3 - 4\mu_2 \le 8\nu_3 + 5\nu_1 + 2\nu_2 + 2\nu_4 - \nu_5 - \nu_6 - 4\nu_7 - 4\nu_9 - 7\nu_8,$$
$$3\lambda_3 - 3\lambda_1 + 5\mu_1 - \mu_2 - 4\mu_3 \le 8\nu_3 + 5\nu_1 + 2\nu_2 + 2\nu_4 - \nu_5 - \nu_6 - 4\nu_7 - 4\nu_9 - 7\nu_8,$$
$$3\lambda_3 - 3\lambda_1 + 5\mu_1 - \mu_2 - 4\mu_3 \le 8\nu_1 + 5\nu_2 + 2\nu_3 + 2\nu_4 - \nu_5 - \nu_8 - 4\nu_6 - 4\nu_9 - 7\nu_7,$$
$$3\lambda_3 - 3\lambda_1 + 5\mu_1 - \mu_3 - 4\mu_2 \le 8\nu_3 + 5\nu_2 + 2\nu_1 + 2\nu_5 - \nu_4 - \nu_6 - 4\nu_7 - 4\nu_8 - 7\nu_9,$$
$$3\lambda_3 - 3\lambda_1 + 5\mu_1 - \mu_3 - 4\mu_2 \le 8\nu_3 + 5\nu_1 + 2\nu_2 + 2\nu_5 - \nu_4 - \nu_7 - 4\nu_6 - 4\nu_8 - 7\nu_9,$$
$$3\lambda_3 - 3\lambda_1 + 5\mu_1 - \mu_3 - 4\mu_2 \le 8\nu_3 + 5\nu_2 + 2\nu_1 + 2\nu_4 - \nu_5 - \nu_7 - 4\nu_6 - 4\nu_8 - 7\nu_9,$$
$$3\lambda_3 - 3\lambda_1 + 5\mu_1 - \mu_3 - 4\mu_2 \le 8\nu_1 + 5\nu_2 + 2\nu_3 + 2\nu_6 - \nu_5 - \nu_7 - 4\nu_4 - 4\nu_8 - 7\nu_9,$$
$$3\lambda_3 - 3\lambda_1 + 5\mu_1 - \mu_3 - 4\mu_2 \le 8\nu_1 + 5\nu_3 + 2\nu_2 + 2\nu_5 - \nu_6 - \nu_7 - 4\nu_4 - 4\nu_8 - 7\nu_9,$$
$$3\lambda_3 - 3\lambda_1 + 5\mu_1 - \mu_3 - 4\mu_2 \le 8\nu_2 + 5\nu_1 + 2\nu_3 + 2\nu_4 - \nu_5 - \nu_9 - 4\nu_6 - 4\nu_7 - 7\nu_8,$$
$$3\lambda_3 - 3\lambda_1 + 5\mu_1 - \mu_3 - 4\mu_2 \le 8\nu_2 + 5\nu_3 + 2\nu_1 + 2\nu_4 - \nu_5 - \nu_7 - 4\nu_6 - 4\nu_9 - 7\nu_8,$$
$$3\lambda_3 - 3\lambda_1 + 5\mu_1 - \mu_3 - 4\mu_2 \le 8\nu_3 + 5\nu_1 + 2\nu_2 + 2\nu_5 - \nu_4 - \nu_6 - 4\nu_7 - 4\nu_9 - 7\nu_8,$$
$$3\lambda_3 - 3\lambda_1 + 5\mu_1 - \mu_3 - 4\mu_2 \le 8\nu_2 + 5\nu_3 + 2\nu_1 + 2\nu_5 - \nu_4 - \nu_6 - 4\nu_7 - 4\nu_9 - 7\nu_8,$$
$$3\lambda_3 - 3\lambda_1 + 5\mu_1 - \mu_3 - 4\mu_2 \le 8\nu_3 + 5\nu_2 + 2\nu_1 + 2\nu_4 - \nu_5 - \nu_6 - 4\nu_7 - 4\nu_9 - 7\nu_8,$$
$$3\lambda_3 - 3\lambda_1 + 5\mu_1 - \mu_3 - 4\mu_2 \le 8\nu_1 + 5\nu_2 + 2\nu_3 + 2\nu_4 - \nu_5 - \nu_9 - 4\nu_6 - 4\nu_8 - 7\nu_7.$$

A.3. **Systems of format** $2 \times n$**.** In this case all cubicles and extremal edges are explicitly known, see Example 4.1.2. The tables below give marginal inequalities for $n = 3, 4$. For $n > 4$ the number of marginal inequalities increase to thousands and can't be published.

A.3.1. *Format* 2×3. In this case there are 41 independent marginal inequalities.

$$\mu_1 - \mu_2 \le \nu_1 + \nu_2 + \nu_3 - \nu_4 - \nu_5 - \nu_6.$$

$$\begin{aligned}
\lambda_1 + \lambda_2 - 2\lambda_3 &\le \nu_1 + \nu_2 + \nu_3 + \nu_4 - 2\nu_5 - 2\nu_6,\\
\lambda_2 + \lambda_3 - 2\lambda_1 &\le \nu_1 + \nu_2 + \nu_3 + \nu_6 - 2\nu_4 - 2\nu_5.
\end{aligned}$$

$$\begin{aligned}
2\lambda_1 - \lambda_2 - \lambda_3 &\le 2\nu_1 + 2\nu_2 - \nu_3 - \nu_4 - \nu_5 - \nu_6,\\
2\lambda_3 - \lambda_1 - \lambda_2 &\le 2\nu_2 + 2\nu_3 - \nu_1 - \nu_4 - \nu_5 - \nu_6.
\end{aligned}$$

$$\begin{aligned}
2\lambda_1 - 2\lambda_3 - \mu_2 + \mu_1 &\le 3\nu_1 + \nu_2 + \nu_3 - \nu_4 - \nu_5 - 3\nu_6,\\
2\lambda_1 - 2\lambda_3 + \mu_2 - \mu_1 &\le 3\nu_2 + \nu_1 + \nu_3 - \nu_4 - \nu_5 - 3\nu_6,\\
2\lambda_2 - 2\lambda_3 - \mu_2 + \mu_1 &\le 3\nu_2 + \nu_1 + \nu_3 - \nu_4 - \nu_5 - 3\nu_6,\\
2\lambda_1 - 2\lambda_2 + \mu_2 - \mu_1 &\le 3\nu_2 + \nu_1 + \nu_4 - \nu_3 - \nu_5 - 3\nu_6,\\
2\lambda_1 - 2\lambda_3 + \mu_2 - \mu_1 &\le 3\nu_1 + \nu_2 + \nu_4 - \nu_3 - \nu_5 - 3\nu_6,\\
2\lambda_1 - 2\lambda_2 - \mu_2 + \mu_1 &\le 3\nu_1 + \nu_2 + \nu_4 - \nu_3 - \nu_5 - 3\nu_6,\\
2\lambda_2 - 2\lambda_3 - \mu_2 + \mu_1 &\le 3\nu_1 + \nu_2 + \nu_4 - \nu_3 - \nu_5 - 3\nu_6,\\
2\lambda_3 - 2\lambda_1 - \mu_2 + \mu_1 &\le 3\nu_3 + \nu_1 + \nu_4 - \nu_2 - \nu_5 - 3\nu_6,\\
2\lambda_3 - 2\lambda_1 + \mu_2 - \mu_1 &\le 3\nu_3 + \nu_2 + \nu_4 - \nu_1 - \nu_5 - 3\nu_6,\\
2\lambda_3 - 2\lambda_2 + \mu_2 - \mu_1 &\le 3\nu_2 + \nu_3 + \nu_4 - \nu_1 - \nu_5 - 3\nu_6,\\
2\lambda_3 - 2\lambda_1 - \mu_2 + \mu_1 &\le 3\nu_2 + \nu_3 + \nu_4 - \nu_1 - \nu_5 - 3\nu_6,\\
2\lambda_2 - 2\lambda_1 + \mu_2 - \mu_1 &\le 3\nu_1 + \nu_2 + \nu_6 - \nu_4 - \nu_3 - 3\nu_5,\\
2\lambda_3 - 2\lambda_1 - \mu_2 + \mu_1 &\le 3\nu_1 + \nu_2 + \nu_6 - \nu_4 - \nu_3 - 3\nu_5,\\
2\lambda_2 - 2\lambda_3 + \mu_2 - \mu_1 &\le 3\nu_1 + \nu_2 + \nu_4 - \nu_3 - \nu_6 - 3\nu_5,\\
2\lambda_1 - 2\lambda_2 + \mu_2 - \mu_1 &\le 3\nu_2 + \nu_1 + \nu_3 - \nu_4 - \nu_6 - 3\nu_5,\\
2\lambda_2 - 2\lambda_3 + \mu_2 - \mu_1 &\le 3\nu_2 + \nu_1 + \nu_3 - \nu_4 - \nu_6 - 3\nu_5,\\
2\lambda_1 - 2\lambda_3 + \mu_2 - \mu_1 &\le 3\nu_1 + \nu_2 + \nu_3 - \nu_6 - \nu_4 - 3\nu_5,\\
2\lambda_1 - 2\lambda_2 - \mu_2 + \mu_1 &\le 3\nu_1 + \nu_2 + \nu_3 - \nu_6 - \nu_4 - 3\nu_5,\\
2\lambda_3 - 2\lambda_1 - \mu_2 + \mu_1 &\le 3\nu_1 + \nu_2 + \nu_5 - \nu_3 - \nu_6 - 3\nu_4,\\
2\lambda_3 - 2\lambda_1 + \mu_2 - \mu_1 &\le 3\nu_1 + \nu_2 + \nu_6 - \nu_3 - \nu_5 - 3\nu_4.
\end{aligned}$$

$$\begin{aligned}
2\lambda_1 + 2\lambda_2 - 4\lambda_3 + 3\mu_1 - 3\mu_2 &\le 5\nu_1 + 5\nu_2 - \nu_3 - \nu_4 - \nu_5 - 7\nu_6,\\
2\lambda_3 + 2\lambda_1 - 4\lambda_2 + 3\mu_1 - 3\mu_2 &\le 5\nu_3 + 5\nu_1 - \nu_2 - \nu_4 - \nu_5 - 7\nu_6,\\
2\lambda_1 + 2\lambda_3 - 4\lambda_2 + 3\mu_2 - 3\mu_1 &\le 5\nu_2 + 5\nu_3 - \nu_1 - \nu_4 - \nu_5 - 7\nu_6,\\
2\lambda_2 + 2\lambda_3 - 4\lambda_1 + 3\mu_1 - 3\mu_2 &\le 5\nu_2 + 5\nu_3 - \nu_1 - \nu_4 - \nu_5 - 7\nu_6,\\
2\lambda_2 + 2\lambda_1 - 4\lambda_3 + 3\mu_2 - 3\mu_1 &\le 5\nu_1 + 5\nu_2 - \nu_3 - \nu_6 - \nu_4 - 7\nu_5,\\
2\lambda_3 + 2\lambda_1 - 4\lambda_2 + 3\mu_1 - 3\mu_2 &\le 5\nu_1 + 5\nu_2 - \nu_3 - \nu_6 - \nu_4 - 7\nu_5,\\
2\lambda_2 + 2\lambda_3 - 4\lambda_1 + 3\mu_1 - 3\mu_2 &\le 5\nu_1 + 5\nu_3 - \nu_2 - \nu_6 - \nu_4 - 7\nu_5,\\
2\lambda_2 + 2\lambda_3 - 4\lambda_1 + 3\mu_1 - 3\mu_2 &\le 5\nu_1 + 5\nu_2 - \nu_3 - \nu_5 - \nu_6 - 7\nu_4.
\end{aligned}$$

$$\begin{aligned}
4\lambda_1 - 2\lambda_2 - 2\lambda_3 + 3\mu_1 - 3\mu_2 &\le 7\nu_1 + \nu_2 + \nu_3 + \nu_4 - 5\nu_5 - 5\nu_6,\\
4\lambda_1 - 2\lambda_2 - 2\lambda_3 + 3\mu_2 - 3\mu_1 &\le 7\nu_2 + \nu_1 + \nu_3 + \nu_4 - 5\nu_5 - 5\nu_6,\\
4\lambda_2 - 2\lambda_1 - 2\lambda_3 + 3\mu_1 - 3\mu_2 &\le 7\nu_2 + \nu_1 + \nu_3 + \nu_4 - 5\nu_5 - 5\nu_6,\\
4\lambda_2 - 2\lambda_1 - 2\lambda_3 + 3\mu_1 - 3\mu_2 &\le 7\nu_1 + \nu_2 + \nu_3 + \nu_5 - 5\nu_4 - 5\nu_6,\\
4\lambda_3 - 2\lambda_1 - 2\lambda_2 + 3\mu_1 - 3\mu_2 &\le 7\nu_3 + \nu_1 + \nu_2 + \nu_4 - 5\nu_5 - 5\nu_6,\\
4\lambda_3 - 2\lambda_1 - 2\lambda_2 + 3\mu_1 - 3\mu_2 &\le 7\nu_2 + \nu_1 + \nu_3 + \nu_5 - 5\nu_4 - 5\nu_6,\\
4\lambda_2 - 2\lambda_1 - 2\lambda_3 + 3\mu_2 - 3\mu_1 &\le 7\nu_1 + \nu_2 + \nu_3 + \nu_6 - 5\nu_4 - 5\nu_5,\\
4\lambda_3 - 2\lambda_1 - 2\lambda_2 + 3\mu_1 - 3\mu_2 &\le 7\nu_1 + \nu_2 + \nu_3 + \nu_6 - 5\nu_4 - 5\nu_5.
\end{aligned}$$

A.3.2. *Format* 2×4. Marginal constraints in this case are given by the following 234 independent inequalities. Note that three qubit case can be reduced to this one.

$$\mu_1-\mu_2\le \nu_1+\nu_2+\nu_3+\nu_4-\nu_5-\nu_6-\nu_7-\nu_8.$$

$$\begin{aligned}
\lambda_1+\lambda_2-\lambda_3-\lambda_4&\le \nu_1+\nu_2+\nu_3+\nu_4-\nu_5-\nu_6-\nu_7-\nu_8,\\
\lambda_1+\lambda_4-\lambda_2-\lambda_3&\le \nu_1+\nu_2+\nu_4+\nu_5-\nu_3-\nu_6-\nu_7-\nu_8,\\
\lambda_2+\lambda_3-\lambda_1-\lambda_4&\le \nu_1+\nu_2+\nu_3+\nu_6-\nu_4-\nu_5-\nu_7-\nu_8,\\
\lambda_3+\lambda_4-\lambda_1-\lambda_2&\le \nu_2+\nu_3+\nu_4+\nu_5-\nu_1-\nu_6-\nu_7-\nu_8,\\
\lambda_3+\lambda_4-\lambda_1-\lambda_2&\le \nu_1+\nu_2+\nu_3+\nu_8-\nu_4-\nu_5-\nu_6-\nu_7.
\end{aligned}$$

$$\begin{aligned}
\lambda_1+\lambda_2+\lambda_3-3\lambda_4&\le \nu_1+\nu_2+\nu_3+\nu_4+\nu_5+\nu_6-3\nu_7-3\nu_8,\\
\lambda_1+\lambda_3+\lambda_4-3\lambda_2&\le \nu_1+\nu_2+\nu_3+\nu_4+\nu_5+\nu_8-3\nu_6-3\nu_7.
\end{aligned}$$

$$\begin{aligned}
3\lambda_1-\lambda_2-\lambda_3-\lambda_4&\le 3\nu_1+3\nu_2-\nu_3-\nu_4-\nu_5-\nu_6-\nu_7-\nu_8,\\
3\lambda_3-\lambda_1-\lambda_2-\lambda_4&\le 3\nu_2+3\nu_3-\nu_1-\nu_4-\nu_5-\nu_6-\nu_7-\nu_8.
\end{aligned}$$

$$\begin{aligned}
\lambda_1+\lambda_2+\lambda_3-3\lambda_4+2\mu_1-2\mu_2&\le 3\nu_1+3\nu_2+3\nu_3-\nu_4-\nu_5-\nu_6-\nu_7-5\nu_8,\\
\lambda_1+\lambda_2+\lambda_4-3\lambda_3+2\mu_1-2\mu_2&\le 3\nu_1+3\nu_2+3\nu_4-\nu_3-\nu_5-\nu_6-\nu_7-5\nu_8,\\
\lambda_1+\lambda_2+\lambda_3-3\lambda_4+2\mu_2-2\mu_1&\le 3\nu_1+3\nu_2+3\nu_3-\nu_4-\nu_5-\nu_6-\nu_8-5\nu_7,\\
\lambda_1+\lambda_2+\lambda_4-3\lambda_3+2\mu_1-2\mu_2&\le 3\nu_1+3\nu_2+3\nu_3-\nu_4-\nu_5-\nu_6-\nu_8-5\nu_7,\\
\lambda_1+\lambda_3+\lambda_4-3\lambda_2+2\mu_1-2\mu_2&\le 3\nu_1+3\nu_3+3\nu_4-\nu_2-\nu_5-\nu_6-\nu_7-5\nu_8,\\
\lambda_1+\lambda_3+\lambda_4-3\lambda_2+2\mu_1-2\mu_2&\le 3\nu_1+3\nu_2+3\nu_4-\nu_3-\nu_5-\nu_6-\nu_8-5\nu_7,\\
\lambda_1+\lambda_3+\lambda_4-3\lambda_2+2\mu_1-2\mu_2&\le 3\nu_1+3\nu_2+3\nu_3-\nu_4-\nu_5-\nu_7-\nu_8-5\nu_6,\\
\lambda_1+\lambda_3+\lambda_4-3\lambda_2+2\mu_2-2\mu_1&\le 3\nu_2+3\nu_3+3\nu_4-\nu_1-\nu_5-\nu_6-\nu_7-5\nu_8,\\
\lambda_2+\lambda_3+\lambda_4-3\lambda_1+2\mu_1-2\mu_2&\le 3\nu_2+3\nu_3+3\nu_4-\nu_1-\nu_5-\nu_6-\nu_7-5\nu_8,\\
\lambda_2+\lambda_3+\lambda_4-3\lambda_1+2\mu_1-2\mu_2&\le 3\nu_1+3\nu_3+3\nu_4-\nu_2-\nu_5-\nu_6-\nu_8-5\nu_7,\\
\lambda_2+\lambda_3+\lambda_4-3\lambda_1+2\mu_1-2\mu_2&\le 3\nu_1+3\nu_2+3\nu_4-\nu_3-\nu_5-\nu_7-\nu_8-5\nu_6,\\
\lambda_2+\lambda_3+\lambda_4-3\lambda_1+2\mu_1-2\mu_2&\le 3\nu_1+3\nu_2+3\nu_3-\nu_4-\nu_6-\nu_7-\nu_8-5\nu_5.
\end{aligned}$$

