

Quantum near-horizon geometry of a black 0-brane

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 We investigate a bunch of D0-branes to reveal their quantum nature from the gravity side. In the classical limit, it is well described by a non-extremal black 0-brane in type IIA supergravity. The solution is uplifted to the eleven dimensions and expressed by a non-extremal M-wave solution. After reviewing the effective action for the M-theory, we explicitly solve the equations of motion for the near-horizon geometry of the M-wave. As a result, we derive a unique solution that includes the effect of the quantum gravity. The thermodynamic properties of the quantum near-horizon geometry of the black 0-brane are also studied by using Wald's entropy formula. Combining our result with that of the Monte Carlo simulation of the dual thermal gauge theory, we find strong evidence for the gauge/gravity duality in the D0-brane system at the level of quantum gravity.

Subject Index B22

1. Introduction

Superstring theory is a promising candidate for the theory of quantum gravity, and it plays an important role in revealing the quantum nature of black holes. The fundamental objects in superstring theory are D-branes as well as strings [1], and in the low-energy limit their dynamics are governed by supergravity. D-branes are described by classical solutions in supergravity, called black branes [2,3]. A special class of these has an event horizon like black holes and its entropy can be evaluated by the area law. Interestingly, the entropy can be statistically explained by counting the number of microstates in the gauge theory on the D-branes [4]. This motivates us to study the black hole thermodynamics from the gauge theory. Furthermore, it is conjectured that the near-horizon geometry of the black brane corresponds to the gauge theory on the D-branes [5]. If this gauge/gravity duality is correct, the strong coupling limit of the gauge theory can be analyzed by supergravity [6,7].

In this paper we consider a bunch of D0-branes in type IIA superstring theory. In the low-energy limit, a bunch of D0-branes with additional internal energy are well described by a non-extremal black 0-brane solution in type IIA supergravity [2,3]. After taking the near-horizon limit, the metric achieves an anti-de Sitter (AdS) black hole-like geometry in 10D space-time [8]. From the gauge/gravity duality, this geometry corresponds to the strong coupling limit of the gauge theory on the D0-branes [8], which is described by $(1+0)$ -dimensional $U(N)$ super Yang–Mills theory [9]. This gauge theory has attracted a great deal of attention as a nonperturbative definition of M-theory [10,11], which is the strong coupling description of the type IIA superstring theory [12,13]. Recently, the nonperturbative aspects of the gauge theory have studied by computer simulation

[14–24] (see Refs. [25,26] for reviews including other topics). In particular, in Ref. [19], the physical quantities of the thermal gauge theory, such as the internal energy, are evaluated numerically, and a direct test of the gauge/gravity duality is performed including the α' correction to the type IIA supergravity. Furthermore, if the internal energy of the black 0-brane can be evaluated precisely from the gravity side including the g_s correction, it is possible to give a direct test for the gauge/gravity duality at the level of quantum gravity [24] ($\alpha' = \ell_s^2$ is the string length squared and g_s is the string coupling constant).

The purpose of this paper is to derive a quantum correction for the near-horizon geometry of the non-extremal black 0-brane directly from the gravity side. In order to do this, we need to know an effective action that includes a quantum correction to the type IIA supergravity. In principle the effective action can be constructed so as to be consistent with the scattering amplitudes in the type IIA superstring theory [27], and it is expressed by double expansion of α' and g_s . For example, since the four-point amplitudes of gravitons at the tree and one-loop levels are nontrivial, there should exist terms like $\alpha'^3 e^{-2\phi} t_8 t_8 R^4$ and $\alpha'^3 g_s^2 t_8 t_8 R^4$ in the effective action, respectively [27–36]. These are called higher derivative terms; t_8 represents the products of four Kronecker's deltas with eight indices. In particular, we are interested in the latter terms, which give nontrivial g_s corrections to the geometry. These higher derivative terms often play important roles in counting the entropy of extremal black holes [37,38].

It is necessary that the effective action of the type IIA superstring should possess local supersymmetry in ten dimensions. Therefore, the supersymmetrization of $\alpha'^3 g_s^2 t_8 t_8 R^4$ is very important [28–30,33,35,36] to understand the structure of the effective action. Although the task is not yet complete, since our interest is in the geometry of the black 0-brane, it is enough to know the terms that contain the metric, dilaton field, and R – R 1-form field only. Notice that these fields are collected into the metric in 11D supergravity [39], and the black 0-brane is expressed by an M-wave solution. Then $\alpha'^3 g_s^2 t_8 t_8 R^4$ and other terms, which include the dilaton and R – R 1-form field, are simply collected into $\ell_p^6 t_8 t_8 R^4$ terms in eleven dimensions. Here $\ell_p = \ell_s g_s^{1/3}$ is the Planck length in eleven dimensions. Thus we consider the effective action for the M-theory and investigate quantum corrections to the near-horizon geometry of the non-extremal M-wave. We show equations of motion for the effective action and explicitly solve them up to the order of g_s^2 . The M-wave geometry receives the quantum corrections, and the thermodynamic quantities for the M-wave are modified. In particular, the internal energy of the M-wave, including the quantum effect of gravity, is obtained quantitatively.

The organization of this paper is as follows. In Sect. 2, we review the classical near-horizon geometry of the black 0-brane in ten dimensions, and uplift it to that of the M-wave in eleven dimensions. In Sect. 3, we discuss the higher derivative corrections in the type IIA superstring theory and the M-theory, and solve the equations of motion for the near-horizon geometry of the non-extremal M-wave in Sect. 4. In Sect. 5, we evaluate the entropy and the energy of the M-wave up to $1/N^2$. We probe the quantum near-horizon geometry by the D0-brane in Sect. 6 and clarify the validity of our analyses in Sect. 7. Section 8 is devoted to the conclusion and discussion. Detailed calculations and discussions on the ambiguities of the higher derivative corrections are collected in the appendices.

2. Classical near-horizon geometry of the black 0-brane

In this section, we briefly review the non-extremal solution of the black 0-brane that carries mass and R – R charge. In particular, we uplift the solution to eleven dimensions and show that the black 0-brane is described by the M-wave solution.

In the low-energy limit, the dynamics of massless modes in type IIA superstring theory are governed by type IIA supergravity. Since we are interested in the black 0-brane that couples to the graviton $g_{\mu\nu}$, the dilaton ϕ , and the R - R 1-form field C_μ , the relevant part of the type IIA supergravity action is given by

$$S_{10}^{(0)} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} \left\{ e^{-2\phi} (R + 4\partial_\mu \phi \partial^\mu \phi) - \frac{1}{4} G_{\mu\nu} G^{\mu\nu} \right\}, \quad (1)$$

where $2\kappa_{10}^2 = (2\pi)^7 \ell_s^8 g_s^2$ and $G_{\mu\nu}$ is the field strength of C_μ . g_s and ℓ_s are the string coupling constant and the string length, respectively. It is possible to solve the equations of motion by making the ansatz that the metric is static and has $SO(9)$ rotation symmetry. Then we obtain the non-extremal solution of the black 0-brane (see, e.g., Ref. [40]):

$$\begin{aligned} ds_{10}^2 &= -\tilde{H}^{-\frac{1}{2}} \tilde{F} dt^2 + \tilde{H}^{\frac{1}{2}} \tilde{F}^{-1} dr^2 + \tilde{H}^{\frac{1}{2}} r^2 d\Omega_8^2, \\ e^\phi &= \tilde{H}^{\frac{3}{4}}, \quad C = \left(\frac{r_+}{r_-} \right)^{\frac{7}{2}} \tilde{H}^{-1} dt, \\ \tilde{H} &= 1 + \frac{r_-^7}{r^7}, \quad \tilde{F} = 1 - \frac{r_+^7 - r_-^7}{r^7}. \end{aligned} \quad (2)$$

The horizon is located at $r_H = (r_+^7 - r_-^7)^{\frac{1}{7}}$. The parameters r_\pm are related to the mass M_0 and the R - R charge Q_0 of the black 0-brane by

$$M_0 = \frac{V_{S^8}}{2\kappa_{10}^2} (8r_+^7 - r_-^7), \quad Q_0 = \frac{N}{\ell_s g_s} = \frac{7V_{S^8}}{2\kappa_{10}^2} (r_+ r_-)^{\frac{7}{2}}, \quad (3)$$

where N is the number of D0-branes and $V_{S^8} = \frac{2\pi^{9/2}}{\Gamma(9/2)} = \frac{2(2\pi)^4}{7 \cdot 15}$ is the volume of S^8 . Now the parameters r_\pm are expressed as

$$r_\pm^7 = (1 + \delta)^{\pm 1} (2\pi)^2 15\pi g_s N \ell_s^7, \quad (4)$$

where δ is a non-negative parameter. The extremal limit $r_+ = r_-$ is saturated when $\delta = 0$.

Let us rewrite the solution (2) in terms of $U = r/\ell_s^2$ and $\lambda = g_s N / (2\pi)^2 \ell_s^3$, which correspond to the typical energy scale and 't Hooft coupling in the dual gauge theory, respectively. The near-horizon limit of the non-extremal black 0-brane is taken by $\ell_s \rightarrow 0$ while U , λ , and δ/ℓ_s^4 are fixed. Then the near-horizon limit of the solution (2) becomes [8]

$$\begin{aligned} ds_{10}^2 &= \ell_s^2 \left(-H^{-\frac{1}{2}} F dt^2 + H^{\frac{1}{2}} F^{-1} dU^2 + H^{\frac{1}{2}} U^2 d\Omega_8^2 \right), \\ e^\phi &= \ell_s^{-3} H^{\frac{3}{4}}, \quad C = \ell_s^4 H^{-1} dt, \\ H &= \frac{(2\pi)^4 15\pi \lambda}{U^7}, \quad F = 1 - \frac{U_0^7}{U^7}, \end{aligned} \quad (5)$$

where $U_0^7 = \frac{2\delta}{\ell_s^4} (2\pi)^4 15\pi \lambda$.

The type IIA supergravity is related to 11D supergravity via circle compactification. In fact, the 11D metric is related to the 10D one as $ds_{11}^2 = e^{-2\phi/3} ds_{10}^2 + e^{4\phi/3} (dz - C_\mu dx^\mu)^2$. The near-horizon limit of the non-extremal solution of the black 0-brane (5) can be uplifted to eleven dimensions as

$$ds_{11}^2 = \ell_s^4 \left(-H^{-1} F dt^2 + F^{-1} dU^2 + U^2 d\Omega_8^2 + (\ell_s^{-4} H^{\frac{1}{2}} dz - H^{-\frac{1}{2}} dt)^2 \right). \quad (6)$$

This represents the near-horizon limit of the non-extremal M-wave solution in eleven dimensions. The solution is purely geometrical and the expressions become simple. Furthermore, in the geometrical part, the quantum corrections to 11D supergravity are under control. This is the reason why we execute analyses of the solution in eleven dimensions.

3. Quantum correction to 11D supergravity

The 11D supergravity is realized as the low-energy limit of the M-theory. A fundamental object in the M-theory is a membrane and, if we could take account of the interaction of membranes, the effective action of the M-theory would become 11D supergravity with some higher derivative terms. Unfortunately, quantization of the membrane has not yet been completed. It is, however, possible to derive the relevant part of the quantum corrections in the M-theory by requiring local supersymmetry. In this section we review the quantum corrections to the 11D supergravity.

The massless fields of 11D supergravity consist of a vielbein $e^a{}_\mu$, a Majorana gravitino ψ_μ , and a 3-form field $A_{\mu\nu\rho}$. Since we are only interested in the M-wave solution, we only need to take account of the action that only depends on the graviton:

$$2\kappa_{11}^2 S_{11}^{(0)} = \int d^{11}x e R, \quad (7)$$

where $2\kappa_{11}^2 = (2\pi)^8 \ell_p^9 = (2\pi)^8 \ell_s^9 g_s^3$. Notice that, after dimensional reduction, this becomes the action (1), which contains the dilation and the R - R 1-form field as well as the graviton in ten dimensions [39].

Of course there are other terms that depend on ψ_μ and $A_{\mu\nu\rho}$, which are completely determined by the local supersymmetry. For example, a variation of the vielbein under the local supersymmetry is given by $\delta[e] = [\bar{\epsilon}\psi]$. Here we use a symbol $[X]$ to abbreviate indices and gamma matrices in X , and ϵ represents a parameter of the local supersymmetry. Then the variation of the scalar curvature is written as $\delta[eR] = [eR\bar{\epsilon}\psi]$. In order to cancel this, we see that a variation of the Majorana gravitino should include $\delta[\psi] = [D\epsilon] + \dots$ and simultaneously there should exist a term like $[e\bar{\psi}\psi_2]$ in the action. Here ψ_2 represents the field strength of the Majorana gravitino. By continuing this process, it is possible to determine the structure of the 11D supergravity completely [39].

