Experimental Mathematics

ISSN: 1058-6458 (Print) 1944-950X (Online) Journal homepage: http://www.tandfonline.com/loi/uexm20

Quantum Periods for Certain Four-Dimensional Fano Manifolds

Tom Coates, Sergey Galkin, Alexander Kasprzyk & Andrew Strangeway

EXPERIMENTAL MATHEMATICS
2018, VOL. 0, NO. 0, 1–39
https://doi.org/10.1080/10586458.2018.1448018

Taylor & Francis
Taylor & Francis Group

Check for updates

Quantum Periods for Certain Four-Dimensional Fano Manifolds

Tom Coates[a], Sergey Galkin [b], Alexander Kasprzyk[a], and Andrew Strangeway[a]

[a]School of Mathematical Sciences, University of Nottingham, Nottingham, UK; [b]Faculty of Mathematics and Laboratory of Algebraic Geometry, National Research University Higher School of Economics, Moscow, Russia

ABSTRACT
We collect a list of known four-dimensional Fano manifolds and compute their quantum periods. This list includes all four-dimensional Fano manifolds of index greater than one, all four-dimensional toric Fano manifolds, all four-dimensional products of lower-dimensional Fano manifolds, and certain complete intersections in projective bundles.

KEYWORDS
Fano manifolds; four-folds; mirror symmetry; quantum period

MATHEMATICS SUBJECT CLASSIFICATION
Primary 14J45, 14N35; Secondary 14J33

1. Introduction

In this paper, we take the first step toward implementing a program, laid out in [Coates et al. 12], to find and classify four-dimensional Fano manifolds using mirror symmetry. We compute quantum periods and quantum differential equations for many known four-dimensional Fano manifolds, using techniques described in [Coates et al. 16]. Our basic reference for the theory of Fano manifolds is the book by Iskovskikh–Prokhorov [Iskovskikh and Prokhorov 99]. Recall that the *index* of a Fano manifold X is the largest integer r such that $-K_X = rH$ for some ample divisor H. A four-dimensional Fano manifold has index at most 5 [Shokurov 85]. Four-dimensional Fano manifolds with index $r > 1$ have been classified. In what follows we compute the quantum periods and quantum differential equations for all four-dimensional Fano manifolds of index $r > 1$, for all four-dimensional Fano toric manifolds, and for certain other four-dimensional Fano manifolds of index 1.

Highlights

We draw the reader's attention to:

- Section 6.2.4, where new tools for computing Gromov–Witten invariants (twisted I-functions for toric complete intersections [Coates et al. 14] and an improved Quantum Lefschetz theorem [Coates 14]) make a big practical difference to the computation of quantum periods. This should be contrasted with [Coates et al. 16, Section 19], where the new techniques were not available.
- Section 6.2.11, which relies on a new construction of Szurek–Wiśniewski's null-correlation bundle [Szurek and Wiśniewski 90] that may be of independent interest.
- The tables of regularized quantum period sequences in Appendix A.
- The numerical calculation of quantum differential operators in Section 9 and Appendix B. This suggests in particular that, for each four-dimensional Fano manifold X with Fano index $r > 1$, the regularized quantum differential equation of X is either extremal or of low ramification.
- Section 6.2.17 and Section B.34, which together give an example of a product such that the regularized quantum differential equation for each factor is extremal, but the regularized quantum differential equation for the product itself is not.

This paper is accompanied by fully commented source code, written in the computational algebra system Magma [Bosma et al. 97]. This will allow the reader to verify the calculations presented here, or to perform similar computations.

2. Methodology

The quantum period G_X of a Fano manifold X is a generating function

$$G_X(t) = 1 + \sum_{d=1}^{\infty} c_d t^d \quad t \in \mathbb{C} \tag{2–1}$$

for certain genus-zero Gromov–Witten invariants c_d of X. A precise definition can be found in [Coates et al. 16,

CONTACT Tom Coates t.coates@imperial.ac.uk Department of Mathematics, Imperial College London, London SW7 2AZ, United Kingdom.