$$\begin{aligned}
3\lambda_1-\lambda_2-\lambda_3-\lambda_4+2\mu_1-2\mu_2&\le 5\nu_1+\nu_2+\nu_3+\nu_4+\nu_5-3\nu_6-3\nu_7-3\nu_8,\\
3\lambda_1-\lambda_2-\lambda_3-\lambda_4+2\mu_2-2\mu_1&\le 5\nu_2+\nu_1+\nu_3+\nu_4+\nu_5-3\nu_6-3\nu_7-3\nu_8,\\
3\lambda_2-\lambda_1-\lambda_3-\lambda_4+2\mu_1-2\mu_2&\le 5\nu_2+\nu_1+\nu_3+\nu_4+\nu_5-3\nu_6-3\nu_7-3\nu_8,\\
3\lambda_2-\lambda_1-\lambda_3-\lambda_4+2\mu_1-2\mu_2&\le 5\nu_1+\nu_2+\nu_3+\nu_4+\nu_6-3\nu_5-3\nu_7-3\nu_8,\\
3\lambda_3-\lambda_1-\lambda_2-\lambda_4+2\mu_1-2\mu_2&\le 5\nu_3+\nu_1+\nu_2+\nu_4+\nu_5-3\nu_6-3\nu_7-3\nu_8,\\
3\lambda_3-\lambda_1-\lambda_2-\lambda_4+2\mu_1-2\mu_2&\le 5\nu_2+\nu_1+\nu_3+\nu_4+\nu_6-3\nu_5-3\nu_7-3\nu_8,\\
3\lambda_3-\lambda_1-\lambda_2-\lambda_4+2\mu_1-2\mu_2&\le 5\nu_1+\nu_2+\nu_3+\nu_4+\nu_7-3\nu_5-3\nu_6-3\nu_8,\\
3\lambda_4-\lambda_1-\lambda_2-\lambda_3+2\mu_1-2\mu_2&\le 5\nu_4+\nu_1+\nu_2+\nu_3+\nu_5-3\nu_6-3\nu_7-3\nu_8,\\
3\lambda_4-\lambda_1-\lambda_2-\lambda_3+2\mu_1-2\mu_2&\le 5\nu_3+\nu_1+\nu_2+\nu_4+\nu_6-3\nu_5-3\nu_7-3\nu_8,\\
3\lambda_4-\lambda_1-\lambda_2-\lambda_3+2\mu_1-2\mu_2&\le 5\nu_2+\nu_1+\nu_3+\nu_4+\nu_7-3\nu_5-3\nu_6-3\nu_8,\\
3\lambda_3-\lambda_1-\lambda_2-\lambda_4+2\mu_2-2\mu_1&\le 5\nu_1+\nu_2+\nu_3+\nu_4+\nu_8-3\nu_5-3\nu_6-3\nu_7,\\
3\lambda_4-\lambda_1-\lambda_2-\lambda_3+2\mu_1-2\mu_2&\le 5\nu_1+\nu_2+\nu_3+\nu_4+\nu_8-3\nu_5-3\nu_6-3\nu_7.
\end{aligned}$$

$$\begin{aligned}
\lambda_1+\lambda_2-\lambda_3-\lambda_4+\mu_1-\mu_2 &\le 2\nu_1+2\nu_2-2\nu_7-2\nu_8,\\
\lambda_1+\lambda_3-\lambda_2-\lambda_4+\mu_1-\mu_2 &\le 2\nu_1+2\nu_3-2\nu_7-2\nu_8,\\
\lambda_1+\lambda_3-\lambda_2-\lambda_4+\mu_1-\mu_2 &\le 2\nu_1+2\nu_2-2\nu_6-2\nu_8,\\
\lambda_1+\lambda_3-\lambda_2-\lambda_4+\mu_2-\mu_1 &\le 2\nu_2+2\nu_3-2\nu_7-2\nu_8,\\
\lambda_2+\lambda_3-\lambda_1-\lambda_4+\mu_1-\mu_2 &\le 2\nu_2+2\nu_3-2\nu_7-2\nu_8,\\
\lambda_1+\lambda_4-\lambda_2-\lambda_3+\mu_1-\mu_2 &\le 2\nu_1+2\nu_4-2\nu_7-2\nu_8,\\
\lambda_1+\lambda_4-\lambda_2-\lambda_3+\mu_1-\mu_2 &\le 2\nu_1+2\nu_3-2\nu_6-2\nu_8,\\
\lambda_2+\lambda_3-\lambda_1-\lambda_4+\mu_1-\mu_2 &\le 2\nu_1+2\nu_3-2\nu_6-2\nu_8,\\
\lambda_2+\lambda_3-\lambda_1-\lambda_4+\mu_1-\mu_2 &\le 2\nu_1+2\nu_2-2\nu_5-2\nu_8,\\
\lambda_1+\lambda_3-\lambda_2-\lambda_4+\mu_2-\mu_1 &\le 2\nu_1+2\nu_2-2\nu_6-2\nu_7,\\
\lambda_1+\lambda_4-\lambda_2-\lambda_3+\mu_1-\mu_2 &\le 2\nu_1+2\nu_2-2\nu_6-2\nu_7,\\
\lambda_1+\lambda_4-\lambda_2-\lambda_3+\mu_2-\mu_1 &\le 2\nu_2+2\nu_4-2\nu_7-2\nu_8,\\
\lambda_2+\lambda_4-\lambda_1-\lambda_3+\mu_1-\mu_2 &\le 2\nu_2+2\nu_4-2\nu_7-2\nu_8,\\
\lambda_2+\lambda_4-\lambda_1-\lambda_3+\mu_1-\mu_2 &\le 2\nu_1+2\nu_4-2\nu_6-2\nu_8,\\
\lambda_1+\lambda_4-\lambda_2-\lambda_3+\mu_2-\mu_1 &\le 2\nu_2+2\nu_3-2\nu_6-2\nu_8,\\
\lambda_2+\lambda_4-\lambda_1-\lambda_3+\mu_1-\mu_2 &\le 2\nu_2+2\nu_3-2\nu_6-2\nu_8,\\
\lambda_2+\lambda_4-\lambda_1-\lambda_3+\mu_1-\mu_2 &\le 2\nu_1+2\nu_3-2\nu_5-2\nu_8,\\
\lambda_2+\lambda_3-\lambda_1-\lambda_4+\mu_2-\mu_1 &\le 2\nu_1+2\nu_2-2\nu_5-2\nu_7,\\
\lambda_2+\lambda_4-\lambda_1-\lambda_3+\mu_1-\mu_2 &\le 2\nu_1+2\nu_2-2\nu_5-2\nu_7,\\
\lambda_2+\lambda_3-\lambda_1-\lambda_4+\mu_2-\mu_1 &\le 2\nu_1+2\nu_3-2\nu_6-2\nu_7,\\
\lambda_2+\lambda_4-\lambda_1-\lambda_3+\mu_1-\mu_2 &\le 2\nu_1+2\nu_3-2\nu_6-2\nu_7,\\
\lambda_3+\lambda_4-\lambda_1-\lambda_2+\mu_1-\mu_2 &\le 2\nu_3+2\nu_4-2\nu_7-2\nu_8,\\
\lambda_3+\lambda_4-\lambda_1-\lambda_2+\mu_1-\mu_2 &\le 2\nu_2+2\nu_4-2\nu_6-2\nu_8,\\
\lambda_3+\lambda_4-\lambda_1-\lambda_2+\mu_1-\mu_2 &\le 2\nu_2+2\nu_3-2\nu_5-2\nu_8,\\
\lambda_3+\lambda_4-\lambda_1-\lambda_2+\mu_1-\mu_2 &\le 2\nu_1+2\nu_3-2\nu_5-2\nu_7,\\
\lambda_3+\lambda_4-\lambda_1-\lambda_2+\mu_1-\mu_2 &\le 2\nu_1+2\nu_4-2\nu_6-2\nu_7,\\
\lambda_3+\lambda_4-\lambda_1-\lambda_2+\mu_1-\mu_2 &\le 2\nu_1+2\nu_2-2\nu_5-2\nu_6.
\end{aligned}$$

$$\begin{aligned}
5\lambda_1+\lambda_2-3\lambda_3-3\lambda_4+2\mu_1-2\mu_2 &\le 7\nu_1+3\nu_2+3\nu_3-\nu_4-\nu_5-\nu_6-5\nu_7-5\nu_8,\\
5\lambda_1+\lambda_2-3\lambda_3-3\lambda_4+2\mu_2-2\mu_1 &\le 7\nu_2+3\nu_1+3\nu_3-\nu_4-\nu_5-\nu_6-5\nu_7-5\nu_8,\\
5\lambda_2+\lambda_1-3\lambda_3-3\lambda_4+2\mu_1-2\mu_2 &\le 7\nu_2+3\nu_1+3\nu_3-\nu_4-\nu_5-\nu_6-5\nu_7-5\nu_8,\\
5\lambda_1+\lambda_2-3\lambda_3-3\lambda_4+2\mu_2-2\mu_1 &\le 7\nu_1+3\nu_2+3\nu_4-\nu_3-\nu_5-\nu_6-5\nu_7-5\nu_8,\\
5\lambda_1+\lambda_3-3\lambda_2-3\lambda_4+2\mu_1-2\mu_2 &\le 7\nu_1+3\nu_2+3\nu_4-\nu_3-\nu_5-\nu_6-5\nu_7-5\nu_8,\\
5\lambda_2+\lambda_1-3\lambda_3-3\lambda_4+2\mu_1-2\mu_2 &\le 7\nu_1+3\nu_2+3\nu_4-\nu_3-\nu_5-\nu_6-5\nu_7-5\nu_8,\\
5\lambda_1+\lambda_3-3\lambda_2-3\lambda_4+2\mu_1-2\mu_2 &\le 7\nu_1+3\nu_2+3\nu_3-\nu_4-\nu_5-\nu_7-5\nu_6-5\nu_8,\\
5\lambda_1+\lambda_3-3\lambda_2-3\lambda_4+2\mu_2-2\mu_1 &\le 7\nu_2+3\nu_1+3\nu_4-\nu_3-\nu_5-\nu_6-5\nu_7-5\nu_8,\\
5\lambda_1+\lambda_4-3\lambda_2-3\lambda_3+2\mu_1-2\mu_2 &\le 7\nu_1+3\nu_2+3\nu_5-\nu_3-\nu_4-\nu_6-5\nu_7-5\nu_8,\\
5\lambda_1+\lambda_4-3\lambda_2-3\lambda_3+2\mu_1-2\mu_2 &\le 7\nu_1+3\nu_2+3\nu_4-\nu_3-\nu_5-\nu_7-5\nu_6-5\nu_8,\\
5\lambda_1+\lambda_3-3\lambda_2-3\lambda_4+2\mu_2-2\mu_1 &\le 7\nu_2+3\nu_1+3\nu_3-\nu_4-\nu_5-\nu_7-5\nu_6-5\nu_8,\\
5\lambda_1+\lambda_3-3\lambda_2-3\lambda_4+2\mu_2-2\mu_1 &\le 7\nu_1+3\nu_2+3\nu_3-\nu_4-\nu_5-\nu_8-5\nu_6-5\nu_7,\\
5\lambda_1+\lambda_4-3\lambda_2-3\lambda_3+2\mu_1-2\mu_2 &\le 7\nu_1+3\nu_2+3\nu_3-\nu_4-\nu_5-\nu_8-5\nu_6-5\nu_7,\\
5\lambda_3+\lambda_2-3\lambda_1-3\lambda_4+2\mu_1-2\mu_2 &\le 7\nu_3+3\nu_1+3\nu_4-\nu_2-\nu_5-\nu_6-5\nu_7-5\nu_8,\\
5\lambda_3+\lambda_1-3\lambda_2-3\lambda_4+2\mu_2-2\mu_1 &\le 7\nu_2+3\nu_3+3\nu_4-\nu_1-\nu_5-\nu_6-5\nu_7-5\nu_8,\\
5\lambda_3+\lambda_2-3\lambda_1-3\lambda_4+2\mu_1-2\mu_2 &\le 7\nu_2+3\nu_3+3\nu_4-\nu_1-\nu_5-\nu_6-5\nu_7-5\nu_8,\\
5\lambda_1+\lambda_4-3\lambda_2-3\lambda_3+2\mu_2-2\mu_1 &\le 7\nu_2+3\nu_1+3\nu_5-\nu_3-\nu_4-\nu_6-5\nu_7-5\nu_8,\\
5\lambda_1+\lambda_4-3\lambda_2-3\lambda_3+2\mu_2-2\mu_1 &\le 7\nu_2+3\nu_1+3\nu_4-\nu_3-\nu_5-\nu_7-5\nu_6-5\nu_8,\\
5\lambda_1+\lambda_4-3\lambda_2-3\lambda_3+2\mu_2-2\mu_1 &\le 7\nu_2+3\nu_1+3\nu_3-\nu_4-\nu_5-\nu_8-5\nu_6-5\nu_7,\\
5\lambda_3+\lambda_2-3\lambda_1-3\lambda_4+2\mu_2-2\mu_1 &\le 7\nu_3+3\nu_2+3\nu_4-\nu_1-\nu_5-\nu_6-5\nu_7-5\nu_8,\\
5\lambda_4+\lambda_3-3\lambda_1-3\lambda_2+2\mu_1-2\mu_2 &\le 7\nu_4+3\nu_1+3\nu_5-\nu_2-\nu_3-\nu_6-5\nu_7-5\nu_8,\\
5\lambda_3+\lambda_4-3\lambda_1-3\lambda_2+2\mu_2-2\mu_1 &\le 7\nu_1+3\nu_2+3\nu_8-\nu_3-\nu_4-\nu_5-5\nu_6-5\nu_7,\\
5\lambda_4+\lambda_3-3\lambda_1-3\lambda_2+2\mu_1-2\mu_2 &\le 7\nu_1+3\nu_2+3\nu_8-\nu_3-\nu_4-\nu_5-5\nu_6-5\nu_7.
\end{aligned}$$

$$3\lambda_1+3\lambda_2-\lambda_3-5\lambda_4+2\mu_1-2\mu_2\le 5\nu_1+5\nu_2+\nu_3+\nu_4+\nu_5-3\nu_6-3\nu_7-7\nu_8,$$
$$3\lambda_1+3\lambda_3-\lambda_2-5\lambda_4+2\mu_1-2\mu_2\le 5\nu_1+5\nu_3+\nu_2+\nu_4+\nu_5-3\nu_6-3\nu_7-7\nu_8,$$
$$3\lambda_1+3\lambda_2-\lambda_3-5\lambda_4+2\mu_2-2\mu_1\le 5\nu_1+5\nu_2+\nu_3+\nu_4+\nu_6-3\nu_5-3\nu_7-7\nu_8,$$
$$3\lambda_1+3\lambda_2-\lambda_4-5\lambda_3+2\mu_1-2\mu_2\le 5\nu_1+5\nu_2+\nu_3+\nu_4+\nu_6-3\nu_5-3\nu_7-7\nu_8,$$
$$3\lambda_1+3\lambda_3-\lambda_2-5\lambda_4+2\mu_1-2\mu_2\le 5\nu_1+5\nu_2+\nu_3+\nu_4+\nu_6-3\nu_5-3\nu_7-7\nu_8,$$
$$3\lambda_1+3\lambda_2-\lambda_3-5\lambda_4+2\mu_2-2\mu_1\le 5\nu_1+5\nu_2+\nu_3+\nu_4+\nu_5-3\nu_6-3\nu_8-7\nu_7,$$
$$3\lambda_1+3\lambda_2-\lambda_4-5\lambda_3+2\mu_1-2\mu_2\le 5\nu_1+5\nu_2+\nu_3+\nu_4+\nu_5-3\nu_6-3\nu_8-7\nu_7,$$
$$3\lambda_1+3\lambda_3-\lambda_2-5\lambda_4+2\mu_2-2\mu_1\le 5\nu_2+5\nu_3+\nu_1+\nu_4+\nu_5-3\nu_6-3\nu_7-7\nu_8,$$
$$3\lambda_2+3\lambda_3-\lambda_1-5\lambda_4+2\mu_1-2\mu_2\le 5\nu_2+5\nu_3+\nu_1+\nu_4+\nu_5-3\nu_6-3\nu_7-7\nu_8,$$
$$3\lambda_2+3\lambda_3-\lambda_1-5\lambda_4+2\mu_1-2\mu_2\le 5\nu_1+5\nu_3+\nu_2+\nu_4+\nu_6-3\nu_5-3\nu_7-7\nu_8,$$
$$3\lambda_2+3\lambda_3-\lambda_1-5\lambda_4+2\mu_1-2\mu_2\le 5\nu_1+5\nu_2+\nu_3+\nu_5+\nu_6-3\nu_4-3\nu_7-7\nu_8,$$
$$3\lambda_1+3\lambda_3-\lambda_2-5\lambda_4+2\mu_2-2\mu_1\le 5\nu_1+5\nu_2+\nu_3+\nu_4+\nu_6-3\nu_5-3\nu_8-7\nu_7,$$
$$3\lambda_1+3\lambda_3-\lambda_2-5\lambda_4+2\mu_2-2\mu_1\le 5\nu_1+5\nu_3+\nu_2+\nu_4+\nu_5-3\nu_6-3\nu_8-7\nu_7,$$
$$3\lambda_1+3\lambda_3-\lambda_4-5\lambda_2+2\mu_2-2\mu_1\le 5\nu_1+5\nu_2+\nu_3+\nu_4+\nu_8-3\nu_5-3\nu_6-7\nu_7,$$
$$3\lambda_1+3\lambda_4-\lambda_3-5\lambda_2+2\mu_1-2\mu_2\le 5\nu_1+5\nu_2+\nu_3+\nu_4+\nu_8-3\nu_5-3\nu_6-7\nu_7,$$
$$3\lambda_2+3\lambda_3-\lambda_1-5\lambda_4+2\mu_2-2\mu_1\le 5\nu_1+5\nu_2+\nu_3+\nu_5+\nu_6-3\nu_4-3\nu_8-7\nu_7,$$
$$3\lambda_2+3\lambda_3-\lambda_1-5\lambda_4+2\mu_2-2\mu_1\le 5\nu_1+5\nu_3+\nu_2+\nu_4+\nu_6-3\nu_5-3\nu_8-7\nu_7,$$
$$3\lambda_2+3\lambda_3-\lambda_1-5\lambda_4+2\mu_2-2\mu_1\le 5\nu_2+5\nu_3+\nu_1+\nu_4+\nu_5-3\nu_6-3\nu_8-7\nu_7,$$
$$3\lambda_1+3\lambda_4-\lambda_3-5\lambda_2+2\mu_1-2\mu_2\le 5\nu_1+5\nu_2+\nu_3+\nu_4+\nu_7-3\nu_5-3\nu_8-7\nu_6,$$
$$3\lambda_1+3\lambda_4-\lambda_3-5\lambda_2+2\mu_2-2\mu_1\le 5\nu_1+5\nu_2+\nu_3+\nu_4+\nu_8-3\nu_5-3\nu_7-7\nu_6,$$
$$3\lambda_3+3\lambda_4-\lambda_1-5\lambda_2+2\mu_2-2\mu_1\le 5\nu_2+5\nu_3+\nu_4+\nu_5+\nu_6-3\nu_1-3\nu_7-7\nu_8,$$
$$3\lambda_3+3\lambda_4-\lambda_2-5\lambda_1+2\mu_1-2\mu_2\le 5\nu_2+5\nu_3+\nu_4+\nu_5+\nu_6-3\nu_1-3\nu_7-7\nu_8,$$
$$3\lambda_3+3\lambda_4-\lambda_2-5\lambda_1+2\mu_1-2\mu_2\le 5\nu_1+5\nu_2+\nu_3+\nu_6+\nu_7-3\nu_4-3\nu_8-7\nu_5.$$

