Lawrence Berkeley National Laboratory
Lawrence Berkeley National Laboratory

Title
QUANTUM NOISE THEORY FOR THE dc SQUID

Permalink
https://escholarship.org/uc/item/7h58999x

Author
Koch, Roger H.

Publication Date
1980-10-01

Lawrence Berkeley Laboratory

UNIVERSITY OF CALIFORNIA

Materials & Molecular Research Division

QUANTUM NOISE THEORY FOR THE dc SQUID

Roger H. Koch, D.J. Van Harlingen, and John Clarke

October 1980

DISCLAIMER

QUANTUM NOISE THEORY FOR THE dc SQUID

Roger H. Koch, D. J. Van Harlingen and John Clarke

Department of Physics, University of California, and
Materials and Molecular Research Division, Lawrence
Berkeley Laboratory, Berkeley, California 94720

Abstract

The noise temperature of a dc SQUID coupled to a tuned input circuit

is computed using the complete quantum expression for the equilibrium noise

in the shunt resistance of each junction. At T=0, where the noise reduces

to zero-point fluctuations, the noise temperature for an optimized system

is $h\nu/k_B \ln 2$, where ν is the signal frequency. The computation is extended

to non-zero temperatures, and it is shown that a SQUID operated at 1K can

approach the quantum limit.

Recent research on dc SQUIDs has been directed toward the fabrication

of a device limited in sensitivity by intrinsic quantum mechanical noise

processes. In this Letter we present a model calculation of the noise in

the dc SQUID that involves the complete quantum expression for the equilib-

rium noise in the shunt resistance of each junction. At T=0, where the

sensitivity is limited by zero-point fluctuations in the shunt resistances,

for an optimized SQUID we find an equivalent noise energy $\varepsilon/1Hz = S_\phi/2L \approx \hbar$,

where S_ϕ is the spectral density of the equivalent flux noise, and L is

the SQUID inductance. When the SQUID is coupled to an optimized series-

resonant input circuit, we find the noise temperature of the amplifier to

be $T_N \approx h\nu/k_B \ln 2$, where ν is the signal frequency. This noise temperature represents
the uncertainty principle limit for any/amplifier and thus the system
linear
operates nearly as an ideal amplifier at T=0. We show that by proper

choice of parameters it should be possible to approach this ideal behavior

with a SQUID operated at 1K.

Tesche and Clarke[1] calculated $\varepsilon/1$Hz for a SQUID incorporating two

resistively shunted Josephson junctions (RSJ) with zero capacitance

assuming that the only source of noise was thermal noise in the resistive

shunts. Their predictions have subsequently been found to be in reasonable

accord with experimental measurements.[2] In the absence of any rigorous

calculation of the limiting voltage noise in a current-biased RSJ at T=0,

they speculated that the ultimate sensitivity should be limited by pair

shot noise in the tunnel junctions. Assuming a shot noise voltage spectral

density $2eI_0R^2$ for a junction with critical current I_0 and shunt resistance R,

they found $\varepsilon/1$Hz $\approx h/2$ for an optimized SQUID. However, subsequent calculations
an overdamped
by Koch et al.[3] suggest that the noise in / RSJ at T=0 is set by zero-point

fluctuations in the shunt resistance rather than by any intrinsic shot

noise in the junction itself.[4] We now apply the zero-point fluctuation

calculation, including the effects of junction capacitance, to the case of

the SQUID.

We begin by considering the isolated symmetric SQUID. Each of the two

junctions has a critical current I_0, capacitance C, and shunt resistance R.

We introduce the dimensionless parameters $\beta_c \equiv 2\pi I_0 R^2 C/\Phi_0$, $\beta \equiv 2LI_0/\Phi_0$,

