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# Quantum Mechanics in Riemannian Manifold

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The quantum mechanics on the hypersurface $V_{N-1}$ embedded in Euclidean space $R_N$ is examined. The quantization is done in accordance with Dirac method, and as a result some additional term proportional to $\hbar^2$ appears in the potential energy. This term has the information of exterior to the hypersurface; thus, the quantum mechanics in Riemann space without any kind of exterior world is different from quantum mechanics on hypersurface embedded in Euclidean space.



## § 1. Introduction

The quantum mechanics in curved space has been examined about the ordering problem for a long time. Many authors have considered this problem in the framework of the canonical quantization[1] and the path-integral method.[2] In the present paper, we introduce the curved space as a hypersurface (H.S.) in higher dimensional Euclidean space and consider a particle motion on H.S., which means that we consider the motion with constraint.

In classical mechanics there is no difference between the equation of motion on H.S. and that in curved space, while in quantum mechanics we show that the Hamiltonian $H=P_aP^a/2+V(x)$ with some constraints is different from $K+V(x(q))$ ($K$ is covariant kinetic operator on H.S.) and there appears an additional term which is not expressed only in terms of quantities on H.S. Due to $P_aP^a/2$ term in $H$ and the structure of the commutator $[P_a, P_b]$, we have substantially no ordering problem. These considerations teach us that the two approaches of treating curved space are essentially different quantum-mechanically. After giving the general formulation, the meaning of the additional term is discussed.

## § 2. Classical mechanics on hypersurface

Consider $N$ dimensional Euclidean space; $\boldsymbol{R}_N$, a point which is specified by a set of Cartesian coordinates; $\{x_1, x_2, \cdots, x_a, \cdots, x_N\}$. Further consider in $\boldsymbol{R}_N$ the $N-1$-dimensional H.S., $V_{N-1}$ subject to the equation $f(x_1, x_2, \cdots, x_N)=0$. Let us examine the motion of a particle on this H.S. with potential $V(x)$. Then we write Lagrangian for this particle as

$$L=\dot{x}^a\dot{x}_a/2-V(x)+\lambda f(x)\,; \qquad (2\cdot1)$$

here, the metric is $\delta_{ab}$; $\lambda$ is a variable which is independent of $x^a$ ($a$; $1, 2, \cdots, N$); the dot denotes the time derivative. The canonical momenta conjugate to $x^a$ and $\lambda$ are

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$$P_a = \partial L/\partial \dot{x}^a = \dot{x}^a\,, \tag{2·2a}$$

$$P_\lambda = \partial L/\partial \dot{\lambda} \approx 0\,. \tag{2·2b}$$

The second equation means the primary constraint. Now, we will treat this constraint system in accordance with Dirac method. This is done in Appendix A, and as the result we have

Hamiltonian: $H = P_a P^a/2 + V(x)\,,$ (2·3)

Brackets: $\{x^a, P_b\}_D = \delta^a{}_b - \boldsymbol{n}^a \boldsymbol{n}_b\,,$ (2·4a)

$\{x^a, x^b\}_D = 0\,,$ (2·4b)

$\{P_a, P_b\}_D = P^c(\boldsymbol{n}_b \partial_c \boldsymbol{n}_a - \boldsymbol{n}_a \partial_c \boldsymbol{n}_b)\,,$ (2·4c)

Constraints: $f(x) = 0\,,$ (2·5a)

$P^a \partial_a f(x) = 0\,.$ (2·5b)

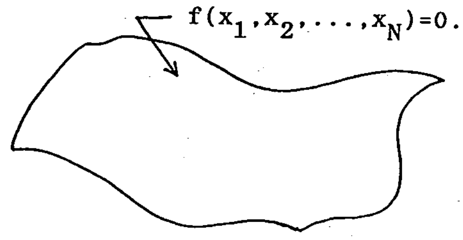


Fig. 1.

Here, $\boldsymbol{n}^a = \boldsymbol{n}_a$ is the unit vector normal to H.S., which is defined as

$$\boldsymbol{n}^a = \boldsymbol{n}_a = \frac{\partial f(x)/\partial x^a}{[\partial_b f(x) \cdot \partial^b f(x)]^{1/2}}\,. \tag{2·6}$$

Then we obtain equation of motion from the Hamilton equation

$$\ddot{x}^a + (\boldsymbol{n}_a \partial_c \boldsymbol{n}_b)\dot{x}^b \dot{x}^c = (\boldsymbol{n}^a \boldsymbol{n}^d - \delta^{ad}) \cdot \partial V(x)/\partial x^d \tag{2·7}$$

with constraints (2·5). This equation is identical with Euler-Lagrange equation derived from (2·1), which is given in Appendix A.