$$2\lambda_1 - 2\lambda_4 + \mu_1 - \mu_2 \le 3\nu_1 + \nu_2 + \nu_3 + \nu_4 - \nu_5 - \nu_6 - \nu_7 - 3\nu_8,$$
$$2\lambda_1 - 2\lambda_4 + \mu_2 - \mu_1 \le 3\nu_2 + \nu_1 + \nu_3 + \nu_4 - \nu_5 - \nu_6 - \nu_7 - 3\nu_8,$$
$$2\lambda_2 - 2\lambda_4 + \mu_1 - \mu_2 \le 3\nu_2 + \nu_1 + \nu_3 + \nu_4 - \nu_5 - \nu_6 - \nu_7 - 3\nu_8,$$
$$2\lambda_1 - 2\lambda_3 + \mu_1 - \mu_2 \le 3\nu_1 + \nu_2 + \nu_3 + \nu_5 - \nu_4 - \nu_6 - \nu_7 - 3\nu_8,$$
$$2\lambda_2 - 2\lambda_4 + \mu_1 - \mu_2 \le 3\nu_1 + \nu_2 + \nu_3 + \nu_5 - \nu_4 - \nu_6 - \nu_7 - 3\nu_8,$$
$$2\lambda_1 - 2\lambda_4 + \mu_2 - \mu_1 \le 3\nu_1 + \nu_2 + \nu_3 + \nu_4 - \nu_5 - \nu_6 - \nu_8 - 3\nu_7,$$
$$2\lambda_1 - 2\lambda_3 + \mu_1 - \mu_2 \le 3\nu_1 + \nu_2 + \nu_3 + \nu_4 - \nu_5 - \nu_6 - \nu_8 - 3\nu_7,$$
$$2\lambda_3 - 2\lambda_4 + \mu_1 - \mu_2 \le 3\nu_3 + \nu_1 + \nu_2 + \nu_4 - \nu_5 - \nu_6 - \nu_7 - 3\nu_8,$$
$$2\lambda_1 - 2\lambda_3 + \mu_2 - \mu_1 \le 3\nu_2 + \nu_1 + \nu_3 + \nu_5 - \nu_4 - \nu_6 - \nu_7 - 3\nu_8,$$
$$2\lambda_2 - 2\lambda_3 + \mu_1 - \mu_2 \le 3\nu_2 + \nu_1 + \nu_3 + \nu_5 - \nu_4 - \nu_6 - \nu_7 - 3\nu_8,$$
$$2\lambda_3 - 2\lambda_4 + \mu_1 - \mu_2 \le 3\nu_2 + \nu_1 + \nu_3 + \nu_5 - \nu_4 - \nu_6 - \nu_7 - 3\nu_8,$$
$$2\lambda_1 - 2\lambda_3 + \mu_2 - \mu_1 \le 3\nu_1 + \nu_2 + \nu_4 + \nu_5 - \nu_3 - \nu_6 - \nu_7 - 3\nu_8,$$
$$2\lambda_1 - 2\lambda_2 + \mu_1 - \mu_2 \le 3\nu_1 + \nu_2 + \nu_4 + \nu_5 - \nu_3 - \nu_6 - \nu_7 - 3\nu_8,$$
$$2\lambda_2 - 2\lambda_3 + \mu_1 - \mu_2 \le 3\nu_1 + \nu_2 + \nu_4 + \nu_5 - \nu_3 - \nu_6 - \nu_7 - 3\nu_8,$$
$$2\lambda_2 - 2\lambda_4 + \mu_2 - \mu_1 \le 3\nu_1 + \nu_2 + \nu_3 + \nu_6 - \nu_4 - \nu_5 - \nu_7 - 3\nu_8,$$
$$2\lambda_2 - 2\lambda_3 + \mu_1 - \mu_2 \le 3\nu_1 + \nu_2 + \nu_3 + \nu_6 - \nu_4 - \nu_5 - \nu_7 - 3\nu_8,$$
$$2\lambda_3 - 2\lambda_4 + \mu_1 - \mu_2 \le 3\nu_1 + \nu_2 + \nu_3 + \nu_6 - \nu_4 - \nu_5 - \nu_7 - 3\nu_8,$$
$$2\lambda_1 - 2\lambda_2 + \mu_1 - \mu_2 \le 3\nu_1 + \nu_2 + \nu_3 + \nu_5 - \nu_4 - \nu_6 - \nu_8 - 3\nu_7,$$
$$2\lambda_2 - 2\lambda_4 + \mu_2 - \mu_1 \le 3\nu_1 + \nu_2 + \nu_3 + \nu_5 - \nu_4 - \nu_6 - \nu_8 - 3\nu_7,$$
$$2\lambda_2 - 2\lambda_3 + \mu_1 - \mu_2 \le 3\nu_1 + \nu_2 + \nu_3 + \nu_5 - \nu_4 - \nu_6 - \nu_8 - 3\nu_7,$$
$$2\lambda_1 - 2\lambda_3 + \mu_2 - \mu_1 \le 3\nu_2 + \nu_1 + \nu_3 + \nu_4 - \nu_5 - \nu_6 - \nu_8 - 3\nu_7,$$
$$2\lambda_2 - 2\lambda_4 + \mu_2 - \mu_1 \le 3\nu_2 + \nu_1 + \nu_3 + \nu_4 - \nu_5 - \nu_6 - \nu_8 - 3\nu_7,$$
$$2\lambda_2 - 2\lambda_3 + \mu_1 - \mu_2 \le 3\nu_2 + \nu_1 + \nu_3 + \nu_4 - \nu_5 - \nu_6 - \nu_8 - 3\nu_7,$$
$$2\lambda_1 - 2\lambda_2 + \mu_1 - \mu_2 \le 3\nu_1 + \nu_2 + \nu_3 + \nu_4 - \nu_5 - \nu_7 - \nu_8 - 3\nu_6,$$
$$2\lambda_1 - 2\lambda_2 + \mu_2 - \mu_1 \le 3\nu_2 + \nu_1 + \nu_4 + \nu_5 - \nu_3 - \nu_6 - \nu_7 - 3\nu_8,$$
$$2\lambda_3 - 2\lambda_4 + \mu_2 - \mu_1 \le 3\nu_1 + \nu_2 + \nu_3 + \nu_6 - \nu_4 - \nu_5 - \nu_8 - 3\nu_7,$$
$$2\lambda_1 - 2\lambda_2 + \mu_2 - \mu_1 \le 3\nu_2 + \nu_1 + \nu_3 + \nu_5 - \nu_4 - \nu_6 - \nu_8 - 3\nu_7,$$
$$2\lambda_3 - 2\lambda_4 + \mu_2 - \mu_1 \le 3\nu_2 + \nu_1 + \nu_3 + \nu_5 - \nu_4 - \nu_6 - \nu_8 - 3\nu_7,$$
$$2\lambda_3 - 2\lambda_4 + \mu_2 - \mu_1 \le 3\nu_3 + \nu_1 + \nu_2 + \nu_4 - \nu_5 - \nu_6 - \nu_8 - 3\nu_7,$$

$$
\begin{aligned}
2\lambda_1 - 2\lambda_2 + \mu_2 - \mu_1 &\le 3\nu_2 + \nu_1 + \nu_3 + \nu_4 - \nu_5 - \nu_7 - \nu_8 - 3\nu_6,\\
2\lambda_4 - 2\lambda_2 + \mu_1 - \mu_2 &\le 3\nu_4 + \nu_1 + \nu_2 + \nu_5 - \nu_3 - \nu_6 - \nu_7 - 3\nu_8,\\
2\lambda_3 - 2\lambda_1 + \mu_1 - \mu_2 &\le 3\nu_3 + \nu_1 + \nu_4 + \nu_5 - \nu_2 - \nu_6 - \nu_7 - 3\nu_8,\\
2\lambda_4 - 2\lambda_2 + \mu_1 - \mu_2 &\le 3\nu_3 + \nu_1 + \nu_4 + \nu_5 - \nu_2 - \nu_6 - \nu_7 - 3\nu_8,\\
2\lambda_3 - 2\lambda_2 + \mu_2 - \mu_1 &\le 3\nu_2 + \nu_3 + \nu_4 + \nu_5 - \nu_1 - \nu_6 - \nu_7 - 3\nu_8,\\
2\lambda_3 - 2\lambda_1 + \mu_1 - \mu_2 &\le 3\nu_2 + \nu_3 + \nu_4 + \nu_5 - \nu_1 - \nu_6 - \nu_7 - 3\nu_8,\\
2\lambda_3 - 2\lambda_2 + \mu_2 - \mu_1 &\le 3\nu_1 + \nu_2 + \nu_3 + \nu_8 - \nu_4 - \nu_5 - \nu_6 - 3\nu_7,\\
2\lambda_4 - 2\lambda_2 + \mu_1 - \mu_2 &\le 3\nu_1 + \nu_2 + \nu_3 + \nu_8 - \nu_4 - \nu_5 - \nu_6 - 3\nu_7,\\
2\lambda_3 - 2\lambda_1 + \mu_1 - \mu_2 &\le 3\nu_1 + \nu_2 + \nu_3 + \nu_7 - \nu_4 - \nu_5 - \nu_8 - 3\nu_6,\\
2\lambda_4 - 2\lambda_2 + \mu_1 - \mu_2 &\le 3\nu_1 + \nu_2 + \nu_3 + \nu_7 - \nu_4 - \nu_5 - \nu_8 - 3\nu_6,\\
2\lambda_3 - 2\lambda_1 + \mu_1 - \mu_2 &\le 3\nu_1 + \nu_2 + \nu_3 + \nu_6 - \nu_4 - \nu_7 - \nu_8 - 3\nu_5,\\
2\lambda_4 - 2\lambda_1 + \mu_1 - \mu_2 &\le 3\nu_4 + \nu_1 + \nu_3 + \nu_5 - \nu_2 - \nu_6 - \nu_7 - 3\nu_8,\\
2\lambda_3 - 2\lambda_1 + \mu_2 - \mu_1 &\le 3\nu_3 + \nu_2 + \nu_4 + \nu_5 - \nu_1 - \nu_6 - \nu_7 - 3\nu_8,\\
2\lambda_4 - 2\lambda_2 + \mu_2 - \mu_1 &\le 3\nu_3 + \nu_2 + \nu_4 + \nu_5 - \nu_1 - \nu_6 - \nu_7 - 3\nu_8,\\
2\lambda_4 - 2\lambda_1 + \mu_1 - \mu_2 &\le 3\nu_3 + \nu_2 + \nu_4 + \nu_5 - \nu_1 - \nu_6 - \nu_7 - 3\nu_8,\\
2\lambda_4 - 2\lambda_2 + \mu_2 - \mu_1 &\le 3\nu_2 + \nu_3 + \nu_4 + \nu_6 - \nu_1 - \nu_5 - \nu_7 - 3\nu_8,\\
2\lambda_4 - 2\lambda_1 + \mu_1 - \mu_2 &\le 3\nu_2 + \nu_3 + \nu_4 + \nu_6 - \nu_1 - \nu_5 - \nu_7 - 3\nu_8,\\
2\lambda_4 - 2\lambda_1 + \mu_1 - \mu_2 &\le 3\nu_3 + \nu_1 + \nu_4 + \nu_6 - \nu_2 - \nu_5 - \nu_7 - 3\nu_8,\\
2\lambda_3 - 2\lambda_1 + \mu_2 - \mu_1 &\le 3\nu_1 + \nu_2 + \nu_4 + \nu_8 - \nu_3 - \nu_5 - \nu_6 - 3\nu_7,\\
2\lambda_4 - 2\lambda_1 + \mu_1 - \mu_2 &\le 3\nu_1 + \nu_2 + \nu_4 + \nu_8 - \nu_3 - \nu_5 - \nu_6 - 3\nu_7,\\
2\lambda_4 - 2\lambda_1 + \mu_1 - \mu_2 &\le 3\nu_3 + \nu_1 + \nu_4 + \nu_5 - \nu_2 - \nu_6 - \nu_8 - 3\nu_7,\\
2\lambda_4 - 2\lambda_1 + \mu_1 - \mu_2 &\le 3\nu_4 + \nu_1 + \nu_2 + \nu_5 - \nu_3 - \nu_6 - \nu_8 - 3\nu_7,\\
2\lambda_4 - 2\lambda_1 + \mu_1 - \mu_2 &\le 3\nu_1 + \nu_2 + \nu_4 + \nu_7 - \nu_3 - \nu_5 - \nu_8 - 3\nu_6,\\
2\lambda_4 - 2\lambda_1 + \mu_1 - \mu_2 &\le 3\nu_2 + \nu_1 + \nu_3 + \nu_7 - \nu_4 - \nu_5 - \nu_8 - 3\nu_6,\\
2\lambda_3 - 2\lambda_1 + \mu_2 - \mu_1 &\le 3\nu_1 + \nu_2 + \nu_3 + \nu_8 - \nu_4 - \nu_5 - \nu_7 - 3\nu_6,\\
2\lambda_4 - 2\lambda_2 + \mu_2 - \mu_1 &\le 3\nu_1 + \nu_2 + \nu_3 + \nu_8 - \nu_4 - \nu_5 - \nu_7 - 3\nu_6,\\
2\lambda_4 - 2\lambda_1 + \mu_1 - \mu_2 &\le 3\nu_1 + \nu_2 + \nu_3 + \nu_8 - \nu_4 - \nu_5 - \nu_7 - 3\nu_6,\\
2\lambda_4 - 2\lambda_1 + \mu_1 - \mu_2 &\le 3\nu_1 + \nu_2 + \nu_3 + \nu_7 - \nu_4 - \nu_6 - \nu_8 - 3\nu_5,\\
2\lambda_4 - 2\lambda_1 + \mu_1 - \mu_2 &\le 3\nu_2 + \nu_1 + \nu_3 + \nu_6 - \nu_4 - \nu_7 - \nu_8 - 3\nu_5.
\end{aligned}
$$

