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Published on: 01 Dec 1987 - Physical Review A (American Physical Society)

Topics: Optical heterodyne detection, Measured quantity, Quantum limit, Open quantum system and Quantum metrology

Quantum measurement theory of optical heterodyne detection

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(Received 26 May 1987)

An analysis is presented of optical heterodyne detection of an intracavity field within the context of quantum measurement theory. We find the intracavity density operator becomes diagonal in the basis which diagonalizes the measured quantity, a quadrature phase amplitude of the field. This representation is the "pointer basis," and this result is the continuous-measurement equivalent of state reduction. The model illustrates a general feature of continuous measurement; one parameter given by the product of the system and measuring device coupling bandwidth and the fluctuations in the measuring device completely characterizes the measurement. This constant determines both the rate of diagonalization of the density operator and the rate of growth of fluctuations in the system quantity conjugate to the measured observable.

I. INTRODUCTION

Since the discovery of squeezed states of light,[1] measurements of phase-dependent properties of the quantized electromagnetic field have become increasingly important. Squeezed states of light are states of the electromagnetic field which have phase-dependent noise. Moreover, the noise in one quadrature phase is reduced below that of the ground state of the field with a consequent increase in the canonically conjugate quadrature. The detection of squeezed states of light is based on optical heterodyne detection.[2–5] A strong coherent local oscillator field of fixed carrier frequency and phase beats with the signal field on a beam splitter. The combined field from one output of the beam splitter is then subjected to standard quantum counting measurements. It is the local oscillator which defines the phase and frequency reference for the determination of phase-dependent properties of the signal field.

However, if the field from only one output port of the beam splitter is measured, an unambiguous determination of phase-dependent properties such as squeezing is difficult, as this output field is "contaminated" with reflected local oscillator intensity fluctuations which are, of course, phase independent. A more refined version of optical heterodyne detection, "balanced detection,"[2] is now employed to overcome this problem. In these schemes the fields from both output ports of the beam splitter are subjected to photon counting measurements and the photoelectron current from both detectors combined before subsequent analysis. In this way local oscillator intensity fluctuations may be subtracted from the measured photoelectron current which then is determined by the state of the signal field alone.

Balanced heterodyne detection realizes a direct measurement of the quadrature phase amplitudes of the signal field. However, as was first pointed out by Yuen and Shapiro,[5] balanced detection is necessary only when coherent fields, perhaps with additional phase-independent noise, are the only local oscillator fields available. If squeezed states of light are used for the local oscillator, balanced detection is unnecessary; single-port heterodyne detection with such states realizes a direct measurement of the quadrature phase amplitudes of the signal field. In the case considered in this paper, heterodyne detection of an intracavity field, this is the only option as the cavity has only one output port.

The description of heterodyne detection at optical frequencies is well understood and is adequate to explain all the experimental results. In this paper we will not raise any questions regarding the adequacy of the standard description. Rather we wish to consider some questions of principle concerning the description of such measurements within the wider context of quantum measurement theory. For example, if in homodyne detection one is making a direct measurement of quadrature phase amplitude as is claimed, can one speak of "state reduction" in the system upon which the measurement is made? Another related question one might ask concerns the concept of a "pointer basis."[6,7] If a system is connected to a measuring apparatus constructed to measure a particular physical quantity then it is required that the system density operator become diagonal in the basis which diagonalizes the measured observable. Indeed, there are measurement models which demonstrate this behavior.[8] It is the purpose of this paper firstly to define in what sense these questions may meaningfully be put in an optical context and secondly to provide some answers. In so doing we hope to provide a concrete example of a recently developed formalism for a complete description of continuous measurement in quantum mechanics.[9]

The paper is organized as follows. In Sec. II we describe the particular experimental scheme we wish to discuss. It is the measurement of the quadrature phase amplitudes for a single-mode intracavity field via squeezed-state heterodyne detection. This system is not the standard heterodyne configuration for which the signal is traveling wave rather than an intracavity field. However, from the point of view of quantum measurement theory it is more instructive to consider the cavity mode signal configuration. A simple analysis based on mode-amplitude transformations illustrates how a squeezed local oscillator field may be used to avoid the reflected local oscillator intensity fluctuations discussed above. In Sec. III we determine the evolution of the measured system during the continuous measurement and the evolution of the measurement results. Our treatment is based on techniques recently presented by Barchielli[9] and related techniques of Gardiner and Collett.[10] We show that the measured photoelectron current does indeed realize a measurement of either canonically conjugate intracavity quadrature phase amplitudes depending on the phase of the local oscillator. Furthermore, we show that the density operator of the system is diagonalized in the basis which diagonalizes the measured quantity. Thus the pointer basis is established and changing the phase of the local oscillator changes the pointer basis. The results illustrate a general feature of continuous measurement, namely, a single parameter given by the product of the fluctuations in the measuring device and the bandwidth of the system-measuring-device coupling completely characterize the measurement process.

II. THE MEASUREMENT MODEL

A schematic outline of the experimental scheme is shown in Fig. 1. A single-mode ring cavity field is coupled to an external field via a partially transparent mirror of transmittivity η. The input field with positive frequency components represented by the operator $b_i(t)$ beats with the cavity field to produce the output field $b_o(t)$ which thus carriers information about the intracavity field. This information is extracted by standard photoelectron detection and analysis of the resulting current $\langle i(t)\rangle$. The input field, that is the local oscillator, is prepared in a broadband squeezed state at the carrier frequency $\Omega_{\rm LO}$, where LO refers to the local oscillator.