Next we consider the general coordinate transformation in $N$-dimensional Euclidean space.

$$\begin{array}{ll} x^a & \longrightarrow \quad q^\mu \\ a{:}1 \sim N & \mu{:}0 \sim N-1 \end{array} \tag{2·8}$$

Taking such a special coordinate frame that the $q^i$'s are the coordinates on H.S. and $q^0$ is the coordinate normal to H.S., we have the relations

$$f(x) = 0 \longleftrightarrow q^0 = \text{constant}\,,$$

$$\dot{x}^a \partial_a f(x) = 0 \longleftrightarrow \dot{x}^a \partial_a q^0 = \dot{q}^0 = 0\,.$$

Here we take the constant $q^0$ to be equal to zero without any loss of generality. Now, instead of (2·5) we have the constraints to the particle motion on H.S. as

$$q^0 = 0\,, \quad \dot{q}^0 = 0\,. \tag{2·9}$$

$dq^i$ is now a tangential vector on H.S., and $dq^0$ is the normal vector. The metric for

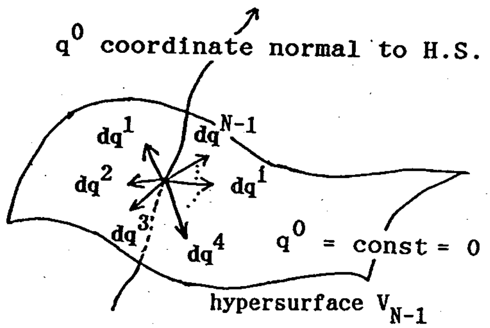


Fig. 2.

$dq^\mu$ is generally given by

$$ds^2=\sum_a dx^a dx^a = dx^a dx_a = g_{\mu\nu} dq^\mu dq^\nu \,, \tag{2·10}$$

where the metric is

$$[g_{\mu\nu}]=\left[\begin{array}{c|c} g_{00} & 0 \\ \hline 0 & g_{ij} \end{array}\right] ; \tag{2·11}$$

$g_{i0}=g_{0i}=0$ means that $dq^0$ is normal to H.S.

The inverse of $g_{\mu\nu}$ can be written as

$$[g^{\mu\nu}]=\left[\begin{array}{c|c} g^{00} & 0 \\ \hline 0 & g^{ij} \end{array}\right] . \tag{2·12}$$

Here,

$$g^{00}=1/g_{00} \quad \text{and} \quad g^{ij}\cdot g_{jk}=\delta^i{}_k \,. \tag{2·13}$$

Under a general coordinate transformation, $\boldsymbol{n}^a$ is transformed into $\boldsymbol{n}^\mu$ as,

$$\boldsymbol{n}^\mu = \partial q^\mu/\partial x^a \cdot \boldsymbol{n}^a \,, \tag{2·14a}$$

$$\boldsymbol{n}_\mu = g_{\mu\nu}\cdot \boldsymbol{n}^\nu = \partial x^a/\partial q^\mu \cdot \boldsymbol{n}_a \,. \tag{2·14b}$$

In our coordinate frame, we obtain from (2·6)

$$\boldsymbol{n}^\mu = \delta^\mu{}_0 (g^{00})^{1/2} \,, \tag{2·15a}$$

$$\boldsymbol{n}_\mu = \delta^0{}_\mu (g_{00})^{1/2} \,. \tag{2·15b}$$

When the present transformation is performed to Eqs. (2·3)~(2·7), we obtain the equations written in terms of the $q^\mu$'s as follows:

$$\text{Eqs. of motion:} \quad \ddot{q}^i + \Gamma^i_{jk}\dot{q}^j\dot{q}^k + g^{ik}\cdot\partial v(q)/\partial q^k = 0 \,, \tag{2·16a}$$

$$\ddot{q}^0 = 0 \,, \tag{2·16b}$$

$$\text{Constraints:} \quad q^0 = \dot{q}^0 = 0 \,. \tag{2·16c}$$

Here we have used the definition $V(x)=V(x(q))=v(q)$.