$$3\lambda_1 + \lambda_2 - \lambda_3 - 3\lambda_4 + \mu_1 - \mu_2 \leq 4\nu_1 + 2\nu_2 + 2\nu_3 - 2\nu_6 - 2\nu_7 - 4\nu_8,$$
$$3\lambda_1 + \lambda_2 - \lambda_3 - 3\lambda_4 + \mu_2 - \mu_1 \leq 4\nu_2 + 2\nu_1 + 2\nu_3 - 2\nu_6 - 2\nu_7 - 4\nu_8,$$
$$3\lambda_2 + \lambda_1 - \lambda_3 - 3\lambda_4 + \mu_1 - \mu_2 \leq 4\nu_2 + 2\nu_1 + 2\nu_3 - 2\nu_6 - 2\nu_7 - 4\nu_8,$$
$$3\lambda_1 + \lambda_2 - \lambda_3 - 3\lambda_4 + \mu_2 - \mu_1 \leq 4\nu_1 + 2\nu_2 + 2\nu_4 - 2\nu_6 - 2\nu_7 - 4\nu_8,$$
$$3\lambda_1 + \lambda_3 - \lambda_2 - 3\lambda_4 + \mu_1 - \mu_2 \leq 4\nu_1 + 2\nu_2 + 2\nu_4 - 2\nu_6 - 2\nu_7 - 4\nu_8,$$
$$3\lambda_2 + \lambda_1 - \lambda_3 - 3\lambda_4 + \mu_1 - \mu_2 \leq 4\nu_1 + 2\nu_2 + 2\nu_4 - 2\nu_6 - 2\nu_7 - 4\nu_8,$$
$$3\lambda_1 + \lambda_2 - \lambda_3 - 3\lambda_4 + \mu_2 - \mu_1 \leq 4\nu_1 + 2\nu_2 + 2\nu_3 - 2\nu_5 - 2\nu_7 - 4\nu_8,$$
$$3\lambda_1 + \lambda_2 - \lambda_4 - 3\lambda_3 + \mu_1 - \mu_2 \leq 4\nu_1 + 2\nu_2 + 2\nu_3 - 2\nu_5 - 2\nu_7 - 4\nu_8,$$
$$3\lambda_1 + \lambda_3 - \lambda_2 - 3\lambda_4 + \mu_1 - \mu_2 \leq 4\nu_1 + 2\nu_2 + 2\nu_3 - 2\nu_5 - 2\nu_7 - 4\nu_8,$$
$$3\lambda_1 + \lambda_2 - \lambda_3 - 3\lambda_4 + \mu_2 - \mu_1 \leq 4\nu_1 + 2\nu_2 + 2\nu_3 - 2\nu_6 - 2\nu_8 - 4\nu_7,$$
$$3\lambda_1 + \lambda_2 - \lambda_4 - 3\lambda_3 + \mu_1 - \mu_2 \leq 4\nu_1 + 2\nu_2 + 2\nu_3 - 2\nu_6 - 2\nu_8 - 4\nu_7,$$
$$3\lambda_1 + \lambda_3 - \lambda_2 - 3\lambda_4 + \mu_2 - \mu_1 \leq 4\nu_2 + 2\nu_1 + 2\nu_4 - 2\nu_6 - 2\nu_7 - 4\nu_8,$$
$$3\lambda_1 + \lambda_2 - \lambda_4 - 3\lambda_3 + \mu_2 - \mu_1 \leq 4\nu_1 + 2\nu_2 + 2\nu_4 - 2\nu_5 - 2\nu_7 - 4\nu_8,$$
$$3\lambda_2 + \lambda_1 - \lambda_3 - 3\lambda_4 + \mu_2 - \mu_1 \leq 4\nu_1 + 2\nu_2 + 2\nu_4 - 2\nu_5 - 2\nu_7 - 4\nu_8,$$
$$3\lambda_2 + \lambda_1 - \lambda_4 - 3\lambda_3 + \mu_1 - \mu_2 \leq 4\nu_1 + 2\nu_2 + 2\nu_4 - 2\nu_5 - 2\nu_7 - 4\nu_8,$$
$$3\lambda_1 + \lambda_2 - \lambda_4 - 3\lambda_3 + \mu_2 - \mu_1 \leq 4\nu_2 + 2\nu_1 + 2\nu_3 - 2\nu_5 - 2\nu_7 - 4\nu_8,$$
$$3\lambda_1 + \lambda_3 - \lambda_2 - 3\lambda_4 + \mu_2 - \mu_1 \leq 4\nu_2 + 2\nu_1 + 2\nu_3 - 2\nu_5 - 2\nu_7 - 4\nu_8,$$
$$3\lambda_2 + \lambda_1 - \lambda_3 - 3\lambda_4 + \mu_2 - \mu_1 \leq 4\nu_2 + 2\nu_1 + 2\nu_3 - 2\nu_5 - 2\nu_7 - 4\nu_8,$$
$$3\lambda_2 + \lambda_1 - \lambda_4 - 3\lambda_3 + \mu_1 - \mu_2 \leq 4\nu_2 + 2\nu_1 + 2\nu_3 - 2\nu_5 - 2\nu_7 - 4\nu_8,$$
$$3\lambda_1 + \lambda_2 - \lambda_4 - 3\lambda_3 + \mu_2 - \mu_1 \leq 4\nu_1 + 2\nu_2 + 2\nu_4 - 2\nu_6 - 2\nu_8 - 4\nu_7,$$
$$3\lambda_1 + \lambda_3 - \lambda_2 - 3\lambda_4 + \mu_2 - \mu_1 \leq 4\nu_1 + 2\nu_2 + 2\nu_3 - 2\nu_5 - 2\nu_8 - 4\nu_7,$$
$$3\lambda_1 + \lambda_3 - \lambda_2 - 3\lambda_4 + \mu_2 - \mu_1 \leq 4\nu_1 + 2\nu_2 + 2\nu_4 - 2\nu_6 - 2\nu_8 - 4\nu_7,$$
$$3\lambda_2 + \lambda_1 - \lambda_3 - 3\lambda_4 + \mu_2 - \mu_1 \leq 4\nu_1 + 2\nu_2 + 2\nu_4 - 2\nu_6 - 2\nu_8 - 4\nu_7,$$
$$3\lambda_2 + \lambda_1 - \lambda_4 - 3\lambda_3 + \mu_1 - \mu_2 \leq 4\nu_1 + 2\nu_2 + 2\nu_4 - 2\nu_6 - 2\nu_8 - 4\nu_7,$$
$$3\lambda_1 + \lambda_2 - \lambda_4 - 3\lambda_3 + \mu_2 - \mu_1 \leq 4\nu_2 + 2\nu_1 + 2\nu_3 - 2\nu_6 - 2\nu_8 - 4\nu_7,$$
$$3\lambda_2 + \lambda_1 - \lambda_3 - 3\lambda_4 + \mu_2 - \mu_1 \leq 4\nu_2 + 2\nu_1 + 2\nu_3 - 2\nu_6 - 2\nu_8 - 4\nu_7,$$
$$3\lambda_2 + \lambda_1 - \lambda_4 - 3\lambda_3 + \mu_1 - \mu_2 \leq 4\nu_2 + 2\nu_1 + 2\nu_3 - 2\nu_6 - 2\nu_8 - 4\nu_7,$$
$$3\lambda_3 + \lambda_2 - \lambda_1 - 3\lambda_4 + \mu_1 - \mu_2 \leq 4\nu_3 + 2\nu_1 + 2\nu_4 - 2\nu_6 - 2\nu_7 - 4\nu_8,$$
$$3\lambda_3 + \lambda_1 - \lambda_2 - 3\lambda_4 + \mu_2 - \mu_1 \leq 4\nu_2 + 2\nu_3 + 2\nu_4 - 2\nu_6 - 2\nu_7 - 4\nu_8,$$
$$3\lambda_3 + \lambda_2 - \lambda_1 - 3\lambda_4 + \mu_1 - \mu_2 \leq 4\nu_2 + 2\nu_3 + 2\nu_4 - 2\nu_6 - 2\nu_7 - 4\nu_8,$$
$$3\lambda_2 + \lambda_3 - \lambda_1 - 3\lambda_4 + \mu_2 - \mu_1 \leq 4\nu_1 + 2\nu_2 + 2\nu_6 - 2\nu_5 - 2\nu_7 - 4\nu_8,$$
$$3\lambda_3 + \lambda_2 - \lambda_1 - 3\lambda_4 + \mu_1 - \mu_2 \leq 4\nu_1 + 2\nu_2 + 2\nu_6 - 2\nu_5 - 2\nu_7 - 4\nu_8,$$
$$3\lambda_1 + \lambda_4 - \lambda_3 - 3\lambda_2 + \mu_1 - \mu_2 \leq 4\nu_1 + 2\nu_2 + 2\nu_5 - 2\nu_4 - 2\nu_7 - 4\nu_8,$$
$$3\lambda_3 + \lambda_2 - \lambda_1 - 3\lambda_4 + \mu_1 - \mu_2 \leq 4\nu_1 + 2\nu_2 + 2\nu_5 - 2\nu_4 - 2\nu_7 - 4\nu_8,$$

$$3\lambda_1+\lambda_4-\lambda_2-3\lambda_3+\mu_2-\mu_1 \le 4\nu_1+2\nu_2+2\nu_4-2\nu_3-2\nu_7-4\nu_8,$$
$$3\lambda_1+\lambda_4-\lambda_3-3\lambda_2+\mu_1-\mu_2 \le 4\nu_1+2\nu_2+2\nu_4-2\nu_3-2\nu_7-4\nu_8,$$
$$3\lambda_1+\lambda_3-\lambda_4-3\lambda_2+\mu_2-\mu_1 \le 4\nu_1+2\nu_2+2\nu_3-2\nu_5-2\nu_6-4\nu_7,$$
$$3\lambda_1+\lambda_4-\lambda_3-3\lambda_2+\mu_1-\mu_2 \le 4\nu_1+2\nu_2+2\nu_3-2\nu_5-2\nu_6-4\nu_7,$$
$$3\lambda_1+\lambda_4-\lambda_3-3\lambda_2+\mu_1-\mu_2 \le 4\nu_1+2\nu_2+2\nu_3-2\nu_5-2\nu_8-4\nu_6,$$
$$3\lambda_3+\lambda_2-\lambda_1-3\lambda_4+\mu_2-\mu_1 \le 4\nu_3+2\nu_2+2\nu_4-2\nu_6-2\nu_7-4\nu_8,$$
$$3\lambda_1+\lambda_4-\lambda_3-3\lambda_2+\mu_2-\mu_1 \le 4\nu_2+2\nu_1+2\nu_5-2\nu_4-2\nu_7-4\nu_8,$$
$$3\lambda_3+\lambda_2-\lambda_1-3\lambda_4+\mu_2-\mu_1 \le 4\nu_1+2\nu_2+2\nu_6-2\nu_4-2\nu_7-4\nu_8,$$
$$3\lambda_1+\lambda_4-\lambda_3-3\lambda_2+\mu_2-\mu_1 \le 4\nu_1+2\nu_2+2\nu_5-2\nu_3-2\nu_7-4\nu_8,$$
$$3\lambda_1+\lambda_4-\lambda_3-3\lambda_2+\mu_2-\mu_1 \le 4\nu_2+2\nu_1+2\nu_4-2\nu_3-2\nu_7-4\nu_8,$$
$$3\lambda_1+\lambda_4-\lambda_3-3\lambda_2+\mu_2-\mu_1 \le 4\nu_2+2\nu_1+2\nu_3-2\nu_5-2\nu_6-4\nu_7,$$
$$3\lambda_3+\lambda_2-\lambda_1-3\lambda_4+\mu_2-\mu_1 \le 4\nu_1+2\nu_2+2\nu_5-2\nu_4-2\nu_8-4\nu_7,$$
$$3\lambda_3+\lambda_2-\lambda_1-3\lambda_4+\mu_2-\mu_1 \le 4\nu_1+2\nu_2+2\nu_6-2\nu_5-2\nu_8-4\nu_7,$$
$$3\lambda_3+\lambda_2-\lambda_1-3\lambda_4+\mu_2-\mu_1 \le 4\nu_2+2\nu_3+2\nu_4-2\nu_6-2\nu_8-4\nu_7,$$
$$3\lambda_3+\lambda_2-\lambda_1-3\lambda_4+\mu_2-\mu_1 \le 4\nu_3+2\nu_1+2\nu_4-2\nu_6-2\nu_8-4\nu_7,$$
$$3\lambda_1+\lambda_4-\lambda_3-3\lambda_2+\mu_2-\mu_1 \le 4\nu_2+2\nu_1+2\nu_3-2\nu_5-2\nu_8-4\nu_6,$$
$$3\lambda_1+\lambda_4-\lambda_3-3\lambda_2+\mu_2-\mu_1 \le 4\nu_1+2\nu_2+2\nu_3-2\nu_5-2\nu_7-4\nu_6,$$
$$3\lambda_3+\lambda_4-\lambda_1-3\lambda_2+\mu_2-\mu_1 \le 4\nu_2+2\nu_3+2\nu_4-2\nu_1-2\nu_7-4\nu_8,$$
$$3\lambda_3+\lambda_4-\lambda_2-3\lambda_1+\mu_1-\mu_2 \le 4\nu_2+2\nu_3+2\nu_4-2\nu_1-2\nu_7-4\nu_8,$$
$$3\lambda_3+\lambda_4-\lambda_1-3\lambda_2+\mu_2-\mu_1 \le 4\nu_1+2\nu_2+2\nu_8-2\nu_5-2\nu_6-4\nu_7,$$
$$3\lambda_4+\lambda_3-\lambda_1-3\lambda_2+\mu_1-\mu_2 \le 4\nu_1+2\nu_2+2\nu_8-2\nu_5-2\nu_6-4\nu_7,$$
$$3\lambda_4+\lambda_3-\lambda_2-3\lambda_1+\mu_1-\mu_2 \le 4\nu_4+2\nu_1+2\nu_5-2\nu_3-2\nu_7-4\nu_8,$$
$$3\lambda_3+\lambda_4-\lambda_2-3\lambda_1+\mu_2-\mu_1 \le 4\nu_2+2\nu_3+2\nu_5-2\nu_1-2\nu_7-4\nu_8,$$
$$3\lambda_4+\lambda_3-\lambda_1-3\lambda_2+\mu_2-\mu_1 \le 4\nu_2+2\nu_3+2\nu_5-2\nu_1-2\nu_7-4\nu_8,$$
$$3\lambda_4+\lambda_3-\lambda_2-3\lambda_1+\mu_1-\mu_2 \le 4\nu_2+2\nu_3+2\nu_5-2\nu_1-2\nu_7-4\nu_8,$$
$$3\lambda_3+\lambda_4-\lambda_2-3\lambda_1+\mu_2-\mu_1 \le 4\nu_3+2\nu_2+2\nu_4-2\nu_1-2\nu_7-4\nu_8,$$
$$3\lambda_4+\lambda_3-\lambda_1-3\lambda_2+\mu_2-\mu_1 \le 4\nu_3+2\nu_2+2\nu_4-2\nu_1-2\nu_7-4\nu_8,$$
$$3\lambda_4+\lambda_3-\lambda_2-3\lambda_1+\mu_1-\mu_2 \le 4\nu_3+2\nu_2+2\nu_4-2\nu_1-2\nu_7-4\nu_8,$$
$$3\lambda_3+\lambda_4-\lambda_2-3\lambda_1+\mu_2-\mu_1 \le 4\nu_1+2\nu_2+2\nu_8-2\nu_4-2\nu_6-4\nu_7,$$
$$3\lambda_4+\lambda_3-\lambda_1-3\lambda_2+\mu_2-\mu_1 \le 4\nu_1+2\nu_2+2\nu_8-2\nu_4-2\nu_6-4\nu_7,$$
$$3\lambda_4+\lambda_3-\lambda_2-3\lambda_1+\mu_1-\mu_2 \le 4\nu_1+2\nu_2+2\nu_8-2\nu_4-2\nu_6-4\nu_7,$$
$$3\lambda_3+\lambda_4-\lambda_2-3\lambda_1+\mu_2-\mu_1 \le 4\nu_1+2\nu_2+2\nu_8-2\nu_5-2\nu_7-4\nu_6,$$
$$3\lambda_4+\lambda_3-\lambda_1-3\lambda_2+\mu_2-\mu_1 \le 4\nu_1+2\nu_2+2\nu_8-2\nu_5-2\nu_7-4\nu_6,$$
$$3\lambda_4+\lambda_3-\lambda_2-3\lambda_1+\mu_1-\mu_2 \le 4\nu_1+2\nu_2+2\nu_8-2\nu_5-2\nu_7-4\nu_6,$$
$$3\lambda_4+\lambda_3-\lambda_2-3\lambda_1+\mu_1-\mu_2 \le 4\nu_1+2\nu_2+2\nu_6-2\nu_4-2\nu_8-4\nu_5.$$

A.4. **System of format** $2\times 2\times 3$**.** In contrast with other systems of rank ≤ 4 in this case there are 9 *redundant extremal edges* for which all the associated inequalities are redundant. For all the other systems basic inequalities are essential and independent.

As usual we skip inequalities obtained from another one by a permutation of qubits. This reduces the number of marginal inequalities to 232 (instead of 442). Under additional constraint $\lambda_1-\lambda_2\ge\mu_1-\mu_2$ they form a complete and independent system.

