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Quantum Mechanical Rigid Rotator with an Arbitrary Deformation. I

——*Dynamical Group Approach to Quadratically Deformed Body*——

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A rigorous definition is given of "rigidity" in the framework of quantum mechanics. It is argued that corresponding to the measurement of the shape of the body—at least within Gedanken experiments—there must exist a set of operators associated with this shape. The "rigidity" of the body is defined such that the shape is in principle measurable without any quantum mechanical uncertainties.

For a quadratically deformed rigid body, the commutation relations among the shape operators and angular momentum form a Lie algebra, which is shown to be the semi-direct product of the $O(3)$ to a five-dimensional Abelian group; $O(3)\times T_5$. The irreducible representations of this dynamical group are explicitly constructed by employing an elementary algebraic method. It is found that the representation can be specified naturally in terms of Bohr and Mottelson's deformation parameters β and γ as $(\beta^2, \beta^3\cos 3\gamma)$ and that every member of all the possible rotation bands of the rigid rotator is contained in our single irreducible representation once and only once.

Several remarks are added concerning the relation of our dynamical group to the more general rotation-vibration group $SL(3R)$.

§ 1. Introduction

The quantum mechanical treatment of the symmetric and asymmetric tops[1] has been well known since the late thirties with wide applications in molecular and nuclear physics. The quantum mechanics of rigid body with an arbitrary deformation, however, has not been forthcoming, although such a theory seems to be desired in a phenomenological description of nuclear states.

In a series of papers, we shall develop a full quantum mechanical description of rotational motion of rigid body with an arbitrary deformation. The present paper, which is the first of the series, is devoted to the reformulation of quantum mechanics of the symmetric and asymmetric tops from the standpoint of dynamical group recently developed in high energy physics.[2]

Since the concept of "rigidity" is not necessarily clear in quantum mechanics, it is first necessary to discuss this point in some detail. The concept of a body with a non-spherical shape presupposes that this shape is in some sense measurable. This, in turn, requires—by the uncertainty relations for angular momentum—that infinitely many angular momenta are necessary in order to specify the shape.[3] If the shape of the body is rigid, these large angular momenta must not dynamically affect the shape being measured, so that the shape can be considered as fixed over the entire range of energy. In order that the shape is measurable in quantum mechanical framework, there must exist a set of quantum mechanical operators associated with the shape of the body. The eigenvalues of these operators will correspond to the result of measurement of the shape.

From the above consideration, it is clear that these shape operators do not commute with the angular momentum operators nor even with the Hamiltonian of the system. Therefore, the shape operators cannot be the generators of any symmetry groups of the system, but rather constitute those of the dynamical group.

If any of the shape operators do not commute with each other, it is of course impossible to diagonalize these operators simultaneously. This implies that the shape cannot accurately be determined without uncertainty in this case. Thus, it is natural to define the concept of the "rigid" shape as being the condition that the quantum mechanical operators associated with the shape of the body can be simultaneously diagonalized. In other words, we will call the shape "quantum mechanically rigid", if and only if the non-spherical shape is *in principle* measurable without any quantum mechanical uncertainties.

As the quantum mechanical shape operators, we shall take (mass) multipole operators which may be obtained from the density distribution of the body in the usual way. Then, for a quadratically deformed shape such as the symmetric and asymmetric tops, the five components of the (mass) quadrupole moment are sufficient to define the shape. In order that the shape is rigid,*) every component of the operator must commute with each other. Since the quadrupole moment will transform as an irreducible tensor of rank 2 under rotation of ordinary space, the commutation relations among the shape operators and angular momentum can be written down explicitly. It is to be noted that the dynamical variables which enter into our problem are only the shape operators and the angular momentum.**) Thus, the commutation relations among them define the Lie algebra of the dynamical group of the system.

In the next section, the dynamical group of a quadratically deformed rigid body is shown to be the $O(3)\times T_5$, where $\times$ denotes the semi-direct product. It should be noted that the dynamical group of the rigid rotator is usually believed to be the $O(3,1)$ or the $SL(3R)$.[4] The relation between the $SL(3R)$ and our dynamical group $O(3)\times T_5$ is discussed in some detail. The irreducible unitary representations of the dynamical group are obtained in § 4 in such a way

*) More correctly, the mass distribution including the whole shape of the body.

**) In this connection, we note that, in contrast to the case of linear momentum, the angular momentum operator has no well-defined canonical conjugate operator in the usual Hilbert space. The same statement is also true for the multipole operators.

that every shape operator is diagonal. By expanding it into angular momentum basis, it is shown that every member of all the possible rotational bands of the rigid rotator is actually contained in our single irreducible representation once and only once.

Although some parts of the present work are group theoretical, we treat our problem elementarily by employing an algebraic method. A mathematically rigorous formulation will certainly be possible by using Mackey's induced representation,[5] since the group under consideration is a semi-direct product of the $O(3)$ and an Abelian group.

As an illustrative example of our algebraic method, the $O(3)\times T_3$ is treated in § 3 to derive the helicity representation. An intuitive geometrical interpretation of our results are given in § 5.

§ 2. Formulation of the problem

We shall begin with a slightly general case of the body with a non-spherical shape which can be defined by a single (mass) multipole moment of order l. Writing the μ-th (spherical) component of the multipole operator as $Q_\mu{}^{(l)}$, the commutation relations of $Q_\mu{}^{(l)}$ with the angular momentum can be written down explicitly,

$$[L_z, Q_\mu{}^{(l)}]=\mu Q_\mu{}^{(l)}$$

and

$$[L_\pm, Q_\mu{}^{(l)}]=\sqrt{(l\mp\mu)(l\pm\mu+1)}\,Q_{\mu\pm1}^{(l)},$$

which simply states that the $Q^{(l)}$ transforms as a rank l irreducible tensor under rotation of ordinary space. As is discussed in § 1, every component of the multipole operator must commute with each other, when the shape of the body is rigid. Then, we have the following set of commutation relations:

$$[L_z, L_\pm]=\pm L_\pm \quad\text{and}\quad [L_+, L_-]=2L_z\,, \tag{2·1a}$$

$$[L_z, Q_\mu{}^{(l)}]=\mu Q_\mu{}^{(l)}, \tag{2·1b}$$

$$[L_\pm, Q_\mu{}^{(l)}]=\sqrt{(l\mp\mu)(l\pm\mu+1)}\,Q_{\mu\pm1}^{(l)}$$

and

$$[Q_\mu{}^{(l)}, Q_{\mu'}^{(l)}]=0 \tag{2·1c}$$

which define a Lie algebra of the semi-direct product of $O(3)$ to the $(2l+1)$-dimensional Abelian group.

Since the dynamical variables of our system can be taken to be L and $Q^{(l)}$, the above commutation relations can be interpreted as those of the dynamical group, which contains almost all informations on the dynamics of the system.

In the subsequent sections, we shall calculate the representations of (2·1) for $l=1$ and 2, in which every component of $Q^{(l)}$ is diagonal. It will be shown that every member of rotation bands is actually contained in a single irreducible (infinite dimensional) representation once and only once.