Now let us discuss quantum corrections to the 11D supergravity. Since the M-theory is related to the type IIA superstring theory by dimensional reduction, the effective action of the M-theory should contain that of the type IIA superstring theory. The latter can be obtained so as to be consistent with the scattering amplitudes of strings, and it is well known that leading corrections to the type IIA supergravity include terms like $[eR^4]$. This is directly uplifted to the eleven dimensions and we see that the effective action of the M-theory should include terms like $B_1 = [eR^4]_7$. The subscript 7 indicates that there are potentially 7 independent terms if we consider the possible contractions of 16 indices out of 4 Riemann tensors. (To be more precise, we have excluded terms that contain Ricci tensor or scalar curvature, since these can be eliminated by redefinition of the graviton. Discussions of these terms can be found in Appendix C.) As in the case of 11D supergravity, it is possible to determine other corrections by requiring the local supersymmetry. For example, variations of B_1 under the local supersymmetry contain terms like $V_1 = [eR^4\bar{\epsilon}\psi]$. In order to cancel these terms, $B_{11} = [e\epsilon_{11}AR^4]_2$ and $F_1 = [eR^3\bar{\psi}\psi_2]_{92}$ should exist in the action. The structures of B_1 , B_{11} , and F_1 are severely restricted

by the local supersymmetry. By continuing this process, it is possible to show that a combination of terms in B_1 is completely determined up to an overall factor [35,36]. The result is as follows:

$$\begin{aligned}
2\kappa_{11}^2 S_{11}^{(1)} &= \frac{\pi^2 \ell_p^6}{3 \cdot 2^{84}!} \int d^{11}x e \left(t_8 t_8 R^4 - \frac{1}{4!} \epsilon_{11} \epsilon_{11} R^4 \right) \\
&= \frac{\pi^2 \ell_p^6}{3 \cdot 2^{84}!} \int d^{11}x e \left\{ 24(R_{abcd} R_{abcd} R_{efgh} R_{efgh} - 64 R_{abcd} R_{aefg} R_{bcdh} R_{efgh} \right. \\
&\quad \left. + 2 R_{abcd} R_{abef} R_{cdgh} R_{efgh} + 16 R_{acbd} R_{aebf} R_{cgdh} R_{efgh} \right. \\
&\quad \left. - 16 R_{abcd} R_{aefg} R_{befh} R_{cdgh} - 16 R_{abcd} R_{aefg} R_{bfhe} R_{cdgh} \right\}. \quad (8)
\end{aligned}$$

Here t_8 are the products of four Kronecker's deltas with eight indices and ϵ_{11} is an antisymmetric tensor with eleven indices. Local Lorentz indices are labeled by $a, b, \dots = 0, 1, \dots, 10$. Although all indices are lowered, it is understood those are contracted by the flat metric η_{ab} . The Riemann tensor with local Lorentz indices is defined by $R_{abcd} = e^\mu{}_c e^\nu{}_d (\partial_\mu \omega_{\nu ab} - \partial_\nu \omega_{\mu ab} + \omega_{\mu a}{}^e \omega_{\nu eb} - \omega_{\nu a}{}^e \omega_{\mu eb})$, where $\omega_{\mu ab}$ is a spin connection and μ, ν are space-time indices. The overall factor in Eq. (8) is determined by employing the result of the 1-loop four-graviton amplitude in the type IIA superstring theory.

Since the near-horizon limit of the M-wave solution (6) is purely geometrical, it is possible to examine the leading quantum corrections to it from the action (8). Other terms that depend on the 3-form field are irrelevant to the analyses for the M-wave. In summary, the effective action of the M-theory is described by

$$S_{11} = S_{11}^{(0)} + S_{11}^{(1)} = \frac{1}{2\kappa_{11}^2} \int d^{11}x e \left\{ R + \gamma \ell_s^{12} \left(t_8 t_8 R^4 - \frac{1}{4!} \epsilon_{11} \epsilon_{11} R^4 \right) \right\}, \quad (9)$$

where $\gamma = \frac{\pi^2}{3 \cdot 2^{84}!} \frac{g_s^2}{\ell_s^6} = \frac{\pi^6}{2^7 3^2} \frac{\lambda^2}{N^2}$. Notice that the parameter γ remains finite after the decoupling limit is taken. After the dimensional reduction, the action (9) becomes the effective action of the type IIA superstring theory, which includes the 1-loop effect of gravity.

Now we derive equations of motion for the action (9). Although the derivation is straightforward, we need to labor at many calculations because of the higher derivative terms in the action. Therefore, in practice, we use the Mathematica code for the calculations. Below we show the points of the calculations to build the code.

First of all we list the variations of the fields with respect to the vielbein:

$$\begin{aligned}
\delta e &= -e^i{}_\mu \delta e^\mu{}_i = -e \eta_{ij} \delta e^{ij}, \\
\delta \omega_{cab} &= e^\rho{}_c \delta \omega_{\rho ab} = (\delta_{[a}^k \eta_{b]i} \eta_{cj} + \delta_{[a}^k \eta_{b]j} \eta_{ci} + \delta_c^k \eta_{i[a} \eta_{b]j}) D_k \delta e^{ij}, \\
\delta R_{abcd} &= \delta e^\mu{}_c R_{ab\mu d} + \delta e^\mu{}_d R_{abc\mu} + e^\mu{}_c e^\nu{}_d \delta R_{ab\mu\nu} = -2\delta e^{ij} R_{abi[c} \eta_{d]j} + 2D_{[c} \delta \omega_{d]ab}, \quad (10) \\
\delta R_{ab} &= -\delta e^{ij} R_{ajib} + \delta e^{ij} R_{ai} \eta_{bj} + D_b \delta \omega^c{}_{ac} - D_c \delta \omega_{ba}{}^c,
\end{aligned}$$

where $\delta e^{ij} \equiv e^i{}_\mu \delta e^{\mu j}$. Then variations of the higher derivative terms are evaluated as

$$\begin{aligned}
 & e \delta \left(t_8 t_8 R^4 - \frac{1}{4!} \epsilon_{11} \epsilon_{11} R^4 \right) \\
 &= 24e \left\{ 4(\delta R_{abcd}) R_{abcd} R_{efgh} R_{efgh} - 64(\delta R_{abcd}) R_{abce} R_{dfgh} R_{efgh} \right. \\
 &\quad + 8(\delta R_{abcd}) R_{abef} R_{cdgh} R_{efgh} + 64(\delta R_{abcd}) R_{aecg} R_{bf dh} R_{efgh} \\
 &\quad - 64(\delta R_{abcd}) R_{abeg} R_{cf eh} R_{dfgh} - 64(\delta R_{abcd}) R_{efag} R_{ef ch} R_{gbhd} \\
 &\quad \left. + 32(\delta R_{abcd}) R_{abef} R_{cegh} R_{dfgh} \right\} \\
 &= e(\delta R_{abcd}) X_{abcd} \\
 &= 2e \delta e^{ij} R_{abci} X^{abc}{}_j - 2e X^{abcd} D_d \delta \omega_{cab} \\
 &\cong 2e \delta e^{ij} R_{abci} X^{abc}{}_j - 2e (\delta_a^k \eta_{bi} \eta_{cj} + \delta_a^k \eta_{bj} \eta_{ci} + \delta_c^k \eta_{ia} \eta_{bj}) D_k D_d X^{abcd} \delta e^{ij} \\
 &= 2e R_{abci} X^{abc}{}_j \delta e^{ij} - 2e D^c D^d (X_{cijd} + X_{cjid} + X_{ijcd}) \delta e^{ij} \\
 &= e(3R_{abci} X^{abc}{}_j - R_{abcj} X^{abc}{}_i) \delta e^{ij} - 2e D^c D^d (X_{cijd} + X_{cjid}) \delta e^{ij} \\
 &= e(3R_{abci} X^{abc}{}_j - R_{abcj} X^{abc}{}_i - 4D_{(a} D_{b)} X^a{}_{ij}{}^b) \delta e^{ij}, \tag{11}
 \end{aligned}$$

where we have defined

$$\begin{aligned}
 X_{abcd} &= \frac{1}{2} \left(X'_{[ab][cd]} + X'_{[cd][ab]} \right), \tag{12} \\
 X'_{abcd} &= 96(R_{abcd} R_{efgh} R_{efgh} - 16R_{abce} R_{dfgh} R_{efgh} + 2R_{abef} R_{cdgh} R_{efgh} \\
 &\quad + 16R_{aecg} R_{bf dh} R_{efgh} - 16R_{abeg} R_{cf eh} R_{dfgh} - 16R_{efag} R_{ef ch} R_{gbhd} \\
 &\quad + 8R_{abef} R_{cegh} R_{dfgh}).
 \end{aligned}$$

Finally we obtain the equations of motion for the effective action (9):

$$\begin{aligned}
 E_{ij} \equiv R_{ij} - \frac{1}{2} \eta_{ij} R + \gamma \ell_s^{12} \left\{ -\frac{1}{2} \eta_{ij} \left(t_8 t_8 R^4 - \frac{1}{4!} \epsilon_{11} \epsilon_{11} R^4 \right) \right. \\
 \left. + \frac{3}{2} R_{abci} X^{abc}{}_j - \frac{1}{2} R_{abcj} X^{abc}{}_i - 2D_{(a} D_{b)} X^a{}_{ij}{}^b \right\} = 0. \tag{13}
 \end{aligned}$$

As mentioned before, the action (9) is not unique due to the ambiguity of field redefinitions, such as $g_{\mu\nu} \rightarrow g'_{\mu\nu} = g_{\mu\nu} + \gamma \ell_s^{12} R^2 R_{\mu\nu}$. Therefore, the equations of motion are not unique either. We will discuss, however, that the physical quantities of the M-wave do not depend on these ambiguities (see Appendix D).

4. Quantum near-horizon geometry of the black 0-brane

In the previous section, we explained the effective action of the M-theory (9), and derived the equations of motion (13). In this section we solve them up to the linear order of γ and obtain the non-extremal solution of the M-wave with quantum gravity correction.

In order to obtain the solution of (13), we relax the ansatz for the M-wave as

$$\begin{aligned}
 ds_{11}^2 &= \ell_s^4 \left(-H_1^{-1} F_1 dt^2 + F_1^{-1} U_0^2 dx^2 + U_0^2 x^2 d\Omega_8^2 + (\ell_s^{-4} H_2^{\frac{1}{2}} dz - H_3^{-\frac{1}{2}} dt)^2 \right), \tag{14} \\
 H_i &= \frac{(2\pi)^4 15\pi\lambda}{U_0^7} \left(\frac{1}{x^7} + \frac{\gamma}{U_0^6} h_i \right), \quad F_1 = 1 - \frac{1}{x^7} + \frac{\gamma}{U_0^6} f_1,
 \end{aligned}$$

where $i = 1, 2, 3$, and h_i and f_1 are functions of a dimensionless variable $x = \frac{U}{U_0}$. This ansatz is static and possesses $SO(9)$ rotation symmetry, and if we take $N = \infty$, the metric just becomes the

classical solution (6). By solving the equations of motion (13), we determine the functions $h_i(x)$ and $f_1(x)$.

The calculations are straightforward but complicated, so we use the Mathematica code to explicitly write down the equations of motion. Some of the results are listed in Appendices A and B. From the output we find that there are five nontrivial equations, which are given by

$$E_1 = -63x^{34} f_1 - 9x^{35} f_1' - 49x^{41} h_1 + 49x^{34}(1 - x^7)h_2 + 23x^{35}(1 - x^7)h_2' + 2x^{36}(1 - x^7)h_2'' + 98x^{41}h_3 + 7x^{42}h_3' - 63\,402\,393\,600x^{14} + 70\,230\,343\,680x^7 + 1062\,512\,640 = 0, \quad (15)$$

$$E_2 = 63x^{34} f_1 + 9x^{35} f_1' + 7x^{34}(9 - 2x^7)h_1 + 9x^{35}(1 - x^7)h_1' - 112x^{34}(1 - x^7)h_2 - 16x^{35}(1 - x^7)h_2' - 98x^{41}h_3 - 7x^{42}h_3' - 2159\,861\,760x^7 - 5730\,600\,960 = 0, \quad (16)$$

$$E_3 = 133x^{34} f_1 + 35x^{35} f_1' + 2x^{36} f_1'' + 28x^{34}(3 - 10x^7)h_1 + 7x^{35}(4 - 7x^7)h_1' + 2x^{36}(1 - x^7)h_1'' - 7x^{34}(5 - 26x^7)h_2 - 21x^{35}(1 - 2x^7)h_2' - 2x^{36}(1 - x^7)h_2'' + 98x^{41}h_3 + 7x^{42}h_3' + 5669\,637\,120x^7 - 8626\,383\,360 = 0, \quad (17)$$

$$E_4 = 259x^{34} f_1 + 53x^{35} f_1' + 2x^{36} f_1'' + 147x^{34}(1 - 3x^7)h_1 + x^{35}(37 - 58x^7)h_1' + 2x^{36}(1 - x^7)h_1'' + 147x^{41}h_2 + 21x^{42}h_2' + 294x^{41}h_3 + 21x^{42}h_3' - 63\,402\,393\,600x^{14} + 133\,632\,737\,280x^7 - 71\,292\,856\,320 = 0, \quad (18)$$

$$E_5 = 49x^{34} h_1 + 7x^{35} h_1' + 49x^{34} h_2 - x^{35} h_2' - x^{36} h_2'' - 98x^{34} h_3 - 22x^{35} h_3' - x^{36} h_3'' - 63\,402\,393\,600x^7 + 70\,230\,343\,680 = 0. \quad (19)$$

Here we have defined $E_1 = 4U_0^8 \ell_s^4 x^{36} \gamma^{-1} E_{00}$, $E_2 = 4U_0^8 \ell_s^4 x^{36} \gamma^{-1} E_{11}$, $E_3 = 4U_0^8 \ell_s^4 x^{36} \gamma^{-1} E_{22}$, $E_4 = 4U_0^8 \ell_s^4 x^{36} \gamma^{-1} E_{1010}$, and $E_5 = 4U_0^8 \ell_s^4 x^{\frac{65}{2}} (-1 + x^7)^{-\frac{1}{2}} \gamma^{-1} E_{010}$. Note that the above equations are derived up to the order of γ , and part of γ^0 is zero since the ansatz (14) is a fluctuation around the classical solution (6).

Now we solve these equations to obtain h_i and f_1 . We will see that h_i and f_1 are uniquely determined as functions of x by imposing reasonable boundary conditions. Because the calculations below are a bit tedious, the results are summarized at the end of this section.