The above equations mean that there is no motion in the normal direction to H.S. and the equation of motion is quite the same as Euler-Lagrange equation obtained from Lagrangian $L=1/2\cdot g_{ij}\dot{q}^i\dot{q}^j - v(q)$. This implies that classically we have the same equation of motion in a curved space as well as on a H.S. In terms of the $q$-variables, the set of relations in Hamilton formalism becomes as follows:

Hamiltonian: $\boldsymbol{H}=1/2\cdot g^{ij}(q)p_ip_j+v(q)\,,$ (2·17)

Brackets: $\{q^i, p_j\}_D=\delta^i{}_j\,,$ (2·18a)

$\{q^i, q^j\}_D=0\,,$ (2·18b)

$\{p_i, p_j\}_D=0\,,$ (2·18c)

Constraints: $q^0=p_0=0\,;$ (2·19)

here we have used the transformation for $p_\mu$ from $P_a$ as

$$p_\mu=\partial x^a/\partial q^\mu\cdot P_a\,,$$

and the property of Dirac bracket,

$$\{F(x), A\}_D=\partial_a F(x)\cdot\{x^a, A\}_D\,.$$

As we have shown in (2·19), in classical mechanics we could completely eliminate the degree of freedom in the direction normal to H.S. While, in quantum theory, it is not a trivial problem because of existence of the uncertainty principle. Naively speaking, if we take $\Delta q^0$ tend to 0, $\Delta p_0$ goes to infinity, which means Hamiltonian involves a divergent term; furthermore, the surface $q^0=0$ is not a plane generally, which makes the consideration more complicated. We will discuss this problem in the next section.

## § 3. Quantum mechanical treatment

Our classical system under consideration is defined by the set of equations; (2·3) ~(2·5). Now we proceed to consider the quantization problem through the replacement as usual, that is,

$$\{A, B\}_P\longrightarrow(i\hbar)^{-1}\cdot[A, B]\,.$$

But now, instead of Poisson bracket we have the Dirac bracket, which satisfies the Jacobi Identity and the same algebraic relations as Poisson bracket. So, we take the quantization rule, according to Dirac's prescription, as

$$\{A, B\}_D\longrightarrow(i\hbar)^{-1}\cdot[A, B]\,. \tag{3·1}$$

Then we have the ordering problem on the right-hand side of (2·4c), and the left-hand side of (2·5b). By requiring hermiticity, we symmetrize them as

$$\boldsymbol{P}_a\boldsymbol{n}^a=0\longrightarrow\{\boldsymbol{P}_a, \boldsymbol{n}^a\}=0\,, \tag{3·2a}$$

$$\boldsymbol{P}^c(\boldsymbol{n}_b\partial_c\boldsymbol{n}_a-\boldsymbol{n}_a\partial_c\boldsymbol{n}_b)\longrightarrow\{\boldsymbol{P}^c, (\boldsymbol{n}_b\partial_c\boldsymbol{n}_a-\boldsymbol{n}_a\partial_c\boldsymbol{n}_b)\}\,, \tag{3·2b}$$

where

$$\{A, B\}=1/2\cdot(AB+BA)\,.$$

These are the most important hypotheses when we consider the ordering problem. It should be stressed, however, that the symmetrized form (3·2b) is almost unique since this is rewritten in another symmetrized form, e.g.,

$$\{\boldsymbol{n}_b, \{\boldsymbol{P}^c, \partial_c \boldsymbol{n}_a\}\} - \{\boldsymbol{n}_a, \{\boldsymbol{P}^c, \partial_c \boldsymbol{n}_b\}\}$$

in so far as $[\boldsymbol{P}^c, \partial_c \boldsymbol{n}_a]$ is a function of the $x^a$'s. Then instead of (2·3)~(2·5) we obtain the following set of quantum-mechanical relations in Hamilton formalism:

$$\text{Hamiltonian:} \quad H = 1/2 \cdot P^a P_a + V(x)\,, \tag{3·3}$$

$$\text{Commutators:} \quad [x^a, P_b] = i\hbar(\delta^a{}_b - \boldsymbol{n}^a \boldsymbol{n}_b)\,, \tag{3·4}$$

$$[x^a, x^b] = 0\,, \tag{3·5}$$

$$[P_a, P_b] = i\hbar\{P^c, (\boldsymbol{n}_b \partial_c \boldsymbol{n}_a - \boldsymbol{n}_a \partial_c \boldsymbol{n}_b)\}\,, \tag{3·6}$$

$$\text{Constraints:} \quad f(x) = 0\,, \tag{3·7}$$

$$\{P_a, \boldsymbol{n}^a\} = 0\,. \tag{3·8}$$

Equation (3·8) is the quantum mechanical expression representing the momentum in the normal direction vanishes. Let us consider this problem on the uncertainty principle. By multiplying $\boldsymbol{n}^b$ on the both sides of (3·4), we obtain $[x^a, \{P_b, \boldsymbol{n}^b\}] = 0$, which means the momentum in the normal direction commutes with all $x^a$'s; therefore there works no uncertainty principle in this direction. This is owing to the structure of the Dirac bracket (3·4).