Before calculating the representations, let us discuss in some detail the Lie algebra of the dynamical group of symmetric and asymmetric tops, which is defined by (2·1) for $l=2$:

$$[L_z, L_\pm]=\pm L_\pm \quad \text{and} \quad [L_+, L_-]=2L_z\,, \tag{2·2a}$$

$$[L_z, Q_\mu^{(2)}]=\mu Q_\mu^{(2)} \quad \text{and} \quad [L_\pm, Q_\mu^{(2)}]=\sqrt{(2\mp\mu)(3\pm\mu)}\,Q_{\mu\pm1}^{(2)}\,, \tag{2·2b}$$

$$[Q_\mu^{(2)}, Q_{\mu'}^{(2)}]=0\,. \tag{2·2c}$$

This Lie algebra defines an eight-parameter (non-compact) Lie group with semi-direct product structure. On the other hand, the well-known Lie algebra of the $SU(3)$ group is expressed in Racah's spherical basis as[7]

$$[L_z, L_\pm]=\pm L_\pm \quad \text{and} \quad [L_+, L_-]=2L_z\,, \tag{2·3a}$$

$$[L_z, Q_\mu^{(C)}]=\mu Q_\mu^{(C)} \quad \text{and} \quad [L_\pm, Q_\mu^{(C)}]=\sqrt{(2\mp\mu)(3\pm\mu)}\,Q_{\mu\pm1}^{(C)}\,, \tag{2·3b}$$

$$[Q_\mu^{(C)}, Q_{\mu'}^{(C)}]=3\sqrt{10}\,(22\mu\mu'|1\mu+\mu')\,L_{\mu\pm\mu'}\,. \tag{2·3c)*}$$

Therefore, (2·2) can be obtained from (2·3) by the procedure[8] of "contraction": first, put $Q_\mu^{(C)}=(1/\varepsilon)T_\mu$ and, next, take the limit $\varepsilon\to0$ keeping T_μ finite.

There is another Lie algebra from which (2·2) can also be obtained by the same contraction.**) This algebra is known as that of the (non-compact) $SL(3R)$ group,***) whose group elements are all three-dimensional unimodular real matrix. Explicitly,

$$[L_z, L_\pm]=\pm L_\pm \quad \text{and} \quad [L_+, L_-]=2L_z\,, \tag{2·4a}$$

$$[L_z, Q_\mu^{(N)}]=\mu Q_\mu^{(N)} \quad \text{and} \quad [L_\pm, Q_\mu^{(N)}]=\sqrt{(2\mp\mu)(3\pm\mu)}\,Q_{\mu\pm1}^{(N)}\,, \tag{2·4b}$$

$$[Q_\mu^{(N)}, Q_{\mu'}^{(N)}]=-3\sqrt{10}\,(22\mu\mu'|1\mu+\mu')\cdot L_{\mu+\mu'}\,. \tag{2·4c}$$

Since the only difference of these Lie algebras of the $SU(3)$ and $SL(3R)$ groups appears in the opposite sign of (2·3c) and (2·4c), it is clear that the Lie algebra of the $O(3)\times T_5$ can also be obtained from (2·3) and (2·4) by putting $Q^{(2)}=Q^{(C)}+Q^{(N)}$.

In order to see more clearly the above situation, we shall adopt the simplest realizations of the form

$$L_x=\frac{1}{i}\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right),\quad L_y=\frac{1}{i}\left(z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}\right),\quad L_z=\frac{1}{i}\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)$$

*) $L_0=L_z$ and $L_{\pm1}=(\mp)(1/\sqrt{2})L_\pm$.

**) We note that another complex extension of the $SU(3)$ Lie algebra, i.e. the $SU(2,1)$,[9] cannot be contracted to our Lie algebra.

***) Although the finite dimensional (non-unitary) representations of the $SL(3R)$ are well known,[10] a few unitary representations are available in a tractable form.[11]

and

$$Q_\mu^{(2)} = r^2 Y_{2\mu}(\theta, \phi).$$

By introducing three kinds of boson operators $a_x^+(a_x)$, $a_y^+(a_y)$ and $a_z^+(a_z)$ through

$$a_x^+ = \frac{1}{\sqrt{2}}\left(x - \frac{\partial}{\partial x}\right), \quad a_x = \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right), \quad \text{etc.}$$

$$[a_x, a_x^+] = 1,$$

we have

$$L_x = (1/i)(a_y^+ a_z - a_z^+ a_y), \qquad L_y = (1/i)(a_z^+ a_x - a_x^+ a_z),$$
$$L_z = (1/i)(a_x^+ a_y - a_y^+ a_x),$$

and

$$\begin{aligned}
Q_0^{(2)} &= Q_0^{(C)} + Q_0^{(N)} \\
&= (2a_z^+ a_z - a_x^+ a_x - a_y^+ a_y) \\
&\quad + (1/2)(2a_z^+ a_z^+ + 2a_z a_z - a_x^+ a_x^+ - a_x a_x - a_y^+ a_y^+ - a_y a_y), \\
Q_{\pm 1}^{(2)} &= Q_{\pm 1}^{(C)} + Q_{\pm 1}^{(N)} \\
&= \mp\sqrt{(3/2)}\{(a_z^+ a_x + a_x^+ a_z) \pm i(a_y^+ a_z + a_z^+ a_y)\} \\
&\quad \mp\sqrt{(3/2)}\{(a_z^+ a_x^+ + a_z a_x) \pm i(a_y^+ a_z^+ + a_y a_z)\}, \\
Q_{\pm 2}^{(2)} &= Q_{\pm 2}^{(C)} + Q_{\pm 2}^{(N)} \\
&= \sqrt{(3/2)}\{(a_x^+ a_x - a_y^+ a_y) \pm i(a_x^+ a_y + a_y^+ a_x)\} \\
&\quad + \sqrt{(3/2)}\{(a_x^+ a_x^+ + a_x a_x - a_y^+ a_y^+ - a_y a_y)/2 \pm i(a_x^+ a_y^+ + a_x a_y)\}.
\end{aligned}$$

It is now clear that all the three groups, the $SU(3)$, $SL(3R)$ and $O(3) \times T_5$, are contained as subgroups in the full dynamical group[12] $Sp(6R)$ of the three-dimensional harmonic oscillator. Of them, the $SU(3)$ is the well-known symmetry group of the harmonic oscillator, while the $SL(3R)$ contains as a subgroup the symmetry group of the many surface phonon state[13]—the twice covering group of the $SU(1, 1)$.

§ 3. Illustrational example: The $O(3) \times T_3$

In the preceding section, we found that the dynamical group of the non-spherical rigid body whose shape is specified by the l-th multipole moment is given by the semi-direct of $O(3)$ to the $(2l+1)$-dimensional Abelian group, $O(3) \times T_{2l+1}$. In this section, we shall deal with the simplest case $l=1$, i.e. $O(3) \times T_3$, as an illustrative example of our algebraic procedure to treat general cases.