First let us evaluate the sum of E_1 and E_2 :

$$\begin{aligned} \frac{1}{9x^{28}(x^7 - 1)}(E_1 + E_2) &= -7x^6 h_1 - x^7 h_1' + 7x^6 h_2 - \frac{7}{9}x^7 h_2' - \frac{2}{9}x^8 h_2'' \\ &\quad + \frac{518\,676\,480}{x^{28}} - \frac{7044\,710\,400}{x^{21}} \\ &= \left(-x^7 h_1 + x^7 h_2 - \frac{2}{9}x^8 h_2' + \frac{352\,235\,520}{x^{20}} - \frac{19\,210\,240}{x^{27}} \right)' = 0. \quad (20) \end{aligned}$$

From this equation h_1 is expressed in terms of h_2 as

$$h_1 = h_2 - \frac{2}{9}x h_2' + \frac{c_1}{x^7} + \frac{352\,235\,520}{x^{27}} - \frac{19\,210\,240}{x^{34}}, \quad (21)$$

where c_1 is an integral constant. Next let us evaluate E_5 :

$$\begin{aligned} \frac{1}{x^{28}}E_5 &= 49x^6h_1 + 7x^7h'_1 + 49x^6h_2 - x^7h'_2 - x^8h''_2 - 98x^6h_3 - 22x^7h'_3 - x^8h''_3 \\ &\quad - \frac{63\,402\,393\,600}{x^{21}} + \frac{70\,230\,343\,680}{x^{28}} \\ &= \left(7x^7h_1 + 7x^7h_2 - x^8h'_2 - 14x^7h_3 - x^8h'_3 + \frac{3170\,119\,680}{x^{20}} - \frac{2601\,123\,840}{x^{27}} \right)' \\ &= \left(14x^7h_2 - \frac{23}{9}x^8h'_2 - 14x^7h_3 - x^8h'_3 + \frac{5635\,768\,320}{x^{20}} - \frac{2735\,595\,520}{x^{27}} \right)' = 0. \end{aligned} \tag{22}$$

In the last line, we have removed h_1 by using Eq. (21). Thus a linear combination of h_3 is expressed in terms of h_2 as

$$14x^7h_3 + x^8h'_3 = 14x^7h_2 - \frac{23}{9}x^8h'_2 + c_2 + \frac{5635\,768\,320}{x^{20}} - \frac{2735\,595\,520}{x^{27}}, \tag{23}$$

where c_2 is an integral constant. From Eqs. (21) and (23), it is possible to remove h_1 and h_3 from E_1 , E_3 , and E_4 . After some calculations, we obtain three equations remaining to be solved:

$$\begin{aligned} E_1 &= -63x^{34}f_1 - 9x^{35}f'_1 + 49x^{34}h_2 + x^{35}(23 - 30x^7)h'_2 + 2x^{36}(1 - x^7)h''_2 \\ &\quad - 49c_1x^{34} + 7c_2x^{34} - 41\,211\,555\,840x^{14} + 52\,022\,476\,800x^7 + 1062\,512\,640 = 0, \end{aligned} \tag{24}$$

$$\begin{aligned} E_3 &= 133x^{34}f_1 + 35x^{35}f'_1 + 2x^{36}f''_1 \\ &\quad + 49x^{34}h_2 - \frac{7}{9}x^{35}(23 - 62x^7)h'_2 - \frac{2}{9}x^{36}(32 - 53x^7)h''_2 - \frac{4}{9}x^{37}(1 - x^7)h'''_2 \\ &\quad - 49c_1x^{34} + 7c_2x^{34} - 125\,748\,080\,640x^{14} + 301\,493\,283\,840x^7 - 37\,672\,266\,240 = 0, \end{aligned} \tag{25}$$

$$\begin{aligned} E_4 &= 259x^{34}f_1 + 53x^{35}f'_1 + 2x^{36}f''_1 \\ &\quad + 147x^{34}h_2 - \frac{7}{9}x^{35}(5 - 26x^7)h'_2 - \frac{2}{9}x^{36}(32 - 53x^7)h''_2 - \frac{4}{9}x^{37}(1 - x^7)h'''_2 \\ &\quad - 147c_1x^{34} + 21c_2x^{34} - 81\,366\,405\,120x^{14} + 324\,970\,168\,320x^7 - 95\,670\,650\,880. \end{aligned} \tag{26}$$

Notice, however, that three functions E_1 , E_3 , and E_4 are not independent because of the identity

$$E_4 = \frac{2}{7}xE'_1 - 9E_1 + \frac{16}{7}E_3. \tag{27}$$

This corresponds to the energy conservation, $D_a E^{ab} = 0$. Thus we only need to solve following two equations:

$$\begin{aligned} -\frac{1}{2}E_1 + \frac{1}{4}(E_3 - E_4) &= -\frac{1}{14}xE'_1 + \frac{7}{4}E_1 - \frac{9}{28}E_3 \\ &= -49x^{34}h_2 - x^{35}(15 - 22x^7)h'_2 - x^{36}(1 - x^7)h''_2 + 7(7c_1 - c_2)x^{34} \\ &\quad + 9510\,359\,040x^{14} - 31\,880\,459\,520x^7 + 13\,968\,339\,840 = 0, \end{aligned} \tag{28}$$

$$\begin{aligned} \frac{1}{2}(E_3 - E_4) &= -\frac{1}{7}xE'_1 + \frac{9}{2}E_1 - \frac{9}{14}E_3 \\ &= -63x^{34}f_1 - 9x^{35}f'_1 - 49x^{34}h_2 - 7x^{35}(1 - 2x^7)h'_2 + 7(7c_1 - c_2)x^{34} \\ &\quad - 22\,190\,837\,760x^{14} - 11\,738\,442\,240x^7 + 28\,999\,192\,320 = 0. \end{aligned} \tag{29}$$

By solving Eq. (28), we finally obtain h_2 as

$$h_2 = \frac{19\,160\,960}{x^{34}} - \frac{58\,528\,288}{x^{27}} + \frac{2213\,568}{13x^{20}} - \frac{1229\,760}{13x^{13}} + c_1 - \frac{c_2}{7} + \frac{2459\,520}{x^6} + \frac{c_4}{3136x^7} + 1054\,080 \left(2 - \frac{1}{x^7}\right) I(x), \tag{30}$$

$$I(x) = \frac{c_3}{944\,455\,680} + \log(x - 1) + \frac{c_4}{6611\,189\,760} \log(1 - x^{-7}) - \sum_{n=1,3,5} \cos \frac{n\pi}{7} \log(x^2 + 2x \cos \frac{n\pi}{7} + 1) - 2 \sum_{n=1,3,5} \sin \frac{n\pi}{7} \tan^{-1} \left(\frac{x + \cos \frac{n\pi}{7}}{\sin \frac{n\pi}{7}} \right), \tag{31}$$

where c_3 and c_4 are integral constants. Although the form of $I(x)$ seems to be complicated, its derivative becomes

$$I'(x) = \frac{7}{x^7 - 1} \left(1 + \frac{c_4 x^{-1}}{6611\,189\,760}\right). \tag{32}$$

So far there are four integral constants, but these will be fixed by appropriate conditions. In fact it is natural to require that $h_i(1)$ are finite and $h_i(x) \sim \mathcal{O}(x^{-8})$ when x goes to infinity. In order to satisfy these conditions, it is necessary to choose $c_2 = 7c_1$, $c_3 = 944\,455\,680\pi(\sin \frac{\pi}{7} + \sin \frac{3\pi}{7} + \sin \frac{5\pi}{7})$, and $c_4 = -6611\,189\,760$. Inserting these values into Eqs. (30), (31), and (32), we obtain

$$h_2 = \frac{19\,160\,960}{x^{34}} - \frac{58\,528\,288}{x^{27}} + \frac{2213\,568}{13x^{20}} - \frac{1229\,760}{13x^{13}} - \frac{2108\,160}{x^7} + \frac{2459\,520}{x^6} + 1054\,080 \left(2 - \frac{1}{x^7}\right) I(x), \tag{33}$$

$$I(x) = \log \frac{x^7(x - 1)}{x^7 - 1} - \sum_{n=1,3,5} \cos \frac{n\pi}{7} \log(x^2 + 2x \cos \frac{n\pi}{7} + 1) - 2 \sum_{n=1,3,5} \sin \frac{n\pi}{7} \left\{ \tan^{-1} \left(\frac{x + \cos \frac{n\pi}{7}}{\sin \frac{n\pi}{7}} \right) - \frac{\pi}{2} \right\}, \tag{34}$$

and

$$I'(x) = \frac{7(1 - x^{-1})}{x^7 - 1}. \tag{35}$$

Note that the function $I(x)$ behaves as

$$I(x) \sim -\frac{7}{6x^6} + \frac{1}{x^7} - \frac{7}{13x^{13}} + \frac{1}{2x^{14}} + \mathcal{O}(x^{-15}), \tag{36}$$

when x goes to infinity.

Now we remove h_2 from Eq. (29), and obtain the differential equation only for f_1 :

$$\begin{aligned} \frac{1}{18x^{28}}(E_3 - E_4) &= -x^7 f_1' - 7x^6 f_1 + 819\,840I' + 3279\,360x^7(x^7 - 1)I' \\ &+ \frac{3624\,512\,640}{x^{28}} - \frac{3228\,113\,280}{x^{21}} - \frac{5738\,880}{x^{14}} - \frac{5738\,880}{x^7} \\ &+ 22\,955\,520x^6 - 22\,955\,520x^7 \\ &= \left(-x^7 f_1 + 819\,840I - \frac{1208\,170\,880}{9x^{27}} \right. \\ &\quad \left. + \frac{161\,405\,664}{x^{20}} + \frac{5738\,880}{13x^{13}} + \frac{956\,480}{x^6} \right)' = 0. \end{aligned} \tag{37}$$

Then f_1 is solved as

$$f_1 = -\frac{1208\,170\,880}{9x^{34}} + \frac{161\,405\,664}{x^{27}} + \frac{5738\,880}{13x^{20}} + \frac{956\,480}{x^{13}} + \frac{819\,840}{x^7} I(x). \tag{38}$$

Here the integral constant is set to zero, because we have imposed the boundary condition that $f_1(x) \sim \mathcal{O}(x^{-8})$ when x goes to infinity. From Eq. (21), h_1 is determined as

$$\begin{aligned} h_1 &= \frac{1302\,501\,760}{9x^{34}} - \frac{57\,462\,496}{x^{27}} + \frac{12\,051\,648}{13x^{20}} - \frac{4782\,400}{13x^{13}} \\ &- \frac{3747\,840}{x^7} + \frac{4099\,200}{x^6} - \frac{1639\,680(x-1)}{(x^7-1)} + 117\,120 \left(18 - \frac{23}{x^7} \right) I(x). \end{aligned} \tag{39}$$

The integral constant c_1 is chosen as zero so as to satisfy $h_1(x) \sim \mathcal{O}(x^{-8})$ when x goes to infinity. Finally, from Eq. (23), we derive

$$\begin{aligned} 0 &= -x^{14}h_3' - 14x^{13}h_3 + (29\,514\,240x^{13} - 33\,613\,440x^6)I(x) \\ &+ (2693\,760x^7 - 5387\,520x^{14})I'(x) + 72\,145\,920x^7 - 67\,226\,880x^6 \\ &- \frac{7222\,208\,000}{9x^{21}} + \frac{777\,920\,416}{x^{14}} + \frac{144\,127\,872}{13x^7} - \frac{58\,072\,000}{13} \\ &= \left(-x^{14}h_3 + (2108\,160x^{14} - 4801\,920x^7)I(x) + 2459\,520x^8 - 2108\,160x^7 \right. \\ &\quad \left. + \frac{361\,110\,400}{9x^{20}} - \frac{59\,840\,032}{x^{13}} - \frac{24\,021\,312}{13x^6} - \frac{58\,072\,000}{13}x \right)'. \end{aligned} \tag{40}$$

Thus h_3 is expressed as

$$\begin{aligned} h_3 &= \frac{361\,110\,400}{9x^{34}} - \frac{59\,840\,032}{x^{27}} - \frac{24\,021\,312}{13x^{20}} - \frac{58\,072\,000}{13x^{13}} \\ &- \frac{2108\,160}{x^7} + \frac{2459\,520}{x^6} + 117\,120 \left(18 - \frac{41}{x^7} \right) I(x). \end{aligned} \tag{41}$$

The integral constant is set to zero, since this term can be removed by a general coordinate transformation in the z direction. This corresponds to the gauge transformation on C_μ in ten dimensions.