A simple description of this system may be given as follows (see Fig. 1). Let $a_i(\omega)$ denote the operator Fourier amplitude of the cavity field at frequency ω just prior to interaction with the beam splitter and let $a_o(\omega)$ denote the corresponding operator just after interaction

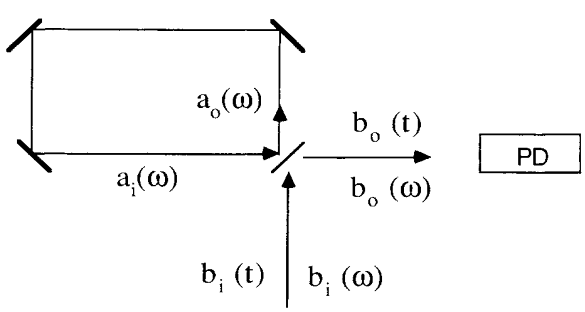

FIG. 1. Schematic representation of the cavity quadrature phase measurement scheme and the beam splitter relations. $a(t)$, $b_i(t)$, and $b_o(t)$ represent the positive frequency components of the internal field, input field, and output field, respectively. PD is a photoelectron detector.

with the beam splitter. These two operators are coupled to the two external amplitudes at the same frequency by the equations

$$b_o(\omega)=\sqrt{\eta}a_i(\omega)-\sqrt{1-\eta}b_i(\omega)\ , \tag{2.1a}$$

$$a_o(\omega)=\sqrt{\eta}b_i(\omega)+\sqrt{1-\eta}a_i(\omega)\ . \tag{2.1b}$$

Due to the phase shift around the cavity we have

$$a_i(\omega)=e^{i\Phi}a_o(\omega)\ , \tag{2.1c}$$

where

$$\Phi=\frac{2\pi}{\Omega_c}(\omega_0-\omega)\ ,$$

with Ω_c the cavity-free spectral range and ω_0 the cavity resonance frequency. Equations (2.1a)–(2.1c) may be used to show

$$a_0(\omega)=\frac{\sqrt{\eta}}{1-e^{i\Phi}\sqrt{1-\eta}}b_i(\omega)\ . \tag{2.2}$$

We now take the good cavity limit; that is, we assume the loss through the output mirror is small. This limit may be stated as $\eta\ll 1$ and we put

$$\eta=\xi\frac{2\pi}{\Omega_c}\ , \tag{2.3}$$

with $\xi\ll\Omega_c/2\pi$. Thus

$$a_0(\omega)\simeq\frac{(\xi\Omega_c/2\pi)^{1/2}}{(\xi/2-i\Delta\omega)}b_i(\omega)\ , \tag{2.4}$$

where $\Delta\omega\equiv\omega_0-\omega$. This approximates the Airy function response of the cavity for $\eta\ll 1$ with a Lorentzian of width $\xi/2$.[11]

Information regarding the cavity field is carried to the photodetector via the output field. To see this consider the first- and second-order moments for the output photon-number operator at frequency $\omega=\Omega_{\rm LO}=\omega_0$ (resonance). The average photon number at this frequency is

$$\langle b_0^\dagger b_0\rangle=\eta\langle a_i^\dagger a_i\rangle+(1-\eta)\langle b_i^\dagger b_i\rangle -\sqrt{\eta(1-\eta)}\langle a_i^\dagger b_i+a_i b_i^\dagger\rangle \tag{2.5}$$

(we have dropped the explicit reference to the frequency $\omega=\Omega_{\rm LO}=\omega_0$). Assume the input field to be in a broadband squeezed state $|\Psi\rangle$ related to the vacuum by the unitary transformation[12] $|\Psi\rangle=\hat{U}|0\rangle$ with

$$U^\dagger b_i(\omega)U=\alpha(\omega)+\mu(\omega)b_i(\omega)+\nu(\omega)b_i^\dagger(2\Omega_{\rm LO}-\omega)\ , \tag{2.6}$$

with $|\mu(\omega)|^2-|\nu(\omega)|^2=1$. Such states are generated by two-photon processes which couple modes at frequencies ω and $2\Omega_{\rm LO}-\omega$, i.e., the symmetric side bands of the carrier frequency. The parameter $\alpha(\omega)$ represents a coherent component on the squeezed field and we assume

$$\alpha(\omega)=\begin{cases}Ae^{i\theta}, & \omega=\Omega_{\rm LO}\\ 0, & \omega\neq\Omega_{\rm LO}\ .\end{cases} \tag{2.7}$$

Using Eq. (2.6) we find

$$\langle b_i^\dagger b_i \rangle = A^2 + |\nu|^2 . \quad (2.8)$$

We further assume that the intensity of the input field is due mainly to the coherent excitation, i.e., $A^2 \gg |\nu|^2$ but that nonetheless $|\nu|^2 \gg 1$, and we assume $|A|^2 \gg \eta \langle a_i^\dagger a_i \rangle$. Using these assumptions we find that

$$\langle b_0^\dagger b_0 \rangle \simeq (1-\eta) A^2 - A^2 \sqrt{\eta(1-\eta)} \langle \hat{X}_\theta \rangle , \quad (2.9)$$

where

$$\hat{X}_\theta = (a_i e^{-i\theta} + a_i^\dagger e^{i\theta}) . \quad (2.10)$$

$\hat{X}_\theta$ and $\hat{X}_{\theta+\pi/2}$ are the canonically conjugate quadrature phase amplitudes of the cavity field. Thus the mean photon number of the output field gives the mean of the cavity quadrature phase amplitudes.