From Eq. (2·1), the Lie algebra of the $O(3)\times T_3$ group*),**) is written as

$$[L_z, L_\pm]=\pm L_\pm, \qquad [L_+, L_-]=2L_z,$$
$$[L_z, P_\mu]=\mu P_\mu, \qquad [L_\pm, P_\mu]=\sqrt{(1\mp\mu)(2\pm\mu)}P_{\mu\pm1},$$
$$[P_\mu, P_{\mu'}]=0,$$

and its invariants are given by

$$\boldsymbol{P}^2=P_0^2-2P_1P_{-1} \tag{3·2a}$$

and

$$(\boldsymbol{L}\cdot\boldsymbol{P})/|P|=\left\{L_zP_0+\frac{1}{\sqrt{2}}(L_+P_{-1}-L_-P_{+1})\right\}\Big/|P|. \tag{3·2b}$$

Let us determine the representation in which all $P_\mu(\mu=0, \pm1)$ are diagonal:***)

$$P_0|\boldsymbol{p}, \lambda\rangle=p_0|\boldsymbol{p}, \lambda\rangle, \tag{3·3a}$$
$$P_{+1}|\boldsymbol{p}, \lambda\rangle=p_{+1}|\boldsymbol{p}, \lambda\rangle, \tag{3·3b}$$
$$P_{-1}|\boldsymbol{p}, \lambda\rangle=p_{-1}|\boldsymbol{p}, \lambda\rangle \tag{3·3c}$$

and

$$\boldsymbol{P}^2|\boldsymbol{p}, \lambda\rangle=p^2|\boldsymbol{p}\cdot\lambda\rangle, \tag{3·4a}$$
$$\{(\boldsymbol{P}\cdot\boldsymbol{L})/|P|\}\cdot|\boldsymbol{p}, \lambda\rangle=\lambda|\boldsymbol{p}, \lambda\rangle. \tag{3·4b}$$

Inserting (3·3) into (3·4a), we have

$$\boldsymbol{p}^2=p_0^2-2p_{+1}p_{-1}$$

which suggests the following parametrizations:

$$p_0=p\cos\theta \quad\text{and}\quad p_{\pm1}=\mp\frac{p}{\sqrt{2}}\sin\theta\cdot e^{\pm i\phi}. \tag{3·5}$$

Next, the state $|\boldsymbol{p}, \lambda\rangle$ is expanded into angular momentum bases:

$$|\boldsymbol{p}, \lambda\rangle=\sum_{lm}\langle lm; p\lambda|\boldsymbol{p}, \lambda\rangle\cdot|lm; p\lambda\rangle, \tag{3·6}$$

where
$$L_z|lm; p\lambda\rangle=m|lm; p\lambda\rangle,$$
$$L_\pm|lm; p\lambda\rangle=\sqrt{(l\mp m)(l\pm m+1)}|lm\pm1; p\lambda\rangle \tag{3·7}$$

and $\langle lm; p\lambda|\boldsymbol{p}, \lambda\rangle$ is the transformation bracket to be determined through (3·4b).

*) The $O(3)\times T_3$ group is isomorphic to E_3, the rotation and translation group in three-dimensional Euclidean space. Its induced representation is well known in scattering theory as the helicity representation of Jacob and Wick.[6] This group appears also as the strong coupling group[14] in isoscalar p-wave meson theory.

**) Throughout this section, $Q_\mu^{(1)}$ will be denoted as P_μ.

***) For the representation in which L_z is diagonal, see Pauli.[15]

Introducing (3·6) into (3·4b), we have

$$\{(\boldsymbol{L}\cdot\boldsymbol{P})/|P|\}|\boldsymbol{p},\lambda\rangle=\lambda|\boldsymbol{p},\lambda\rangle$$
$$=\{\cos\theta\cdot L_z+\tfrac{1}{2}\sin\theta e^{-i\phi}L_+ +\tfrac{1}{2}\sin\theta e^{-i\phi}L_-\}|\boldsymbol{p},\lambda\rangle$$
$$=\sum_{lm}\{m\cos\theta\langle lm;p\lambda|\boldsymbol{p},\lambda\rangle$$
$$+\tfrac{1}{2}\sin\theta e^{-i\phi}\sqrt{(l+m)(l-m+1)}\langle l,m-1;p\lambda|\boldsymbol{p},\lambda\rangle$$
$$+\tfrac{1}{2}\sin\theta e^{i\phi}\sqrt{(l-m)(l+m+1)}\langle l,m+1;p\lambda|\boldsymbol{p},\lambda\rangle\}|lm;p\lambda\rangle. \tag{3·8}$$

Identifing each coefficient of $|lm;p\lambda\rangle$ on both sides of the above equation, we have

$$\lambda\langle lm;p\lambda|\boldsymbol{p},\lambda\rangle=m\cos\theta\cdot\langle lm;p\lambda|\boldsymbol{p},\lambda\rangle$$
$$+\tfrac{1}{2}\sin\theta\{e^{-i\phi}\sqrt{(l+m)(l-m+1)}\langle l,m-1;p\lambda|\boldsymbol{p},\lambda\rangle$$
$$+e^{i\phi}\sqrt{(l-m)(l+m+1)}\langle l,m+1;p\lambda|\boldsymbol{p},\lambda\rangle\},$$

the m-dependence of which can be easily removed by putting

$$\langle lm;p\lambda|\boldsymbol{p},\lambda\rangle=e^{-im\phi}d(lm\lambda;\boldsymbol{p}) \tag{3·9}$$

with the result

$$(\lambda-m\cos\theta)d(lm\lambda;\boldsymbol{p})=\tfrac{1}{2}\sin\theta\{\sqrt{(l+m)(l-m+1)}d(lm-1\lambda;\boldsymbol{p})$$
$$+\sqrt{(l-m)(l+m+1)}d(lm+1\lambda;\boldsymbol{p})\}. \tag{3·10}$$

First, we note that the recurrence formula (3·10) can be solved elementarily for fixed l starting from the maximum value of $m(=l)$. Moreover, if we adopt the following normalizations for $|\boldsymbol{p},\lambda\rangle$ and $|lm;p\lambda\rangle$, viz.

$$\langle\boldsymbol{p}',\lambda|\boldsymbol{p},\lambda\rangle=\delta(\boldsymbol{p}-\boldsymbol{p}')$$

and

$$\langle lm;p\lambda|l'm';p\lambda\rangle=\delta(l,l')\cdot\delta(m,m'), \tag{3·11}$$ *)

we can determine—up to phase—an explicit algebraic form of $d(lm\lambda;\boldsymbol{p})$ for arbitrary values of l and m. It is then found**) that this function can be taken to be identical to Wigner's d-function $d^{(l)}_{m\lambda}(\theta)$ by an appropriate choice of phases:

$$d(lm\lambda;\boldsymbol{p})=d^{(l)}_{m\lambda}(\theta). \tag{3·12}$$

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*) If we think of P as momentum, (3·11) needs no explanations. For later applications, we note that this is a special form of the Peter-Weyl theorem.