Let us summarize the quantum correction to the near-horizon geometry of the non-extremal M-wave and the black 0-brane. By solving Eqs. (15)–(19), we obtained the quantum near-horizon

geometry of the non-extremal M-wave,

$$ds_{11}^2 = \ell_s^4 \left(-H_1^{-1} F_1 dt^2 + F_1^{-1} U_0^2 dx^2 + U_0^2 x^2 d\Omega_8^2 + (\ell_s^{-4} H_2^{\frac{1}{2}} dz - H_3^{-\frac{1}{2}} dt)^2 \right), \quad (42)$$

$$H_i = \frac{(2\pi)^4 15\pi\lambda}{U_0^7} \left(\frac{1}{x^7} + \epsilon \frac{\lambda^2}{U_0^6} h_i \right), \quad F_1 = 1 - \frac{1}{x^7} + \epsilon \frac{\lambda^2}{U_0^6} f_1.$$

Instead of γ , we introduced the dimensionless parameter

$$\epsilon = \frac{\gamma}{\lambda^2} = \frac{\pi^6}{2^7 3^2 N^2} \sim \frac{0.835}{N^2}, \quad (43)$$

and the functions h_i and f_1 were uniquely determined as

$$\begin{aligned} h_1 &= \frac{1302\,501\,760}{9x^{34}} - \frac{57\,462\,496}{x^{27}} + \frac{12\,051\,648}{13x^{20}} - \frac{4782\,400}{13x^{13}} \\ &\quad - \frac{3747\,840}{x^7} + \frac{4099\,200}{x^6} - \frac{1639\,680(x-1)}{(x^7-1)} + 117\,120 \left(18 - \frac{23}{x^7} \right) I(x), \\ h_2 &= \frac{19\,160\,960}{x^{34}} - \frac{58\,528\,288}{x^{27}} + \frac{2213\,568}{13x^{20}} - \frac{1229\,760}{13x^{13}} \\ &\quad - \frac{2108\,160}{x^7} + \frac{2459\,520}{x^6} + 1054\,080 \left(2 - \frac{1}{x^7} \right) I(x), \\ h_3 &= \frac{361\,110\,400}{9x^{34}} - \frac{59\,840\,032}{x^{27}} - \frac{24\,021\,312}{13x^{20}} - \frac{58\,072\,000}{13x^{13}} \\ &\quad - \frac{2108\,160}{x^7} + \frac{2459\,520}{x^6} + 117\,120 \left(18 - \frac{41}{x^7} \right) I(x), \\ f_1 &= -\frac{1208\,170\,880}{9x^{34}} + \frac{161\,405\,664}{x^{27}} + \frac{5738\,880}{13x^{20}} + \frac{956\,480}{x^{13}} + \frac{819\,840}{x^7} I(x). \end{aligned} \quad (44)$$

The function $I(x)$ is defined by Eq. (34). In order to fix the integral constants, we required that $h_i(1)$ are finite and $h_i(x), f_1(x) \sim \mathcal{O}(x^{-8})$ when x goes to infinity. After the dimensional reduction to ten dimensions, we obtain

$$ds_{10}^2 = \ell_s^2 \left(-H_1^{-1} H_2^{\frac{1}{2}} F_1 dt^2 + H_2^{\frac{1}{2}} F_1^{-1} U_0^2 dx^2 + H_2^{\frac{1}{2}} U_0^2 x^2 d\Omega_8^2 \right), \quad (45)$$

$$e^\phi = \ell_s^{-3} H_2^{\frac{3}{4}}, \quad C = \ell_s^4 H_2^{-\frac{1}{2}} H_3^{-\frac{1}{2}} dt.$$

This represents the quantum near-horizon geometry of the non-extremal black 0-brane.

5. Thermodynamics of the quantum near-horizon geometry of the black 0-brane

Since the quantum near-horizon geometry of the non-extremal black 0-brane was derived in the previous section, it is interesting to evaluate its thermodynamics. In this section, we estimate the entropy and the internal energy of the quantum near-horizon geometry of the non-extremal black 0-brane by using Wald's formula [41,42]. These quantities are quite important when we test the gauge/gravity duality.

In the following, quantities are calculated up to $\mathcal{O}(\epsilon^2)$. First of all, let us examine the location of the horizon x_H . This is defined by $F_1(x_H) = 0$ and becomes

$$x_H = 1 - \epsilon \frac{f_1(1)}{7} \tilde{U}_0^{-6}, \quad (46)$$

where $\tilde{U}_0 \equiv U_0/\lambda^{\frac{1}{3}}$ is a dimensionless parameter. The temperature of the black 0-brane is derived by the usual prescription. We consider the Euclidean geometry by changing the time coordinate as $t = -i\tau$ and require smoothness of the geometry at the horizon. This fixes the periodicity of the τ direction and its inverse gives the temperature of the non-extremal black 0-brane. Then the dimensionless temperature $\tilde{T} = T/\lambda^{\frac{1}{3}}$ of the black 0-brane is evaluated as

$$\tilde{T} = \frac{1}{4\pi} U_0^{-1} H_1^{-\frac{1}{2}} F_1' \Big|_{x_H} / \lambda^{\frac{1}{3}} = a_1 \tilde{U}_0^{\frac{5}{2}} (1 + \epsilon a_2 \tilde{U}_0^{-6}), \tag{47}$$

where a_1 and a_2 are numerical constants given by

$$\begin{aligned} a_1 &= \frac{7}{16\pi^3 \sqrt{15\pi}} \sim 0.00206, \\ a_2 &= \frac{9}{14} f_1(1) + \frac{1}{7} f_1'(1) - \frac{1}{2} h_1(1) \sim 937000. \end{aligned} \tag{48}$$

Inversely solving Eq. (47), the dimensionless parameter \tilde{U}_0 is written in terms of the temperature \tilde{T} as

$$\tilde{U}_0 = a_1^{-\frac{2}{5}} \tilde{T}^{\frac{2}{5}} \left(1 - \epsilon \frac{2}{5} a_1^{\frac{12}{5}} a_2 \tilde{T}^{-\frac{12}{5}} \right). \tag{49}$$

By using this replacement, it is always possible to express physical quantities as functions of \tilde{T} .

Next we derive the entropy of the quantum near-horizon geometry of the non-extremal black 0-brane. In practice, we consider the quantum near-horizon geometry of the non-extremal M-wave because of its simple expression. Since the effective action (9) includes higher derivative terms, we should employ Wald's entropy formula, which ensures the first law of black hole thermodynamics. Wald's entropy formula is given by

$$S = -2\pi \int_H d\Omega_8 dz \sqrt{h} \frac{\partial S_{11}}{\partial R_{\mu\nu\rho\sigma}} N_{\mu\nu} N_{\rho\sigma}, \tag{50}$$

where $\sqrt{h} = (\ell_s^2 U_0 x)^8 \ell_s^{-2} H_2^{1/2}$ is the volume factor at the horizon and $N_{\mu\nu}$ is an antisymmetric tensor binormal to the horizon. The binormal tensor satisfies $N_{\mu\nu} N^{\mu\nu} = -2$ and the nonzero component is only $N_{tx} = -\ell_s^4 U_0 H_1^{-1/2}$. The effective action is given by Eq. (9), and in the formula the variation of the action is evaluated as if the Riemann tensor is an independent variable, i.e.,

$$\frac{\partial S_{11}}{\partial R_{\mu\nu\rho\sigma}} = \frac{1}{2\kappa_{11}^2} (g^{\mu[\rho} g^{\sigma]v} + \gamma \ell_s^{12} X^{\mu\nu\rho\sigma}). \tag{51}$$

Now we are ready to evaluate the entropy of the quantum near-horizon geometry of the non-extremal M-wave. Some useful results are collected in Appendix B. By using these, the entropy is

evaluated as

$$\begin{aligned}
S &= \frac{4\pi}{2\kappa_{11}^2} \int_{\text{H}} d\Omega_8 dz \sqrt{h} \left(1 - \frac{1}{2} \gamma \ell_s^{12} X^{\mu\nu\rho\sigma} N_{\mu\nu} N_{\rho\sigma} \right) \\
&= \frac{4\pi}{2\kappa_{11}^2} \int_{\text{H}} d\Omega_8 dz \sqrt{h} (1 - 2\gamma \ell_s^{20} U_0^2 H_1^{-1} X^{txtx}) \\
&= \frac{4\pi}{2\kappa_{11}^2} \int_{\text{H}} d\Omega_8 dz \sqrt{h} \left(1 + 40\,642\,560 \epsilon \frac{1}{\tilde{U}_0^6 x_{\text{H}}^{20}} \right) \\
&= \frac{4}{49} a_1 N^2 \tilde{U}_0^{\frac{9}{5}} \left\{ 1 + \epsilon \left(-\frac{9}{14} f_1(1) + \frac{1}{2} h_2(1) + 40\,642\,560 \right) \tilde{U}_0^{-6} \right\} \\
&= \frac{4}{49} a_1^{-\frac{4}{5}} N^2 \tilde{T}^{\frac{9}{5}} \left\{ 1 + \epsilon a_1^{\frac{12}{5}} \left(-\frac{9}{5} f_1(1) - \frac{9}{35} f_1'(1) + \frac{9}{10} h_1(1) + \frac{1}{2} h_2(1) + 40\,642\,560 \right) \tilde{T}^{-\frac{12}{5}} \right\} \\
&= a_3 N^2 \tilde{T}^{\frac{9}{5}} (1 + \epsilon a_4 \tilde{T}^{-\frac{12}{5}}), \tag{52}
\end{aligned}$$

where the numerical constants a_3 and a_4 are defined as

$$\begin{aligned}
a_3 &= \frac{4}{49} a_1^{-\frac{4}{5}} = 2^{\frac{26}{5}} 15^{\frac{2}{5}} 7^{-\frac{14}{5}} \pi^{\frac{14}{5}} \sim 11.5, \\
a_4 &= a_1^{\frac{12}{5}} \left(-\frac{9}{5} f_1(1) - \frac{9}{35} f_1'(1) + \frac{9}{10} h_1(1) + \frac{1}{2} h_2(1) + 40\,642\,560 \right) \sim 0.400. \tag{53}
\end{aligned}$$

So far we have obtained the entropy for the M-wave. Because of the duality between type IIA string theory and M-theory, this is equivalent to that of the black 0-brane.

Finally let us derive the internal energy of the quantum near-horizon geometry of the non-extremal black 0-brane. Wald's entropy formula is constructed so as to satisfy the thermodynamic laws of black holes. Then, by integrating $d\tilde{E} = \tilde{T} dS$, it is possible to obtain the dimensionless energy $\tilde{E} = E/\lambda^{\frac{1}{3}}$ as

$$\frac{\tilde{E}}{N^2} = \frac{9}{14} a_3 \tilde{T}^{\frac{14}{5}} - \epsilon \frac{3}{2} a_3 a_4 \tilde{T}^{\frac{2}{5}} \sim 7.41 \tilde{T}^{\frac{14}{5}} - \frac{5.77}{N^2} \tilde{T}^{\frac{2}{5}}. \tag{54}$$

This result includes the quantum gravity effect, and it gives quite a nontrivial test of the gauge/gravity duality if we can evaluate the internal energy from the dual gauge theory. In fact, this is possible by employing the Monte Carlo simulation; the result strongly concludes that the duality holds at this order [24].

The specific heat is evaluated as

$$\frac{1}{N^2} \frac{d\tilde{E}}{d\tilde{T}} = \frac{9}{5} a_3 \tilde{T}^{\frac{9}{5}} - \epsilon \frac{3}{5} a_3 a_4 \tilde{T}^{-\frac{3}{5}}. \tag{55}$$

Notice that the specific heat becomes negative in the region where $\tilde{T} < (\epsilon a_4/3)^{5/12} \sim 0.4N^{-5/6}$. In this region the non-extremal black 0-brane behaves like a Schwarzschild black hole and will be unstable. When $N = \infty$ the instability will be suppressed. This result is also verified from the Monte Carlo simulation of the dual gauge theory [24].

6. D0-brane probe

In this section, we probe the quantum near-horizon geometry of the non-extremal black 0-brane (45) via a D0-brane. From the analysis it is possible to study how the test D0-brane is affected by the background field.

The bosonic part of the D0-brane action consists of the Born–Infeld action and the Chern–Simons one. Here we neglect an excitation of the gauge field on the D0-brane, so the Born–Infeld action is simply given by the pull-back of the metric. We also assume that the D0-brane moves only along the radial direction. Then the probe D0-brane action in the background of (45) is written as

$$\begin{aligned} S_{D0} &= -T_0 \int dt e^{-\phi} \sqrt{-g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}} + T_0 \int C \\ &= -T_0 \ell_s^4 \int dt H_2^{-\frac{1}{2}} \sqrt{H_1^{-1} F_1 - F_1^{-1} U_0^2 \dot{x}^2} + T_0 \ell_s^4 \int dt H_2^{-\frac{1}{2}} H_3^{-\frac{1}{2}}. \end{aligned} \quad (56)$$

The momentum conjugate to x is evaluated as

$$p = T_0 \ell_s^4 H_2^{-\frac{1}{2}} \frac{F_1^{-1} U_0^2 \dot{x}}{\sqrt{H_1^{-1} F_1 - F_1^{-1} U_0^2 \dot{x}^2}}, \quad (57)$$

and the energy of the probe D0-brane is given by

$$\begin{aligned} E_{D0} &= p\dot{x} + T_0 \ell_s^4 H_2^{-\frac{1}{2}} \sqrt{H_1^{-1} F_1 - F_1^{-1} U_0^2 \dot{x}^2} - T_0 \ell_s^4 H_2^{-\frac{1}{2}} H_3^{-\frac{1}{2}} \\ &= T_0 \ell_s^4 H_2^{-\frac{1}{2}} \frac{H_1^{-1} F_1}{\sqrt{H_1^{-1} F_1 - F_1^{-1} U_0^2 \dot{x}^2}} - T_0 \ell_s^4 H_2^{-\frac{1}{2}} H_3^{-\frac{1}{2}} \\ &= T_0 \ell_s^4 H_1^{-\frac{1}{2}} H_2^{-\frac{1}{2}} F_1^{\frac{1}{2}} \sqrt{1 + \left(\frac{p F_1^{\frac{1}{2}} H_2^{\frac{1}{2}}}{T_0 \ell_s^4 U_0} \right)^2} - T_0 \ell_s^4 H_2^{-\frac{1}{2}} H_3^{-\frac{1}{2}} \\ &\sim \frac{1}{2} H_1^{-\frac{1}{2}} H_2^{\frac{1}{2}} F_1^{\frac{3}{2}} \frac{p^2}{T_0 \ell_s^4 U_0^2} + T_0 \ell_s^4 \left(H_1^{-\frac{1}{2}} H_2^{-\frac{1}{2}} F_1^{\frac{1}{2}} - H_2^{-\frac{1}{2}} H_3^{-\frac{1}{2}} \right). \end{aligned} \quad (58)$$

In the final line we have taken the non-relativistic limit. From this we see that the potential energy for the probe D0-brane is expressed as

$$V_{D0} = T_0 \ell_s^4 \left(H_1^{-\frac{1}{2}} H_2^{-\frac{1}{2}} F_1^{\frac{1}{2}} - H_2^{-\frac{1}{2}} H_3^{-\frac{1}{2}} \right). \quad (59)$$

The first term corresponds to the gravitational attractive force and the second one to the $R - R$ repulsive force.