$\kappa \equiv eI_0R/k_BT$ and $\Gamma \equiv 2\pi k_BT/I_0\Phi_0$. The phase differences across junctions 1

and 2, δ_1 and δ_2, and the voltage V(t) across the SQUID are related to the

circulating current J(t) and applied flux Φ by the equations

$$J/I_0 = (\delta_1 - \delta_2 - 2\pi\Phi/\Phi_0)/\pi\beta, \tag{1}$$

$$V = (\dot{\delta}_1 + \dot{\delta}_2)\,\Phi_0/4\pi, \tag{2}$$

$$\beta_c \frac{\ddot{\delta}_1}{I_0}\left(\frac{\Phi_0}{2\pi R}\right)^2 + \dot{\delta}_1\left(\frac{\Phi_0}{2\pi R}\right) = \frac{I}{2} - J - I_0 \sin\delta_1 + I_{n1}, \tag{3}$$

$$\beta_c \frac{\ddot{\delta}_2}{I_0}\left(\frac{\Phi_0}{2\pi R}\right)^2 + \dot{\delta}_2\left(\frac{\Phi_0}{2\pi R}\right) = \frac{I}{2} + J - I_0 \sin\delta_2 + I_{n2}. \tag{4}$$

Here, I_{n1} and I_{n2} are the equilibrium noise currents generated in the
shunt resistors, each with a spectral density $(2h\nu/R)\coth(h\nu/2k_BT)$, which
includes zero-point fluctuations. To find the limiting performance, we first
solve Eqs.(1) to (4) numerically to obtain $\varepsilon/1Hz$ and T_N for a SQUID at $T=0$
with $\kappa\Gamma=0.02$ ($R\approx40\Omega$) and $\beta=1$; at $T=0$, the spectral density of the current
noise in each shunt resistor reduces to $2h\nu/R$. The computer techniques used
were similar to those described briefly in our earlier paper.[3] Figure
1 (a)-(f) shows the time-averaged voltage V, the transfer function $\partial V/\partial\Phi$, and
the noise spectral densities S_V, S_J, S_{VJ}, and $\varepsilon/1Hz$ vs. Φ for three values
of β_c, where S_V and S_J are the spectral densities of the voltage noise across
and the current noise around the SQUID; S_{VJ} is the cross-spectral density of
these two noises[5], and $\varepsilon/1Hz = S_\Phi/2L$, where $S_\Phi = S_V/(\partial V/\partial\Phi)^2$. The current
I, which has the same value of 1.63 I_0 for all the curves, has been chosen
so that the maximum in $\partial V/\partial\Phi$ always occurs near $\Phi=\Phi_0/4$. In a separate
investigation, we have found that this procedure produces the optimum performance.
The peaks in the noise spectral densities occur at slightly different values
of applied flux than the peak in $\partial V/\partial\Phi$. As β_c is increased from 0.25 to 1,
the maximum values of $\partial V/\partial\Phi$ and S_V, S_J, and S_{VJ} all increase markedly.
However, the minimum values of $\varepsilon/1Hz$ are remarkably close to each other, reflecting
the fact that $(\partial V/\partial\Phi)^2$ and S_V increase almost proportionately as β_c is changed
with R held constant.

The minima in ε/1Hz, about $\hbar$ in all three cases, are much broader with respect to the applied flux than the peaks in $\partial V/\partial\Phi$ and the noise spectral densities.

The quantity ε/1Hz is a convenient parameter for comparing the performance of isolated SQUIDS, but does not completely specify the sensitivity of the device. In practice, the SQUID must be coupled to an input circuit, and one must take into account not only S_V, but also S_J and S_{VJ}. The current noise of the SQUID couples back to the input coil, producing a real voltage noise at the input terminals, while the voltage noise of the SQUID can be regarded as an effective noise referred to the input. In ref. 6 it was shown that the tuned input circuit shown in Fig. 2 has a higher sensitivity than an untuned circuit. For a given source resistance R_i, the values of L_i and C_i have been optimized[6] for a signal frequency $\nu << 2eV/h$. From Eqs. (15) and (16) of ref. 6, it is straightforward to show that the SQUID introduces a total effective voltage noise referred to the input terminals of the voltmeter with a spectral density

$$S_V^{(s)}(\nu) = \frac{4\pi\nu R_i}{\partial V/\partial\Phi} \ (S_V S_J - S_{VJ}^2)^{1/2}. \tag{5}$$

We define the quantity $n^{(s)}h\nu = S_V^{(s)}(\nu)/4R_i$ as the effective mean photon power per hertz in the input circuit due to intrinsic SQUID noise. Thus

$$n^{(s)} = \pi(S_V S_J - S_{VJ}^2)^{1/2}/h(\partial V/\partial\Phi). \tag{6}$$

In Fig. 1(g), we plot $n^{(s)}$ vs. Φ. We see that for $\beta_c = 0.25$ and 0.5, $n^{(s)}$ has the minimum value of $\sim$1/2, whereas for $\beta_c = 1$, the minimum has a somewhat higher value. Thus, for $\beta_c = 0.25$ and 0.5, an optimized tuned voltmeter at T=0 has a mean input noise due to the intrinsic SQUID noise of 1/2 photon

at the measurement frequency.