**) To see this directly, compare Eq. (3·10) to the familiar recurrence formulas of d-function which determine uniquely d-function—for example, Eqs. (D·13) and (D·14) of Fano and Racah.[16] Note that the definition of d-function of Fano and Racah is different from ours:

$$d^{(l)}_{mn}(\theta)=\langle lm|e^{i\theta J_y}|ln\rangle, \quad \text{(Fano and Racah)}$$
$$d^{(l)}_{mn}(\theta)=\langle lm|e^{-i\theta J_y}|ln\rangle. \quad \text{(ours)}$$

In summary, we get

$$\langle lm; p\lambda | \boldsymbol{p}, \lambda\rangle = e^{-im\phi} d^{(l)}_{m\lambda}(\theta). \qquad (3\cdot13)$$

Finally, the matrix element of P between the angular momentum basis may be determined as follows: Since P transforms as a unit rank irreducible tensor, Wigner and Eckart theorem*) states that

$$P_\mu | lm; p\lambda\rangle = \sum_{l'} \frac{(l1m\mu | l'm+\mu)}{\sqrt{2l'+1}} (l' \| P \| l) \, | l'm+\mu; p\lambda\rangle \qquad (3\cdot14)$$

where $(l'\|P\|l)$ is reduced matrix element. Inserting (3·14) into

$$P_\mu | \boldsymbol{p}, \lambda\rangle = p_\mu | \boldsymbol{p}, \lambda\rangle$$

and using (3·5) and (3·12), we have a set of equations to determine the reduced matrix elements of P:

$$\begin{aligned} p_\mu \langle lm; p\lambda | \boldsymbol{p}\lambda\rangle = \frac{1}{\sqrt{2l+1}} \{ & (l-11m-\mu\mu | lm)\, (l\|P\|l-1) \langle l-1m-\mu; p\lambda | \boldsymbol{p}\lambda\rangle \\ & + (l1m-\mu\mu | lm)\, (l\|P\|l) \langle lm-\mu; p\lambda | \boldsymbol{p}\lambda\rangle \\ & + (l+11m-\mu\mu | lm)\, (l\|P\|l-1) \langle l+1m-\mu; p\lambda | \boldsymbol{p}\lambda\rangle \}. \end{aligned}$$

By comparing this with Clebsch-Gordan formula ($j_1=1$, $m_1=\mu$, $n_1=0$),

$$D^{(j_1)}_{m_1 n_1} D^{(j_2)}_{m_2 n_2} = \sum\nolimits_J (j_1 j_2 m_1 m_2 | JM)\, (j_1 j_2 n_1 n_2 | JN) \cdot D^{(J)}_{MN}, \qquad (3\cdot15)$$

one sees immediately

$$(l'\|P\|l) = \sqrt{2l+1}\, (l1\lambda 0 | l'\lambda) \cdot p. \qquad (3\cdot16)$$

Thus, we get the complete solution of (3·1) for $p \neq 0$ purely algebraically.

§ 4. Symmetric and asymmetric tops

We now proceed to the dynamical group approach of the rigid body whose shape is specified by the quadratic deformation. The shape operator in this case is taken to be the mass quadrupole moment $Q^{(2)}$. The Lie algebra of our system is given by (2·2);

$$[L_z, L_\pm] = \pm L_\pm \quad \text{and} \quad [L_+, L_-] = 2L_z, \qquad (2\cdot2\text{a})$$

$$[L_z, Q_\mu{}^{(2)}] = \mu Q_\mu{}^{(2)} \quad \text{and} \quad [L_\pm, Q_\mu{}^{(2)}] = \sqrt{(2\mp\mu)(3\pm\mu)}\, Q^{(2)}_{\mu\pm1}, \qquad (2\cdot2\text{b})$$

$$[Q_\mu{}^{(2)}, Q^{(2)}_{\mu'}] = 0, \qquad (2\cdot2\text{c})$$

*) Throughout this paper, we use the Wigner-Eckart theorem of the form

$$\langle j'm' | T_\mu{}^{(k)} | jm\rangle = \frac{(jkm\mu | j'm')}{\sqrt{2j'+1}} (j' \| T^{(k)} \| j).$$

and its invariants are taken to be*)

$$C_2=\sum_\mu (22\mu-\mu|00)\, Q^{(2)}_\mu Q^{(2)}_{-\mu} \tag{4·1}$$

and

$$C_3=\sum_{\mu_1\mu_2\mu_3} (22\mu_1\mu_2|2\mu_3)\,(22\mu_3-\mu_3|00)\, Q^{(2)}_{\mu_1} Q^{(2)}_{\mu_2} Q^{(2)}_{-\mu_3}\,. \tag{4·2}$$

Let us calculate the representation in which all the components of $Q^{(2)}$ are diagonal:

$$Q_\mu{}^{(2)}|\boldsymbol{q}, \alpha\rangle = q_\mu|\boldsymbol{q}, \alpha\rangle. \tag{4·3}$$

The hermiticity property of $Q_\mu{}^{(2)}$

$$Q_\mu{}^{(2)+}=(-)^\mu Q^{(2)}_{-\mu} \tag{4·4a}$$

implies

$$q_\mu=(-)^\mu \bar{q}_{-\mu}\,. \tag{4·4b}$$

The representation $|\boldsymbol{q}, \alpha\rangle$ will be specified by the eigenvalues of the invariants c_2 and c_3,

$$C_2|\boldsymbol{q}, \alpha\rangle = c_2|\boldsymbol{q}, \alpha\rangle \tag{4·5a}$$

and

$$C_3|\boldsymbol{q}, \alpha\rangle = c_3|\boldsymbol{q}, \alpha\rangle \tag{4·5b}$$

which may be expressed in terms of q_μ as

$$c_2=\sum_\mu (22\mu-\mu|00)\, q_\mu q_{-\mu} \tag{4·6a}$$

and

$$c_3=\sum_{\mu_1\mu_2\mu_3} (22\mu_1\mu_2|2\mu_3)\,(22\mu_3-\mu_3|00)\, q_{\mu_1} q_{\mu_2} q_{-\mu_3}\,. \tag{4·6b}$$

As is seen from the previous example of the $O(3)\times T_3$, the first important step is to parametrize the above formula in such a way that Eqs. (4·5a) and (4·5b) are automatically satisfied. For this purpose, we recall the orthogonality and contraction properties of D-function

$$\sum_m (-)^m D^{(2)}_{-m-n}(\Omega)\, D^{(2)}_{mn'}(\Omega) = (-)^n \delta(n, n')$$

and

$$\sum_{m_1m_2m_3} (-)^{m_3}(22m_1m_2|2-m_3)\, D^{(2)}_{m_1n_1}(\Omega)\, D^{(2)}_{m_2n_2}(\Omega)\, D^{(2)}_{m_3n_3}(\Omega) = (-)^{n_3}(22n_1n_2|2-n_3)\,,$$

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*) These invariants can be obtained from the Casimir operators[17] of the $SU(3)$ group

$$C_2=(\boldsymbol{L}\cdot\boldsymbol{L})+(1/3)\,(\boldsymbol{Q}^{(2)}\cdot\boldsymbol{Q}^{(2)}),$$

$$C_3=[\boldsymbol{Q}^{(2)}\times\boldsymbol{Q}^{(2)}\times\boldsymbol{Q}^{(2)}]_0{}^{(0)}+3\sqrt{3/7}[\boldsymbol{Q}^{(2)}\times\boldsymbol{L}\times\boldsymbol{L}]_0{}^{(0)}$$

by the Wigner contraction.