$$\lambda_1-\lambda_2\le\rho_1+\rho_2+\rho_3+\rho_4+\rho_5+\rho_6-\rho_7-\rho_8-\rho_9-\rho_{10}-\rho_{11}-\rho_{12}.$$

$$2\nu_1-\nu_2-\nu_3\le 2\rho_1+2\rho_2+2\rho_3+2\rho_4-\rho_5-\rho_6-\rho_7-\rho_8-\rho_9-\rho_{10}-\rho_{11}-\rho_{12}.$$

$$\nu_1+\nu_2-2\nu_3\le\rho_1+\rho_2+\rho_3+\rho_4+\rho_5+\rho_6+\rho_7+\rho_8-2\rho_9-2\rho_{10}-2\rho_{11}-2\rho_{12}.$$

$$\lambda_1-\lambda_2+\mu_1-\mu_2\le 2\rho_1+2\rho_2+2\rho_3-2\rho_{10}-2\rho_{11}-2\rho_{12}.$$

$$\lambda_1 - \lambda_2 + \nu_1 - \nu_3 \le 2\rho_1 + 2\rho_2 + \rho_3 + \rho_4 - \rho_9 - \rho_{10} - 2\rho_{11} - 2\rho_{12}.$$

$$3\lambda_1 - 3\lambda_2 + 2\nu_1 - \nu_2 - \nu_3 \le 5\rho_1 + 5\rho_2 + 2\rho_3 + 2\rho_4 + 2\rho_5 + 2\rho_6 - \rho_7 - \rho_8 - 4\rho_9 - 4\rho_{10} - 4\rho_{11} - 4\rho_{12},$$
$$3\lambda_1 - 3\lambda_2 + 2\nu_3 - \nu_1 - \nu_2 \le 5\rho_2 + 5\rho_3 + 2\rho_1 + 2\rho_4 + 2\rho_5 + 2\rho_6 - \rho_7 - \rho_8 - 4\rho_9 - 4\rho_{10} - 4\rho_{11} - 4\rho_{12},$$
$$3\lambda_1 - 3\lambda_2 + 2\nu_3 - \nu_1 - \nu_2 \le 5\rho_1 + 5\rho_2 + 2\rho_3 + 2\rho_4 + 2\rho_5 + 2\rho_6 - \rho_8 - \rho_9 - 4\rho_7 - 4\rho_{10} - 4\rho_{11} - 4\rho_{12}.$$

$$3\lambda_1 - 3\lambda_2 + \nu_1 + \nu_2 - 2\nu_3 \le 4\rho_1 + 4\rho_2 + 4\rho_3 + 4\rho_4 + \rho_5 + \rho_6 - 2\rho_7 - 2\rho_8 - 2\rho_9 - 2\rho_{10} - 5\rho_{11} - 5\rho_{12},$$
$$3\lambda_1 - 3\lambda_2 + \nu_2 + \nu_3 - 2\nu_1 \le 4\rho_1 + 4\rho_2 + 4\rho_3 + 4\rho_6 + \rho_4 + \rho_5 - 2\rho_7 - 2\rho_8 - 2\rho_9 - 2\rho_{10} - 5\rho_{11} - 5\rho_{12},$$
$$3\lambda_1 - 3\lambda_2 + \nu_2 + \nu_3 - 2\nu_1 \le 4\rho_1 + 4\rho_2 + 4\rho_3 + 4\rho_4 + \rho_5 + \rho_6 - 2\rho_7 - 2\rho_8 - 2\rho_9 - 2\rho_{12} - 5\rho_{10} - 5\rho_{11}.$$

$$\lambda_1 - \lambda_2 + \mu_1 - \mu_2 + 2\nu_1 - 2\nu_3 \le 4\rho_1 + 2\rho_2 + 2\rho_3 + 2\rho_4 - 2\rho_9 - 2\rho_{10} - 2\rho_{11} - 4\rho_{12},$$
$$\lambda_1 - \lambda_2 + \mu_1 - \mu_2 + 2\nu_2 - 2\nu_3 \le 4\rho_2 + 2\rho_1 + 2\rho_3 + 2\rho_4 - 2\rho_9 - 2\rho_{10} - 2\rho_{11} - 4\rho_{12},$$
$$\lambda_1 - \lambda_2 + \mu_2 - \mu_1 + 2\nu_1 - 2\nu_3 \le 4\rho_2 + 2\rho_1 + 2\rho_3 + 2\rho_4 - 2\rho_9 - 2\rho_{10} - 2\rho_{11} - 4\rho_{12},$$
$$\lambda_1 - \lambda_2 + \mu_1 - \mu_2 + 2\nu_1 - 2\nu_2 \le 4\rho_1 + 2\rho_2 + 2\rho_3 + 2\rho_5 - 2\rho_9 - 2\rho_{10} - 2\rho_{11} - 4\rho_{12},$$
$$\lambda_1 - \lambda_2 + \mu_1 - \mu_2 + 2\nu_2 - 2\nu_3 \le 4\rho_1 + 2\rho_2 + 2\rho_3 + 2\rho_4 - 2\rho_8 - 2\rho_{10} - 2\rho_{11} - 4\rho_{12},$$
$$\lambda_1 - \lambda_2 + \mu_1 - \mu_2 + 2\nu_1 - 2\nu_2 \le 4\rho_1 + 2\rho_2 + 2\rho_3 + 2\rho_4 - 2\rho_9 - 2\rho_{10} - 2\rho_{12} - 4\rho_{11},$$
$$\lambda_1 - \lambda_2 + \mu_2 - \mu_1 + 2\nu_1 - 2\nu_3 \le 4\rho_1 + 2\rho_2 + 2\rho_3 + 2\rho_4 - 2\rho_9 - 2\rho_{10} - 2\rho_{12} - 4\rho_{11},$$
$$\lambda_1 - \lambda_2 + \mu_2 - \mu_1 + 2\nu_2 - 2\nu_3 \le 4\rho_1 + 2\rho_2 + 2\rho_3 + 2\rho_4 - 2\rho_8 - 2\rho_{10} - 2\rho_{12} - 4\rho_{11},$$
$$\lambda_1 - \lambda_2 + \mu_2 - \mu_1 + 2\nu_1 - 2\nu_2 \le 4\rho_2 + 2\rho_1 + 2\rho_3 + 2\rho_5 - 2\rho_9 - 2\rho_{10} - 2\rho_{11} - 4\rho_{12},$$
$$\lambda_1 - \lambda_2 + \mu_2 - \mu_1 + 2\nu_1 - 2\nu_2 \le 4\rho_2 + 2\rho_1 + 2\rho_3 + 2\rho_4 - 2\rho_9 - 2\rho_{10} - 2\rho_{12} - 4\rho_{11},$$
$$\lambda_1 - \lambda_2 + \mu_2 - \mu_1 + 2\nu_2 - 2\nu_3 \le 4\rho_2 + 2\rho_1 + 2\rho_3 + 2\rho_4 - 2\rho_9 - 2\rho_{10} - 2\rho_{12} - 4\rho_{11},$$
$$\lambda_1 - \lambda_2 + \mu_2 - \mu_1 + 2\nu_1 - 2\nu_2 \le 4\rho_1 + 2\rho_2 + 2\rho_4 + 2\rho_5 - 2\rho_9 - 2\rho_{10} - 2\rho_{11} - 4\rho_{12},$$
$$\lambda_1 - \lambda_2 + \mu_2 - \mu_1 + 2\nu_2 - 2\nu_3 \le 4\rho_1 + 2\rho_2 + 2\rho_3 + 2\rho_4 - 2\rho_8 - 2\rho_9 - 2\rho_{11} - 4\rho_{12},$$
$$\lambda_1 - \lambda_2 + \mu_2 - \mu_1 + 2\nu_3 - 2\nu_1 \le 4\rho_2 + 2\rho_3 + 2\rho_4 + 2\rho_5 - 2\rho_9 - 2\rho_{10} - 2\rho_{11} - 4\rho_{12},$$
$$\lambda_1 - \lambda_2 + \mu_2 - \mu_1 + 2\nu_3 - 2\nu_1 \le 4\rho_1 + 2\rho_2 + 2\rho_3 + 2\rho_4 - 2\rho_8 - 2\rho_9 - 2\rho_{10} - 4\rho_{11}.$$

$$3\lambda_1 - 3\lambda_2 + 3\mu_1 - 3\mu_2 + 4\nu_1 - 2\nu_2 - 2\nu_3 \le 10\rho_1 + 4\rho_2 + 4\rho_3 + 4\rho_4 + 4\rho_5 - 2\rho_6 - 2\rho_7 - 2\rho_8 - 2\rho_9 - 2\rho_{10} - 8\rho_{11} - 8\rho_{12},$$
$$3\lambda_1 - 3\lambda_2 + 3\mu_1 - 3\mu_2 + 4\nu_2 - 2\nu_1 - 2\nu_3 \le 10\rho_2 + 4\rho_1 + 4\rho_3 + 4\rho_4 + 4\rho_5 - 2\rho_6 - 2\rho_7 - 2\rho_8 - 2\rho_9 - 2\rho_{10} - 8\rho_{11} - 8\rho_{12},$$
$$3\lambda_1 - 3\lambda_2 + 3\mu_2 - 3\mu_1 + 4\nu_1 - 2\nu_2 - 2\nu_3 \le 10\rho_2 + 4\rho_1 + 4\rho_3 + 4\rho_4 + 4\rho_5 - 2\rho_6 - 2\rho_7 - 2\rho_8 - 2\rho_9 - 2\rho_{10} - 8\rho_{11} - 8\rho_{12},$$
$$3\lambda_1 - 3\lambda_2 + 3\mu_1 - 3\mu_2 + 4\nu_2 - 2\nu_1 - 2\nu_3 \le 10\rho_1 + 4\rho_2 + 4\rho_3 + 4\rho_4 + 4\rho_5 - 2\rho_6 - 2\rho_7 - 2\rho_8 - 2\rho_9 - 2\rho_{11} - 8\rho_{10} - 8\rho_{12},$$
$$3\lambda_1 - 3\lambda_2 + 3\mu_1 - 3\mu_2 + 4\nu_3 - 2\nu_1 - 2\nu_2 \le 10\rho_3 + 4\rho_1 + 4\rho_2 + 4\rho_4 + 4\rho_5 - 2\rho_6 - 2\rho_7 - 2\rho_8 - 2\rho_9 - 2\rho_{10} - 8\rho_{11} - 8\rho_{12},$$
$$3\lambda_1 - 3\lambda_2 + 3\mu_1 - 3\mu_2 + 4\nu_3 - 2\nu_1 - 2\nu_2 \le 10\rho_2 + 4\rho_1 + 4\rho_3 + 4\rho_4 + 4\rho_5 - 2\rho_6 - 2\rho_7 - 2\rho_8 - 2\rho_9 - 2\rho_{11} - 8\rho_{10} - 8\rho_{12},$$
$$3\lambda_1 - 3\lambda_2 + 3\mu_1 - 3\mu_2 + 4\nu_3 - 2\nu_1 - 2\nu_2 \le 10\rho_1 + 4\rho_2 + 4\rho_3 + 4\rho_5 + 4\rho_6 - 2\rho_4 - 2\rho_7 - 2\rho_8 - 2\rho_9 - 2\rho_{10} - 8\rho_{11} - 8\rho_{12},$$
$$3\lambda_1 - 3\lambda_2 + 3\mu_1 - 3\mu_2 + 4\nu_3 - 2\nu_1 - 2\nu_2 \le 10\rho_1 + 4\rho_2 + 4\rho_3 + 4\rho_4 + 4\rho_5 - 2\rho_6 - 2\rho_7 - 2\rho_8 - 2\rho_9 - 2\rho_{12} - 8\rho_{10} - 8\rho_{11},$$
$$3\lambda_1 - 3\lambda_2 + 3\mu_2 - 3\mu_1 + 4\nu_2 - 2\nu_1 - 2\nu_3 \le 10\rho_1 + 4\rho_2 + 4\rho_3 + 4\rho_4 + 4\rho_5 - 2\rho_6 - 2\rho_7 - 2\rho_8 - 2\rho_9 - 2\rho_{12} - 8\rho_{10} - 8\rho_{11}.$$

$$3\lambda_1 - 3\lambda_2 + 3\mu_1 - 3\mu_2 + 2\nu_1 + 2\nu_2 - 4\nu_3 \le 8\rho_1 + 8\rho_2 + 2\rho_3 + 2\rho_4 + 2\rho_5 + 2\rho_6 + 2\rho_7 - 4\rho_8 - 4\rho_9 - 4\rho_{10} - 4\rho_{11} - 10\rho_{12},$$
$$3\lambda_1 - 3\lambda_2 + 3\mu_1 - 3\mu_2 + 2\nu_1 + 2\nu_3 - 4\nu_2 \le 8\rho_1 + 8\rho_3 + 2\rho_2 + 2\rho_4 + 2\rho_5 + 2\rho_6 + 2\rho_7 - 4\rho_8 - 4\rho_9 - 4\rho_{10} - 4\rho_{11} - 10\rho_{12},$$
$$3\lambda_1 - 3\lambda_2 + 3\mu_1 - 3\mu_2 + 2\nu_1 + 2\nu_3 - 4\nu_2 \le 8\rho_1 + 8\rho_2 + 2\rho_3 + 2\rho_4 + 2\rho_5 + 2\rho_6 + 2\rho_7 - 4\rho_8 - 4\rho_9 - 4\rho_{10} - 4\rho_{12} - 10\rho_{11},$$
$$3\lambda_1 - 3\lambda_2 + 3\mu_2 - 3\mu_1 + 2\nu_1 + 2\nu_2 - 4\nu_3 \le 8\rho_1 + 8\rho_2 + 2\rho_3 + 2\rho_4 + 2\rho_5 + 2\rho_6 + 2\rho_7 - 4\rho_8 - 4\rho_9 - 4\rho_{10} - 4\rho_{12} - 10\rho_{11},$$
$$3\lambda_1 - 3\lambda_2 + 3\mu_1 - 3\mu_2 + 2\nu_2 + 2\nu_3 - 4\nu_1 \le 8\rho_1 + 8\rho_2 + 2\rho_3 + 2\rho_4 + 2\rho_5 + 2\rho_6 + 2\rho_9 - 4\rho_7 - 4\rho_8 - 4\rho_{10} - 4\rho_{11} - 10\rho_{12},$$
$$3\lambda_1 - 3\lambda_2 + 3\mu_1 - 3\mu_2 + 2\nu_2 + 2\nu_3 - 4\nu_1 \le 8\rho_2 + 8\rho_3 + 2\rho_1 + 2\rho_4 + 2\rho_5 + 2\rho_6 + 2\rho_7 - 4\rho_8 - 4\rho_9 - 4\rho_{10} - 4\rho_{11} - 10\rho_{12},$$
$$3\lambda_1 - 3\lambda_2 + 3\mu_2 - 3\mu_1 + 2\nu_1 + 2\nu_3 - 4\nu_2 \le 8\rho_2 + 8\rho_3 + 2\rho_1 + 2\rho_4 + 2\rho_5 + 2\rho_6 + 2\rho_7 - 4\rho_8 - 4\rho_9 - 4\rho_{10} - 4\rho_{11} - 10\rho_{12},$$
$$3\lambda_1 - 3\lambda_2 + 3\mu_1 - 3\mu_2 + 2\nu_2 + 2\nu_3 - 4\nu_1 \le 8\rho_1 + 8\rho_3 + 2\rho_2 + 2\rho_4 + 2\rho_5 + 2\rho_6 + 2\rho_7 - 4\rho_8 - 4\rho_9 - 4\rho_{10} - 4\rho_{12} - 10\rho_{11},$$
$$3\lambda_1 - 3\lambda_2 + 3\mu_1 - 3\mu_2 + 2\nu_2 + 2\nu_3 - 4\nu_1 \le 8\rho_1 + 8\rho_2 + 2\rho_3 + 2\rho_4 + 2\rho_5 + 2\rho_6 + 2\rho_7 - 4\rho_8 - 4\rho_9 - 4\rho_{11} - 4\rho_{12} - 10\rho_{10}.$$

$$6\lambda_1-6\lambda_2+3\mu_1-3\mu_2+2\nu_1-\nu_2-\nu_3\le 11\rho_1+8\rho_2+8\rho_3+5\rho_4+2\rho_5+2\rho_6-\rho_7-4\rho_8-4\rho_9-7\rho_{10}-10\rho_{11}-10\rho_{12},$$
$$6\lambda_1-6\lambda_2+3\mu_1-3\mu_2+2\nu_2-\nu_1-\nu_3\le 11\rho_2+8\rho_1+8\rho_3+5\rho_4+2\rho_5+2\rho_6-\rho_7-4\rho_8-4\rho_9-7\rho_{10}-10\rho_{11}-10\rho_{12},$$
$$6\lambda_1-6\lambda_2+3\mu_1-3\mu_2+2\nu_2-\nu_1-\nu_3\le 11\rho_1+8\rho_2+8\rho_3+5\rho_5+2\rho_4+2\rho_6-\rho_7-4\rho_8-4\rho_9-7\rho_{10}-10\rho_{11}-10\rho_{12},$$
$$6\lambda_1-6\lambda_2+3\mu_1-3\mu_2+2\nu_2-\nu_1-\nu_3\le 11\rho_1+8\rho_2+8\rho_3+5\rho_4+2\rho_5+2\rho_6-\rho_8-4\rho_7-4\rho_9-7\rho_{10}-10\rho_{11}-10\rho_{12},$$
$$6\lambda_1-6\lambda_2+3\mu_1-3\mu_2+2\nu_2-\nu_1-\nu_3\le 11\rho_1+8\rho_2+8\rho_3+5\rho_4+2\rho_5+2\rho_6-\rho_7-4\rho_8-4\rho_9-7\rho_{11}-10\rho_{10}-10\rho_{12},$$
$$6\lambda_1-6\lambda_2-3\mu_2+3\mu_1+2\nu_3-\nu_1-\nu_2\le 11\rho_1+8\rho_2+8\rho_3+5\rho_4+2\rho_5+2\rho_6-\rho_9-4\rho_7-4\rho_8-7\rho_{10}-10\rho_{11}-10\rho_{12},$$
$$6\lambda_1-6\lambda_2-3\mu_2+3\mu_1+2\nu_3-\nu_1-\nu_2\le 11\rho_3+8\rho_1+8\rho_2+5\rho_4+2\rho_5+2\rho_6-\rho_7-4\rho_8-4\rho_9-7\rho_{10}-10\rho_{11}-10\rho_{12},$$
$$6\lambda_1-6\lambda_2-3\mu_2+3\mu_1+2\nu_3-\nu_1-\nu_2\le 11\rho_2+8\rho_1+8\rho_3+5\rho_5+2\rho_4+2\rho_6-\rho_7-4\rho_8-4\rho_9-7\rho_{10}-10\rho_{11}-10\rho_{12},$$
$$6\lambda_1-6\lambda_2-3\mu_2+3\mu_1+2\nu_3-\nu_1-\nu_2\le 11\rho_2+8\rho_1+8\rho_3+5\rho_4+2\rho_5+2\rho_6-\rho_8-4\rho_7-4\rho_9-7\rho_{10}-10\rho_{11}-10\rho_{12},$$
$$6\lambda_1-6\lambda_2-3\mu_2+3\mu_1+2\nu_3-\nu_1-\nu_2\le 11\rho_2+8\rho_1+8\rho_3+5\rho_4+2\rho_5+2\rho_6-\rho_7-4\rho_8-4\rho_9-7\rho_{11}-10\rho_{10}-10\rho_{12},$$
$$6\lambda_1-6\lambda_2-3\mu_2+3\mu_1+2\nu_3-\nu_1-\nu_2\le 11\rho_1+8\rho_2+8\rho_3+5\rho_6+2\rho_4+2\rho_5-\rho_7-4\rho_8-4\rho_9-7\rho_{10}-10\rho_{11}-10\rho_{12},$$
$$6\lambda_1-6\lambda_2-3\mu_2+3\mu_1+2\nu_3-\nu_1-\nu_2\le 11\rho_1+8\rho_2+8\rho_3+5\rho_5+2\rho_4+2\rho_6-\rho_8-4\rho_7-4\rho_9-7\rho_{10}-10\rho_{11}-10\rho_{12},$$
$$6\lambda_1-6\lambda_2-3\mu_2+3\mu_1+2\nu_3-\nu_1-\nu_2\le 11\rho_1+8\rho_2+8\rho_3+5\rho_5+2\rho_4+2\rho_6-\rho_7-4\rho_8-4\rho_9-7\rho_{11}-10\rho_{10}-10\rho_{12},$$
$$6\lambda_1-6\lambda_2-3\mu_2+3\mu_1+2\nu_3-\nu_1-\nu_2\le 11\rho_1+8\rho_2+8\rho_3+5\rho_4+2\rho_5+2\rho_6-\rho_8-4\rho_7-4\rho_9-7\rho_{11}-10\rho_{10}-10\rho_{12},$$
$$6\lambda_1-6\lambda_2-3\mu_2+3\mu_1+2\nu_3-\nu_1-\nu_2\le 11\rho_1+8\rho_2+8\rho_3+5\rho_4+2\rho_5+2\rho_6-\rho_7-4\rho_8-4\rho_9-7\rho_{12}-10\rho_{10}-10\rho_{11}.$$