The intrinsic noise can be expressed alternatively as a noise temperature.[7,8] At a signal frequency ν, we add the SQUID noise in the input circuit, $\sim h\nu/2$, to the zero-point fluctuations of the input resistor, $h\nu/2$, and equate the sum to the thermal noise power per Hz available from the resistor at an effective temperature T_N, namely $h\nu/[\exp(h\nu/k_B T_N)-1]$.

We thus obtain

$$T_N \approx h\nu/k_B \ln 2 , \tag{7}$$

the optimum value for any linear amplifier.[7]

We now consider briefly the experimentally realistic case $T>0$. In Table I we list values of $\partial V/\partial \Phi$, $\varepsilon/1Hz$ and $n^{(s)}$ for a SQUID with an inductance of 1pH and a capacitance at 0.5pF per junction (corresponding to a junction area of about $10\mu m^2$), for specified values of T, β, I_0, β_c, and R. We also list $\kappa V/I_0 R$, the parameter characterizing the relative importance of the quantum and thermal noise sources, and $\Delta\Phi/\Phi_0$, a measure of the applied flux over which the quoted sensitivity can be obtained. For the computations at 4.2K we used the classical limit $4k_B T/R$ for the spectral density of the current noise in each shunt, whereas at the lower temperatures we used the full expression $(2h\nu/R)$ coth $(h\nu/2k_B T)$. At 4.2K, we find for the fixed values of SQUID inductance and junction capacitance chosen, the best $\varepsilon/1Hz$ occurs at $\beta \approx \beta_c \approx 1$. At 4.2K, the values of $\varepsilon/1Hz$ range from about $3\hbar$ to $11\hbar$,

while the values of $n^{(s)}$ range from about 4.5 to 9. It is interesting to note that the lowest values of $\varepsilon/1Hz$ do not necessarily correspond to the lowest values of $n^{(s)}$ due to the effects of current noise. We have

also computed the sensitivity at T=0.9K and 0.45K for the values

$\beta = \beta_c = 1$, and find $n^{(s)} = 2.0$ and 1.5, respectively. It is interesting

to note that the SQUID approaches the quantum limit when $\kappa V/I_o R \gtrsim 1$, as we

would expect from our analysis of the single junction. As one goes to lower

temperatures, the range of applied flux, $\Delta\Phi/\Phi_o$, over which the best

sensitivity is obtained narrows from 0.005 at 4.2K to 0.001 at the lower

temperatures.

In conclusion, we have shown that at T=0 a dc SQUID coupled to a

tuned input circuit has a noise temperature of $\sim h\nu/k_B \ln 2$, corresponding to

an intrinsic photon power per Hz of $\sim h\nu$, when the parameters are properly

optimized. At a temperature of 1K, one should be able to achieve a photon

power per Hz of about $2h\nu$, using junctions with a critical current density

of $10^8 Am^{-2}$. Junctions with a smaller area and a higher critical current

density should enable one to achieve an even better performance, provided

that heating or other non-equilibrium effects do not become important.

The near ideal performance for a given bias current is obtained over a

relatively narrow range of flux, typically 10^{-3} to $10^{-2}\Phi_o$. However, since

the flux noise is typically $10^{-8}\Phi_o Hz^{-1/2}$, the dynamic range of the amplifier

can still be considerable, at least 10^5 in a unit bandwidth. The limited

working range of the SQUID makes it most attractive as a small-signal

amplifier, without flux-modulation or negative feedback, although operation

in a flux-locked loop with near-ideal sensitivity may not be out of the

question. In our calculations we have entirely neglected the contributions

of 1/f noise which will, of course, degrade the performance at low

frequencies. On the other hand, the quoted performance should be obtainable up to high frequencies, say 1/10 of the Josephson frequency or about 10 GHz for the devices listed in Table I.