When we take $N = \infty$, the potential energy becomes $V_{D0} = T_0 \ell_s^4 H^{-1} (\sqrt{F} - 1)$. The part $(\sqrt{F} - 1)$ shows that the gravitational attractive force overcomes the $R - R$ repulsive force. Similarly, when N is finite, we regard $\sqrt{F_1}$ as the gravitational attractive force to the probe D0-brane. The function of $\sqrt{F_1}$ is plotted in Fig. 1. From this we see that the gravitational force becomes repulsive near the horizon x_H .

7. Validity of the analyses on the quantum near-horizon geometry

Our analyses so far are based on the effective action (9), which becomes the 1-loop effective action of the type IIA superstring theory after the dimensional reduction. Since the superstring theory is defined by the perturbative expansions of α' and g_s , terms with higher powers of these parameters

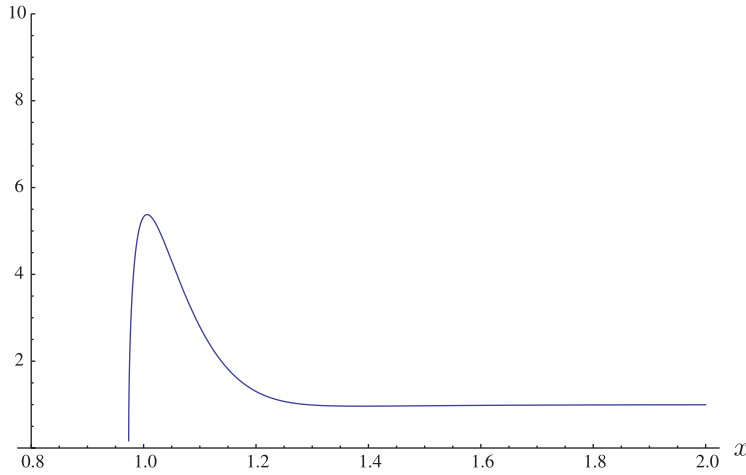


Fig. 1. The function $\sqrt{F_1(x)}$ with $F_1(x) = 1 - 1/x^7 + 0.000\,001 f_1(x)$.

also contribute to the effective action. Then our results are valid when the 1-loop effect becomes dominant compared to other stringy or loop effects. In this section we clarify the valid parameter region of our analyses.

First let us consider the validity of the type IIA supergravity approximation. From Eq. (5), the curvature radius ρ and the effective string coupling $g_s e^\phi$ at the event horizon $U = U_0$ are evaluated as

$$\frac{\alpha'}{\rho^2} \sim \tilde{U}_0^{\frac{3}{2}} \sim \tilde{T}^{\frac{3}{5}}, \quad g_s e^\phi \sim \frac{\tilde{U}_0^{-\frac{21}{4}}}{N} \sim \frac{\tilde{T}^{-\frac{21}{10}}}{N}. \tag{60}$$

Here we have used the relation Eq. (49) by setting $\epsilon = 0$. Then the supergravity approximation is valid when the string length $\sqrt{\alpha'}$ is quite small compared to the curvature radius ρ and the effective string coupling $g_s e^\phi$ is also quite small, i.e., $\tilde{T} \sim 0$ and $N \sim \infty$.

Now we consider the validity of our 1-loop analyses. From the effective action (9), we derived the internal energy (54) of the black 0-brane. However, if we include other higher derivative terms in the effective action, the Lagrangian is expected to be

$$\begin{aligned} \mathcal{L} \sim & R + (\alpha'^3 R^4 + \alpha'^5 \partial^4 R^4 + \dots) + g_s^2 (\alpha'^3 R^4 + \alpha'^6 \partial^6 R^4 + \dots) \\ & + g_s^4 (\alpha'^5 \partial^4 R^4 + \dots) + \dots + g_s^{2n} (\alpha'^{3+n} \partial^{2n} R^4 + \dots) + \dots, \end{aligned} \tag{61}$$

where R is the abbreviation of the Riemann tensor. The existence of these terms can be found in Ref. [43]. By following the calculation of Eq. (52) and using the dimensional analyses (60), the internal energy will be modified as

$$\begin{aligned} \frac{\tilde{E}}{N^2} \sim & 7.41 \tilde{T}^{\frac{14}{5}} \left\{ 1 + \left(\tilde{T}^{\frac{9}{5}} + \tilde{T}^3 + \dots \right) + \frac{1}{N^2} \left(-\frac{0.779}{\tilde{T}^{\frac{12}{5}}} + \frac{1}{\tilde{T}^{\frac{3}{5}}} + \dots \right) \right. \\ & \left. + \frac{1}{N^4} \left(\frac{c_2}{\tilde{T}^{\frac{27}{5}}} + \dots \right) + \dots + \frac{1}{N^{2n}} \left(\frac{1}{\tilde{T}^{\frac{18}{5}n - \frac{9}{5}}} + \dots \right) + \dots \right\}. \end{aligned} \tag{62}$$

$\tilde{T}^{\frac{9}{5}}$ and \tilde{T}^3 come from the α'^3 and α'^5 terms at tree level, and $\frac{1}{N^{2n}}$ correspond to n -loop amplitudes. Numerical constants are assumed to be $\mathcal{O}(1)$ and this is at least true for the 1-loop result. The coefficient c_2 at 2-loop will be discussed later. From the above estimation, the 1-loop contribution of

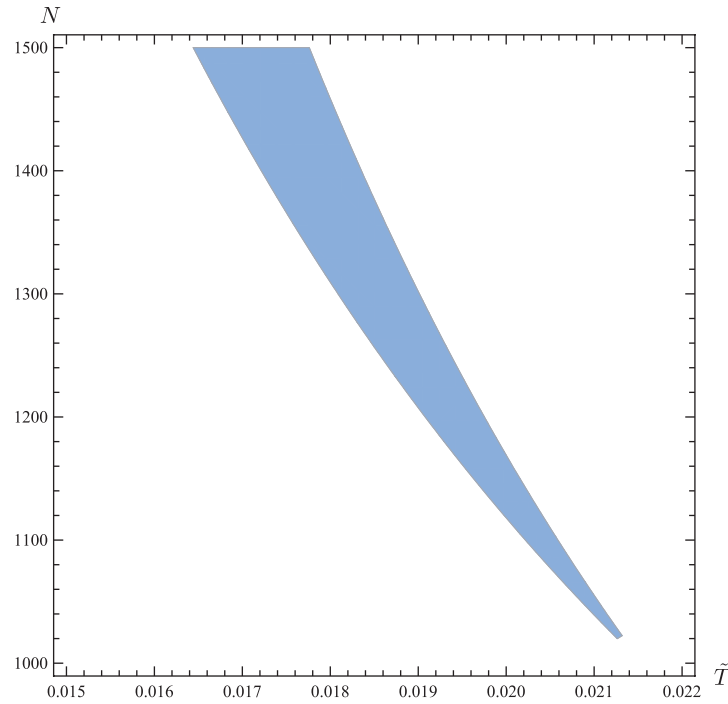


Fig. 2. Region of $\frac{\sqrt{10}}{\tilde{T}^{3/2}} \leq N \leq \frac{1}{\sqrt{10}\tilde{T}^{21/10}}$.

$-0.779/(N^2\tilde{T}^{12/5})$ becomes subleading when the following conditions are satisfied:

$$\frac{\tilde{T}^9}{1/N^2\tilde{T}^{12/5}} = N^2\tilde{T}^{21/5} \leq s, \quad \tilde{T}^9 \leq s, \quad \frac{1/N^4\tilde{T}^{27/5}}{1/N^2\tilde{T}^{12/5}} = \frac{1}{N^2\tilde{T}^3} \leq s, \quad (63)$$

where $s < 1$. Note that the terms at the n -loop ($n > 2$) are well suppressed from both the second and third inequalities. The above inequalities are equivalent to

$$\frac{1}{\sqrt{s}\tilde{T}^{3/2}} \leq N \leq \frac{\sqrt{s}}{\tilde{T}^{21/10}}. \quad (64)$$

Thus our analyses are estimated to be valid in this parameter region.

The case of $s = 0.1$ is shown in Fig. 2 where, e.g., $(\tilde{T}, N) = (0.02, 1140)$ is located inside the region. Then from Eq. (49), we obtain $U_0 = 2.48$ and $F_1(x) = 1 - 1/x^7 + 0.00357f_1(x)$. This shows that the quantum effect becomes important near the event horizon (see Fig. 1).

Notice that the validity of the parameter region obtained in Eq. (64) is roughly estimated. In order to identify a more precise one, we should determine the coefficient c_2 at the 2-loop. Although this is beyond the scope of this paper, if we suppose that $c_2 \sim 0.005$, the lower bound in Eq. (64) is enlarged as $0.0801/\sqrt{s}\tilde{T}^{3/2} \leq N$. This overlaps with the region $N < 0.334/\tilde{T}^{6/5}$ where the specific heat (55) becomes negative. For example, $(\tilde{T}, N) = (0.02, 30)$ is inside the overlap region when we choose $s \sim 1$. On the other hand, if we suppose that $c_2 \sim 1$, the parameter region (64) does not overlap with that of the negative specific heat. However, as the temperature decreases from region (64) with fixed N , the 2-loop term dominates the internal energy. Then, if c_2 is negative, the internal energy takes a large negative value because of the negative power in $c_2\tilde{T}^{-13/5}$. Thus we expect c_2 to be positive, and

again the specific heat becomes negative. As a reference, we mention that the numerical simulation suggests that the 2-loop coefficient becomes $c_2 = +0.00459$ [24].

8. Conclusion and discussion

In this paper we have studied the quantum nature of the bunch of D0-branes in the type IIA superstring theory. In the classical limit, it is well described by the non-extremal black 0-brane in type IIA supergravity. The quantum correction to the non-extremal black 0-brane is investigated after taking the near-horizon limit.

In order to manage the quantum effect of the gravity, we uplifted the near-horizon geometry of the non-extremal black 0-brane into that of the M-wave solution in 11D supergravity. These two are equivalent via the duality between the type IIA superstring theory and the M-theory, but the latter is purely geometrical and the calculations become rather simple. The geometrical part of the effective action for the M-theory (9) is derived so as to be consistent with the 1-loop amplitudes in the type IIA superstring theory, and the quantum correction to the M-wave solution is taken into account by explicitly solving the equations of motion (13). The solution is uniquely determined and its explicit form is given by Eq. (45). It is interesting to note that a probe D0-brane moving in this background would feel a repulsive force near the horizon. This means that the solution includes the back-reaction of the Hawking radiation.

We also investigated the thermodynamic property of the quantum near-horizon geometry of the non-extremal black 0-brane. Since the effective action contains higher derivative terms, we examined the thermodynamic property of the black 0-brane by employing Wald's formula. The entropy and the internal energy of the black 0-brane are evaluated up to $1/N^2$. The quantum correction to the internal energy becomes important when N is small. In Ref. [24], the internal energy is also calculated from the dual thermal gauge theory by using the Monte Carlo simulation, and it agrees with Eq. (54) very well. This gives strong evidence for the gauge/gravity duality at the level of quantum gravity.

Finally, we give an important remark on the effective action for the M-theory. It contains higher derivative terms, but these cannot be determined uniquely because of the field redefinitions. In the appendices we have considered all possible higher derivative terms and have shown that the ambiguities of the effective action have nothing to do with the thermodynamic properties of the near-horizon geometry of the non-extremal black 0-brane.

For future work, it will be important to derive the quantum geometry of the non-extremal black 0-brane and obtain the solution (45) by taking the near-horizon limit. The result will be reported elsewhere, but it is really possible. It will also be interesting to examine the quantum correction to the black 6-brane, which is also described by a purely geometrical object, called the Kaluza–Klein monopole, in 11D supergravity. To find connections between our results and other approaches to the field theory on D0-branes is also important [44,45]. Other approaches to probe curvature corrections by the black brane will also be related to our results in Sect. 6 [46,47]. Since now we can capture the quantum nature of the near-horizon geometry of the black 0-brane, it would be interesting to consider a recent proposal to resolve the information paradox on the black hole [48–50].