Now we will transform $\{x^a\}$ into the general coordinates $\{q^\mu\}$ as we have done in (2·8). Here we use the following hypothesis and definition:

$$df(x)/dt = \{dx^a/dt, \partial f(x)/\partial x^a\}\,, \tag{3·9}$$

and the momentum in the general coordinate

$$p_\mu = \{e^a{}_\mu, P_a\}\,, \tag{3·10}$$

where $e^a{}_\mu = \partial x^a / \partial q^\mu$ is the vielbein. By using $P_a = \dot{x}^a$, the constraints $f(x) = 0$ and $\{P^a, \boldsymbol{n}_a\} = 0$ lead to (2·9) again, i.e.,

$$q^0 = 0\,, \qquad \dot{q}^0 = 0\,; \tag{3·11}$$

here we have used the definition of $\boldsymbol{n}^a$(i.e., (2·6)), (3·4) and (3·9). Hamiltonian and commutation relations for $p_\mu$, $q^\nu$ are obtained from (3·3)~(3·6) by using (3·9) and (3·10). This is done in Appendix B, the results of which are

$$H = 1/2 \cdot g^{-1/4} p_i g^{1/2} g^{ij} p_j g^{-1/4} + v(q) + \hbar^2/8 \cdot g^{00} \Gamma^i_{i0} \Gamma^j_{j0}\,, \tag{3·12}$$

$$[q^i, p_j] = i\hbar\delta^i{}_j\,, \quad [q^i, q^j] = [p_i, p_j] = 0\,, \tag{3·13}$$

$$q^0 = p_0 = 0\,, \tag{3·14}$$

where $g$ is the determinant of $g_{ij}$ which is the metric on H.S. The first term of Hamiltonian is the kinetic operator which is form invariant under general coordinate transformation on the curved manifold and corresponds to Laplace Beltrami operator. The last term of Hamiltonian is including the index "0", which means that this term appears only when $V_{N-1}$ is treated as a hypersurface embedded in $R_N$. This term is rewritten in the form as

$$\Delta V=\hbar^2/8\cdot g^{00}(1/2\cdot\partial\ln g/\partial q^0)^2\,. \tag{3·15}$$

Here we must note that we cannot take $\Delta V=0$ by using constraint (3·14). Because, constraint (3·14) should be only used when we construct the equation of motion from the Hamiltonian, but not in constructing the Hamiltonian itself. Then the meaning of Eq. (3·15) is to take $q^0=0$ after the derivative of $q^0$. The interpretation of $\Delta V$ will be discussed in the next section.

## § 4. Interpretation of the additional term

The additional term in Hamiltonian is expressed in terms of the quantities such as $g^{00}$ and $\partial_0 g_{ij}$ which do not exist when we consider the curved space from the beginning. Through the following consideration, we can recognize this term cannot be written in terms of quantities on $V_{N-1}$. The only equation which requires the relations between $\{g^{00}, \partial_0 g_{ij}\}$ and $\{g_{ij}, \partial_k g_{ij}, \partial_k\partial_l g_{ij}\}$ is the vanishing of the total curvature tensor, i.e.,

$$R^{\alpha}{}_{\mu\nu\lambda}=0\,, \tag{4·1}$$

which implies the physical system lies totally in Euclidean space. By using the symmetrical property, Eq. (4·1) is decomposed into the three-type equations:

$$R^{m}{}_{ijk}=0\,, \tag{4·2a}$$

$$R^{0}{}_{ijk}=0\,, \tag{4·2b}$$

$$R^{0}{}_{i0j}=0\,. \tag{4·2c}$$

These are written by using the new type tensor $b_{ij}$ in the following forms:

$$\tilde{R}_{mijk}=b_{ij}b_{mk}-b_{ik}b_{jm}\,, \tag{4·3}$$

$$\nabla_k b_{ij}-\nabla_j b_{ik}=0\,, \tag{4·4}$$

$$\partial_0 b_{ij}+(g_{00})^{1/2}g^{km}b_{ik}b_{jm}=\nabla_j[\partial_i(g_{00})^{1/2}]\,, \tag{4·5}$$

where the definition of $b_{ij}$ is

$$b_{ij}=-1/2\cdot(g^{00})^{1/2}\cdot\partial_0 g_{ij}\,; \tag{4·6}$$

$\nabla_j$ is the covariant derivative on the H.S. such as

$$\nabla_j A_i=\partial_j A_i-\Gamma^{m}{}_{ji}A_m\,.$$

$\tilde{R}_{mijk}$ is the curvature tensor on the H.S. which is of course expressed by using only $\{g_{jk}, \partial_i g_{jk}, \partial_i\partial_m g_{jk}\}$. $b_{jk}$ is a tensor only on the H.S. and is not expressed by using only the quantities on the H.S. such as $\{g_{jk}, \partial_i g_{jk}, \partial_i\partial_m g_{jk}\}$ since our tools are only (4·3) and (4·4).[3),4)]