Finally, we remark that it may be difficult to achieve a high coupling coefficient, α, between a SQUID with an inductance as low as 1pH and an input inductance large enough to be useful in many applications. In principle, since α does not appear in Eq. (6), a low coupling efficiency does not have a deleterious effect on the noise/temperature, but, in practice, a low value of α may lead to undesirable values of R_i and/or C_i[6]. However, if one were to reduce the junction area to $0.1\mu m^2$, one could increase L to the more useful value of 100pH and still achieve comparable performances to those listed in Table I.
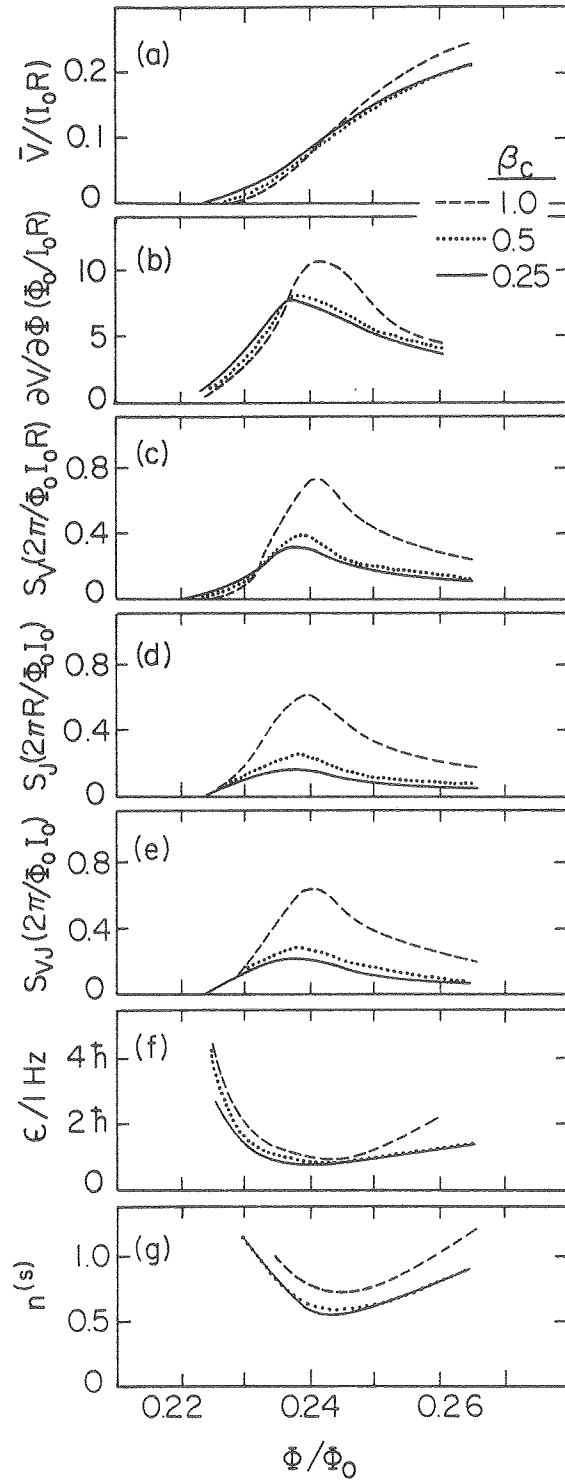
References

1. C. D. Tesche and J. Clarke, J. Low Temp. Phys. 29, 301 (1977).

2. For a recent review, see J. Clarke, to be published in IEEE Trans. on Electron Devices, October 1980.

3. R. H. Koch, D. J. Van Harlingen, and J. Clarke, submitted to Phys. Rev. Lett.

4. The possibility that zero point fluctuations provided the limiting sensitivity has been discussed by J. C. Gallop and B. W. Petley, J. Phys. E 9, 417 (1976).

5. C. D. Tesche and J. Clarke, J. Low Temp. Phys. 37, 397 (1979).

6. J. Clarke, C. D. Tesche, and R. P. Giffard, J. Low Temp. Phys. 37, 405 (1979).

7. W. H. Louisell, A. Yariv, and A. E. Siegman, Phys. Rev. 124, 1646 (1961).

8. J. P. Gordon, W. H. Louisell, and L. R. Walker, Phys. Rev. 129, 481 (1963).

Table I. Parameters for SQUID with L=1pH and C=0.5pF. Computed values accurate ±15%.