which suggest the following parametrization for q_μ,

$$q_\mu = q^{(0)} D^{(2)}_{\mu 0}(\Omega) + \frac{q^{(2)}}{\sqrt{2}} \{ D^{(2)}_{\mu 2}(\Omega) + D^{(2)}_{\mu -2}(\Omega) \}, \tag{4·7}$$

where $q^{(0)}$ and $q^{(2)}$ are real constants.*) The eigenvalues of the second and third order Casimir operators can then be written as

$$c^{(2)} = (1/\sqrt{5}) \cdot \{ (q^{(0)})^2 + (q^{(2)})^2 \} \tag{4·8a}$$

and

$$c^{(3)} = (\sqrt{2}/\sqrt{35})\, q^{(0)} \{ -(q^{(0)})^2 + 3 (q^{(2)})^2 \}. \tag{4·8b}$$

Further, if we adopt Bohr and Mottelson[18] type parametrizations**) for $q^{(0)}$ and $q^{(2)}$, we have

$$c^{(2)} = (1/\sqrt{5}) \beta^2 \quad \text{and} \quad c^{(3)} = -(\sqrt{2}/\sqrt{35}) \beta^3 \cos 3\gamma , \tag{4·9}$$

where $q^{(0)} = \beta \cos \gamma$ and $q^{(2)} = \beta \sin \gamma$. Next, the state $|\boldsymbol{q}, \alpha\rangle$ will be expanded into the angular momentum basis:

$$|\boldsymbol{q}, \alpha\rangle = \sum_{lm} \langle lm ; \alpha | \boldsymbol{q}, \alpha \rangle \cdot | lm ; \alpha \rangle, \tag{4·10}$$

where

$$L_z | lm ; \alpha \rangle = m | lm ; \alpha \rangle \quad \text{and} \quad L_\pm | lm ; \alpha \rangle = \sqrt{(l \mp m)(l \pm m + 1)}\, | lm \pm 1 ; \alpha \rangle.$$

Wigner and Eckart theorem states that

$$Q_\mu{}^{(2)} | lm ; \alpha \rangle = \sum_{l'} \frac{(l2m\mu | l'm+\mu)}{\sqrt{2l'+1}} (l', \alpha \| Q^{(2)} \| l, \alpha)\, | l'm' ; \alpha \rangle. \tag{4·11}$$

Using the above results, (4·3) can now be written as

$$\begin{aligned} Q_\mu{}^{(2)} | \boldsymbol{q}, \alpha \rangle &= q_u \sum_{lm} \langle \alpha, lm | \boldsymbol{q}, \alpha \rangle \cdot | lm ; \alpha \rangle \\ &= \sum_{l, l', m} \frac{(l2m\mu | l'm+\mu)}{\sqrt{2l'+1}} (l' \| Q^{(2)} \| l) \cdot \langle lm ; \alpha | \boldsymbol{q} \cdot \alpha \rangle \\ &\quad \times | l'm+\mu ; \alpha \rangle, \end{aligned} \tag{4·12}$$

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*) Although the most general parametrization is

$$q_\mu = q^{(0)} D^{(2)}_{\mu 0} + \frac{q^{(1)}}{\sqrt{2}} \{ D^{(2)}_{\mu 1} - D^{(2)}_{\mu -1} \} + \frac{q^{(2)}}{\sqrt{2}} \{ D^{(2)}_{\mu 2} + D^{(2)}_{\mu -2} \}; \tag{4·7'}$$

this can be brought into the form (4·7) by an appropriate choice of Euler angle. Since the D-function is unitary, the representation obtained by (4·7′) is clearly unitary equivalent to that by (4·7).

It is further noted that, even if we specify the reference axis to define the Euler angle, the relabelings of x, y and z axes does not change any essential contents of our representation. This arbitrariness of relabeling leads to the well-known symmetry under the point group D_2. It is clear that this D_2 group plays a role of the (discrete) little group in the $O(3) \times T_5$.

**) As is well known, the Bohr-Mottelson parametrization is valid only for a small deformation. On the other hand, (4·9) hold for large deformation as well.

from which we get the recurrence formula to determine the transformation bracket $\langle lm;\alpha|\boldsymbol{q},\alpha\rangle$;

$$\langle lm;\alpha|\boldsymbol{q},\alpha\rangle q_\mu=\sum_{l'}\frac{(l'2m-\mu\mu|lm)}{\sqrt{2l+1}}(l\|Q^{(2)}\|l')\cdot\langle l'm-\mu;\alpha|\boldsymbol{q},\alpha\rangle. \qquad (4\cdot13)$$

It is to be noted that, in contrast to the case of $O(3)\times T_3$ treated in the previous section,*) the recurrence formula to determine the transformation bracket contains the reduced matrix elements of $Q^{(2)}$. We, therefore, need to determine the reduced matrix elements first.

The set of equations to determine the reduced matrix elements of $Q^{(2)}$ may be obtained as follows: First, consider

$$\begin{aligned}&\sum_{\mu\mu'}(22\mu\mu'|k\nu)\,(Q^{(2)}_\mu Q^{(2)}_{\mu'}-Q^{(2)}_{\mu'}Q^{(2)}_\mu)\\ &\quad=\sum_{\mu\mu'}\{(22\mu\mu'|k\nu)\,Q^{(2)}_\mu Q^{(2)}_{\mu'}-(-)^k(22\mu'\mu|k\nu)\,Q^{(2)}_{\mu'}Q^{(2)}_\mu\}\\ &\quad=\{1-(-)^k\}\,[Q^{(2)}\times Q^{(2)}]_\nu^{(k)},\end{aligned}$$

which gives us non-trivial results only if $k=$odd. Since the every components of $Q^{(2)}$ will commute with each other, the matrix element of $[Q^{(2)}\times Q^{(2)}]^{(k)}$ between $|l_1m_1\rangle$ and $|l_2m_2\rangle$ states must vanish for odd k, viz.

$$\langle l_2m_2|\,[Q^{(2)}\times Q^{(2)}]_\nu^{(k)}|l_1m_1\rangle=\frac{(l_1km_1\nu|l_2m_2)}{\sqrt{2l_2+1}}(l_2\|\,[Q^{(2)}\times Q^{(2)}]^{(k)}\|l_1)=0\,. \quad (4\cdot14)$$

Therefore, we have

$$(l_2\|\,[Q^{(2)}\times Q^{(2)}]^{(k)}\|l_1)=\sum_{l'}\sqrt{2k+1}\,(l_2\|Q^{(2)}\|l')\,(l'\|Q^{(2)}\|l_1)\cdot W(l_1l_222;kl')=0\,,$$

$$(k=1 \text{ and } 3) \qquad (4\cdot15)$$

where W is the Racah coefficient. The matrix elements of the Casimir operators (4·1) and (4·2) can be written also in terms of the reduced matrix elements as

$$\beta^2=\sum_{l'}\frac{(-)^{l'-l_2}}{(2l_2+1)}(l_2\|Q^{(2)}\|l')\,(l'\|Q^{(2)}\|l_1)\cdot\delta(l_1,l_2)\cdot\delta(m_1,m_2) \qquad (4\cdot16)$$

and

$$\begin{aligned}-\sqrt{2/35}\,\beta^3\cos\gamma=&\sum_{l'l''}\frac{(-)^{l'-l_2}}{(2l_2+1)}W(l_22l'2;l''2)\\ &\times(l_2\|Q^{(2)}\|l'')\,(l''\|Q^{(2)}\|l')\,(l'\|Q^{(2)}\|l_1)\cdot\delta(l_1,l_2)\cdot\delta(m_1,m_2)\,. \qquad (4\cdot17)\end{aligned}$$

Equations (4·15)$\sim$(4·17) are sufficient to determine the reduced matrix element uniquely up to phase. Instead of solving these equations explicitly, we shall first seek the consistent solution of Eqs. (4·13), (4·15), (4·16) and (4·17) for the simple case of the symmetric top.