We also find the variance in the output field photon number to be given by

$$V(b_0^\dagger b_0) \simeq 2A^2(1-\eta)^2 \{ |\mu||\nu| \cos[2(\theta+\phi)] + |\nu|^2 + \tfrac{1}{2} \} + \eta(1-\eta) A^2 V(\hat{X}_\theta) , \quad (2.11)$$

where $V(\hat{B}) \equiv \langle \hat{B}^2 \rangle - \langle \hat{B} \rangle^2$ and ϕ is defined by $\mu\nu = |\mu||\nu| e^{-2i\phi}$. The first term in Eq. (2.11) is due to reflected local oscillator intensity fluctuations. The advantage of using a squeezed local oscillator is due to the fact that this term may be made small for an appropriate choice of phases θ and ϕ. To see this, we redefine the quantities μ and ν in terms of the squeeze parameter r by $|\mu| = \cosh r$, $|\nu| = \sinh r$. We also define the input field quadrature phase amplitudes

$$\hat{Y}_\phi = (b_i e^{i\phi} + b_i^\dagger e^{-i\phi}) .$$

(In fact, $\hat{Y}_0$ and $\hat{Y}_{\pi/2}$ are the dc components of the oscillating in phase and quadrature phase amplitudes defined with respect to the carrier frequency Ω_{LO}.) One then easily establishes that

$$\langle b_i \rangle = A e^{i\theta} ,$$

$$V(\hat{Y}_\phi) = 2|\mu||\nu| + 2|\nu|^2 + 1 = e^{2r} ,$$

$$V(\hat{Y}_{\phi+\pi/2}) = 2|\nu|^2 - 2|\mu||\nu| + 1 = e^{-2r} .$$

These quantities are represented in Figs. 2(a)–2(c). The squeezed state $|\Psi\rangle$ is represented in a complex plane with a vector of length A and phase θ with "error ellipses" indicating the quadrature phase variances. In Figs. 2(b) and 2(c) we represent the state with $\theta+\phi = \pi/2$ and $A \gg |\nu|^2 \gg 1$. It is clear that for both cases this choice of phase corresponds to a state with reduced intensity fluctuations and increased phase fluctuations.

Returning to Eq. (2.11) we see that the first term may be made small with $\theta+\phi=\pi/2$. In view of the discussion of the previous paragraph this ensures that the local oscillator is in a state with reduced intensity fluctuations. Changing θ through $\pi/2$ radians while ensuring $\theta+\phi = \pi/2$ enables one to measure unambiguously both quadrature phases of the intracavity field.

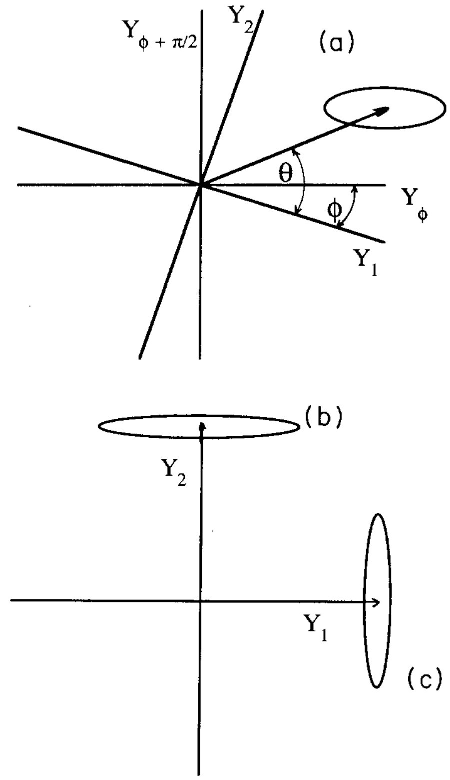

FIG. 2. Complex amplitude diagram for the squeezed state of the input field at the carrier frequency. $\hat{Y}_1 \equiv \hat{Y}_{\phi=0}$, $\hat{Y}_2 \equiv \hat{Y}_{\phi=\pi/2}$. (a) θ,ϕ arbitrary; (b) $\theta=\pi/2$, $\phi=0$; (c) $\theta=0$, $\phi=\pi/2$.

III. DYNAMICS OF THE CONTINUOUS MEASUREMENT

In Sec. II we specified the arrangement for optical quadrature phase measurements that we wish to discuss from the point of view of quantum measurement theory. The measured system is the single-mode intracavity field. The measuring device or "meter" is separated into two stages. The first stage coupled directly to the measured system is the external field. This is coupled to the second stage represented by the photoelectron detector and associated circuitry. Our description will be confined to an analysis of the dynamics of the measured system and the dynamics of the measurement result, that is, the photoelectron current. It is only the coupling of the system and the first meter stage which can directly effect the system. Preparing the local oscillator in a squeezed state is to be regarded as an essential element in specifying the nature of the measuring device. That is, the measuring device is so "constructed" as to provide an accurate determination of the system quadrature phase amplitudes.

There are two complementary aspects to a measurement theoretical analysis of this experiment. Firstly one must determine the time evolution of the measured system and secondly one must show what system variables determine the measurement results. These two aspects must finally be shown to be consistent in a sense to be defined below.

Before presenting an analysis of this measurement it will be useful to make some comments regarding measurements in general. Consider a situation where one makes an instantaneous measurement of some physical quantity represented by an operator $\hat{A}$ (for simplicity we will assume $\hat{A}$ to have a discrete spectrum). Let $\hat{\rho}(0)$ denote the initial density operator for the system. There are two ways in which one may think about this measurement.[13] Firstly consider the ensemble of identically prepared systems described by $\hat{\rho}(0)$. Subject each system to the measurement and record the result. The result of each measurement is, of course, unpredictable in general but must be one of the eigenvalues of $\hat{A}$. The relative frequency with which a particular eigenvalue occurs is determined by $P(a)=\mathrm{tr}[\hat{\rho}(0)|a\rangle\langle a|]$ and the resolution of the measuring device. If this device is so constructed as to resolve the neighboring eigenvalues the relative frequency of results is determined by $P(a)$ alone. One may now form a new ensemble by selecting those elements of the initial ensemble which upon interaction with the measurement device gave the result a. This new ensemble is assumed to be described by the density operator $\hat{\rho}^{(s)}=|a\rangle\langle a|$. This assumption is often referred to as the projection postulate or state reduction. This first sense of a measurement has been called a selective measurement.[13] It is an appropriate sense when the description of a measurement includes statements about a particular experimental result. However, one may think of the measurement in another sense which may be referred to as the nonselective sense of measurement. In this case we retain all members of the initial ensemble after interaction with the measuring device to form the new ensemble. If we are to be consistent with the projection postulate this new ensemble must be described by the density operator $\hat{\rho}^{(\mathrm{NS})}=\sum_a P(a)|a\rangle\langle a|$. In the sense of a nonselective measurement the density operator has become diagonal in the basis which diagonalizes the measured quantity. This preferred basis has been called the pointer basis.[6,7] We see that the projection postulate and the existence of a pointer basis are equivalent assumptions regarding the interaction of a system with a measuring device. The nonselective sense of measurement is appropriate when the description of the measurement concerns only the state of the measured system and no statements about measurement results.