It may be worthwhile to note that our $\Delta V$ is rewritten as

$$\Delta V=\hbar^2/8\cdot(b_{ij}g^{ij})^2\,. \tag{4·7}$$

By utilizing (4·3), $\Delta V$ is reexpressed as

$$\Delta V=\hbar^2/8\cdot[g^{mj}g^{ik}b_{ij}b_{mk}]-\hbar^2/8\cdot\tilde{R} \tag{4·8}$$

with

$$\tilde{R}=g^{mj}g^{ik}b_{ij}b_{mk}-(b_{ij}g^{ij})^2\,.$$

The existence of $\Delta V$ which is written in terms of $b_{ij}$ shows the difference between the ordinary curved space and the H.S. embedded in Euclidean space. Because $b_{ij}$ includes informations about space structures exterior to our curved space (H.S.), the above statement means $b_{ij}$ appears only when we consider the embedded curved space. Let us consider the meaning of $\Delta V$. From Eq. (3·12) we obtain

$$\Delta V=\hbar^2/8\cdot[(g^{00})^{1/2}\Gamma^i_{i0}]^2=1/8\cdot\|[P_a,\boldsymbol{n}^a]\|^2\,. \tag{4·9}$$

The last equality is seen to hold by noting

$$\begin{aligned}[P_a,\boldsymbol{n}^a]&=[P_a,x^d]\cdot\partial_d\boldsymbol{n}^a=-i\hbar(\delta_a{}^d-\boldsymbol{n}_a\boldsymbol{n}^d)\cdot\partial_d\boldsymbol{n}^a\\&=-i\hbar e^\mu{}_a\cdot\partial(e^a{}_\nu\boldsymbol{n}^\nu)/\partial q^\mu=-i\hbar(g^{00})^{1/2}\Gamma^i_{i0}\,.\end{aligned} \tag{4·10}$$

Further by using

$$[P_a,\boldsymbol{n}^a]=[\{e^i{}_a,p_i\},\boldsymbol{n}^a]=e^i{}_a[p_i,\boldsymbol{n}^a]\,, \tag{4·11}$$

$\Delta V$ is rewritten as

$$\begin{aligned}\Delta V&=1/8\cdot|e^i{}_a\cdot[p_i,\boldsymbol{n}^a]|^2\\&=\hbar^2/8\cdot\{e^i{}_a\cdot\partial\boldsymbol{n}^a/\partial q^i\}^2\,.\end{aligned} \tag{4·12}$$

The last equality can be also obtained from (4·7) by utilizing Weingarten equation.[4] $p_i$ is the momentum which is tangential to H.S., and $\boldsymbol{n}^a$ is the normal unit vector satisfying $\boldsymbol{n}^a\cdot e^i{}_a=0$. When the H.S. is curved, $\delta\boldsymbol{n}^a$ exists for $\delta q^i\neq 0$, and $\delta\boldsymbol{n}^a\cdot e^i{}_a\equiv(\delta\boldsymbol{n})^i$ does not vanish; in other words, particle fluctuation on H.S. causes a fluctuation of the normal unit vector $\boldsymbol{n}^a$, leading to $\Delta V$.

## § 5. Conclusion

We have investigated the quantum mechanical treatment of a particle motion in a curved space $V_{N-1}$ embedded in a Euclidean space $R_N$, and have pointed out that in the Hamiltonian there appears $\Delta V$ in addition to the covariant kinetic term on H.S. $V_{N-1}$. This $\Delta V$ is caused by the geometrical structure of the embedding, and may affect the particle motion on H.S. $V_{N-1}$. The same $\Delta V$ is also obtained without employing explicitly the coordinate $q^0$, which is explained in another paper.[5] It is worthwhile to remark that we can also derive the commutation relations among the $x^a$'s and the $P_a$'s (3·4)~(3·6), and the same Hamiltonian along the quantization procedure of nonlinear theory given in Ref. 1), which is explained shortly as follows: Employing $\dot{x}^a=\{\partial x^a/\partial q^j,\dot{q}^j\}$ and Ansatz: $[q^j,\dot{q}^k]=i\hbar f^{jk}(q)$, we have

$$L=1/2\cdot\dot{x}^a\dot{x}_a-V(x)=1/2\cdot\dot{q}^ig_{ij}(q)\dot{q}^j+\text{function of }(q)\,,$$

$$g_{jk}\equiv\sum_a e^a{}_je^a{}_k\,,\quad e^a{}_j(q)\equiv\partial x^a/\partial q^j\quad\text{with}\quad e^j{}_a\equiv g^{jk}\cdot e^a{}_k\,.$$