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Appendix A. Calculations of Ricci tensor and scalar curvature

By using the ansatz (14) for the metric, each component of the Ricci tensor up to the linear order of γ is calculated as

$$\begin{aligned}
 R_{00} &= \frac{\gamma}{4U_0^8 x^2 \ell_s^4} \{98f_1 + 30xf_1' + 2x^2 f_1'' + 49(2 - 7x^7)h_1 + 3x(10 - 17x^7)h_1' + 2x^2(1 - x^7)h_1'' \\
 &\quad + 147x^7 h_2 + 21x^8 h_2' + 196x^7 h_3 + 14x^8 h_3'\}, \\
 R_{11} &= \frac{\gamma}{4U_0^8 x^2 \ell_s^4} \{ -98f_1 - 30xf_1' - 2x^2 f_1'' - 35(1 - 8x^7)h_1 - 21x(1 - 2x^7)h_1' \\
 &\quad - 2x^2(1 - x^7)h_1'' - 7(9 + 12x^7)h_2 + 7x(1 - 4x^7)h_2' + 2x^2(1 - x^7)h_2'' \\
 &\quad - 196x^7 h_3 - 14x^8 h_3'\}, \tag{A.1} \\
 R_{\bar{a}\bar{a}} &= \frac{\gamma}{2U_0^8 x^2 \ell_s^4} \{ -14f_1 - 2xf_1' - 7(1 - x^7)h_1 - x(1 - x^7)h_1' + 7(1 - x^7)h_2 + x(1 - x^7)h_2'\}, \\
 R_{\natural\natural} &= \frac{\gamma}{4U_0^8 x^2 \ell_s^4} \{98f_1 + 14xf_1' + 49(1 - 3x^7)h_1 + 7x(1 - x^7)h_1' \\
 &\quad + 49(1 - x^7)h_2 + 23x(1 - x^7)h_2' + 2x^2(1 - x^7)h_2'' + 196x^7 h_3 + 14x^8 h_3'\}, \\
 R_{0\natural} &= \frac{\gamma x^{3/2} \sqrt{x^7 - 1}}{4U_0^8 \ell_s^4} \{49h_1 + 7xh_1' + 49h_2 - xh_2' - x^2 h_2'' - 98h_3 - 22xh_3' - x^2 h_3''\}.
 \end{aligned}$$

Here we have used \natural instead of 10 and $\bar{a} = 2, \dots, 9$. The Ricci scalar up to the linear order of γ becomes:

$$\begin{aligned}
 R &= \frac{\gamma}{2U_0^8 x^2 \ell_s^4} \{ -161f_1 - 39xf_1' - 2x^2 f_1'' - 98(1 - 3x^7)h_1 - 3x(10 - 17x^7)h_1' \\
 &\quad - 2x^2(1 - x^7)h_1'' + 49(1 - 4x^7)h_2 + x(23 - 44x^7)h_2' + 2x^2(1 - x^7)h_2'' - 98x^7 h_3 - 7x^8 h_3'\}. \tag{A.2}
 \end{aligned}$$

Appendix B. Calculations of higher derivative terms

In this appendix we summarize the values of the higher derivative terms appearing in Eq. (13). Note that we only need to evaluate these terms by using the ansatz (14) with $\gamma = 0$, because the equations of motion are solved up to the linear order of γ . First of all, each component of R_{abcd} is calculated as

$$\begin{aligned}
 R_{0101} &= -\frac{28}{U_0^2 x^2 \ell_s^4}, & R_{0\bar{a}0\bar{a}} &= \frac{7}{2U_0^2 x^2 \ell_s^4}, \\
 R_{011\natural} &= \frac{28\sqrt{x^7 - 1}}{U_0^2 x^{\frac{11}{2}} \ell_s^4}, & R_{0\bar{a}\bar{a}\natural} &= -\frac{7\sqrt{x^7 - 1}}{2U_0^2 x^{\frac{11}{2}} \ell_s^4}, \\
 R_{1\natural1\natural} &= -\frac{28(x^7 - 1)}{U_0^2 x^9 \ell_s^4}, & R_{1\bar{a}1\bar{a}} &= -\frac{7}{2U_0^2 x^9 \ell_s^4}, \tag{B.1} \\
 R_{\bar{a}\natural\bar{a}\natural} &= \frac{7(x^7 - 1)}{2U_0^2 x^9 \ell_s^4}, & R_{\bar{a}\bar{b}\bar{a}\bar{b}} &= \frac{1}{U_0^2 x^9 \ell_s^4}.
 \end{aligned}$$

We have used \natural instead of 10 and $\bar{a}, \bar{b} = 2, \dots, 9$. The scalar curvature and each component of the Ricci tensor become zero, and each component of X_{abcd} in Eq. (12) is evaluated as

$$\begin{aligned} X_{0101} &= -\frac{20\,321\,280}{U_0^6 x^{20} \ell_s^{12}}, & X_{0\bar{a}0\bar{a}} &= -\frac{1270\,080}{U_0^6 x^{20} \ell_s^{12}}, \\ X_{011\natural} &= \frac{20\,321\,280\sqrt{x^7-1}}{U_0^6 x^{\frac{47}{2}} \ell_s^{12}}, & X_{0\bar{a}\bar{a}\natural} &= \frac{1270\,080\sqrt{x^7-1}}{U_0^6 x^{\frac{47}{2}} \ell_s^{12}}, \\ X_{1\natural1\natural} &= -\frac{20\,321\,280(x^7-1)}{U_0^6 x^{27} \ell_s^{12}}, & X_{1\bar{a}1\bar{a}} &= \frac{1270\,080}{U_0^6 x^{27} \ell_s^{12}}, \\ X_{\bar{a}\natural\bar{a}\natural} &= -\frac{1270\,080(x^7-1)}{U_0^6 x^{27} \ell_s^{12}}, & X_{\bar{a}\bar{b}\bar{a}\bar{b}} &= \frac{1192\,320}{U_0^6 x^{27} \ell_s^{12}}. \end{aligned} \tag{B.2}$$

Using these results, we are ready to calculate the higher derivative terms in Eq. (13). The R^4 terms are calculated as

$$t_8 t_8 R^4 - \frac{1}{4!} \epsilon_{11\epsilon 11} R^4 = \frac{531\,256\,320}{U_0^8 x^{36} \ell_s^{16}}. \tag{B.3}$$

The RX terms become

$$\begin{aligned} R_{abc0} X^{abc}_0 &= -\frac{1066\,867\,200}{U_0^8 x^{29} \ell_s^{16}}, & R_{abc1} X^{abc}_1 &= \frac{1066\,867\,200}{U_0^8 x^{36} \ell_s^{16}}, \\ R_{abc\natural} X^{abc}_\natural &= -\frac{1066\,867\,200(x^7-1)}{U_0^8 x^{36} \ell_s^{16}}, & R_{abc\bar{a}} X^{abc}_{\bar{a}} &= -\frac{1088\,640}{U_0^8 x^{36} \ell_s^{16}} \delta_{\bar{a}\bar{b}}, \\ R_{abc0} X^{abc}_\natural &= R_{abc\natural} X^{abc}_0 = -\frac{1066\,867\,200\sqrt{x^7-1}}{U_0^8 x^{\frac{65}{2}} \ell_s^{16}}, \end{aligned} \tag{B.4}$$

and the DDX terms are evaluated as

$$\begin{aligned} D_{(a} D_b) X^a_{00}{}^b &= \frac{198\,132\,480(-47+40x^7)}{U_0^8 x^{29} \ell_s^{16}}, & D_{(a} D_b) X^a_{11}{}^b &= \frac{21\,772\,80(513+124x^7)}{U_0^8 x^{36} \ell_s^{16}}, \\ D_{(a} D_b) X^a_{\natural\natural}{}^b &= \frac{198\,132\,480(47-87x^7+40x^{14})}{U_0^8 x^{36} \ell_s^{16}}, & D_{(a} D_b) X^a_{\bar{a}\bar{b}}{}^b &= \frac{236\,234\,880(4-3x^7)}{U_0^8 x^{36} \ell_s^{16}} \delta_{\bar{a}\bar{b}}, \\ D_{(a} D_b) X^a_{0\natural}{}^b &= D_{(a} D_b) X^a_{\natural 0}{}^b = \frac{198\,132\,480(-47+40x^7)\sqrt{x^7-1}}{U_0^8 x^{\frac{65}{2}} \ell_s^{16}}. \end{aligned} \tag{B.5}$$

By inserting these results into Eq. (13), we obtain Eqs. (15)–(19).

Appendix C. Generic R^4 terms, equations of motion, and solution

In this appendix, we classify the independent R^4 terms that consist of four products of the Riemann tensor, the Ricci tensor, or the scalar curvature. The R^4 terms that include the Ricci tensor or the scalar curvature cannot be determined from the scattering amplitudes in the type IIA superstring theory. So in general the effective action and equations of motion are affected by these ambiguities.

First let us review the R^4 terms that only consist of the Riemann tensor. Since there are 16 indices, we have 8 pairs to be contracted. Naively, it seems that there are so many possible patterns. However,

carefully using the properties of the Riemann tensor, such as $R_{abcd} = -R_{bcad} - R_{cabd}$, it is possible to show that there are only 7 independent terms:

$$\begin{aligned}
 B_1 &= R_{abcd}R_{abcd}R_{efgh}R_{efgh}, & B_2 &= R_{abcd}R_{aefg}R_{bcdh}R_{efgh}, \\
 B_3 &= R_{abcd}R_{abef}R_{cdgh}R_{efgh}, & B_4 &= R_{acbd}R_{aebf}R_{cgdh}R_{efgh}, \\
 B_5 &= R_{abcd}R_{aefg}R_{befh}R_{cdgh}, & B_6 &= R_{abcd}R_{aefg}R_{bfhe}R_{cdgh}, \\
 B_7 &= R_{acbd}R_{aefg}R_{befh}R_{cgdh}.
 \end{aligned} \tag{C.1}$$

In the main part of this paper we considered the R^4 terms $t_8 t_8 R^4 - \frac{1}{4!} \epsilon_{11} \epsilon_{11} R^4 = 24(B_1 - 64B_2 + 2B_3 + 16B_4 - 16B_5 - 16B_6)$ that are explicitly written in Eq. (8). In order to derive the equations of motion, we need to calculate variations of (C.1). These are evaluated as

$$\begin{aligned}
 \delta B_1 &= 4(\delta R_{abcd})R_{abcd}R_{efgh}R_{efgh}, & \delta B_2 &= (\delta R_{abcd})R_{abce}R_{dfgh}R_{efgh}, \\
 \delta B_3 &= 4(\delta R_{abcd})R_{abef}R_{cdgh}R_{efgh}, & \delta B_4 &= 4(\delta R_{abcd})R_{aecg}R_{bfdh}R_{efgh}, \\
 \delta B_5 &= 2(\delta R_{abcd})R_{abeg}R_{cfhe}R_{dfgh} + 2(\delta R_{abcd})R_{efag}R_{efch}R_{gbhd}, \\
 \delta B_6 &= 2(\delta R_{abcd})R_{abeg}R_{cfhe}R_{dfgh} + 2(\delta R_{abcd})R_{efag}R_{efch}R_{gbhd} - 2(\delta R_{abcd})R_{abef}R_{cegh}R_{dfgh}, \\
 \delta B_7 &= 4(\delta R_{abcd})R_{aefg}R_{cefh}R_{gbhd}.
 \end{aligned} \tag{C.2}$$

By using these results, we evaluated Eq. (11) and derived the equations of motion (13).

Next let us consider the R^4 terms that necessarily depend on the Ricci tensor or the scalar curvature. Since the procedure for the classification is straightforward, we employ a Mathematica code. As a result, these are classified into 19 terms:

$$\begin{aligned}
 B_8 &= R_{abcd}R_{abcd}R_{ef}R_{ef}, & B_9 &= R_{abcd}R_{abcd}R^2, & B_{10} &= R_{abcd}R_{bcd}R_{ef}R_{ae}, \\
 B_{11} &= R_{abcd}R_{aefg}R_{bcd}R_{ef}, & B_{12} &= R_{abcd}R_{bcde}R_{ae}R, & B_{13} &= R_{acbd}R_{ced}R_{ef}R_{ab}, \\
 B_{14} &= R_{abcd}R_{abeg}R_{cdf}R_{ef}, & B_{15} &= R_{acbd}R_{aebg}R_{cdf}R_{ef}, & B_{16} &= R_{abcd}R_{abef}R_{ced}R, \\
 B_{17} &= R_{acbd}R_{aebf}R_{ced}R, & B_{18} &= R_{acbd}R_{ab}R_{cd}R, & B_{19} &= R_{abcd}R_{ced}R_{ae}R_{bf}, \\
 B_{20} &= R_{acbd}R_{ced}R_{ae}R_{bf}, & B_{21} &= R_{acbd}R_{ae}R_{be}R_{cd}, & B_{22} &= R_{ab}R_{ab}R_{cd}R_{cd}, \\
 B_{23} &= R_{ab}R_{ab}R^2, & B_{24} &= R_{ab}R_{cd}R_{ac}R_{bd}, & B_{25} &= R_{ab}R_{ac}R_{bc}R, \\
 B_{26} &= R^4.
 \end{aligned} \tag{C.3}$$

Then the effective action (9) is generalized into the form of

$$S_{11} = \frac{1}{2\kappa_{11}^2} \int d^{11}x e \left\{ R + \gamma \ell_s^{12} \left(t_8 t_8 R^4 - \frac{1}{4!} \epsilon_{11} \epsilon_{11} R^4 + \sum_{n=8}^{26} b_n B_n \right) \right\}. \tag{C.4}$$

The coefficients $b_n (n = 8, \dots, 26)$ cannot be determined from the results of scattering amplitudes in the type IIA superstring theory, since we can remove or add these terms by appropriate field redefinitions of the metric. Therefore it is expected that these terms do not affect physical quantities such as the internal energy of the black 0-brane. We will confirm this in Appendix D.