Section B], but roughly speaking c_d is the "virtual number" of degree-d rational curves C in X that pass through a given point and satisfy certain constraints on their complex structure. (The degree of a curve C here is the quantity $\langle -K_X, C\rangle$.) The quantum period is discussed in detail in [Coates et al. 12, Coates et al. 16]; one property that will be important in what follows is that the regularized quantum period

$$\widehat{G}_X(t) = 1 + \sum_{d=1}^{\infty} d! c_d t^d \quad t \in \mathbb{C}, |t| \ll \infty \qquad (2\text{–}2)$$

satisfies a differential equation called the *regularized quantum differential equation* of X:

$$L_X \widehat{G}_X \equiv 0 \quad L_X = \sum_{m=0}^{m=N} p_m(t) D^m, \qquad (2\text{–}3)$$

where the p_m are polynomials and $D = t\frac{d}{dt}$. It is expected that the regularized quantum differential equation for a Fano manifold X is *extremal* or *of low ramification*, as described in Section 9 below. This is a strong constraint on the Gromov–Witten invariants c_d of X.

Quantum periods for a broad class of toric complete intersections can be computed using Givental's mirror theorem [Givental 96].

Theorem 2.1 ([Coates et al. 16, Corollary C.2]). *Let X be a toric Fano manifold and let $D_1, \ldots, D_N \in H^2(X; \mathbb{Q})$ be the cohomology classes Poincaré-dual to the torus-invariant divisors on X. The quantum period of X is*

$$G_X(t) = \sum_{\substack{\beta \in H_2(X;\mathbb{Z}): \\ \langle \beta, D_i\rangle \geq 0\, \forall i}} \frac{t^{\langle \beta, -K_X\rangle}}{\prod_{i=1}^{N} \langle \beta, D_i\rangle!}.$$

Theorem 2.2 ([Coates et al. 16, Corollary D.5]). *Let Y be a toric Fano manifold, and let $D_1, \ldots, D_N \in H^2(Y; \mathbb{Q})$ be the cohomology classes Poincaré-dual to the torus-invariant divisors on Y. Let X be the complete intersection in Y defined by a regular section of $E = L_1 \oplus \cdots \oplus L_s$ where each L_i is a nef line bundle, and let $\rho_i = c_1(L_i)$, $1 \leq i \leq s$. Suppose that the class $c_1(Y) - \Lambda$ is ample on Y, where $\Lambda = c_1(L_1) + \cdots + c_1(L_s)$. Then X is Fano, and the quantum period of X is*

$$G_X(t) = e^{-ct} \sum_{\substack{\beta \in H_2(Y;\mathbb{Z}): \\ \langle \beta, D_i\rangle \geq 0\, \forall i}} t^{\langle \beta, -K_Y - \Lambda\rangle} \frac{\prod_{j=1}^{s} \langle \beta, \rho_j\rangle!}{\prod_{i=1}^{N} \langle \beta, D_i\rangle!}$$

where c is the unique rational number such that the right-hand side has the form $1 + O(t^2)$.

An analogous mirror theorem holds for certain complete intersections in toric Deligne–Mumford stacks, but we will need only the case where the ambient stack is a weighted projective space.

Theorem 2.3 ([Coates et al. 16, Proposition D.9]). *Let Y be the weighted projective space $\mathbb{P}(w_0, \ldots, w_n)$, let X be a smooth Fano manifold given as a complete intersection in Y defined by a section of $E = \mathcal{O}(d_1) \oplus \cdots \oplus \mathcal{O}(d_m)$, and let $-k = w_0 + \cdots + w_n - d_1 - \cdots - d_m$. Suppose that each d_i is a positive integer, that $-k > 0$, and that w_i divides d_j for all i, j such that $0 \leq i \leq n$ and $1 \leq j \leq m$. Then the quantum period of X is*

$$G_X(t) = e^{-ct} \sum_{d=0}^{\infty} t^{-kd} \frac{\prod_{j=1}^{m} (dd_j)!}{\prod_{i=1}^{n} (dw_i)!}$$

where c is the unique rational number such that the right-hand side has the form $1 + O(t^2)$.

The quantum period of a product is the product of the quantum periods.

Theorem 2.4 ([Coates et al. 16, Corollary E.4]). *Let X and Y be smooth projective complex manifolds. Then:*

$$G_{X\times Y}(t) = G_X(t)\, G_Y(t).$$

As we will see below, another powerful tool for computing quantum periods is the Abelian/non-Abelian Correspondence of Ciocan-Fontanine–Kim–Sabbah [Ciocan-Fontanine et al. 08]. We now proceed to the calculation of quantum periods.