Thus the momentum conjugate to $q^j$ is given by $p_j=\{g_{ij}, \dot{q}^k\}$. Then require the canonical commutation relations:

$$[p_j, q^k]=-i\hbar\delta^k{}_j\,, \quad \text{others}=0\,.$$

The momentum $P_a=\dot{x}_a$ is given by $P_a=\{e^a{}_j, \{p_k, g^{kj}\}\}=\{e^j{}_a, p_j\}$. Using the canonical commutation relations among the $q^j$'s and the $p_i$'s, and introducing the coordinate $q^0$ as $q^0=f(x)=0$, with $\dot{q}^0=\{\partial q^0/\partial x^a, P_a\}=0$, we obtain the desired relations and also the same Hamiltonian.

The total kinetic energy $\dot{x}^a\dot{x}_a/2$ can be divided into three parts such as the kinetic energy on H.S., the kinetic energy on a normal direction and the quantum mechanical fluctuation parts. Through the embedding procedure we can eliminate the second one as well as the quantum fluctuation along the normal direction, but not the quantum fluctuation of the normal unit vector itself. Thus, $\Delta V$ is interpreted as the quantum fluctuation energy of the normal unit vector, as noted in the last paragraph in § 4. Though we can give a physical interpretation of $\Delta V$ as mentioned above, the existence of $\Delta V$ makes us somewhat confused. This is because, due to $\Delta V$, the motion of a particle on a hypersurface depends on the geometrical structure of the space exterior to that hypersurface. Thus, it might possibly be said that our embedding procedure is not perfect. If it is so, we must improve the constraint dynamics in a quantum mechanical version to make such a term vanish. On the other hand, it may be possible to take a viewpoint that our procedure of quantization gives a way of choosing a quantum system among possible ones which correspond to the same classical dynamical system. Then, we have to check the physical effect of $\Delta V$ in some real physical model. It is a future task to clarify these problems mentioned above.

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## Appendix A

We consider the problem with

Hamiltonian: $$H=1/2\cdot P_aP^a+V(x)-\lambda f(x)\,, \tag{A·1}$$

Primary constraint: $$P_\lambda\approx 0\,. \tag{A·2}$$

To obtain all the constraints, our new hamiltonian is

$$H_T=1/2\cdot P_aP^a+V(x)-\lambda f(x)+uP_\lambda\,; \tag{A·3}$$

here, $u$ is a multiplier. Consistency conditions are

$$\dot{P}_\lambda=\{P_\lambda, H_T\}_P=f(x)\approx 0\,,$$
$$\ddot{P}_\lambda=\{f(x), H_T\}_P=P^a\cdot\partial f(x)/\partial x^a\approx 0\,,$$
$$P_\lambda^{(3)}=\{P^a\cdot\partial f/\partial x^a, H_T\}_P$$
$$\doteq P^aP^b\partial_a\partial_b f(x)-(\partial_b V(x)-\lambda\partial_b f(x))\cdot\partial^b f(x)\approx 0\,.$$

$P_\lambda^{(4)}$ determines $u$, and the procedure is over. All the constraints are as follows:

$$\Phi_1=P_\lambda\approx 0\,, \tag{A·4a}$$
$$\Phi_2=f(x)\approx 0\,, \tag{A·4b}$$
$$\Phi_3=Df(x)\approx 0\,, \tag{A·4c}$$
$$\Phi_4=D^2f(x)-\partial_b\Phi\cdot\partial^b f(x)\approx 0\,; \tag{A·4d}$$

here, $D=P^a\partial_a$, $\Phi=V(x)-\lambda f(x)$.
These constraints are all belonging to the second class and we can construct a Dirac bracket as