Let us derive equations of motion for the effective action (C.4). The variations of the 19 terms in (C.3) are evaluated as

$$\begin{aligned}
 \delta B_8 &= (\delta R_{abcd})(2R_{abcd}R_{ef}R_{ef} + 2R_{efgh}R_{efgh}R_{ac}\eta_{bd}), \\
 \delta B_9 &= (\delta R_{abcd})(2R_{abcd}R^2 + 2R_{efgh}R_{efgh}\eta_{ac}\eta_{bd}R), \\
 \delta B_{10} &= (\delta R_{abcd})(R_{ebcd}R_{af}R_{ef} + R_{afgh}R_{efgh}R_{ce}\eta_{bd}), \\
 \delta B_{11} &= (\delta R_{abcd})(-R_{ebcd}R_{afeg}R_{fg} - \frac{1}{2}R_{aefg}R_{cefg}R_{bd} - \frac{1}{2}R_{eghi}R_{fghi}R_{eafc}\eta_{bd}), \\
 \delta B_{12} &= (\delta R_{abcd})(R_{ebcd}R_{ae}R + \frac{1}{2}R_{aefg}R_{cefg}\eta_{bd}R + \frac{1}{2}R_{eghi}R_{fghi}R_{ef}\eta_{ac}\eta_{bd}), \\
 \delta B_{13} &= (\delta R_{abcd})(2R_{ebfd}R_{ac}R_{ef} + R_{agch}R_{efgh}R_{ef}\eta_{bd}), \\
 \delta B_{14} &= (\delta R_{abcd})(R_{abeg}R_{cdfg}R_{ef} + 2R_{abef}R_{efgd}R_{cg} + R_{efgh}R_{efai}R_{ghci}\eta_{bd}), \\
 \delta B_{15} &= (\delta R_{abcd})(R_{aecg}R_{bfdg}R_{ef} + 2R_{aecf}R_{egfd}R_{bg} + R_{efgh}R_{eagi}R_{fchi}\eta_{bd}), \\
 \delta B_{16} &= (\delta R_{abcd})(3R_{abef}R_{cdef}R + R_{efgh}R_{efij}R_{ghij}\eta_{ac}\eta_{bd}), \\
 \delta B_{17} &= (\delta R_{abcd})(3R_{aecf}R_{bedf}R + R_{efgh}R_{eigj}R_{fihj}\eta_{ac}\eta_{bd}), \\
 \delta B_{18} &= (\delta R_{abcd})(R_{ac}R_{bd}R + 2R_{aecf}R_{ef}\eta_{bd}R + R_{efgh}R_{eg}R_{fh}\eta_{ac}\eta_{bd}), \\
 \delta B_{19} &= (\delta R_{abcd})(2R_{cdef}R_{ae}R_{bf} + 2R_{aegh}R_{cfgh}R_{ef}\eta_{bd}), \\
 \delta B_{20} &= (\delta R_{abcd})(2R_{ebfd}R_{ae}R_{cf} + 2R_{ageh}R_{cgfh}R_{ef}\eta_{bd}), \\
 \delta B_{21} &= (\delta R_{abcd})(R_{ae}R_{ce}R_{bd} + 2R_{afeg}R_{ce}R_{fg}\eta_{bd} + R_{ebfd}R_{eg}R_{fg}\eta_{ac}), \\
 \delta B_{22} &= 4(\delta R_{abcd})R_{ac}R_{ef}R_{ef}\eta_{bd}, \\
 \delta B_{23} &= (\delta R_{abcd})(2R_{ac}\eta_{bd}R^2 + 2R_{ef}R_{ef}\eta_{ac}\eta_{bd}R), \\
 \delta B_{24} &= 4(\delta R_{abcd})R_{ef}R_{ae}R_{cf}\eta_{bd}, \\
 \delta B_{25} &= (\delta R_{abcd})(3R_{ae}R_{ce}\eta_{bd}R + R_{fg}R_{ef}R_{eg}\eta_{ac}\eta_{bd}), \\
 \delta B_{26} &= 4(\delta R_{abcd})\eta_{ac}\eta_{bd}R^3.
 \end{aligned} \tag{C.5}$$

Also, as in Eq. (12), we define the Y tensor as

$$\begin{aligned}
 Y_{abcd} &= \frac{1}{2}(Y'_{[ab][cd]} + Y'_{[cd][ab]}), \\
 Y'_{abcd} &= b_8(2R_{abcd}R_{ef}R_{ef} + 2R_{efgh}R_{efgh}R_{ac}\eta_{bd}) + b_9(2R_{abcd}R^2 + 2R_{efgh}R_{efgh}\eta_{ac}\eta_{bd}R) \\
 &\quad + b_{10}(R_{ebcd}R_{af}R_{ef} + R_{afgh}R_{efgh}R_{ce}\eta_{bd}) \\
 &\quad + b_{11}(-R_{ebcd}R_{afeg}R_{fg} - \frac{1}{2}R_{aefg}R_{cefg}R_{bd} - \frac{1}{2}R_{eghi}R_{fghi}R_{eafc}\eta_{bd}) \\
 &\quad + b_{12}(R_{ebcd}R_{ae}R + \frac{1}{2}R_{aefg}R_{cefg}\eta_{bd}R + \frac{1}{2}R_{eghi}R_{fghi}R_{ef}\eta_{ac}\eta_{bd}) \\
 &\quad + b_{13}(2R_{ebfd}R_{ac}R_{ef} + R_{agch}R_{efgh}R_{ef}\eta_{bd}) \\
 &\quad + b_{14}(R_{abeg}R_{cdfg}R_{ef} + 2R_{abef}R_{efgd}R_{cg} + R_{efgh}R_{efai}R_{ghci}\eta_{bd}) \\
 &\quad + b_{15}(R_{aecg}R_{bfdg}R_{ef} + 2R_{aecf}R_{egfd}R_{bg} + R_{efgh}R_{eagi}R_{fchi}\eta_{bd}) \\
 &\quad + b_{16}(3R_{abef}R_{cdef}R + R_{efgh}R_{efij}R_{ghij}\eta_{ac}\eta_{bd}) \\
 &\quad + b_{17}(3R_{aecf}R_{bedf}R + R_{efgh}R_{eigj}R_{fihj}\eta_{ac}\eta_{bd}) \\
 &\quad + b_{18}(R_{ac}R_{bd}R + 2R_{aecf}R_{ef}\eta_{bd}R + R_{efgh}R_{eg}R_{fh}\eta_{ac}\eta_{bd})
 \end{aligned} \tag{C.6}$$

$$\begin{aligned}
 &+ b_{19}(2R_{cdef}R_{ae}R_{bf} + 2R_{aegh}R_{cfg}R_{ef}\eta_{bd}) \\
 &+ b_{20}(2R_{ebfd}R_{ae}R_{cf} + 2R_{ageh}R_{cgh}R_{ef}\eta_{bd}) \\
 &+ b_{21}(R_{ae}R_{ce}R_{bd} + 2R_{afeg}R_{ce}R_{fg}\eta_{bd} + R_{ebfd}R_{eg}R_{fg}\eta_{ac}) \\
 &+ 4b_{22}R_{ac}R_{ef}R_{ef}\eta_{bd} + b_{23}(2R_{ac}\eta_{bd}R^2 + 2R_{ef}R_{ef}\eta_{ac}\eta_{bd}R) \\
 &+ 4b_{24}R_{ef}R_{ae}R_{cf}\eta_{bd} + b_{25}(3R_{ae}R_{ce}\eta_{bd}R + R_{fg}R_{ef}R_{eg}\eta_{ac}\eta_{bd}) + 4b_{26}\eta_{ac}\eta_{bd}R^3.
 \end{aligned}$$

Following the similar calculations in Eq. (11), we finally obtain the generic equations of motion

$$\begin{aligned}
 E_{ij} \equiv R_{ij} - \frac{1}{2}\eta_{ij}R + \gamma\ell_s^{12} \left\{ -\frac{1}{2}\eta_{ij} \left(t_8 t_8 R^4 - \frac{1}{4!} \epsilon_{11} \epsilon_{11} R^4 + \sum_{n=8}^{26} b_n B_n \right) \right. \\
 \left. + \frac{3}{2}R_{abci}X^{abc}{}_j - \frac{1}{2}R_{abcj}X^{abc}{}_i - 2D_{(a}D_{b)}X^a{}_{ij}{}^b \right. \\
 \left. + \frac{3}{2}R_{abci}Y^{abc}{}_j - \frac{1}{2}R_{abcj}Y^{abc}{}_i - 2D_{(a}D_{b)}Y^a{}_{ij}{}^b \right\} = 0. \tag{C.7}
 \end{aligned}$$

In order to evaluate these equations, we need to insert the values of the Riemann tensor (B.1) into the above. Since the Ricci tensor and the scalar curvature become zero, we obtain $B_n = 0$, and parts of b_{11} , b_{14} , b_{15} , b_{16} , and b_{17} in the Y tensor only contribute to the above equations of motion.

Below we repeat the similar calculations found in Appendix B. Each component of Y_{abcd} is evaluated as

$$\begin{aligned}
 Y_{0101} &= \frac{1}{U_0^6 x^{27} \ell_s^{12}} \left\{ \frac{11\,907}{2} b_{11}(1+x^7) - 21\,609 b_{14}(1+x^7) \right. \\
 &\quad \left. - \frac{3087}{2} b_{15}(1+x^7) - 85\,176 b_{16} - 10\,458 b_{17} \right\}, \\
 Y_{0\bar{a}0\bar{a}} &= \frac{1}{U_0^6 x^{27} \ell_s^{12}} \left\{ \frac{11\,907}{8} (-1+4x^7) b_{11} + \frac{63}{4} (5-1372x^7) b_{14} \right. \\
 &\quad \left. - \frac{63}{4} (17+98x^7) b_{15} - 85\,176 b_{16} - 10\,458 b_{17} \right\}, \\
 Y_{011\bar{1}} &= \frac{\sqrt{x^7-1}}{U_0^6 x^{\frac{47}{2}} \ell_s^{12}} \left\{ -\frac{11\,907}{2} b_{11} + 21\,609 b_{14} + \frac{3087}{2} b_{15} \right\}, \\
 Y_{0\bar{a}\bar{a}\bar{1}} &= \frac{\sqrt{x^7-1}}{U_0^6 x^{\frac{47}{2}} \ell_s^{12}} \left\{ -\frac{11\,907}{2} b_{11} + 21\,609 b_{14} + \frac{3087}{2} b_{15} \right\}, \\
 Y_{0\bar{1}0\bar{1}} &= \frac{1}{U_0^6 x^{27} \ell_s^{12}} \left\{ \frac{11\,907}{2} b_{11} - 21\,609 b_{14} - \frac{3087}{2} b_{15} - 85\,176 b_{16} - 10\,458 b_{17} \right\} \tag{C.8} \\
 Y_{1\bar{1}1\bar{1}} &= \frac{1}{U_0^6 x^{27} \ell_s^{12}} \left\{ -\frac{11\,907}{2} b_{11}(2-x^7) + 21\,609(2-x^7) b_{14} \right. \\
 &\quad \left. + \frac{3087}{2} b_{15}(2-x^7) + 85\,176 b_{16} + 10\,458 b_{17} \right\}, \\
 Y_{1\bar{a}1\bar{a}} &= \frac{1}{U_0^6 x^{27} \ell_s^{12}} \left\{ -\frac{35\,721}{8} b_{11} + \frac{86\,121}{4} b_{14} + \frac{7245}{4} b_{15} + 85\,176 b_{16} + 10\,458 b_{17} \right\},
 \end{aligned}$$

$$\begin{aligned}
 Y_{\bar{a}\bar{b}\bar{a}\bar{b}} &= \frac{1}{U_0^6 x^{27} \ell_s^{12}} \left\{ \frac{11\,907}{8} b_{11} (-3 + 4x^7) + \frac{63}{4} b_{14} (1367 - 1372x^7) \right. \\
 &\quad \left. + \frac{63}{4} b_{15} (115 - 98x^7) + 85\,176 b_{16} + 10\,458 b_{17}, \right. \\
 Y_{\bar{a}\bar{b}\bar{a}\bar{b}} &= \frac{1}{U_0^6 x^{27} \ell_s^{12}} \left\{ \frac{11\,907}{4} b_{11} - \frac{315}{2} b_{14} + \frac{1071}{2} b_{15} + 85\,176 b_{16} + 10\,458 b_{17} \right\},
 \end{aligned}$$

where $\bar{a}, \bar{b} = 2, \dots, 9$. By using these results it is possible to evaluate the higher derivative terms that depend on the Y tensor in Eq. (C.7). The RY terms are calculated as

$$\begin{aligned}
 R_{abc0} Y^{abc}_0 &= \frac{1}{U_0^8 x^{29} \ell_s^{16}} \left\{ 416\,745 b_{11} - 1214\,514 b_{14} - 71\,442 b_{15} \right\}, \\
 R_{abc1} Y^{abc}_1 &= \frac{1}{U_0^8 x^{36} \ell_s^{16}} \left\{ -416\,745 b_{11} + 1214\,514 b_{14} + 71\,442 b_{15} \right\}, \\
 R_{abc\bar{a}} Y^{abc}_{\bar{a}} &= \frac{x^7 - 1}{U_0^8 x^{36} \ell_s^{16}} \left\{ 416\,745 b_{11} - 1214\,514 b_{14} - 71\,442 b_{15} \right\}, \\
 R_{abc\bar{a}} Y^{abc}_{\bar{b}} &= \frac{1}{4U_0^8 x^{36} \ell_s^{16}} \delta_{\bar{a}\bar{b}} \left\{ 416\,745 b_{11} - 1214\,514 b_{14} - 71\,442 b_{15} \right\}, \\
 R_{abc0} Y^{abc}_{\bar{a}} &= R_{abc\bar{a}} Y^{abc}_0 = \frac{\sqrt{x^7 - 1}}{U_0^8 x^{\frac{65}{2}} \ell_s^{16}} \left\{ 416\,745 b_{11} - 1214\,514 b_{14} - 71\,442 b_{15} \right\},
 \end{aligned} \tag{C.9}$$

and DDY terms become

$$\begin{aligned}
 D_{(a} D_b) Y^a{}_{00}{}^b &= \frac{1701}{U_0^8 x^{36} \ell_s^{16}} \left\{ -\frac{7}{2} (-459 - 235x^7 + 540x^{14}) b_{11} \right. \\
 &\quad + (-6507 - 2397x^7 + 6860x^{14}) b_{14} + \frac{1}{2} (-999 - 282x^7 + 980x^{14}) b_{15} \\
 &\quad \left. + (-36\,504 + 31\,772x^7) b_{16} + (-4482 + 3901x^7) b_{17} \right\}, \\
 D_{(a} D_b) Y^a{}_{11}{}^b &= \frac{1701}{U_0^8 x^{36} \ell_s^{16}} \left\{ -7(31 + 46x^7) b_{11} + 4(6 + 505x^7) b_{14} \right. \\
 &\quad \left. + \frac{1}{2} (-75 + 376x^7) b_{15} + 676(-9 + 16x^7) b_{16} + 83(-9 + 16x^7) b_{17} \right\}, \\
 D_{(a} D_b) Y^a{}_{\bar{a}\bar{a}}{}^b &= \frac{1701}{U_0^8 x^{36} \ell_s^{16}} \left\{ -\frac{7}{2} (1034 - 1455x^7 + 540x^{14}) b_{11} \right. \\
 &\quad + (13\,724 - 18\,897x^7 + 6860x^{14}) b_{14} + \frac{1}{2} (2021 - 2742x^7 + 980x^{14}) b_{15} \\
 &\quad \left. + 676(47 - 33x^7) b_{16} + 83(47 - 33x^7) b_{17} \right\},
 \end{aligned} \tag{C.10}$$

$$D_{(a}D_b)Y^a{}_{\bar{a}\bar{b}}{}^b = \frac{-4 + 3x^7}{U_0^8 x^{36} \ell_s^{16}} \delta_{\bar{a}\bar{b}} \left\{ \frac{1917027}{4} b_{11} - \frac{6013035}{2} b_{14} - \frac{559629}{2} b_{15} \right. \\ \left. - 16098264 b_{16} - 1976562 b_{17} \right\},$$

$$D_{(a}D_b)Y^a{}_{0\bar{a}}{}^b = D_{(a}D_b)Y^a{}_{\bar{0}}{}^b \\ = \frac{\sqrt{x^7 - 1}}{U_0^8 x^{\frac{65}{2}} \ell_s^{16}} \left\{ \frac{59535}{2} (115 - 108x^7) b_{11} - 11907(1031 - 980x^7) b_{14} \right. \\ \left. - 11907(73 - 70x^7) b_{15} + 8049132 b_{16} + 988281 b_{17} \right\}.$$

As mentioned before, only $b_{11}, b_{14}, b_{15}, b_{16},$ and b_{17} appear in the calculations.