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*) The difference between the $O(3)\times T_3$ and $O(3)\times T_5$ comes from the fact that the $O(3)\times T_5$ does not contain the (continuous) little group—the stability group in mathematician's terminology—in it.

4A. *The symmetric top ——a special solution——*

We shall treat in this subsection the simple case of $q^{(2)}=0$. In terms of Bohr and Mottelson parametrization, we expect that this case will correspond to the prolate (axially symmetric) deformation.

In order to obtain a special solution of Eqs. (4·15)~(4·17), we first note the well-known identity involving the Racah and Clebsch-Gordan coefficients:

$$\sum_e \sqrt{(2e+1)(2f+1)}\,(ab\alpha\beta|e\varepsilon)\,(ed\varepsilon\delta|c\gamma)\,W(abcd;ef)$$
$$=(af\alpha\varphi|c\gamma)\,(bd\beta\delta|f\varphi). \tag{4·18}$$

Putting $a=l_1$, $c=l_2$, $b=d=2$, $e=l'$ and $f=k$, we get

$$\sum_{l'} \sqrt{(2l'+1)(2k+1)}\,(l_12K_1\lambda|l'K')\,(l'2K'\lambda'|l_2K_2)\,W(l_12l_22;l'k)$$
$$=(l_1kK_1\varphi|l_2K_2)\,(22\lambda\lambda'|k\varphi). \tag{4·19}$$

Since $(2200|k0)=0$ for $k=$odd, the right-hand side of the above equation should vanish for odd k, if we put $\lambda=\lambda'=\varphi=0$:

$$\sum_{l'} \sqrt{(2l'+1)(2k+1)}\,(l_1'2K0|l'K)\,(l'2K0|l_2K)\,W(l_12l_22;l'k)=0$$

if k is odd.

By comparing this equation to (4·15), we get a special solution of (4·15)

$$(l_2\|Q^{(2)}\|l_1)=\sqrt{2l_1+1}\,(l_12K0|l_2K)\cdot\beta\,. \tag{4·20}$$

Inserting this into (4·16), one sees immediately that Eq. (4·16) is reduced to the well-known orthogonality relation of Clebsch-Gordan coefficient:

$$\sum_j (j_1j_2m_1m_2|jm)\,(j_1j_2m_1'm_2'|jm)=\delta(m_1,m_1')\cdot\delta(m_2,m_2'). \tag{4·21}$$

To examine Eq. (4·17), we note the following identity*)

$$\sum_{eb}\frac{\sqrt{(2e+1)(2b+1)(2c+1)}}{\sqrt{2d+1}}(-)^{a+b-\varepsilon}(ba\beta\alpha|e\varepsilon)\,(ec\varepsilon-\gamma|d-\delta)$$
$$\times(df-\delta\varphi|b\beta)\,W(abcd;ef)=(-)^{d+\delta}(af\alpha\varphi|c\gamma)$$

which leads to

$$\sum_{l'l''}\frac{\sqrt{(2l'+1)(2l''+1)}}{\sqrt{2l_2+1}}(-)^{l'-l_2}(l'2K'\lambda|l''K'')\,(l''2K''-\lambda''|l_2K_2)$$
$$\times(l_22K_2\lambda'|l'K')\,W(l_22l'2;l''2)=\frac{(-)^{\lambda''}}{\sqrt{5}}(22\lambda\lambda'|2\lambda''). \tag{4·22}$$

It is then clear that (4·20)

*) This identity can be derived from (4·18) together with (4·21).

$$(l_2\|Q^{(2)}\|l_1) = \sqrt{2l_1+1}\,(l_1 2K0|l_2 K)\cdot\beta$$

satisfies (4·17) if we put $\cos 3\gamma=1$. Since the simultaneous solution of (4·15) ~ (4·17) should be unique, it is concluded that the above reduced matrix element is the unique solution for the special case of $\cos 3\gamma=1$: i.e. $c_2=(1/\sqrt{5})\beta^2$ and $c_3=-(\sqrt{2/35})\beta^3$.

Finally, the transformation bracket $\langle lm;\alpha|\boldsymbol{q},\alpha\rangle$ will be determined through Eq. (4·13). For this purpose, we note the following identity:

$$D^{(2)}_{\mu 0}(\Omega)\,D^{(l)*}_{mK}(\Omega)=\sum_{l'}\frac{(2l'+1)}{(2l+1)}(l'2m-\mu\mu|lm)\,(l'2K0|lK)\,D^{(l)*}_{m-\mu K}(\Omega), \qquad (4\cdot23)$$

which can be derived from the Clebsch-Gordan formula (3·15) as a special case.

By comparing (4·23) to (4·13), the transformation bracket is now determined as

$$\langle lm;\alpha|\boldsymbol{q},\alpha\rangle=\sqrt{\frac{2l+1}{8\pi^2}}D^{(l)*}_{mK}(\Omega). \qquad (4\cdot24)$$

It is noted that K appears naturally as an additional quantum number to label the states within our representation specified by (β^2, $\cos 3\gamma=1$). In the next section, it will be shown that this K is the z-component of angular momentum in the rotating (body fixed) axis.

4B. *The general solution——asymmetric top*

In the preceding subsection, we obtain the additional quantum number K labeling the states within the given representation. Further, K is found to be diagonal for this special representation. In order to obtain the general solution of (4·13), (4·15), (4·16) and (4·17), it is necessary to write explicitly the quantum number K in these equations and, if needed, perform summation over K in intermediate states. That is, (4·10) may be written as

$$|\boldsymbol{q},\alpha\rangle=\sum_{lKm}\langle lmK;\alpha|\boldsymbol{q},\alpha\rangle\cdot|lmK;\alpha\rangle, \qquad (4\cdot10)$$

where the quantum number α denotes a set of the eigenvalues of the Casimir operators C_2 and C_3: $\alpha=(c_2, c_3)$ or equivalently $\alpha=(\beta^2, \beta^3\cos 3\gamma)$.