Varying opinions are held regarding the projection postulate (equivalently, the existence of a pointer basis). Are these assumptions necessary or merely a consequence of the detailed quantum dynamics of the coupled system–measuring-device complex? We wish to show only that a pointer basis appropriate for a quadrature phase measurement (nonselective sense) arises naturally given certain not unreasonable assumptions about the external field states. At this level of description we need not concern ourselves with what happens further down the measurement chain.

A theory of continuous measurement must show firstly how the coupling of the measured system to the measurement device leads to a diagonalization of the system density operator in a preferred basis, the pointer basis. Secondly, it must show that the system quantity about which information is obtained is indeed diagonal in the pointer basis. This is the required consistency of description mentioned in the first paragraph of this section. Finally, it must show that fluctuations in all those variables which do not commute with the measured quantity grow in accordance with the appropriate uncertainty relation. All of these features are demonstrated in the idealized models of Refs. 14 and 15. We now show how they arise in the more realistic model of this paper.

The Hamiltonian describing the interaction of the external field will be taken to be

$$H=\hbar\omega_0 a^\dagger a+i\hbar\kappa\left[\frac{\hbar}{2\Omega_{\mathrm{LO}}}\right]^{1/2}[ab_i^\dagger(t)-a^\dagger b_i(t)] \tag{3.1}$$

(we are working in an interaction picture with respect to the external fields). The positive frequency components of the external field at the input are

$$b_i(t)=\int_B\frac{d\omega}{2\pi}\left[\frac{\hbar\omega}{2}\right]^{1/2}b_i(\omega)e^{-i\omega t}\,, \tag{3.2}$$

where B is some appropriate bandwidth of integration. As the cavity only "sees" external modes centered around ω_0 we will assume the bandwidth to be

$$B:\Omega_{\mathrm{LO}}-\Omega_c/2\le\omega\le\Omega_{\mathrm{LO}}+\Omega_c/2\,, \tag{3.3}$$

with $\Omega_{\mathrm{LO}}\sim\omega_0$, and Ω_c is the cavity-free spectral range. Using Eq. (3.2) we assume that the input field may be approximated by a one-dimensional traveling wave of fixed polarization. The coupling of the internal fields has been chosen so that κ has units of frequency.

The commutation relations for the operators $b_i(\omega)$ and $b_i^\dagger(\omega)$ may be taken as

$$[b_i(\omega),b_i^\dagger(\omega)]=\frac{4\pi^2}{\Omega_c}\delta(\omega-\omega')\,. \tag{3.4}$$

The state of the input field is taken to be a broadband squeezed state with coherent excitation at the carrier frequency Ω_{LO}. This state is defined in Eq. (2.6), and one easily verifies that

$$\langle b_i(\omega)\rangle=2\pi Ae^{i\theta}\delta(\omega-\omega')\,, \tag{3.5a}$$

$$\langle\Delta b_i(\omega)\Delta b_i(\omega')\rangle=4\pi^2\frac{M(\omega)}{\Omega_c}\delta(2\Omega_{\mathrm{LO}}-\omega-\omega')\,, \tag{3.5b}$$

$$\langle\Delta b_i^\dagger(\omega)\Delta b_i(\omega')\rangle=4\pi^2\frac{N(\omega)}{\Omega_c}\delta(\omega-\omega')\,, \tag{3.5c}$$

where we have put

$$\mu(\omega)\nu(\omega)=4\pi^2\frac{M(\omega)}{\Omega_c},\quad|\nu(\omega)|^2=4\pi^2N(\omega)/\Omega_c\,. \tag{3.6}$$

We have also redefined the magnitude of the coherent amplitude for convenience. The units have been chosen so that $M(\omega)$ and $N(\omega)$ are dimensionless, with $N(\omega)/\Omega_c$ representing the average number of quanta per unit bandwidth.

We now derive an evolution equation for the intracavity field. In as much as such an equation describes the effect of the measurement on the system when no attempt is made to record the results, such an equation describes a nonselective measurement. Our derivation will follow the treatment of Barchielli.[9]

The change in the state of intracavity field in time dt is given by

$$d\hat{\rho}(t)=\hat{\rho}(t+dt)-\hat{\rho}(t)\ , \tag{3.7}$$

where

$$\hat{\rho}(t+dt)=\mathrm{tr}_{\mathrm{LO}}[\hat{U}(dt)\hat{\rho}(t)\otimes\hat{\rho}_{\mathrm{LO}}(t)\hat{U}^{\dagger}(dt)]\ , \tag{3.8}$$

with $\hat{\rho}_{\mathrm{LO}}(t)$ being the state of the input field at time t, $\mathrm{tr}_{\mathrm{LO}}$ denoting a trace over all external field variables, and

$$\hat{U}(dt)\equiv\exp\left[\frac{-i}{\hbar}\hat{H}_0dt+\kappa[adB^{\dagger}(t)-a^{\dagger}dB(t)]\right]\ . \tag{3.9}$$