By using the ansatz (14) and inserting the values of the X and Y tensors into the equations of motion (C.7), we obtain five independent equations with parameters $b_{11}, b_{14}, b_{15}, b_{16},$ and b_{17} :

$$E_1 = -63x^{34} f_1 - 9x^{35} f_1' - 49x^{41} h_1 + 49x^{34} (1 - x^7) h_2 + 23x^{35} (1 - x^7) h_2' + 2x^{36} (1 - x^7) h_2'' \\ + 98x^{41} h_3 + 7x^{42} h_3' - 63402393600x^{14} + 70230343680x^7 + 1062512640 \\ + (25719120b_{11} - 93350880b_{14} - 6667920b_{15})x^{14} \tag{C.11} \\ + (-9525600b_{11} + 27760320b_{14} + 1632960b_{15} - 432353376b_{16} - 53084808b_{17})x^7 \\ - 21861252b_{11} + 88547256b_{14} + 6797196b_{15} + 496746432b_{16} + 60991056b_{17} = 0,$$

$$E_2 = 63x^{34} f_1 + 9x^{35} f_1' + 7x^{34} (9 - 2x^7) h_1 + 9x^{35} (1 - x^7) h_1' - 112x^{34} (1 - x^7) h_2 \\ - 16x^{35} (1 - x^7) h_2' - 98x^{41} h_3 - 7x^{42} h_3' - 2159861760x^7 - 5730600960 \\ + (4381776b_{11} - 27488160b_{14} - 2558304b_{15} - 147184128b_{16} - 18071424b_{17})x^7 \\ + 1285956b_{11} + 4531464b_{14} + 796068b_{15} + 82791072b_{16} + 10165176b_{17} = 0, \tag{C.12}$$

$$E_3 = 133x^{34} f_1 + 35x^{35} f_1' + 2x^{36} f_1'' + 28x^{34} (3 - 10x^7) h_1 + 7x^{35} (4 - 7x^7) h_1' \\ + 2x^{36} (1 - x^7) h_1'' - 7x^{34} (5 - 26x^7) h_2 - 21x^{35} (1 - 2x^7) h_2' - 2x^{36} (1 - x^7) h_2'' \\ + 98x^{41} h_3 + 7x^{42} h_3' + 5669637120x^7 - 8626383360 \\ + (-11502162b_{11} + 72156420b_{14} + 6715548b_{15} + 386358336b_{16} + 47437488b_{17})x^7 \\ + 15752961b_{11} - 97423074b_{14} - 9025506b_{15} - 515144448b_{16} - 63249984b_{17} = 0, \tag{C.13}$$

$$E_4 = 259x^{34} f_1 + 53x^{35} f_1' + 2x^{36} f_1'' + 147x^{34} (1 - 3x^7) h_1 + x^{35} (37 - 58x^7) h_1' \\ + 2x^{36} (1 - x^7) h_1'' + 147x^{41} h_2 + 21x^{42} h_2' + 294x^{41} h_3 + 21x^{42} h_3' \\ - 63402393600x^{14} + 133632737280x^7 - 71292856320 \\ + x^{14} (25719120b_{11} - 93350880b_{14} - 6667920b_{15}) + x^7 (-67631760b_{11} \\ + 252292320b_{14} + 18370800b_{15} + 303567264b_{16} + 37272312b_{17}) \\ + 47580372b_{11} - 181898136b_{14} - 13465116b_{15} - 432353376b_{16} - 53084808b_{17} = 0, \tag{C.14}$$

$$E_5 = 49x^{34} h_1 + 7x^{35} h_1' + 49x^{34} h_2 - x^{35} h_2' - x^{36} h_2'' - 98x^{34} h_3 - 22x^{35} h_3' - x^{36} h_3'' \\ - 63402393600x^7 + 70230343680 \\ + x^7 (25719120b_{11} - 93350880b_{14} - 6667920b_{15}) \\ - 25719120b_{11} + 93350880b_{14} + 6667920b_{15} - 64393056b_{16} - 7906248b_{17} = 0. \tag{C.15}$$

Here we have defined $E_1 = 4U_0^8 \ell_s^4 x^{36} \gamma^{-1} E_{00}$, $E_2 = 4U_0^8 \ell_s^4 x^{36} \gamma^{-1} E_{11}$, $E_3 = 4U_0^8 \ell_s^4 x^{36} \gamma^{-1} E_{22}$, $E_4 = 4U_0^8 \ell_s^4 x^{36} \gamma^{-1} E_{33}$, and $E_5 = 4U_0^8 \ell_s^4 x^{\frac{65}{2}} (-1 + x^7)^{-\frac{1}{2}} \gamma^{-1} E_{01}$.

Equations (C.11)–(C.15) can be solved by following the details in Sect. 4, and the final form of the solution becomes

$$\begin{aligned}
 h_1 = & \left(-\frac{440\,559}{4} b_{11} + \frac{768\,775}{2} b_{14} + \frac{53\,333}{2} b_{15} + 927\,472 b_{16} + 113\,876 b_{17} \right. \\
 & \left. + \frac{1302\,501\,760}{9} \right) \frac{1}{x^{34}} + (23\,814 b_{11} - 86\,436 b_{14} - 6174 b_{15} - 57\,462\,496) \frac{1}{x^{27}} \\
 & + \frac{12\,051\,648}{13x^{20}} - \frac{4782\,400}{13x^{13}} - \frac{3747\,840}{x^7} + \frac{4099\,200}{x^6} - \frac{1639\,680(x-1)}{(x^7-1)} \\
 & + 117\,120 \left(18 - \frac{23}{x^7} \right) I(x), \\
 h_2 = & \left(-\frac{11\,907}{4} b_{11} + \frac{315}{2} b_{14} - \frac{1071}{2} b_{15} - 170\,352 b_{16} - 20\,916 b_{17} + 19\,160\,960 \right) \frac{1}{x^{34}} \\
 & + (23\,814 b_{11} - 86\,436 b_{14} - 6174 b_{15} - 58\,528\,288) \frac{1}{x^{27}} + \frac{2213\,568}{13x^{20}} - \frac{1229\,760}{13x^{13}} \\
 & - \frac{2108\,160}{x^7} + \frac{2459\,520}{x^6} + 1054\,080 \left(2 - \frac{1}{x^7} \right) I(x), \tag{C.16} \\
 h_3 = & \left(-\frac{11\,907}{4} b_{11} + \frac{76\,027}{2} b_{14} + \frac{8225}{2} b_{15} - 94\,640 b_{16} - 11\,620 b_{17} + \frac{361\,110\,400}{9} \right) \frac{1}{x^{34}} \\
 & + (23\,814 b_{11} - 86\,436 b_{14} - 6174 b_{15} - 59\,840\,032) \frac{1}{x^{27}} - \frac{24\,021\,312}{13x^{20}} \\
 & - \frac{58\,072\,000}{13x^{13}} - \frac{2108\,160}{x^7} + \frac{2459\,520}{x^6} + 117\,120 \left(18 - \frac{41}{x^7} \right) I(x), \\
 f_1 = & \left(\frac{440\,559}{4} b_{11} - \frac{730\,919}{2} b_{14} - \frac{48\,685}{2} b_{15} - 889\,616 b_{16} - 109\,228 b_{17} \right. \\
 & \left. - \frac{1208\,170\,880}{9} \right) \frac{1}{x^{34}} + (-130\,977 b_{11} + 432\,810 b_{14} + 28\,728 b_{15} + 1022\,112 b_{16} \\
 & + 125\,496 b_{17} + 161\,405\,664) \frac{1}{x^{27}} + \frac{5738\,880}{13x^{20}} + \frac{956\,480}{x^{13}} + \frac{819\,840}{x^7} I(x).
 \end{aligned}$$

The function $I(x)$ is given by Eq. (34) and integral constants are determined so as to satisfy that $h_i(1)$ are finite and $h_i(x), f_1(x) \sim \mathcal{O}(x^{-8})$ when x goes to infinity. Notice that $b_{11}, b_{14}, b_{15}, b_{16}$, and b_{17} only appear in the coefficients of x^{-27} and x^{-34} . The solution is reliable up to $\mathcal{O}(\epsilon^2)$.

Appendix D. Thermodynamics of the black 0-brane with generic R^4 terms

In this appendix, we examine the thermodynamics of the quantum near-horizon geometry of the black 0-brane (C.16) by following the arguments in Sect. 5. Although the solution is modified, the results obtained up to Eq. (50) do not change. Since the effective action is modified as in Eq. (C.4), Eq. (51) should be replaced with

$$\frac{\partial S_{11}}{\partial R_{\mu\nu\rho\sigma}} = \frac{1}{2\kappa_{11}^2} \{ g^{\mu[\rho} g^{\sigma]v} + \gamma \ell_s^{12} (X^{\mu\nu\rho\sigma} + Y^{\mu\nu\rho\sigma}) \}. \tag{D.1}$$

The entropy of the quantum near-horizon geometry of the black 0-brane is evaluated as

$$\begin{aligned}
 S &= \frac{4\pi}{2\kappa_{11}^2} \int_{\text{H}} d\Omega_8 dz \sqrt{h} \left(1 - \frac{1}{2} \gamma \ell_s^{12} (X^{\mu\nu\rho\sigma} + Y^{\mu\nu\rho\sigma}) N_{\mu\nu} N_{\rho\sigma} \right) \\
 &= \frac{4\pi}{2\kappa_{11}^2} \int_{\text{H}} d\Omega_8 dz \sqrt{h} \left(1 - 2\gamma \ell_s^{20} U_0^2 H_1^{-1} (X^{txtx} + Y^{txtx}) \right) \\
 &= \frac{4\pi}{2\kappa_{11}^2} \int_{\text{H}} d\Omega_8 dz \sqrt{h} \left\{ 1 + \epsilon U_0^{-6} (40\,642\,560 \right. \\
 &\quad \left. - 23\,814b_{11} + 86\,436b_{14} + 61\,74b_{15} + 170\,352b_{16} + 20\,916b_{17}) \right\} \\
 &= \frac{4}{49} a_1 N^2 \tilde{U}_0^{\frac{9}{2}} \left\{ 1 + \epsilon \left(-\frac{9}{14} f_1(1) + \frac{1}{2} h_2(1) + 40\,642\,560 \right. \right. \\
 &\quad \left. \left. - 23\,814b_{11} + 86\,436b_{14} + 61\,74b_{15} + 170\,352b_{16} + 20\,916b_{17} \right) \tilde{U}_0^{-6} \right\} \\
 &= \frac{4}{49} a_1^{-\frac{4}{5}} N^2 \tilde{T}^{\frac{9}{5}} \left\{ 1 + \epsilon a_1^{\frac{12}{5}} \left(-\frac{9}{5} f_1(1) - \frac{9}{35} f_1'(1) + \frac{9}{10} h_1(1) + \frac{1}{2} h_2(1) + 40\,642\,560 \right. \right. \\
 &\quad \left. \left. - 23\,814b_{11} + 86\,436b_{14} + 61\,74b_{15} + 170\,352b_{16} + 20\,916b_{17} \right) \tilde{T}^{-\frac{12}{5}} \right\} \\
 &= a_3 N^2 \tilde{T}^{\frac{9}{5}} (1 + \epsilon a_5 \tilde{T}^{-\frac{12}{5}}). \tag{D.2}
 \end{aligned}$$

Notice that $f_1(1)$, $f_1'(1)$, $h_1(1)$, and $h_2(1)$ depend on b_{11} , b_{14} , b_{15} , b_{16} , and b_{17} . The value of a_3 is given in Sect. 5, and a_5 is given by

$$\begin{aligned}
 a_5 &= a_1^{\frac{12}{5}} \left(-\frac{9}{5} f_1(1) - \frac{9}{35} f_1'(1) + \frac{9}{10} h_1(1) + \frac{1}{2} h_2(1) + 40\,642\,560 \right. \\
 &\quad \left. - 23\,814b_{11} + 86\,436b_{14} + 61\,74b_{15} + 170\,352b_{16} + 20\,916b_{17} \right). \tag{D.3}
 \end{aligned}$$

It seems that a_5 depends on b_{11} , b_{14} , b_{15} , b_{16} , and b_{17} . The explicit calculation, however, shows that $a_5 = a_4$ and the result does not depend on the ambiguities of the effective action. Thus the physical quantities of the black 0-brane are free from the ambiguities and are uniquely determined.

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