The set of equations—Eqs. (4·15) ~ (4·17)—to determine the reduced matrix element of $Q^{(2)}$ can now be written as

$$\sum_{l'K'}\sqrt{2k+1}\,(l_2K_2\|Q^{(2)}\|l'K')\,(l'K'\|Q^{(2)}\|l_1K_1)\,W(l_1l_2 22;kl')=0 \qquad (4\cdot15')$$

$$k=1 \text{ and } 3,$$

$$\beta^2=\sum_{l'K'}\frac{(-)^{l'-l_2}}{(2l_2+1)}(l_2K_2\|Q^{(2)}\|l'K')\,(l'K'\|Q^{(2)}\|l_1K_1)\cdot\delta(l_1,l_2)\cdot\delta(m_1,m_2) \qquad (4\cdot16')$$

and

$$-\sqrt{\frac{2}{35}}\beta^3 \cos 3\gamma = \sum_{l'l''K'K''} \frac{(-)^{l'-l_2}}{(2l_2+1)} W(l_2 2 l_1 2; l'' 2)$$
$$\times (l_2 K_2 \| Q^{(2)} \| l'' K'')(l'' K'' \| Q^{(2)} \| l' K')$$
$$\times (l' K' \| Q^{(2)} \| l_1 K_1) \cdot \delta(l_1, l_2) \cdot \delta(m_1, m_2). \qquad (4 \cdot 17)'$$

In analogy to the case of the symmetric top, we found the non-vanishing reduced matrix elements of $Q^{(2)}$ for a given values of $c_2 = (1/\sqrt{5})\beta^2$ and $c_3 = -(\sqrt{2/35})\beta^3 \cos 3\gamma$ as

$$(l'K \| Q^{(2)} \| lK) = \beta \cos \gamma \sqrt{2l+1}\,(l2K0|l'K), \qquad (4 \cdot 25\text{a})$$

$$(l'K+2 \| Q^{(2)} \| lK) = \frac{\beta \sin \gamma}{\sqrt{2}} \sqrt{2l+1}\,(l2K2|l'K+2), \qquad (4 \cdot 25\text{b})$$

$$(l'K-2 \| Q^{(2)} \| lK) = \frac{\beta \sin \gamma}{\sqrt{2}} \sqrt{2l+1}\,(l2K-2|l'K-2). \qquad (4 \cdot 25\text{c})$$

The above result can be checked directly by using the identities (4·19), (4·21) and (4·22). It is noted that a factor $\delta(K_1, K_2)$ appears on the right-hand side of both Eqs. (4·16) and (4·17). In other words, the Casimir operators C_2 and C_3 are found to be diagonal in K:

$$\langle l'm'K'; \alpha | C_2 | lmK; \alpha \rangle = c_2 \delta(l, l') \cdot \delta(m, m') \cdot \delta(K, K')$$

and

$$\langle l'm'K'; \alpha | C_3 | lmK; \alpha \rangle = c_3 \delta(l, l') \cdot \delta(m, m') \cdot \delta(K, K'),$$

which ensure our previous statement that the quantum number K can be used to label the states within a given representation.

The recurrence formula to determine the transformation bracket reads as

$$q_\mu \langle lmK; \alpha | \boldsymbol{q}, \alpha \rangle = \sum_{l'K'} \frac{(l'2m-\mu\mu|lm)}{\sqrt{2l+1}} (lK \| Q^{(2)} \| l'K') \langle l'm-\mu K'; \alpha | \boldsymbol{q}, \alpha \rangle, \qquad (4 \cdot 13')$$

where q_μ is given by (4·7);

$$q_\mu = q^{(0)} D^{(2)}_{\mu 0}(\Omega) + \frac{q^{(2)}}{\sqrt{2}} \{ D^{(2)}_{\mu 2}(\Omega) + D^{(2)}_{\mu -2}(\Omega) \}.$$

The transformation bracket can now be determined as

$$\langle lmK; \alpha | \boldsymbol{q}, \alpha \rangle = \sqrt{\frac{2l+1}{16\pi^2 \{1+\delta(K, 0)\}}} \{ D^{(l)*}_{mK}(\Omega) \pm (-)^l D^{(l)*}_{m-K}(\Omega) \}, \qquad (4 \cdot 26)$$

by comparing (4·13′) with the following identity:

$$[q^{(0)} D^{(2)}_{\mu 0} + \frac{q^{(2)}}{\sqrt{2}} \{ D^{(2)}_{\mu 2} + D^{(2)}_{\mu -2} \}] \{ D^{(l)*}_{mK} \pm (-)^l D^{(l)*}_{m-K} \}$$

$$=\sum_{l'}\frac{2l'+1}{2l+1}(l'2K0|lK)\,[q^{(0)}(l'2K0|lK)\,\{D^{(l')*}_{m-\mu K}\pm(-)^{l'}D^{(l')*}_{m-\mu,-K}\}$$

$$+\frac{q^{(2)}}{\sqrt{2}}\{(l'2K-22|lK)\,(D^{(l')*}_{m-\mu,K-2}\pm(-)^{l'}D^{(l')*}_{m-\mu,-(K-2)})$$

$$+(l'2K+2-2|lK)\,(D^{(l')*}_{m-\mu,K+2}\pm(-)^{l'}D^{(l')*}_{m-\mu,-(K+2)})\}]. \qquad (4\cdot27)$$

Thus, we get a general solution of the Lie algebra (2·2abc) purely algebraically. In the next section, we shall prove the irreducibility of our representation by looking at that of the (discrete) little group.

§ 5. Several discussions

This section is divided into three parts; in the first part, the quantum number K is shown to be the z-component of angular momentum in the rotating (body-fixed) reference axis. The irreducibility of our representation is proved in § 5B and an intuitive geometrical interpretation of our formalism is given in § 5C.

5A. *The quantum number K*

In order to examine the physical meaning of K, let us introduce operator I, whose spherical components are defined by

$$I_\lambda=\sum_\mu D^{(1)*}_{\mu\lambda}(\Omega)\cdot L_\mu\,, \qquad (\lambda,\ \mu=0,\ \pm1) \qquad (5\cdot1)$$

where L_μ are the spherical components of angular momentum operator. It can easily be proved that

$$[I_\lambda,L_\mu]=0 \qquad \text{for all } \lambda \text{ and } \mu$$

and

$$[I_\lambda,I_{\lambda'}]=\sqrt{2}(11\lambda\lambda'|1\lambda+\lambda')I_{\lambda+\lambda'}\,, \qquad (5\cdot2)$$

which, written in vector notation, reads

$$I\times I=-iI\,.$$

From the above results, I_λ can be interpreted naturally as the (spherical) component of angular momentum in (rotating) body fixed axis.[19),3)]

The action of I_λ on the state $|lmK;\alpha\rangle$ may be determined by operating I_λ on both-sides of Eq. (4·10):

$$I_\lambda|\boldsymbol{q},\alpha\rangle=\sum_{lmK}I_\lambda\cdot\langle lmK;\alpha|\boldsymbol{q},\alpha\rangle\cdot|lmK;\alpha\rangle$$

$$=\sum_{lmK\mu}D^{(1)*}_{\mu\lambda}(\Omega)\cdot\langle lmK;\alpha|\boldsymbol{q},\alpha\rangle\cdot L_\mu|lmK;\alpha\rangle$$

$$=\sum D^{(1)*}_{\mu\lambda}(\Omega)\langle lmK;\alpha|\boldsymbol{q},\alpha\rangle\cdot\sqrt{l(l+1)}\,(l1m\mu|lm+\mu)\,|lm+\mu K;\alpha\rangle$$

$$=\sum\sqrt{l(l+1)}\langle lm-\mu K;\alpha|\boldsymbol{q};\alpha\rangle D^{(1)*}_{\mu\lambda}(\Omega)\,(l1m-\mu\mu|lm)\,|lmK;\alpha\rangle, \qquad (5\cdot3)$$

where the Wigner-Eckart theorem has been employed in the third equation together with

$$(l'\|L\|l) = \sqrt{(2l+1)l(l+1)}\cdot\delta(l,l').$$

Introducing the explicit form of $\langle lmK;\alpha|\boldsymbol{q},\alpha\rangle$, Eq. (4·26), into the last line of (5·3) and using the Clebsch-Gordan formula of the following form,

$$\sum_{\mu}(l1m-\mu\mu|l'm)D^{(l)*}_{m-\mu K}(\Omega)D^{(1)*}_{\mu\lambda}(\Omega)=\sum_{l'}(l1K\lambda|l'K+\lambda)D^{(l')*}_{m,K+\lambda}(\Omega),$$

we get

$$I_0|lmK;\alpha\rangle = K|lmK;\alpha\rangle \tag{5·4}$$

for the 0-th component of $\boldsymbol{I}$. From (5·4), it follows that K is the eigenvalue of the 0-th component of angular momentum in the rotating (body fixed) axis.