Equation (3.9) contains the quantum Wiener process $dB(t)$ defined by[9,10]

$$d\hat{B}(t)\equiv\hat{B}(t+dt)-\hat{B}(t)\ , \tag{3.10}$$

with

$$\hat{B}(t)\equiv\left[\frac{2}{\hbar\Omega_{\mathrm{LO}}}\right]^{1/2}\int_0^t ds\, b_i(s)\ . \tag{3.11}$$

Using Eqs. (3.4), (3.5a)–(3.5c), (3.10), and (3.11) one finds

$$\langle d\hat{B}(t)\rangle=Ae^{i(\theta-\Omega_{\mathrm{LO}}t)}dt\ , \tag{3.12a}$$

$$\langle d\hat{B}(t)d\hat{B}(t)\rangle=2\pi\frac{M}{\Omega_c}e^{-2i\Omega_{\mathrm{LO}}t}dt\ , \tag{3.12b}$$

$$\langle d\hat{B}^{\dagger}(t)d\hat{B}^{\dagger}(t)\rangle=2\pi\frac{M^*}{\Omega_c}e^{2i\Omega_{\mathrm{LO}}t}dt\ , \tag{3.12c}$$

$$\langle d\hat{B}^{\dagger}(t)d\hat{B}(t)\rangle=2\pi\frac{N}{\Omega_c}dt\ , \tag{3.12d}$$

$$\langle d\hat{B}(t)d\hat{B}^{\dagger}(t)\rangle=2\pi\frac{(N+1)}{\Omega_c}dt\ , \tag{3.12e}$$

where $N=N(\Omega_{\mathrm{LO}})$, $M=M(\Omega_{\mathrm{LO}})$. In deriving Eqs. (3.12a)–(3.12e) we have assumed that $M(\omega)$ and $N(\omega)$ are slowly varying around $\Omega_{\mathrm{LO}}\sim\omega_0$ over the bandwidth of integration. This is a reasonable assumption if the cavity resonance is sufficiently narrow ($\xi\ll\Omega_c/2\pi$) and constitutes a Markov approximation. We are assuming that the fluctuations in the external field occur on a much shorter time scale (and thus the spectrum of the external field is relatively broad) than the time scale of the cavity field which is determined by ξ the width of the cavity resonance. This assumption is also implicit in Eq. (3.8) where we have factorized the state of the total system at the start of the time interval dt. The Markov approximation is vital to all of what follows, and in particular leads directly to the existence of a pointer basis. The extent to which one is prepared to accept the latter is determined by the extent to which the former assumption is justified. Inspection of Eqs. (3.12) make it readily apparent why $dB(t)$ is referred to as a quantum Wiener process; both first- and second-order moments are linear in dt.[16]

Expanding $U(dt)$ in Eq. (3.8) to second order in κ and evaluating the trace over the external field using Eqs. (3.12a)–(3.12e) we find

$$\begin{aligned}\hat{\rho}(t+dt)=\hat{\rho}(t)&-\frac{i}{\hbar}[\hat{H}_0,\hat{\rho}(t)]dt+\kappa[\epsilon^*(t)a-\epsilon(t)a^{\dagger},\hat{\rho}(t)]dt\\&+\frac{\pi\kappa^2(N+1)}{\Omega_c}[2a\hat{\rho}(t)a^{\dagger}-a^{\dagger}a\hat{\rho}(t)-\hat{\rho}(t)a^{\dagger}a]dt+\pi\kappa^2\frac{N}{\Omega_c}[2a^{\dagger}\hat{\rho}(t)a-aa^{\dagger}\hat{\rho}(t)-\hat{\rho}(t)aa^{\dagger}]dt\\&-\pi\kappa^2\frac{M^*}{\Omega_c}e^{2i\Omega_{\mathrm{LO}}t}[2a\hat{\rho}(t)a-a^2\hat{\rho}(t)-\hat{\rho}(t)a^2]dt\\&-\pi\kappa^2\frac{M}{\Omega_c}e^{-2i\Omega_{\mathrm{LO}}t}[2a^{\dagger}\hat{\rho}(t)a^{\dagger}-a^{\dagger}a^{\dagger}\hat{\rho}(t)-\hat{\rho}(t)a^{\dagger}a^{\dagger}]dt\ ,\end{aligned} \tag{3.13}$$

where $\epsilon(t)=Ae^{i(\theta-\Omega_{\mathrm{LO}}t)}$.

To make a connection with the discussion of Sec. II we note that the transformations of Eqs. (2.1a) and (2.1b) are defined by the unitary operator

$$R=\exp[g(a_ib_i^{\dagger}-a_i^{\dagger}b_i)]$$

as

$$b_0=R^{\dagger}b_iR,\quad a_0=R^{\dagger}a_iR\ ,$$

with $\sqrt{\eta}=\sin g$. In the good cavity case, $\eta\ll1$ and thus $g^2\simeq\eta=\xi2\pi/\Omega_c$. If we write $\kappa=g\Omega_c/2\pi$, then the good cavity approximation consists in taking

$$\kappa\simeq\left[\frac{\xi\Omega_c}{2\pi}\right]^{1/2}. \tag{3.14}$$

Thus in the good cavity case the evolution equation for the cavity field may be written