3. Four-dimensional Fano manifolds of index 5

The only example here is $\mathbb{P}^4$ [Kobayashi and Ochiai 73, Kollár 81, Serpico 80]. This is a toric variety. Theorem 2.1 yields

$$G_{\mathbb{P}^4}(t) = \sum_{d=0}^{\infty} \frac{t^{5d}}{(d!)^5}$$

×[regularized quantum period p. 15, operator p. 25].

4. Four-dimensional Fano manifolds of index 4

The only example here is the quadric $Q^4 \subset \mathbb{P}^5$ [Kollár 81, Serpico 80]. This is a complete intersection in a toric variety. Theorem 2.2 yields

$$G_{Q^4}(t) = \sum_{d=0}^{\infty} \frac{(2d)!}{(d!)^6} t^{4d}$$

×[regularized quantum period p. 15, operator p. 25].

5. Four-dimensional Fano manifolds of index 3

There are six examples [Fujita 80, Fujita 81, Fujita 84, Fujita 90, Iskovskih 77, Iskovskih 79, Iskovskikh and Prokhorov 99], which are known as del Pezzo fourfolds:

- a sextic hypersurface FI^4_1 in the weighted projective space $\mathbb{P}^5(1^4, 2, 3)$;
- a quartic hypersurface FI^4_2 in the weighted projective space $\mathbb{P}^5(1^5, 2)$;
- a cubic hypersurface $\mathrm{FI}^4_3 \subset \mathbb{P}^5$;
- a complete intersection $\mathrm{FI}^4_4 \subset \mathbb{P}^6$ of type $(2H) \cap (2H)$, where $H = \mathcal{O}_{\mathbb{P}^6}(1)$;
- a complete intersection $\mathrm{FI}^4_5 \subset \mathrm{Gr}(2, 5)$ of type $H \cap H$, where H is the hyperplane bundle; and
- $\mathrm{FI}^4_6 = \mathbb{P}^2 \times \mathbb{P}^2$.

The first four examples here are complete intersections in weighted projective spaces. Theorem 2.3 yields

$$G_{\mathrm{FI}^4_1}(t) = \sum_{d=0}^{\infty} \frac{(6d)!}{(3d)!(2d)!(d!)^4} t^{3d}$$

$$G_{\mathrm{FI}^4_2}(t) = \sum_{d=0}^{\infty} \frac{(4d)!}{(2d)!(d!)^5} t^{3d}$$

$$G_{\mathrm{FI}^4_3}(t) = \sum_{d=0}^{\infty} \frac{(3d)!}{(d!)^6} t^{3d}$$

$$G_{\mathrm{FI}^4_4}(t) = \sum_{d=0}^{\infty} \frac{(2d)!(2d)!}{(d!)^7} t^{3d}.$$

For $\mathrm{FI}^4_5 \subset \mathrm{Gr}(2, 5)$, we use the Abelian/non-Abelian Correspondence, applying Theorem F.1 in [Coates et al. 16] with $a = 2$, $b = c = d = e = 0$. This yields

$$G_{\mathrm{FI}^4_5}(t) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} (-1)^{l+m} t^{3l+3m} \frac{(l+m)!(l+m)!}{(l!)^5(m!)^5} \times (1 - 5(m-l)H_m),$$

where H_m is the mth harmonic number. For $\mathbb{P}^2 \times \mathbb{P}^2$, combining Theorem 2.4 with [Coates et al. 16, Example G.2] yields

$$G_{\mathbb{P}^2\times\mathbb{P}^2}(t) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{t^{3l+3m}}{(l!)^3(m!)^3}$$

X	$\widehat{G}_X$	L_X	X	$\widehat{G}_X$	L_X
FI^4_1	p. 15	p. 26	FI^4_4	p. 15	p. 27
FI^4_2	p. 15	p. 26	FI^4_5	p. 15	p. 27
FI^4_3	p. 15	p. 26	$\mathbb{P}^2 \times \mathbb{P}^2$	p. 15	p. 27

6. Four-dimensional Fano manifolds of index 2

Consider now a four-dimensional Fano manifold with index $r = 2$ and Picard rank ρ.