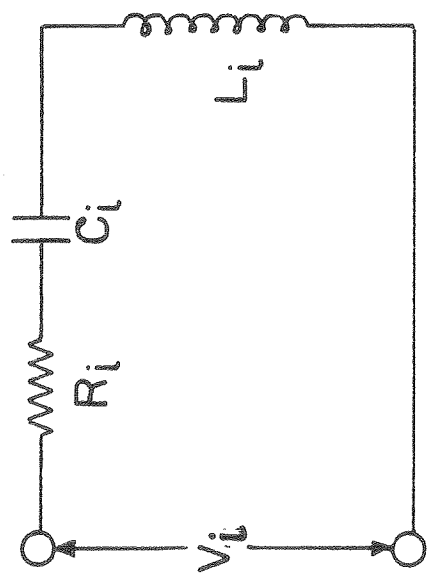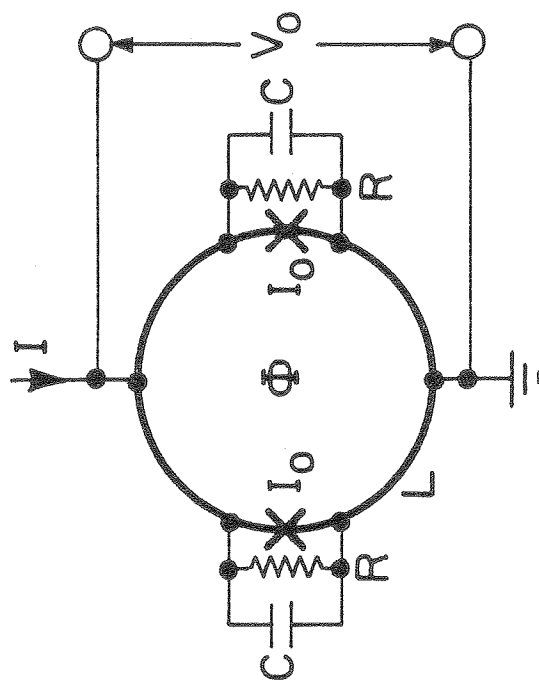| T (K)	β	I_0 (mA)	β_c	R (Ω)	$\kappa V/I_0 R$	$\partial V/\partial \Phi$ (mV/Φ_0)	$\varepsilon/1\text{Hz}$ ($\hbar$)	$n^{(s)}$	$\Delta\Phi/\Phi_0$
4.2	0.5	0.5	1.0	1.13	0.23	27	6	6.0	0.003
4.2	0.5	0.5	0.5	0.80	0.07	8	6	7.5	~0.003
4.2	0.5	0.5	0.25	0.56	0.05	5	7	9.0	0.003
4.2	1.0	1.0	1.0	0.80	0.19	27	3	5.0	0.005
4.2	1.0	1.0	0.5	0.56	0.09	9	7	6.5	0.005
4.2	1.0	1.0	0.25	0.40	0.06	6	9	8.0	0.005
4.2	3.0	3.0	1.0	0.46	0.25	16	6	4.5	0.01
4.2	3.0	3.0	0.5	0.33	0.15	9	7	5.5	0.01
4.2	3.0	3.0	0.25	0.23	0.11	6	11	7.0	0.01
0.9	1.0	1.0	1.00	0.80	0.60	57	1.1	2.0	0.001
0.45	1.0	1.0	1.00	0.80	1.20	80	0.6	1.5	0.001

Figure Captions

Fig. 1   Computed values of V, $\partial V/\partial \Phi$, S_V, S_J, S_{VJ}, $S\Phi/2L$, and $n^{(s)}$ vs.
Φ/Φ_0 for a dc SQUID at $T=0$ with $\kappa\Gamma=0.02$, $\beta=1$, a bias current of
$1.63I_0$, and for $\beta_c=0.25$, 0.5, and 1. Computed values are accurate to $\pm 15\%$.

Fig. 2   Schematic of dc SQUID coupled to tuned input circuit; V_i and V_o
are the input and output voltages.

Fig. 1

XBL 8010-6148

Fig. 2