$$\frac{d\hat{\rho}(t)}{dt}=\frac{-i}{\hbar}[\hat{H}_0,\rho(t)]+\left[\frac{\xi\Omega_c}{2\pi}\right]^{1/2}[\epsilon^*(t)a-\epsilon(t)a^\dagger,\hat{\rho}(t)]$$
$$+\left[\frac{\xi}{2}\right](N+1)[2a\hat{\rho}(t)a^\dagger-a^\dagger a\hat{\rho}(t)-\hat{\rho}(t)a^\dagger a]+\left[\frac{\xi}{2}\right]N[2a^\dagger\hat{\rho}(t)a-aa^\dagger\hat{\rho}(t)-\hat{\rho}(t)aa^\dagger]$$
$$-\left[\frac{\xi}{2}\right]M^*e^{2i\Omega_{\rm LO}t}[2a\hat{\rho}(t)a-a^2\hat{\rho}(t)-\hat{\rho}(t)a^2]-\left[\frac{\xi}{2}\right]Me^{-2i\Omega_{\rm LO}t}[2a^\dagger\hat{\rho}(t)a^\dagger-a^\dagger a^\dagger\hat{\rho}(t)-\hat{\rho}(t)a^\dagger a^\dagger]\ , \tag{3.15}$$

where $\hat{H}_0=\hbar\omega_0 a^\dagger a$. We now transform to an interaction picture for the intracavity fields via

$$\hat{V}(t)\equiv e^{-i\Omega_{\rm LO}a^\dagger at}\ .$$

The evolution equation in this interaction picture becomes

$$\frac{d\hat{\rho}_I(t)}{dt}=-i\Delta[a^\dagger a,\hat{\rho}(t)]+\left[\frac{\xi\Omega_c}{2\pi}\right]^{1/2}A\,[ae^{-i\theta}-a^\dagger e^{i\theta},\hat{\rho}_I(t)]$$
$$+\left[\frac{\xi}{2}\right](N+1)[2a\hat{\rho}_I(t)a^\dagger-a^\dagger a\hat{\rho}_I(t)-\hat{\rho}_I(t)a^\dagger a]+\left[\frac{\xi}{2}\right]N[2a^\dagger\hat{\rho}_I(t)a-aa^\dagger\hat{\rho}_I(t)-\hat{\rho}_I(t)aa^\dagger]$$
$$-\left[\frac{\xi}{2}\right]M^*[2a\hat{\rho}_I(t)a-a^2\hat{\rho}_I(t)-\hat{\rho}_I(t)a^2]-\left[\frac{\xi}{2}\right]M[2a^\dagger\hat{\rho}_I(t)a^\dagger-a^\dagger a^\dagger\hat{\rho}_I(t)-\hat{\rho}_I(t)a^\dagger a^\dagger]\ , \tag{3.16}$$

where $\Delta=\omega_0-\Omega_{\rm LO}$.

This is the master equation in the interaction picture for a single mode interacting with a squeezed-state bath and was first obtained in a quantum optical context by Gardiner and Collett[10] (with $\Delta=0$). The first term in Eq. (3.16) represents the free evolution of the cavity field. The second term represents a coherent driving of the cavity. Note that the field seen by the cavity is enhanced by the factor $(\xi\Omega_c/2\pi)^{1/2}$.

Writing $M=|M|e^{-2i\phi}$, Eq. (3.16) may be written in terms of the quadrature phase amplitudes; thus

$$\frac{d\hat{\rho}_I(t)}{dt}=-i\Delta[\hat{X}^2_\phi+\hat{X}^2_{\phi+\pi/2},\hat{\rho}_I(t)]-2iA\left[\frac{\xi\Omega_c}{2\pi}\right]^{1/2}[\hat{X}_\phi\sin(\theta+\phi)+\hat{X}_{\phi+\pi/2}\cos(\theta+\phi),\hat{\rho}_I(t)]$$
$$-i\xi/2[\hat{X}_\phi,\{\hat{X}_{\phi+\pi/2},\hat{\rho}_I(t)\}]+i\xi/2[\hat{X}_{\phi+\pi/2},\{\hat{X}_\phi,\hat{\rho}_I(t)\}] \tag{3.17}$$
$$-\xi e^{-2r}[\hat{X}_\phi,[\hat{X}_\phi,\hat{\rho}_I(t)]]-\xi e^{2r}[\hat{X}_{\phi+\pi/2},[\hat{X}_{\phi+\pi/2},\hat{\rho}_I(t)]]\ ,$$

where

$$\hat{X}_\phi\equiv\tfrac{1}{2}(ae^{i\phi}+a^\dagger e^{-i\phi}) \tag{3.18}$$

and we have used $|M|=\sinh r\cosh r$, $N=\sinh^2 r$. The first and second terms in Eqs. (3.17) have been discussed in the previous paragraph. The third and fourth terms represent damping of the mean quadrature phase amplitudes $\langle\hat{X}_{\phi+\pi/2}\rangle$ and $\langle\hat{X}_\phi\rangle$, respectively. The fifth and sixth terms have two very important effects from the point of view of measurement theory. To see what these are let us imagine for the moment that they are the only terms in the master equation. The evolution equations for the second-order moments are then easily shown to be

$$\frac{d}{dt}\langle\hat{X}^2_{\phi+\pi/2}\rangle=\xi e^{-2r}\equiv D_{\phi+\pi/2}\ , \tag{3.19a}$$

$$\frac{d}{dt}\langle\hat{X}^2_\phi\rangle=\xi e^{2r}\equiv D_\phi\ . \tag{3.19b}$$

Thus both quadratures experience a diffusion with diffusion constants $D_{\phi+\pi/2}$ and D_ϕ for $\hat{X}_{\phi+\pi/2}$ and $\hat{X}_\phi$, respectively. If the squeezing of the local oscillator is significant $D_{\phi+\pi/2}=0$, while D_ϕ becomes quite large. In this case the measuring process is causing fluctuations in $\hat{X}_\phi$ to grow at a considerable rate. If there is no squeezing in the external field ($r=0$) the fluctuations in each quadrature grow at the same rate. Thus the first effect of these double commutator terms is to drive the fluctuations in the cavity quadrature phase amplitudes.