$$6\lambda_1-6\lambda_2+3\mu_1-3\mu_2+\nu_1+\nu_2-2\nu_3\le 10\rho_1+10\rho_2+7\rho_3+4\rho_4+4\rho_5+\rho_6-2\rho_7-2\rho_8-5\rho_9-8\rho_{10}-8\rho_{11}-11\rho_{12},$$
$$6\lambda_1-6\lambda_2+3\mu_1-3\mu_2+\nu_1+\nu_3-2\nu_2\le 10\rho_1+10\rho_3+7\rho_2+4\rho_4+4\rho_5+\rho_6-2\rho_7-2\rho_8-5\rho_9-8\rho_{10}-8\rho_{11}-11\rho_{12},$$
$$6\lambda_1-6\lambda_2+3\mu_1-3\mu_2+\nu_1+\nu_3-2\nu_2\le 10\rho_1+10\rho_2+7\rho_3+4\rho_4+4\rho_6+\rho_5-2\rho_7-2\rho_8-5\rho_9-8\rho_{10}-8\rho_{11}-11\rho_{12},$$
$$6\lambda_1-6\lambda_2+3\mu_1-3\mu_2+\nu_1+\nu_3-2\nu_2\le 10\rho_1+10\rho_2+7\rho_3+4\rho_4+4\rho_5+\rho_6-2\rho_7-2\rho_9-5\rho_8-8\rho_{10}-8\rho_{11}-11\rho_{12},$$
$$6\lambda_1-6\lambda_2+3\mu_1-3\mu_2+\nu_1+\nu_3-2\nu_2\le 10\rho_1+10\rho_2+7\rho_3+4\rho_4+4\rho_5+\rho_6-2\rho_7-2\rho_8-5\rho_9-8\rho_{10}-8\rho_{12}-11\rho_{11},$$
$$6\lambda_1-6\lambda_2+3\mu_1-3\mu_2+\nu_2+\nu_3-2\nu_1\le 10\rho_1+10\rho_2+7\rho_3+4\rho_4+4\rho_5+\rho_6-2\rho_7-2\rho_9-5\rho_8-8\rho_{10}-8\rho_{12}-11\rho_{11},$$
$$6\lambda_1-6\lambda_2+3\mu_1-3\mu_2+\nu_2+\nu_3-2\nu_1\le 10\rho_2+10\rho_3+7\rho_1+4\rho_4+4\rho_5+\rho_6-2\rho_7-2\rho_8-5\rho_9-8\rho_{10}-8\rho_{11}-11\rho_{12},$$
$$6\lambda_1-6\lambda_2+3\mu_1-3\mu_2+\nu_2+\nu_3-2\nu_1\le 10\rho_1+10\rho_3+7\rho_2+4\rho_4+4\rho_6+\rho_5-2\rho_7-2\rho_8-5\rho_9-8\rho_{10}-8\rho_{11}-11\rho_{12},$$
$$6\lambda_1-6\lambda_2+3\mu_1-3\mu_2+\nu_2+\nu_3-2\nu_1\le 10\rho_1+10\rho_3+7\rho_2+4\rho_4+4\rho_5+\rho_6-2\rho_7-2\rho_9-5\rho_8-8\rho_{10}-8\rho_{11}-11\rho_{12},$$
$$6\lambda_1-6\lambda_2+3\mu_1-3\mu_2+\nu_2+\nu_3-2\nu_1\le 10\rho_1+10\rho_3+7\rho_2+4\rho_4+4\rho_5+\rho_6-2\rho_7-2\rho_8-5\rho_9-8\rho_{10}-8\rho_{12}-11\rho_{11},$$
$$6\lambda_1-6\lambda_2+3\mu_1-3\mu_2+\nu_2+\nu_3-2\nu_1\le 10\rho_1+10\rho_2+7\rho_3+4\rho_5+4\rho_6+\rho_4-2\rho_7-2\rho_8-5\rho_9-8\rho_{10}-8\rho_{11}-11\rho_{12},$$
$$6\lambda_1-6\lambda_2+3\mu_1-3\mu_2+\nu_2+\nu_3-2\nu_1\le 10\rho_1+10\rho_2+7\rho_3+4\rho_4+4\rho_6+\rho_5-2\rho_7-2\rho_9-5\rho_8-8\rho_{10}-8\rho_{11}-11\rho_{12},$$
$$6\lambda_1-6\lambda_2+3\mu_1-3\mu_2+\nu_2+\nu_3-2\nu_1\le 10\rho_1+10\rho_2+7\rho_3+4\rho_4+4\rho_6+\rho_5-2\rho_7-2\rho_8-5\rho_9-8\rho_{10}-8\rho_{12}-11\rho_{11},$$
$$6\lambda_1-6\lambda_2+3\mu_1-3\mu_2+\nu_2+\nu_3-2\nu_1\le 10\rho_1+10\rho_2+7\rho_3+4\rho_4+4\rho_5+\rho_6-2\rho_8-2\rho_9-5\rho_7-8\rho_{10}-8\rho_{11}-11\rho_{12},$$
$$6\lambda_1-6\lambda_2+3\mu_1-3\mu_2+\nu_2+\nu_3-2\nu_1\le 10\rho_1+10\rho_2+7\rho_3+4\rho_4+4\rho_5+\rho_6-2\rho_7-2\rho_8-5\rho_9-8\rho_{11}-8\rho_{12}-11\rho_{10}.$$

$$6\lambda_1-6\lambda_2+3\mu_1-3\mu_2+4\nu_1+\nu_2-5\nu_3\le 13\rho_1+10\rho_2+7\rho_3+4\rho_4+4\rho_5+\rho_6-2\rho_7-2\rho_8-5\rho_9-8\rho_{10}-8\rho_{11}-14\rho_{12},$$
$$6\lambda_1-6\lambda_2+3\mu_1-3\mu_2+4\nu_2+\nu_1-5\nu_3\le 13\rho_2+10\rho_1+7\rho_3+4\rho_4+4\rho_5+\rho_6-2\rho_7-2\rho_8-5\rho_9-8\rho_{10}-8\rho_{11}-14\rho_{12},$$
$$6\lambda_1-6\lambda_2+3\mu_1-3\mu_2+4\nu_2+\nu_1-5\nu_3\le 13\rho_1+10\rho_2+7\rho_4+4\rho_3+4\rho_5+\rho_6-2\rho_7-2\rho_8-5\rho_9-8\rho_{10}-8\rho_{11}-14\rho_{12},$$
$$6\lambda_1-6\lambda_2+3\mu_1-3\mu_2+4\nu_2+\nu_1-5\nu_3\le 13\rho_1+10\rho_2+7\rho_3+4\rho_4+4\rho_5+\rho_7-2\rho_6-2\rho_8-5\rho_9-8\rho_{10}-8\rho_{11}-14\rho_{12},$$
$$6\lambda_1-6\lambda_2+3\mu_1-3\mu_2+4\nu_2+\nu_1-5\nu_3\le 13\rho_1+10\rho_2+7\rho_3+4\rho_4+4\rho_5+\rho_6-2\rho_7-2\rho_8-5\rho_{10}-8\rho_9-8\rho_{11}-14\rho_{12},$$
$$6\lambda_1-6\lambda_2+3\mu_1-3\mu_2+4\nu_1+\nu_3-5\nu_2\le 13\rho_1+10\rho_2+7\rho_3+4\rho_4+4\rho_5+\rho_6-2\rho_7-2\rho_8-5\rho_9-8\rho_{10}-8\rho_{12}-14\rho_{11},$$
$$6\lambda_1-6\lambda_2+3\mu_2-3\mu_1+4\nu_1+\nu_2-5\nu_3\le 13\rho_1+10\rho_2+7\rho_3+4\rho_4+4\rho_5+\rho_6-2\rho_7-2\rho_8-5\rho_9-8\rho_{10}-8\rho_{12}-14\rho_{11},$$
$$6\lambda_1-6\lambda_2+3\mu_2-3\mu_1+4\nu_2+\nu_1-5\nu_3\le 13\rho_2+10\rho_1+7\rho_3+4\rho_4+4\rho_5+\rho_6-2\rho_7-2\rho_8-5\rho_9-8\rho_{10}-8\rho_{12}-14\rho_{11},$$
$$6\lambda_1-6\lambda_2+3\mu_2-3\mu_1+4\nu_2+\nu_1-5\nu_3\le 13\rho_1+10\rho_2+7\rho_4+4\rho_3+4\rho_5+\rho_6-2\rho_7-2\rho_8-5\rho_9-8\rho_{10}-8\rho_{12}-14\rho_{11},$$
$$6\lambda_1-6\lambda_2+3\mu_2-3\mu_1+4\nu_2+\nu_1-5\nu_3\le 13\rho_1+10\rho_2+7\rho_3+4\rho_4+4\rho_6+\rho_7-2\rho_5-2\rho_8-5\rho_9-8\rho_{10}-8\rho_{11}-14\rho_{12},$$
$$6\lambda_1-6\lambda_2+3\mu_2-3\mu_1+4\nu_2+\nu_1-5\nu_3\le 13\rho_1+10\rho_2+7\rho_3+4\rho_4+4\rho_5+\rho_7-2\rho_6-2\rho_8-5\rho_9-8\rho_{10}-8\rho_{12}-14\rho_{11},$$
$$6\lambda_1-6\lambda_2+3\mu_2-3\mu_1+4\nu_2+\nu_1-5\nu_3\le 13\rho_1+10\rho_2+7\rho_3+4\rho_4+4\rho_5+\rho_6-2\rho_7-2\rho_8-5\rho_{10}-8\rho_9-8\rho_{12}-14\rho_{11},$$
$$6\lambda_1-6\lambda_2+3\mu_1-3\mu_2+4\nu_3+\nu_2-5\nu_1\le 13\rho_1+10\rho_2+7\rho_3+4\rho_4+4\rho_5+\rho_6-2\rho_7-2\rho_8-5\rho_{11}-8\rho_9-8\rho_{12}-14\rho_{10},$$
$$6\lambda_1-6\lambda_2+3\mu_1-3\mu_2+4\nu_3+\nu_2-5\nu_1\le 13\rho_2+10\rho_3+7\rho_4+4\rho_1+4\rho_5+\rho_6-2\rho_7-2\rho_8-5\rho_9-8\rho_{10}-8\rho_{11}-14\rho_{12},$$
$$6\lambda_1-6\lambda_2+3\mu_2-3\mu_1+4\nu_3+\nu_1-5\nu_2\le 13\rho_2+10\rho_3+7\rho_4+4\rho_1+4\rho_5+\rho_6-2\rho_7-2\rho_8-5\rho_9-8\rho_{10}-8\rho_{11}-14\rho_{12}.$$