We may define the raising and lowering operators, I_+ and I_-, through

$$I_{\pm} = \mp\sqrt{2}I_{\pm1}.$$

The action of $I_{\pm}$ on the state $|lmK;\alpha\rangle$ can be determined also by using (5·3). It is found that the resulting state is very complicated except for the case of $\langle lmK;\alpha|\boldsymbol{q};\alpha\rangle \backsim D^{(l)}_{mK}(\Omega)$ in Eq. (4·24):

$$I_{\pm}|lmK;\alpha\rangle = \sqrt{(l\pm K)(l\mp K+1)}\,|lmK\mp1;\alpha\rangle \tag{5·5}$$
$$\{\text{only if } \langle lmK;\alpha|\boldsymbol{q};\alpha\rangle \backsim D^{(l)*}_{mK}(\Omega)\}.$$

5B. *Irreducibility of our representation*

We have obtained the representation $|\boldsymbol{q},\alpha\rangle$ of the $O(3)\times T_5$ Lie algebra, in which every $Q_\mu{}^{(2)}$ is diagonal. Expanding it into angular momentum basis, our final result is represented as

$$|\boldsymbol{q},\alpha\rangle = \sum_{lmK}\sqrt{\frac{2l+1}{16\pi^2\{1+\delta(K,0)\}}}\{D^{(l)*}_{mK}(\Omega)\pm(-)^l D^{(l)*}_{m-K}(\Omega)\}\,|lmK;\alpha\rangle. \tag{5·6}$$

The non-vanishing reduced matrix elements of $Q^{(2)}$ are determined in (4·25a)～(4·25c).

Here, we shall prove the irreducibility of our representation. As already noted, the group $O(3)\times T_5$ has no continuous little group in it. Instead, the point group D_2 is contained as the discrete little group. The irreducibility of our representation will be assured by examining that of the little group—due to Mackey's general theorem—in just the same way as in the case of well-known Wigner's construction of the irreducible representation of the quantum mechanical Poincaré group. The irreducible representations of our little group D_2 are usually classified into four representations, A, B_1, B_2 and B_3, which, in terms of l and K, may be summarized as

$$A\text{-type} = \text{even parity and even } K;$$

B_1-type = odd parity and even K;

B_2-type = even parity and odd K;

B_3-type = odd parity and odd K.

Since $Q^{(2)}$ connects only the states with $\Delta K=0$ and ± 2, our representation splits clearly into even and odd K parts. Further, (5·6) has a definite parity; plus sign in the curly bracket has even parity, minus one odd parity. Thus, one sees that our representation splits into the four irreducible representations of the point group D_2. Since the D_2 constitutes the (discrete) little group of the whole group, it follows that (5·6) is the irreducible representation of the $O(3)\times T_5$ group.

It is amusing to note that in the conventional treatment of the asymmetric top the D_2 group appears as an unusual symmetry[20] of its Hamiltonian, while in our formalism the D_2 arises naturally as a little group of the dynamical group.

5C. *A geometrical interpretation*

Once the physical meaning of K becomes clear, a geometrical interpretation of our procedure can be made straightforwardly.

For example, in analogy to (5·1), we may define the (mass) quadrupole moment operator in the rotating body-fixed axis by the following formula:

$$Q_\lambda^{(2)}(\text{body}) = \sum_\mu D_{\mu\lambda}^{(2)*}(\Omega)\cdot Q_\mu^{(2)}. \tag{5·7}$$

It can quickly be checked that every component of $Q^{(2)}(\text{body})$ commutes with each other when $[Q_\mu^{(2)}, Q_{\mu'}^{(2)}]=0$.

Now, consider the state—in the body-fixed axis—where all components of $Q^{(2)}(\text{body})$ are diagonal. From the hermiticity of $Q^{(2)}$, it is always possible to choose the body-fixed axis in such a way that the eigenvalues of $Q_{+1}^{(2)}(\text{body})$ and $Q_{-1}^{(2)}(\text{body})$ vanish. By comparing (4·7) to (5·7), one sees that in our parametrization (4·7) $q^{(0)}=\beta\cos\gamma$ and $q^{(2)}/\sqrt{2}=(\beta/\sqrt{2})\sin\gamma$ are just the eigenvalues of $Q_0^{(2)}(\text{body})$ and $Q_{\pm 2}^{(2)}(\text{body})$, respectively, in this body-fixed axis.

That is, the state in this body-fixed system may be written as

$$Q_\lambda^{(2)}(\text{body})\,|\boldsymbol{q},\alpha\rangle_{\text{body}} = q_\lambda(\text{body})\,|\boldsymbol{q},\alpha\rangle_{\text{body}},$$

where

$$q_0(\text{body})=\beta\cos\gamma,\quad q_{\pm1}(\text{body})=0 \quad\text{and}\quad q_{\pm2}(\text{body})=(\beta/\sqrt{2})\sin\gamma\,.$$

Expanding it into angular momentum basis in the body-fixed system and rotating back to the space-fixed one, we would obtain our final result (5·7). The D-function in (5·7) may be interpreted as arising from this rotation.

The above procedure is found actually to work well to get precisely the same result in the case of the symmetric top. For the general case of asymmetric top, however, we would have a different, although equivalent, formula, as can be expected from (4·26) and (5·6). Nevertheless, the intuitive geometrical interpretation will be helpful to compare our approach to the conventional treatment of the symmetric and asymmetric tops.

§ 6. Concluding remarks

Starting from the discussion on the concept of "rigidity" in quantum mechanics, we have developed a dynamical group approach to the quadratically deformed rigid rotator. By examining the possible range of all the quantum numbers, one sees that the irreducible representation of our dynamical group does actually contain every member of all the possible rotational bands of the rigid rotator once and only once. Since we have not yet introduce the Hamiltonian of the system, all the states are degenerated. In a next publication, we shall discuss how to introduce the Hamiltonian into the framework of our formalism.

The microscopic description of nuclear rotational states has been one of the most interesting problem in nuclear structure. If we prefer an algebraic approach to this problem, we must first seek for the set of operators which, operated repeatedly on the ground state, generate all members of the ground state rotational band. From the results of the previous sections, it is now clear that the rotation model of Bohr and Mottelson can be obtained directly if we could construct the set of operators which satisfies the commutation relations (2·2). The relation between the $SU(3)$ model of Elliott[7] and Bohr and Mottelson's model[18] is also clear from (2·2) and (2·3).

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