To understand the second effect let $|x\rangle$ and $|y\rangle$ denote the representations which diagonalize $\hat{X}_\phi$ and $\hat{X}_{\phi+\pi/2}$, respectively. If $D_{\phi+\pi/2}\simeq 0$, then

$$\frac{\partial}{\partial t}\langle y|\hat{\rho}|y'\rangle=-D_\phi(y-y')^2\langle y|\hat{\rho}|y'\rangle\ . \tag{3.20}$$

It is immediately apparent that there is a rapid decay of the off-diagonal terms (i.e., the "coherences") in the basis $|y\rangle$, which diagonalizes $\hat{X}_{\phi+\pi/2}$. This decay depends on D_ϕ and the square of the separation of the superposed states forming the off-diagonal terms. This latter dependence is characteristic of coherence decay in open systems and provides an automatic semiclassical limit.[17–19] It would appear that the pointer basis is the basis which diagonalizes $\hat{X}_{\phi+\pi/2}$ and thus $\hat{X}_{\phi+\pi/2}$ is the measured quantity. Similar statements would hold for $\hat{X}_\phi$ in the case that $r<0$ with $|r|\gg 1$. If there is no squeezing in the external field $D_\phi = D_{\phi+\pi/2}$ and the evolution tries to diagonalize simultaneously in the basis $|x\rangle, |y\rangle$. Thus the second effect of the fifth and sixth terms in Eq. (3.17) is to diagonalize $\hat{\rho}$ in the basis which diagonalizes $\hat{X}_\phi$ (fifth term) and $\hat{X}_{\phi+\pi/2}$ (sixth term).

Let us summarize the results obtained above for the case that the external field is prepared in a highly squeezed state. The evolution of the cavity field is such as to diagonalize the density operator in the basis which diagonalizes $\hat{X}_{\phi+\pi/2}$ and to cause the fluctuations in the conjugate variable $\hat{X}_\phi$ to grow. The rate of both these processes is determined by the constant $D_\phi = \xi e^{2r}$. The fluctuations in $\hat{X}_{\phi+\pi/2}$ grow at the much smaller rate ξe^{-2r}. These facts suggest that the model realizes a measurement of $\hat{X}_{\phi+\pi/2}$. To verify this conclusion we now turn to an analysis of the measured result.

Information about the intracavity field is carried by the output modes $b_0(t)$ and is "extracted" via standard photon counting. As discussed in Ref. 9 the output field operators are simply the Heisenberg picture operators for the external field, i.e.,

$$b_0(t) = \hat{U}^\dagger(t) b_i(t) \hat{U}(t) , \tag{3.21}$$

where $U(t)$ is the unitary time evolution operator for the coupled cavity and external fields. To analyze the photon counting process we define the photon-number stochastic process $\hat{N}(t)$,

$$\hat{N}_i(t) \equiv \left[\frac{2}{\hbar\Omega_{\rm LO}}\right] \int_0^t ds\, b_i^\dagger(s) b_i(s) . \tag{3.22}$$

The corresponding output operator is

$$\hat{N}_0(t) = \hat{U}^\dagger(t) \hat{N}_i(t) \hat{U}(t) . \tag{3.23}$$

Using a result from Barchielli,[9]

$$\hat{N}_0(t) = \hat{U}_T^\dagger \hat{N}_i(t) \hat{U}_T, \quad \forall T \ge t$$

we find that

$$d\hat{N}_0(t) = \hat{U}^\dagger(t) \hat{U}^\dagger(dt) d\hat{N}_i(t) \hat{U}(dt) \hat{U}(t) . \tag{3.24}$$

The mean output current from the photoelectron counter is given by[16]

$$\langle i(t)\rangle = \int_{-\infty}^t F(t-t') \langle d\hat{N}_0(t)\rangle , \tag{3.25}$$

where $F(t-t')$ is the detector response function. For a detector with an instantaneous response

$$\langle i(t)\rangle = \left\langle \frac{d\hat{N}_0(t)}{dt} \right\rangle \tag{3.26}$$

(factors of electronic charge, etc., have been set to unity). Using Eq. (3.9) we find that in the good cavity case,

$$\langle d\hat{N}_0(t)\rangle = \langle d\hat{N}_i(t)\rangle + \kappa\langle a(t) d\hat{B}^\dagger(t) + a^\dagger(t) d\hat{B}(t)\rangle + \xi\langle a^\dagger(t) a(t)\rangle dt - \xi\langle d\hat{B}^\dagger(t) d\hat{B}(t)\rangle . \tag{3.27}$$

The averages in Eq. (3.27) may be found using Eqs. (3.10), (3.11), and (2.6). The final result is

$$\left\langle \frac{d\hat{N}_0(t)}{dt} \right\rangle = (1-\xi)N + A^2 + \left[\frac{\xi\Omega_c}{2\pi}\right]^{1/2} A\langle \hat{X}_\theta(t)\rangle + \xi\langle a^\dagger a\rangle , \tag{3.28}$$

where

$$\hat{X}_\theta(t) \equiv a(t) e^{-i(\theta-\Omega_{\rm LO}t)} + a^\dagger(t) e^{i(\theta-\Omega_{\rm LO}t)} \tag{3.29}$$

is the intracavity quadrature phase at the reference frequency $\Omega_{\rm LO}$. If we assume that the cavity enhancement is such as to make the third term in Eq. (3.28) the dominant term [i.e., $(\xi\Omega_c/2\pi)^{1/2} \gg 1$, $A^2 \gg N$], then

$$\langle i(t)\rangle \simeq \left[\frac{\xi\Omega_c}{2\pi}\right]^{1/2} A\langle \hat{X}_\theta(t)\rangle . \tag{3.30}$$

The mean photocurrent is directly proportional to the intracavity quadrature phase defined with respect to the local oscillator carrier frequency.