$$\{A, B\}_D=\{A, B\}_P-\{A, \Phi_k\}_P\Delta_{kl}^{-1}\{\Phi_l, B\}_P\,;$$

here, $\Delta_{kl}$ is defined as

$$\Delta_{kl}=\{\Phi_k, \Phi_l\}_P\,.$$

Then we obtain the matrices $\Delta_{kl}$ and $\Delta_{kl}^{-1}$ as

$$(\Delta_{kl})=\begin{pmatrix}0 & 0 & 0 & -\alpha\\ 0 & 0 & \alpha & -\beta\\ 0 & -\alpha & 0 & -\gamma\\ \alpha & \beta & \gamma & 0\end{pmatrix},\qquad (\Delta_{kl}^{-1})=\begin{pmatrix}0 & -\gamma/\alpha^2 & \beta/\alpha^2 & 1/\alpha\\ \gamma/\alpha^2 & 0 & -1/\alpha & 0\\ -\beta/\alpha^2 & 1/\alpha & 0 & 0\\ -1/\alpha & 0 & 0 & 0\end{pmatrix};$$

here, $\alpha=\partial_a f\cdot\partial^a f$; $\beta=-P^a\partial_a\alpha$; $\gamma$ is the complicated function of $f$ and $\Phi$, which does not appear in the final result, so we do not write here its explicit form. In the Dirac bracket, constraint is automatically satisfied; therefore we do not have to calculate all the Dirac brackets. Nontrivial brackets are calculated by using the above matrices, the results of which are

$$\{x^a, P_b\}_D=\delta^a{}_b-\boldsymbol{n}^a\boldsymbol{n}_b\,, \tag{A·5a}$$
$$\{x^a, x^b\}_D=0\,, \tag{A·5b}$$
$$\{P_a, P_b\}_D=\boldsymbol{n}_bD\boldsymbol{n}_a-\boldsymbol{n}_aD\boldsymbol{n}_b=P^c(\boldsymbol{n}_b\partial_c\boldsymbol{n}_a-\boldsymbol{n}_a\partial_c\boldsymbol{n}_b)\,, \tag{A·5c}$$
$$\{x^a, \lambda\}_D=2(\nabla f)^{-2}(\boldsymbol{n}_a\boldsymbol{n}^b-\delta_a{}^b)D\partial_b f\,, \tag{A·6a}$$
$$\{P_a, \lambda\}_D=\text{some function of }\{f, \nabla f, \Phi\}\,. \tag{A·6b}$$

Other brackets vanish. Now our equation of motion is defined with Hamiltonian (A·3), the constraints (A·4) and the brackets (A·5) and (A·6). Then, the constraints (A·4) are treated as strong equations, and our Hamiltonian becomes

$$H=1/2\cdot P^aP_a+V(x)\,,$$

which does not include variable $\lambda$. The necessary constraints are only two:

$$f(x)=0\,, \qquad P_a\cdot \boldsymbol{n}^a=0\,,$$

and necessary brackets are (A·5) only. These are the complete set of Hamilton formalism. Hamilton equation gives

$$\dot{x}^a=\{x^a, H\}_D=P^b(\delta^a{}_b-\boldsymbol{n}^a\boldsymbol{n}_b)\,,$$

$$\dot{P}_a=\{P_a, H\}_D=P^bP^c(\boldsymbol{n}_b\partial_c\boldsymbol{n}_a-\boldsymbol{n}_a\partial_c\boldsymbol{n}_b)+(\boldsymbol{n}_a\boldsymbol{n}^d-\delta^d{}_a)\partial_d V(x)$$

with the constraint $P_a\boldsymbol{n}^a=0$; these equations lead to

$$\ddot{x}^a+(\boldsymbol{n}^a\partial_c\boldsymbol{n}_b)\dot{x}^b\dot{x}^c=(\boldsymbol{n}^a\boldsymbol{n}_d-\delta^a{}_d)\cdot\partial V(x)/\partial x^d\,, \tag{A·7a}$$

$$f(x)=0\,, \quad \dot{x}^a\cdot\boldsymbol{n}_a=0\,. \tag{A·7b}$$

and are also obtained from Euler-Lagrange equation as follows:
For

$$L=1/2\cdot\dot{x}^a\dot{x}_a-V(x)+\lambda f(x)\,,$$

$$\delta_X[L]=0 \quad \text{gives} \quad \ddot{x}^a=-\partial V(x)/\partial x^a+\lambda\partial f(x)/\partial x^a\,; \tag{A·8a}$$

$$\delta_\lambda[L]=0 \quad \text{gives} \quad f(x)=0\,. \tag{A·8b}$$

The second equation should be satisfied at all times. Then we have $\dot{f}(x)=0$, i.e.,

$$\partial f(x)/\partial x^a\cdot\dot{x}^a=0\,. \tag{A·9}$$

Multiply $\dot{x}^a$ on both sides of (A·8a) and, using (A·9), we have

$$d(1/2\cdot\dot{x}^a\dot{x}_a+V(x))/dt=0\,.$$

Thus, we obtain the total conserved energy as

$$E=1/2\cdot\dot{x}^a\dot{x}_a+V(x)\,. \tag{A·10}$$

The second time derivative of $f(x)=0$ gives

$$\partial_a\partial_b f(x)\cdot\dot{x}^a\dot{x}^b+\partial_a f(x)\cdot\ddot{x}^a=0\,. \tag{A·11}$$