$$
\begin{aligned}
6\lambda_1 - 6\lambda_2 + 3\mu_1 - 3\mu_2 + 5\nu_1 - \nu_2 - 4\nu_3 &\le 14\rho_1 + 8\rho_2 + 8\rho_3 + 5\rho_4 + 2\rho_5 + 2\rho_6 - \rho_7 - 4\rho_8 - 4\rho_9 - 7\rho_{10} - 10\rho_{11} - 13\rho_{12},\\
6\lambda_1 - 6\lambda_2 + 3\mu_1 - 3\mu_2 + 5\nu_2 - \nu_1 - 4\nu_3 &\le 14\rho_2 + 8\rho_1 + 8\rho_3 + 5\rho_4 + 2\rho_5 + 2\rho_6 - \rho_7 - 4\rho_8 - 4\rho_9 - 7\rho_{10} - 10\rho_{11} - 13\rho_{12},\\
6\lambda_1 - 6\lambda_2 + 3\mu_2 - 3\mu_1 + 5\nu_1 - \nu_2 - 4\nu_3 &\le 14\rho_2 + 8\rho_1 + 8\rho_3 + 5\rho_4 + 2\rho_5 + 2\rho_6 - \rho_7 - 4\rho_8 - 4\rho_9 - 7\rho_{10} - 10\rho_{11} - 13\rho_{12},\\
6\lambda_1 - 6\lambda_2 + 3\mu_1 - 3\mu_2 + 5\nu_1 - \nu_3 - 4\nu_2 &\le 14\rho_1 + 8\rho_2 + 8\rho_4 + 5\rho_3 + 2\rho_5 + 2\rho_6 - \rho_7 - 4\rho_8 - 4\rho_9 - 7\rho_{10} - 10\rho_{11} - 13\rho_{12},\\
6\lambda_1 - 6\lambda_2 + 3\mu_1 - 3\mu_2 + 5\nu_1 - \nu_3 - 4\nu_2 &\le 14\rho_1 + 8\rho_2 + 8\rho_3 + 5\rho_4 + 2\rho_5 + 2\rho_7 - \rho_6 - 4\rho_8 - 4\rho_9 - 7\rho_{10} - 10\rho_{11} - 13\rho_{12},\\
6\lambda_1 - 6\lambda_2 + 3\mu_1 - 3\mu_2 + 5\nu_1 - \nu_3 - 4\nu_2 &\le 14\rho_1 + 8\rho_2 + 8\rho_3 + 5\rho_4 + 2\rho_5 + 2\rho_6 - \rho_7 - 4\rho_8 - 4\rho_{10} - 7\rho_9 - 10\rho_{11} - 13\rho_{12},\\
6\lambda_1 - 6\lambda_2 + 3\mu_1 - 3\mu_2 + 5\nu_1 - \nu_3 - 4\nu_2 &\le 14\rho_1 + 8\rho_2 + 8\rho_3 + 5\rho_4 + 2\rho_5 + 2\rho_6 - \rho_7 - 4\rho_8 - 4\rho_9 - 7\rho_{10} - 10\rho_{12} - 13\rho_{11},\\
6\lambda_1 - 6\lambda_2 + 3\mu_2 - 3\mu_1 + 5\nu_1 - \nu_3 - 4\nu_2 &\le 14\rho_2 + 8\rho_1 + 8\rho_4 + 5\rho_3 + 2\rho_5 + 2\rho_6 - \rho_7 - 4\rho_8 - 4\rho_9 - 7\rho_{10} - 10\rho_{11} - 13\rho_{12},\\
6\lambda_1 - 6\lambda_2 + 3\mu_2 - 3\mu_1 + 5\nu_1 - \nu_3 - 4\nu_2 &\le 14\rho_2 + 8\rho_1 + 8\rho_3 + 5\rho_4 + 2\rho_5 + 2\rho_7 - \rho_6 - 4\rho_8 - 4\rho_9 - 7\rho_{10} - 10\rho_{11} - 13\rho_{12},\\
6\lambda_1 - 6\lambda_2 + 3\mu_2 - 3\mu_1 + 5\nu_1 - \nu_3 - 4\nu_2 &\le 14\rho_2 + 8\rho_1 + 8\rho_3 + 5\rho_4 + 2\rho_5 + 2\rho_6 - \rho_7 - 4\rho_8 - 4\rho_{10} - 7\rho_9 - 10\rho_{11} - 13\rho_{12},\\
6\lambda_1 - 6\lambda_2 + 3\mu_2 - 3\mu_1 + 5\nu_1 - \nu_3 - 4\nu_2 &\le 14\rho_2 + 8\rho_1 + 8\rho_3 + 5\rho_4 + 2\rho_5 + 2\rho_6 - \rho_7 - 4\rho_8 - 4\rho_9 - 7\rho_{10} - 10\rho_{12} - 13\rho_{11},\\
6\lambda_1 - 6\lambda_2 + 3\mu_2 - 3\mu_1 + 5\nu_1 - \nu_3 - 4\nu_2 &\le 14\rho_1 + 8\rho_2 + 8\rho_3 + 5\rho_4 + 2\rho_5 + 2\rho_8 - \rho_6 - 4\rho_7 - 4\rho_9 - 7\rho_{10} - 10\rho_{11} - 13\rho_{12},\\
6\lambda_1 - 6\lambda_2 + 3\mu_1 - 3\mu_2 + 5\nu_3 - \nu_2 - 4\nu_1 &\le 14\rho_3 + 8\rho_1 + 8\rho_4 + 5\rho_2 + 2\rho_5 + 2\rho_6 - \rho_7 - 4\rho_8 - 4\rho_9 - 7\rho_{10} - 10\rho_{11} - 13\rho_{12},\\
6\lambda_1 - 6\lambda_2 + 3\mu_1 - 3\mu_2 + 5\nu_3 - \nu_2 - 4\nu_1 &\le 14\rho_1 + 8\rho_2 + 8\rho_3 + 5\rho_4 + 2\rho_5 + 2\rho_6 - \rho_7 - 4\rho_8 - 4\rho_{12} - 7\rho_9 - 10\rho_{10} - 13\rho_{11},\\
6\lambda_1 - 6\lambda_2 + 3\mu_2 - 3\mu_1 + 5\nu_2 - \nu_3 - 4\nu_1 &\le 14\rho_1 + 8\rho_2 + 8\rho_3 + 5\rho_4 + 2\rho_5 + 2\rho_6 - \rho_7 - 4\rho_8 - 4\rho_{12} - 7\rho_9 - 10\rho_{10} - 13\rho_{11}.
\end{aligned}
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\begin{aligned}
6\lambda_1 - 6\lambda_2 + 3\mu_1 - 3\mu_2 + 7\nu_1 - 2\nu_2 - 5\nu_3 &\le 16\rho_1 + 10\rho_2 + 7\rho_3 + 4\rho_4 + 4\rho_5 + \rho_6 - 2\rho_7 - 2\rho_8 - 5\rho_9 - 8\rho_{10} - 11\rho_{11} - 14\rho_{12},\\
6\lambda_1 - 6\lambda_2 + 3\mu_2 - 3\mu_1 + 7\nu_1 - 2\nu_2 - 5\nu_3 &\le 16\rho_2 + 10\rho_1 + 7\rho_3 + 4\rho_4 + 4\rho_5 + \rho_6 - 2\rho_7 - 2\rho_8 - 5\rho_9 - 8\rho_{10} - 11\rho_{11} - 14\rho_{12},\\
6\lambda_1 - 6\lambda_2 + 3\mu_1 - 3\mu_2 + 7\nu_1 - 2\nu_3 - 5\nu_2 &\le 16\rho_1 + 10\rho_2 + 7\rho_4 + 4\rho_3 + 4\rho_5 + \rho_6 - 2\rho_7 - 2\rho_8 - 5\rho_9 - 8\rho_{10} - 11\rho_{11} - 14\rho_{12},\\
6\lambda_1 - 6\lambda_2 + 3\mu_2 - 3\mu_1 + 7\nu_1 - 2\nu_2 - 5\nu_3 &\le 16\rho_1 + 10\rho_2 + 7\rho_4 + 4\rho_3 + 4\rho_5 + \rho_6 - 2\rho_7 - 2\rho_8 - 5\rho_9 - 8\rho_{10} - 11\rho_{11} - 14\rho_{12},\\
6\lambda_1 - 6\lambda_2 + 3\mu_1 - 3\mu_2 + 7\nu_1 - 2\nu_3 - 5\nu_2 &\le 16\rho_1 + 10\rho_2 + 7\rho_3 + 4\rho_4 + 4\rho_5 + \rho_7 - 2\rho_6 - 2\rho_8 - 5\rho_9 - 8\rho_{10} - 11\rho_{11} - 14\rho_{12},\\
6\lambda_1 - 6\lambda_2 + 3\mu_1 - 3\mu_2 + 7\nu_1 - 2\nu_3 - 5\nu_2 &\le 16\rho_1 + 10\rho_2 + 7\rho_3 + 4\rho_4 + 4\rho_5 + \rho_6 - 2\rho_7 - 2\rho_8 - 5\rho_{10} - 8\rho_9 - 11\rho_{11} - 14\rho_{12},\\
6\lambda_1 - 6\lambda_2 + 3\mu_2 - 3\mu_1 + 7\nu_1 - 2\nu_2 - 5\nu_3 &\le 16\rho_1 + 10\rho_2 + 7\rho_3 + 4\rho_4 + 4\rho_5 + \rho_6 - 2\rho_7 - 2\rho_8 - 5\rho_{10} - 8\rho_9 - 11\rho_{11} - 14\rho_{12},\\
6\lambda_1 - 6\lambda_2 + 3\mu_1 - 3\mu_2 + 7\nu_1 - 2\nu_3 - 5\nu_2 &\le 16\rho_1 + 10\rho_2 + 7\rho_3 + 4\rho_4 + 4\rho_5 + \rho_6 - 2\rho_7 - 2\rho_8 - 5\rho_9 - 8\rho_{10} - 11\rho_{12} - 14\rho_{11},\\
6\lambda_1 - 6\lambda_2 + 3\mu_2 - 3\mu_1 + 7\nu_1 - 2\nu_2 - 5\nu_3 &\le 16\rho_1 + 10\rho_2 + 7\rho_3 + 4\rho_4 + 4\rho_5 + \rho_6 - 2\rho_7 - 2\rho_8 - 5\rho_9 - 8\rho_{10} - 11\rho_{12} - 14\rho_{11},\\
6\lambda_1 - 6\lambda_2 + 3\mu_2 - 3\mu_1 + 7\nu_1 - 2\nu_3 - 5\nu_2 &\le 16\rho_2 + 10\rho_1 + 7\rho_4 + 4\rho_3 + 4\rho_5 + \rho_6 - 2\rho_7 - 2\rho_8 - 5\rho_9 - 8\rho_{10} - 11\rho_{11} - 14\rho_{12},\\
6\lambda_1 - 6\lambda_2 + 3\mu_2 - 3\mu_1 + 7\nu_1 - 2\nu_3 - 5\nu_2 &\le 16\rho_2 + 10\rho_1 + 7\rho_3 + 4\rho_4 + 4\rho_5 + \rho_7 - 2\rho_6 - 2\rho_8 - 5\rho_9 - 8\rho_{10} - 11\rho_{11} - 14\rho_{12},\\
6\lambda_1 - 6\lambda_2 + 3\mu_2 - 3\mu_1 + 7\nu_1 - 2\nu_3 - 5\nu_2 &\le 16\rho_2 + 10\rho_1 + 7\rho_3 + 4\rho_4 + 4\rho_5 + \rho_6 - 2\rho_7 - 2\rho_8 - 5\rho_{10} - 8\rho_9 - 11\rho_{11} - 14\rho_{12},\\
6\lambda_1 - 6\lambda_2 + 3\mu_2 - 3\mu_1 + 7\nu_1 - 2\nu_3 - 5\nu_2 &\le 16\rho_2 + 10\rho_1 + 7\rho_3 + 4\rho_4 + 4\rho_5 + \rho_6 - 2\rho_7 - 2\rho_8 - 5\rho_9 - 8\rho_{10} - 11\rho_{12} - 14\rho_{11},\\
6\lambda_1 - 6\lambda_2 + 3\mu_2 - 3\mu_1 + 7\nu_1 - 2\nu_3 - 5\nu_2 &\le 16\rho_1 + 10\rho_2 + 7\rho_3 + 4\rho_4 + 4\rho_5 + \rho_8 - 2\rho_6 - 2\rho_7 - 5\rho_9 - 8\rho_{10} - 11\rho_{11} - 14\rho_{12}.
\end{aligned}
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\begin{aligned}
6\lambda_1 - 6\lambda_2 + 3\mu_1 - 3\mu_2 + 5\nu_1 + 2\nu_2 - 7\nu_3 &\le 14\rho_1 + 11\rho_2 + 8\rho_3 + 5\rho_4 + 2\rho_5 + 2\rho_6 - \rho_7 - 4\rho_8 - 4\rho_9 - 7\rho_{10} - 10\rho_{11} - 16\rho_{12},\\
6\lambda_1 - 6\lambda_2 + 3\mu_1 - 3\mu_2 + 5\nu_2 + 2\nu_1 - 7\nu_3 &\le 14\rho_2 + 11\rho_1 + 8\rho_3 + 5\rho_4 + 2\rho_5 + 2\rho_6 - \rho_7 - 4\rho_8 - 4\rho_9 - 7\rho_{10} - 10\rho_{11} - 16\rho_{12},\\
6\lambda_1 - 6\lambda_2 + 3\mu_2 - 3\mu_1 + 5\nu_1 + 2\nu_2 - 7\nu_3 &\le 14\rho_2 + 11\rho_1 + 8\rho_3 + 5\rho_4 + 2\rho_5 + 2\rho_6 - \rho_7 - 4\rho_8 - 4\rho_9 - 7\rho_{10} - 10\rho_{11} - 16\rho_{12},\\
6\lambda_1 - 6\lambda_2 + 3\mu_1 - 3\mu_2 + 5\nu_2 + 2\nu_1 - 7\nu_3 &\le 14\rho_1 + 11\rho_2 + 8\rho_4 + 5\rho_3 + 2\rho_5 + 2\rho_6 - \rho_7 - 4\rho_8 - 4\rho_9 - 7\rho_{10} - 10\rho_{11} - 16\rho_{12},\\
6\lambda_1 - 6\lambda_2 + 3\mu_2 - 3\mu_1 + 5\nu_1 + 2\nu_2 - 7\nu_3 &\le 14\rho_1 + 11\rho_2 + 8\rho_4 + 5\rho_3 + 2\rho_5 + 2\rho_6 - \rho_7 - 4\rho_8 - 4\rho_9 - 7\rho_{10} - 10\rho_{11} - 16\rho_{12},\\
6\lambda_1 - 6\lambda_2 + 3\mu_1 - 3\mu_2 + 5\nu_2 + 2\nu_1 - 7\nu_3 &\le 14\rho_1 + 11\rho_2 + 8\rho_3 + 5\rho_4 + 2\rho_5 + 2\rho_7 - \rho_6 - 4\rho_8 - 4\rho_9 - 7\rho_{10} - 10\rho_{11} - 16\rho_{12},\\
6\lambda_1 - 6\lambda_2 + 3\mu_1 - 3\mu_2 + 5\nu_2 + 2\nu_1 - 7\nu_3 &\le 14\rho_1 + 11\rho_2 + 8\rho_3 + 5\rho_4 + 2\rho_5 + 2\rho_6 - \rho_7 - 4\rho_8 - 4\rho_{10} - 7\rho_9 - 10\rho_{11} - 16\rho_{12},\\
6\lambda_1 - 6\lambda_2 + 3\mu_2 - 3\mu_1 + 5\nu_1 + 2\nu_2 - 7\nu_3 &\le 14\rho_1 + 11\rho_2 + 8\rho_3 + 5\rho_4 + 2\rho_5 + 2\rho_6 - \rho_7 - 4\rho_8 - 4\rho_{10} - 7\rho_9 - 10\rho_{11} - 16\rho_{12},\\
6\lambda_1 - 6\lambda_2 + 3\mu_2 - 3\mu_1 + 5\nu_1 + 2\nu_2 - 7\nu_3 &\le 14\rho_1 + 11\rho_2 + 8\rho_3 + 5\rho_4 + 2\rho_5 + 2\rho_6 - \rho_7 - 4\rho_8 - 4\rho_9 - 7\rho_{10} - 10\rho_{12} - 16\rho_{11},\\
6\lambda_1 - 6\lambda_2 + 3\mu_2 - 3\mu_1 + 5\nu_2 + 2\nu_1 - 7\nu_3 &\le 14\rho_2 + 11\rho_1 + 8\rho_3 + 5\rho_4 + 2\rho_5 + 2\rho_6 - \rho_7 - 4\rho_8 - 4\rho_9 - 7\rho_{10} - 10\rho_{12} - 16\rho_{11},\\
6\lambda_1 - 6\lambda_2 + 3\mu_2 - 3\mu_1 + 5\nu_2 + 2\nu_1 - 7\nu_3 &\le 14\rho_1 + 11\rho_2 + 8\rho_4 + 5\rho_3 + 2\rho_5 + 2\rho_6 - \rho_7 - 4\rho_8 - 4\rho_9 - 7\rho_{10} - 10\rho_{12} - 16\rho_{11},\\
6\lambda_1 - 6\lambda_2 + 3\mu_2 - 3\mu_1 + 5\nu_2 + 2\nu_1 - 7\nu_3 &\le 14\rho_1 + 11\rho_2 + 8\rho_3 + 5\rho_4 + 2\rho_6 + 2\rho_7 - \rho_5 - 4\rho_8 - 4\rho_9 - 7\rho_{10} - 10\rho_{11} - 16\rho_{12},\\
6\lambda_1 - 6\lambda_2 + 3\mu_2 - 3\mu_1 + 5\nu_2 + 2\nu_1 - 7\nu_3 &\le 14\rho_1 + 11\rho_2 + 8\rho_3 + 5\rho_4 + 2\rho_5 + 2\rho_7 - \rho_6 - 4\rho_8 - 4\rho_9 - 7\rho_{10} - 10\rho_{12} - 16\rho_{11},\\
6\lambda_1 - 6\lambda_2 + 3\mu_2 - 3\mu_1 + 5\nu_2 + 2\nu_1 - 7\nu_3 &\le 14\rho_1 + 11\rho_2 + 8\rho_3 + 5\rho_4 + 2\rho_5 + 2\rho_6 - \rho_7 - 4\rho_8 - 4\rho_{10} - 7\rho_9 - 10\rho_{12} - 16\rho_{11}.
\end{aligned}
$$

$$\begin{gathered}
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_1-\nu_3\le 4\rho_1+3\rho_2+2\rho_3+2\rho_4+\rho_5-\rho_8-2\rho_9-2\rho_{10}-3\rho_{11}-4\rho_{12},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_2-\nu_3\le 4\rho_2+3\rho_1+2\rho_3+2\rho_4+\rho_5-\rho_8-2\rho_9-2\rho_{10}-3\rho_{11}-4\rho_{12},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_1-\nu_2\le 4\rho_1+3\rho_3+2\rho_2+2\rho_4+\rho_5-\rho_8-2\rho_9-2\rho_{10}-3\rho_{11}-4\rho_{12},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_2-\nu_3\le 4\rho_1+3\rho_2+2\rho_3+2\rho_5+\rho_4-\rho_8-2\rho_9-2\rho_{10}-3\rho_{11}-4\rho_{12},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_1-\nu_2\le 4\rho_1+3\rho_2+2\rho_3+2\rho_4+\rho_6-\rho_8-2\rho_9-2\rho_{10}-3\rho_{11}-4\rho_{12},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_2-\nu_3\le 4\rho_1+3\rho_2+2\rho_3+2\rho_4+\rho_5-\rho_7-2\rho_9-2\rho_{10}-3\rho_{11}-4\rho_{12},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_1-\nu_2\le 4\rho_1+3\rho_2+2\rho_3+2\rho_4+\rho_5-\rho_9-2\rho_8-2\rho_{10}-3\rho_{11}-4\rho_{12},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_2-\nu_3\le 4\rho_1+3\rho_2+2\rho_3+2\rho_4+\rho_5-\rho_8-2\rho_9-2\rho_{11}-3\rho_{10}-4\rho_{12},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_1-\nu_2\le 4\rho_1+3\rho_2+2\rho_3+2\rho_4+\rho_5-\rho_8-2\rho_9-2\rho_{10}-3\rho_{12}-4\rho_{11},\\
\lambda_1-\lambda_2-2\mu_2+2\mu_1+\nu_3-\nu_2\le 4\rho_1+3\rho_2+2\rho_3+2\rho_4+\rho_5-\rho_7-2\rho_8-2\rho_{10}-3\rho_{11}-4\rho_{12},\\
\lambda_1-\lambda_2-2\mu_2+2\mu_1+\nu_3-\nu_2\le 4\rho_3+3\rho_1+2\rho_2+2\rho_4+\rho_5-\rho_8-2\rho_9-2\rho_{10}-3\rho_{11}-4\rho_{12},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_2-\nu_1\le 4\rho_2+3\rho_3+2\rho_1+2\rho_4+\rho_5-\rho_8-2\rho_9-2\rho_{10}-3\rho_{11}-4\rho_{12},\\
\lambda_2-\lambda_1+2\mu_1-2\mu_2+\nu_1-\nu_2\le 4\rho_2+3\rho_3+2\rho_1+2\rho_4+\rho_5-\rho_8-2\rho_9-2\rho_{10}-3\rho_{11}-4\rho_{12},\\
\lambda_1-\lambda_2-2\mu_2+2\mu_1+\nu_3-\nu_2\le 4\rho_1+3\rho_2+2\rho_3+2\rho_6+\rho_4-\rho_8-2\rho_9-2\rho_{10}-3\rho_{11}-4\rho_{12},\\
\lambda_2-\lambda_1+2\mu_1-2\mu_2+\nu_2-\nu_3\le 4\rho_1+3\rho_2+2\rho_3+2\rho_6+\rho_4-\rho_8-2\rho_9-2\rho_{10}-3\rho_{11}-4\rho_{12},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_2-\nu_1\le 4\rho_1+3\rho_2+2\rho_3+2\rho_5+\rho_6-\rho_8-2\rho_9-2\rho_{10}-3\rho_{11}-4\rho_{12},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_2-\nu_1\le 4\rho_1+3\rho_2+2\rho_3+2\rho_4+\rho_5-\rho_9-2\rho_7-2\rho_{10}-3\rho_{11}-4\rho_{12},\\
\lambda_2-\lambda_1+2\mu_1-2\mu_2+\nu_1-\nu_2\le 4\rho_1+3\rho_2+2\rho_3+2\rho_4+\rho_5-\rho_9-2\rho_7-2\rho_{10}-3\rho_{11}-4\rho_{12},\\
\lambda_1-\lambda_2-2\mu_2+2\mu_1+\nu_3-\nu_2\le 4\rho_1+3\rho_2+2\rho_3+2\rho_4+\rho_5-\rho_8-2\rho_9-2\rho_{12}-3\rho_{10}-4\rho_{11},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_3-\nu_1\le 4\rho_1+3\rho_2+2\rho_3+2\rho_4+\rho_5-\rho_8-2\rho_9-2\rho_{12}-3\rho_{11}-4\rho_{10},\\
\lambda_2-\lambda_1+2\mu_1-2\mu_2+\nu_2-\nu_3\le 4\rho_1+3\rho_2+2\rho_3+2\rho_4+\rho_5-\rho_8-2\rho_9-2\rho_{12}-3\rho_{10}-4\rho_{11},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_2-\nu_1\le 4\rho_1+3\rho_2+2\rho_3+2\rho_4+\rho_5-\rho_8-2\rho_9-2\rho_{11}-3\rho_{12}-4\rho_{10},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_3-\nu_1\le 4\rho_1+3\rho_2+2\rho_3+2\rho_4+\rho_5-\rho_7-2\rho_9-2\rho_{11}-3\rho_{12}-4\rho_{10},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_3-\nu_1\le 4\rho_1+3\rho_2+2\rho_3+2\rho_5+\rho_6-\rho_8-2\rho_9-2\rho_{11}-3\rho_{10}-4\rho_{12},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_3-\nu_1\le 4\rho_1+3\rho_2+2\rho_3+2\rho_4+\rho_6-\rho_7-2\rho_8-2\rho_{10}-3\rho_{11}-4\rho_{12},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_3-\nu_1\le 4\rho_1+3\rho_2+2\rho_3+2\rho_5+\rho_6-\rho_7-2\rho_9-2\rho_{10}-3\rho_{11}-4\rho_{12},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_3-\nu_1\le 4\rho_1+3\rho_2+2\rho_3+2\rho_4+\rho_5-\rho_9-2\rho_7-2\rho_{11}-3\rho_{10}-4\rho_{12},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_3-\nu_1\le 4\rho_1+3\rho_2+2\rho_3+2\rho_4+\rho_5-\rho_9-2\rho_8-2\rho_{12}-3\rho_{10}-4\rho_{11},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_3-\nu_1\le 4\rho_1+3\rho_2+2\rho_3+2\rho_6+\rho_5-\rho_8-2\rho_9-2\rho_{10}-3\rho_{11}-4\rho_{12},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_3-\nu_1\le 4\rho_1+3\rho_2+2\rho_3+2\rho_6+\rho_4-\rho_9-2\rho_8-2\rho_{10}-3\rho_{11}-4\rho_{12},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_3-\nu_1\le 4\rho_1+3\rho_2+2\rho_3+2\rho_4+\rho_6-\rho_8-2\rho_9-2\rho_{12}-3\rho_{10}-4\rho_{11},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_3-\nu_1\le 4\rho_2+3\rho_3+2\rho_1+2\rho_4+\rho_5-\rho_8-2\rho_9-2\rho_{11}-3\rho_{10}-4\rho_{12},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_3-\nu_1\le 4\rho_1+3\rho_2+2\rho_3+2\rho_6+\rho_4-\rho_8-2\rho_9-2\rho_{10}-3\rho_{12}-4\rho_{11},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_3-\nu_1\le 4\rho_1+3\rho_2+2\rho_3+2\rho_4+\rho_5-\rho_8-2\rho_7-2\rho_{10}-3\rho_{11}-4\rho_{12},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_3-\nu_1\le 4\rho_1+3\rho_2+2\rho_3+2\rho_5+\rho_4-\rho_9-2\rho_7-2\rho_{10}-3\rho_{11}-4\rho_{12},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_3-\nu_1\le 4\rho_3+3\rho_2+2\rho_1+2\rho_4+\rho_5-\rho_8-2\rho_9-2\rho_{10}-3\rho_{11}-4\rho_{12},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_3-\nu_1\le 4\rho_3+3\rho_1+2\rho_2+2\rho_4+\rho_6-\rho_8-2\rho_9-2\rho_{10}-3\rho_{11}-4\rho_{12},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_3-\nu_1\le 4\rho_3+3\rho_1+2\rho_2+2\rho_4+\rho_5-\rho_9-2\rho_8-2\rho_{10}-3\rho_{11}-4\rho_{12},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_3-\nu_1\le 4\rho_3+3\rho_1+2\rho_2+2\rho_4+\rho_5-\rho_8-2\rho_9-2\rho_{10}-3\rho_{12}-4\rho_{11},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_3-\nu_1\le 4\rho_2+3\rho_1+2\rho_3+2\rho_5+\rho_6-\rho_8-2\rho_9-2\rho_{10}-3\rho_{11}-4\rho_{12},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_3-\nu_1\le 4\rho_2+3\rho_1+2\rho_3+2\rho_4+\rho_5-\rho_9-2\rho_7-2\rho_{10}-3\rho_{11}-4\rho_{12},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_3-\nu_1\le 4\rho_2+3\rho_1+2\rho_3+2\rho_4+\rho_5-\rho_8-2\rho_9-2\rho_{11}-3\rho_{12}-4\rho_{10},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_3-\nu_1\le 4\rho_2+3\rho_3+2\rho_1+2\rho_5+\rho_4-\rho_8-2\rho_9-2\rho_{10}-3\rho_{11}-4\rho_{12},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_3-\nu_1\le 4\rho_2+3\rho_3+2\rho_1+2\rho_4+\rho_5-\rho_7-2\rho_9-2\rho_{10}-3\rho_{11}-4\rho_{12},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_3-\nu_1\le 4\rho_1+3\rho_3+2\rho_2+2\rho_6+\rho_4-\rho_8-2\rho_9-2\rho_{10}-3\rho_{11}-4\rho_{12},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_3-\nu_1\le 4\rho_1+3\rho_2+2\rho_3+2\rho_5+\rho_4-\rho_8-2\rho_9-2\rho_{11}-3\rho_{12}-4\rho_{10},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_3-\nu_1\le 4\rho_1+3\rho_3+2\rho_2+2\rho_4+\rho_5-\rho_7-2\rho_8-2\rho_{10}-3\rho_{11}-4\rho_{12},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_3-\nu_1\le 4\rho_1+3\rho_2+2\rho_3+2\rho_4+\rho_5-\rho_7-2\rho_8-2\rho_{10}-3\rho_{12}-4\rho_{11},\\
\lambda_1-\lambda_2+2\mu_1-2\mu_2+\nu_3-\nu_1\le 4\rho_1+3\rho_3+2\rho_2+2\rho_4+\rho_5-\rho_8-2\rho_9-2\rho_{12}-3\rho_{10}-4\rho_{11}.
\end{gathered}$$