We now consider fluctuations in the measured result as determined by the current two-time correlation function,

$$\langle \Delta i(t_1) \Delta i(t_2)\rangle \equiv \langle i(t_1) i(t_2)\rangle - \langle i(t_1)\rangle\langle i(t_2)\rangle , \tag{3.31}$$

with $t_1 \ge t_2$. This function is given by

$$\langle \Delta i(t_1)\Delta i(t_2)\rangle = \int_{-\infty}^{t_1} ds_1 \int_{-\infty}^{t_2} ds_2 F(t_1-s_1) F(t_2-s_2) [\langle d\hat{N}_0(s_1) d\hat{N}_0(s_2)\rangle - \langle d\hat{N}_0(s_1)\rangle\langle d\hat{N}_0(s_2)\rangle] . \tag{3.32}$$

Let us again assume that the photoelectron response function is instantaneous. The analysis of Sec. II further suggests we assume $A^2 \gg |M|, N$. With these two assumptions we find

$$\langle \Delta i(t_1)\Delta i(t_2)\rangle = 2A^2\{N + |M|\cos[2(\theta+\phi)] + \tfrac{1}{2}\}\delta(t_1,t_2) + A^2\kappa^2\langle \Delta\hat{X}_\theta(t_1)\Delta\hat{X}_\theta(t_2)\rangle , \tag{3.33}$$

where

$$\delta(t_1,t_2) \equiv \left[\frac{\sin[\Omega_c(t_2-t_1)/2]}{\frac{\Omega_c(t_2-t_1)}{2}}\right]^2 .$$

The first term in Eq. (3.33) represents the white-noise local oscillator intensity fluctuations reflected from the cavity into the beam splitter. The last term in Eq. (3.33) is simply the two-time correlation function for the cavity field quadrature defined in Eq. (3.29). Clearly for this scheme to realize an unambiguous measurement of $\hat{X}_\theta(t)$ we need to minimize the first term in Eq. (3.33). As in Sec. II this is achieved by the choice $\theta+\phi=\pi/2$, for which the first term reduces to

$$A^2 e^{-2r}\delta(t_1,t_2) .$$

We conclude that this scheme realizes a measurement of $\hat{X}_\theta(t)$, provided $\theta+\phi=\pi/2$ and $r \gg 1$ with $A^2 \gg |M|, N$.

We must now check for consistency as discussed at the beginning of this section; that is, we must verify that the cavity field density operator is diagonalized in the basis which diagonalizes $\hat{X}_\theta(t)$. Firstly we note that the master equation (3.14) is in the interaction picture defined by the frequency $\Omega_{\rm LO}$ and thus the quadrature phase operators appearing in this equation are defined with respect to this frequency. There are two cases to consider.

(a) $\theta=0$, $\phi=\pi/2$. In this case,

$$\hat{X}_{\theta=0}(t)=[a(t)e^{i\Omega_{\rm LO}t}+a^\dagger(t)e^{-i\Omega_{\rm LO}\tau}] ,$$

which in the interaction picture becomes

$$\hat{X}_{\theta=0}=(a+a^\dagger)=\hat{X}_1 .$$

The density operator is diagonalized in the basis of $\hat{X}_{\phi+\pi/2}$ which in this case is the basis of $\hat{X}_1$. Thus the density operator is indeed diagonalized in the basis which diagonalizes the measured result and consistency is established.

(b) $\theta=\pi/2$, $\phi=0$.

$$\hat{X}_{\theta=\pi/2}(t)=-i[a(t)e^{i\Omega_{\rm LO}t}-a^\dagger(t)e^{-i\Omega_{\rm LO}t}] ,$$

which, in the interaction picture, becomes

$$\hat{X}_2=-i(a-a^\dagger) .$$

The density operator is diagonalized in the basis of $\hat{X}_{\pi/2}=\hat{X}_2$. Thus consistency is established.

IV. CONCLUSION AND DISCUSSION

We have given a quantum measurement theoretical description of squeezed-state heterodyne detection of an intracavity quadrature phase amplitude. The results illustrate some important features of the quantum theory of continuous measurement. An analysis of the measured result reveals what system observable has been measured. An analysis of the measured system shows that the system density operator becomes diagonal in the basis which diagonalizes the operator representing the measured quantity. Furthermore, fluctuations in the canonically conjugate quantity are driven by a diffusion process. One parameter D suffices to describe both the diagonalization and the diffusion process. In the example of this paper $D=D_\phi$ [Eq. (3.19b)]. The rate of diffusion is proportional to D and the rate of diagonalization is proportional to DX^2, where X is the separation of the superposed states. That coherences decay in this way is a typical property of open quantum systems.[17–19]

The essential assumptions which lead to these results are as follows. Firstly, we have assumed there are two widely separated time scales: (i) τ_R the response time of the measurement and (ii) τ_N the correlation time of the meter states. In the frequency domain we have the bandwidth of the response $B_R=(\tau_R)^{-1}$ and the bandwidth of the meter noise $B_N=(\tau_N)^{-1}$. These time scales are such that $\tau_R \gg \tau_N$ ($B_R \ll B_N$). Secondly, we have assumed that over a bandwidth B_R the meter noise is essentially white. Finally, we require for a "good" measurement the noise level of the meter Σ over the bandwidth B_R may be made very small. In our model $B_R=\xi$ and $\Sigma=e^{-2r}$. When these assumptions are satisfied, the evolution of the system is given by a Markovian master equation characterized by a parameter $D=B_R\Sigma^{-1}$. For a good measurement we require D to be very large.

A formal description of continuous measurement which leads to this behavior has been developed by Barchielli and co-workers.[9,14] The results of this paper show that optical heterodyne detection is a measurement which falls under this general description. The assumptions required to establish the formal description are quite clearly exemplified in the model of this paper, and perhaps clarify the class of measurements which may be described by the general formalism.

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