Introducing (A·8a) to (A·11), we obtain the equation determining $\lambda$, which coincides with $\Phi_4=0$. Then $\ddot{f}(x)=0$ is trivial, and it is not necessary to take time derivative any more. Introducing the definition of $\lambda$ into (A·8a) and by using (A·9), we obtain (A·7a) again.

## Appendix B

We start from Euclidean space

$$\text{Hamiltonian:} \quad H=1/2\cdot P^aP_a+V(x)\,, \tag{B·1}$$

$$\text{Constraints:} \quad f(x)=0\,, \tag{B·2a}$$

$$\{P_a, \boldsymbol{n}^a\}=0\,, \tag{B·2b}$$

Brackets: $[x^a, P_b]=i\hbar(\delta^a{}_b-\boldsymbol{n}^a\boldsymbol{n}_b)$, (B·3a)

$$[x^a, x^b]=0\,, \tag{B·3b}$$

$$[P_a, P_b]=i\hbar\{P^c, (\boldsymbol{n}_b\partial_c\boldsymbol{n}_a-\boldsymbol{n}_a\partial_c\boldsymbol{n}_b)\}\,, \tag{B·3c}$$

and consider the general coordinate transformation

$$\begin{array}{ll} x^a \longrightarrow & q^\mu \\ a{:}1\sim N & \mu{:}0\sim N-1 \end{array} \tag{B·4}$$

Suppose $df(x)/dt=\{\dot{x}^a, \partial_a f(x)\}$. (B·5)

Define $p_\mu=\{e^a{}_\mu, P_a\}$, (B·6)

which is consistent with $P_a=\{e^\mu{}_a, p_\mu\}$. Here $e^a{}_\mu$ and $e^\mu{}_a$ are vielbeins. Then we obtain from (B·3b)

$$[q^\mu, q^\nu]=0\,, \tag{B·7}$$

and $[x^a, P_b]$ can be rewritten as

$$[x^a, P_b]=[x^a(q), \{e^\mu{}_b, p_\mu\}]=\{e^\mu{}_b e^a{}_\nu, [q^\nu, p_\mu]\},$$

which should be equal to $i\hbar e^\mu{}_b e^a{}_\nu\{\delta^\nu{}_\mu-\boldsymbol{n}_\mu\boldsymbol{n}^\nu\}$ by (B·3a). Then we obtain

$$[q^\nu, p_\mu]=i\hbar(\delta^\nu{}_\mu-\boldsymbol{n}^\nu\boldsymbol{n}_\mu)\,. \tag{B·8}$$

$[p_\mu, p_\nu]$ can be calculated by using the relations (B·3) and

$$[p_\mu, p_\nu]=[\{e^a{}_\mu, P_a\}, \{e^b{}_\nu, P_b\}]\,. \tag{B·9}$$

We obtain after the long calculations

$$[p_i, p_j]=-\hbar^2/2\cdot(\partial_i\Gamma^l_{lj}-\partial_j\Gamma^l_{il})=0\,. \tag{B·10}$$

Here $p_i=p_{\mu=i}$. While, $p_{\mu=0}$ should be equal to zero due to the constraint, because

$$p_{\mu=0}=\{e^a{}_{\mu=0}, P_a\}=\{(g_{00})^{1/2}\cdot\boldsymbol{n}^a, P_a\}=(g_{00})^{1/2}\cdot\{\boldsymbol{n}^a, P_a\}=0\,.$$

This is consistent with commutation relations

$$[p_0, p_i]=[p_0, p_0]=0\,, \tag{B·11}$$

which are directly obtained from (B·9).

Hamiltonian can be rewritten by using (B·6); then we have

$$H=1/2\cdot P_aP^a+V(x)=1/2\cdot\{e^i{}_a, p_i\}\cdot\{e^j{}_b, p_j\}+v(q)$$

$$=1/2\cdot g^{-1/4}p_ig^{1/2}g^{ij}p_jg^{-1/4}+v(q)+\hbar^2/8\cdot g^{00}\Gamma^i_{i0}\Gamma^j_{j0}\,. \tag{B·12}$$

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