$$\lambda_1 - \lambda_2 + 3\mu_1 - 3\mu_2 + 2\nu_1 - 2\nu_3 \le 6\rho_1 + 4\rho_2 + 4\rho_3 + 2\rho_4 + 2\rho_5 - 2\rho_8 - 2\rho_9 - 4\rho_{10} - 4\rho_{11} - 6\rho_{12},$$
$$\lambda_1 - \lambda_2 + 3\mu_1 - 3\mu_2 + 2\nu_2 - 2\nu_3 \le 6\rho_2 + 4\rho_1 + 4\rho_3 + 2\rho_4 + 2\rho_5 - 2\rho_8 - 2\rho_9 - 4\rho_{10} - 4\rho_{11} - 6\rho_{12},$$
$$\lambda_2 - \lambda_1 + 3\mu_1 - 3\mu_2 + 2\nu_1 - 2\nu_3 \le 6\rho_2 + 4\rho_1 + 4\rho_3 + 2\rho_4 + 2\rho_5 - 2\rho_8 - 2\rho_9 - 4\rho_{10} - 4\rho_{11} - 6\rho_{12},$$
$$\lambda_1 - \lambda_2 + 3\mu_1 - 3\mu_2 + 2\nu_1 - 2\nu_2 \le 6\rho_1 + 4\rho_2 + 4\rho_4 + 2\rho_3 + 2\rho_5 - 2\rho_8 - 2\rho_9 - 4\rho_{10} - 4\rho_{11} - 6\rho_{12},$$
$$\lambda_1 - \lambda_2 + 3\mu_1 - 3\mu_2 + 2\nu_2 - 2\nu_3 \le 6\rho_1 + 4\rho_2 + 4\rho_4 + 2\rho_3 + 2\rho_5 - 2\rho_8 - 2\rho_9 - 4\rho_{10} - 4\rho_{11} - 6\rho_{12},$$
$$\lambda_2 - \lambda_1 + 3\mu_1 - 3\mu_2 + 2\nu_1 - 2\nu_3 \le 6\rho_1 + 4\rho_2 + 4\rho_4 + 2\rho_3 + 2\rho_5 - 2\rho_8 - 2\rho_9 - 4\rho_{10} - 4\rho_{11} - 6\rho_{12},$$
$$\lambda_1 - \lambda_2 + 3\mu_1 - 3\mu_2 + 2\nu_1 - 2\nu_2 \le 6\rho_1 + 4\rho_2 + 4\rho_3 + 2\rho_4 + 2\rho_6 - 2\rho_8 - 2\rho_9 - 4\rho_{10} - 4\rho_{11} - 6\rho_{12},$$
$$\lambda_2 - \lambda_1 + 3\mu_1 - 3\mu_2 + 2\nu_1 - 2\nu_3 \le 6\rho_1 + 4\rho_2 + 4\rho_3 + 2\rho_4 + 2\rho_6 - 2\rho_8 - 2\rho_9 - 4\rho_{10} - 4\rho_{11} - 6\rho_{12},$$
$$\lambda_1 - \lambda_2 + 3\mu_1 - 3\mu_2 + 2\nu_2 - 2\nu_3 \le 6\rho_1 + 4\rho_2 + 4\rho_3 + 2\rho_4 + 2\rho_5 - 2\rho_7 - 2\rho_9 - 4\rho_{10} - 4\rho_{11} - 6\rho_{12},$$
$$\lambda_2 - \lambda_1 + 3\mu_1 - 3\mu_2 + 2\nu_1 - 2\nu_3 \le 6\rho_1 + 4\rho_2 + 4\rho_3 + 2\rho_4 + 2\rho_5 - 2\rho_7 - 2\rho_9 - 4\rho_{10} - 4\rho_{11} - 6\rho_{12},$$
$$\lambda_1 - \lambda_2 + 3\mu_1 - 3\mu_2 + 2\nu_1 - 2\nu_2 \le 6\rho_1 + 4\rho_2 + 4\rho_3 + 2\rho_4 + 2\rho_5 - 2\rho_8 - 2\rho_{10} - 4\rho_9 - 4\rho_{11} - 6\rho_{12},$$
$$\lambda_1 - \lambda_2 + 3\mu_1 - 3\mu_2 + 2\nu_2 - 2\nu_3 \le 6\rho_1 + 4\rho_2 + 4\rho_3 + 2\rho_4 + 2\rho_5 - 2\rho_8 - 2\rho_{10} - 4\rho_9 - 4\rho_{11} - 6\rho_{12},$$
$$\lambda_2 - \lambda_1 + 3\mu_1 - 3\mu_2 + 2\nu_1 - 2\nu_3 \le 6\rho_1 + 4\rho_2 + 4\rho_3 + 2\rho_4 + 2\rho_5 - 2\rho_8 - 2\rho_{10} - 4\rho_9 - 4\rho_{11} - 6\rho_{12},$$
$$\lambda_1 - \lambda_2 + 3\mu_1 - 3\mu_2 + 2\nu_1 - 2\nu_2 \le 6\rho_1 + 4\rho_2 + 4\rho_3 + 2\rho_4 + 2\rho_5 - 2\rho_8 - 2\rho_9 - 4\rho_{10} - 4\rho_{12} - 6\rho_{11},$$
$$\lambda_2 - \lambda_1 + 3\mu_1 - 3\mu_2 + 2\nu_1 - 2\nu_3 \le 6\rho_1 + 4\rho_2 + 4\rho_3 + 2\rho_4 + 2\rho_5 - 2\rho_8 - 2\rho_9 - 4\rho_{10} - 4\rho_{12} - 6\rho_{11},$$
$$\lambda_2 - \lambda_1 + 3\mu_1 - 3\mu_2 + 2\nu_1 - 2\nu_2 \le 6\rho_2 + 4\rho_1 + 4\rho_4 + 2\rho_3 + 2\rho_5 - 2\rho_8 - 2\rho_9 - 4\rho_{10} - 4\rho_{11} - 6\rho_{12},$$
$$\lambda_2 - \lambda_1 + 3\mu_1 - 3\mu_2 + 2\nu_1 - 2\nu_2 \le 6\rho_2 + 4\rho_1 + 4\rho_3 + 2\rho_4 + 2\rho_6 - 2\rho_8 - 2\rho_9 - 4\rho_{10} - 4\rho_{11} - 6\rho_{12},$$
$$\lambda_2 - \lambda_1 + 3\mu_1 - 3\mu_2 + 2\nu_2 - 2\nu_3 \le 6\rho_2 + 4\rho_1 + 4\rho_3 + 2\rho_4 + 2\rho_6 - 2\rho_8 - 2\rho_9 - 4\rho_{10} - 4\rho_{11} - 6\rho_{12},$$
$$\lambda_2 - \lambda_1 + 3\mu_1 - 3\mu_2 + 2\nu_1 - 2\nu_2 \le 6\rho_2 + 4\rho_1 + 4\rho_3 + 2\rho_4 + 2\rho_5 - 2\rho_8 - 2\rho_{10} - 4\rho_9 - 4\rho_{11} - 6\rho_{12},$$
$$\lambda_2 - \lambda_1 + 3\mu_1 - 3\mu_2 + 2\nu_1 - 2\nu_2 \le 6\rho_2 + 4\rho_1 + 4\rho_3 + 2\rho_4 + 2\rho_5 - 2\rho_8 - 2\rho_9 - 4\rho_{10} - 4\rho_{12} - 6\rho_{11},$$
$$\lambda_2 - \lambda_1 + 3\mu_1 - 3\mu_2 + 2\nu_2 - 2\nu_3 \le 6\rho_2 + 4\rho_1 + 4\rho_3 + 2\rho_4 + 2\rho_5 - 2\rho_8 - 2\rho_9 - 4\rho_{10} - 4\rho_{12} - 6\rho_{11},$$
$$\lambda_2 - \lambda_1 + 3\mu_1 - 3\mu_2 + 2\nu_2 - 2\nu_3 \le 6\rho_1 + 4\rho_2 + 4\rho_4 + 2\rho_3 + 2\rho_6 - 2\rho_8 - 2\rho_9 - 4\rho_{10} - 4\rho_{11} - 6\rho_{12},$$
$$\lambda_2 - \lambda_1 + 3\mu_1 - 3\mu_2 + 2\nu_1 - 2\nu_2 \le 6\rho_1 + 4\rho_2 + 4\rho_4 + 2\rho_3 + 2\rho_5 - 2\rho_7 - 2\rho_9 - 4\rho_{10} - 4\rho_{11} - 6\rho_{12},$$
$$\lambda_2 - \lambda_1 + 3\mu_1 - 3\mu_2 + 2\nu_2 - 2\nu_3 \le 6\rho_1 + 4\rho_2 + 4\rho_4 + 2\rho_3 + 2\rho_5 - 2\rho_8 - 2\rho_9 - 4\rho_{10} - 4\rho_{12} - 6\rho_{11},$$
$$\lambda_2 - \lambda_1 + 3\mu_1 - 3\mu_2 + 2\nu_1 - 2\nu_2 \le 6\rho_1 + 4\rho_2 + 4\rho_3 + 2\rho_4 + 2\rho_6 - 2\rho_7 - 2\rho_9 - 4\rho_{10} - 4\rho_{11} - 6\rho_{12},$$
$$\lambda_2 - \lambda_1 + 3\mu_1 - 3\mu_2 + 2\nu_2 - 2\nu_3 \le 6\rho_1 + 4\rho_2 + 4\rho_3 + 2\rho_4 + 2\rho_6 - 2\rho_7 - 2\rho_9 - 4\rho_{10} - 4\rho_{11} - 6\rho_{12},$$
$$\lambda_2 - \lambda_1 + 3\mu_1 - 3\mu_2 + 2\nu_2 - 2\nu_3 \le 6\rho_1 + 4\rho_2 + 4\rho_3 + 2\rho_4 + 2\rho_6 - 2\rho_8 - 2\rho_{10} - 4\rho_9 - 4\rho_{11} - 6\rho_{12},$$
$$\lambda_2 - \lambda_1 + 3\mu_1 - 3\mu_2 + 2\nu_1 - 2\nu_2 \le 6\rho_1 + 4\rho_2 + 4\rho_3 + 2\rho_4 + 2\rho_5 - 2\rho_7 - 2\rho_{10} - 4\rho_9 - 4\rho_{11} - 6\rho_{12},$$
$$\lambda_2 - \lambda_1 + 3\mu_1 - 3\mu_2 + 2\nu_1 - 2\nu_2 \le 6\rho_1 + 4\rho_2 + 4\rho_3 + 2\rho_4 + 2\rho_5 - 2\rho_7 - 2\rho_9 - 4\rho_{10} - 4\rho_{12} - 6\rho_{11},$$
$$\lambda_2 - \lambda_1 + 3\mu_1 - 3\mu_2 + 2\nu_2 - 2\nu_3 \le 6\rho_1 + 4\rho_2 + 4\rho_3 + 2\rho_4 + 2\rho_5 - 2\rho_7 - 2\rho_9 - 4\rho_{10} - 4\rho_{12} - 6\rho_{11},$$
$$\lambda_2 - \lambda_1 + 3\mu_1 - 3\mu_2 + 2\nu_2 - 2\nu_3 \le 6\rho_1 + 4\rho_2 + 4\rho_3 + 2\rho_4 + 2\rho_5 - 2\rho_8 - 2\rho_{10} - 4\rho_9 - 4\rho_{12} - 6\rho_{11},$$
$$\lambda_2 - \lambda_1 - 3\mu_2 + 3\mu_1 + 2\nu_3 - 2\nu_1 \le 6\rho_1 + 4\rho_2 + 4\rho_3 + 2\rho_4 + 2\rho_5 - 2\rho_8 - 2\rho_{12} - 4\rho_9 - 4\rho_{11} - 6\rho_{10},$$
$$\lambda_1 - \lambda_2 + 3\mu_1 - 3\mu_2 + 2\nu_3 - 2\nu_1 \le 6\rho_1 + 4\rho_2 + 4\rho_3 + 2\rho_4 + 2\rho_5 - 2\rho_7 - 2\rho_{10} - 4\rho_8 - 4\rho_{11} - 6\rho_{12},$$
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$$\lambda_2 - \lambda_1 - 3\mu_2 + 3\mu_1 + 2\nu_3 - 2\nu_2 \le 6\rho_1 + 4\rho_2 + 4\rho_3 + 2\rho_4 + 2\rho_5 - 2\rho_9 - 2\rho_{10} - 4\rho_7 - 4\rho_{11} - 6\rho_{12},$$
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$$\lambda_2 - \lambda_1 - 3\mu_2 + 3\mu_1 + 2\nu_3 - 2\nu_1 \le 6\rho_2 + 4\rho_3 + 4\rho_4 + 2\rho_1 + 2\rho_5 - 2\rho_7 - 2\rho_9 - 4\rho_{10} - 4\rho_{11} - 6\rho_{12},$$
$$\lambda_2 - \lambda_1 - 3\mu_2 + 3\mu_1 + 2\nu_3 - 2\nu_1 \le 6\rho_1 + 4\rho_2 + 4\rho_3 + 2\rho_4 + 2\rho_5 - 2\rho_8 - 2\rho_{10} - 4\rho_7 - 4\rho_{11} - 6\rho_{12},$$
$$\lambda_2 - \lambda_1 - 3\mu_2 + 3\mu_1 + 2\nu_3 - 2\nu_1 \le 6\rho_3 + 4\rho_2 + 4\rho_4 + 2\rho_1 + 2\rho_5 - 2\rho_8 - 2\rho_9 - 4\rho_{10} - 4\rho_{11} - 6\rho_{12}.$$

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