6.1. The case $\rho = 1$

Four-dimensional Fano manifolds with index $r = 2$ and Picard rank $\rho = 1$ have been classified [Mukai 89, Wilson 87], [Iskovskikh and Prokhorov 99, Chapter 5]. Up to deformation, there are nine examples: the "linear unsections" of smooth three-dimensional Fano manifolds with $\rho = 1$, $r = 1$, and degree at most 144. We compute the quantum periods of these examples using the constructions in [Coates et al. 16, Sections 8–16], writing V^4_k for a four-dimensional Fano manifold with $\rho = 1$, $r = 2$, and degree $16k$.

6.1.1. V^4_2. [regularized quantum period p. 16, operator p. 28]

This is a sextic hypersurface in $\mathbb{P}^5(1^5, 3)$. Proposition D.9 in [Coates et al. 16] yields

$$G_{V^4_2}(t) = \sum_{d=0}^{\infty} \frac{(6d)!}{(d!)^5(3d)!} t^{2d}.$$

6.1.2. V^4_4. [regularized quantum period p. 16, operator p. 28]

This is a quartic hypersurface in $\mathbb{P}^5$. Theorem 2.2 yields

$$G_{V^4_4}(t) = \sum_{d=0}^{\infty} \frac{(4d)!}{(d!)^6} t^{2d}.$$

6.1.3. V^4_6. [regularized quantum period p. 16, operator p. 28]

This is a complete intersection of type $(2H) \cap (3H)$ in $\mathbb{P}^6$, where $H = \mathcal{O}_{\mathbb{P}^6}(1)$. Theorem 2.2 yields

$$G_{V^4_6}(t) = \sum_{d=0}^{\infty} \frac{(2d)!(3d)!}{(d!)^7} t^{2d}.$$

6.1.4. V^4_8. [regularized quantum period p. 16, operator p. 29]

This is a complete intersection of type $(2H) \cap (2H) \cap (2H)$ in $\mathbb{P}^7$, where $H = \mathcal{O}_{\mathbb{P}^7}(1)$. Theorem 2.2 yields

$$G_{V^4_8}(t) = \sum_{d=0}^{\infty} \frac{((2d)!)^3}{(d!)^8} t^{2d}.$$

6.1.5. V^4_{10}. [regularized quantum period p. 16, operator p. 29]

This is a complete intersection in $\mathrm{Gr}(2, 5)$, cut out by a regular section of $\mathcal{O}(1) \oplus \mathcal{O}(2)$ where $\mathcal{O}(1)$ is the pullback of $\mathcal{O}(1)$ on projective space under the Plücker embedding. We apply Theorem F.1 in [Coates et al. 16] with $a = b = 1$ and $c = d = e = 0$. This yields

$$G_{V^4_{10}}(t) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{l+m}t^{2l+2m}\frac{(l+m)!(2l+2m)!}{(l!)^5(m!)^5} \times (1-5(m-l)H_m),$$

where H_m is the mth harmonic number.

6.1.6. V^4_{12}. [regularized quantum period p. 16, operator p. 29]

This is the subvariety of $\mathrm{Gr}(2,5)$ cut out by a regular section of $S^\star \otimes \det S^\star$, where S is the universal bundle of subspaces on $\mathrm{Gr}(2,5)$. We apply Theorem F.1 in [Coates et al. 16] with $c = 1$ and $a = b = d = e = 0$. This yields

$$G_{V^4_{12}}(t) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{l+m}t^{2l+2m}\frac{(2l+m)!(l+2m)!}{(l!)^5(m!)^5} \times (1+(m-l)(H_{2l+m}+2H_{l+2m}-5H_m)).$$

6.1.7. V^4_{14}. [regularized quantum period p. 16, operator p. 30]

This is a complete intersection in $\mathrm{Gr}(2,6)$, cut out by a regular section of $\mathcal{O}(1)^{\oplus 4}$ where $\mathcal{O}(1)$ is the pullback of $\mathcal{O}(1)$ on projective space under the Plücker embedding. We apply Theorem F.1 in [Coates et al. 16] with $a = 4$ and $b = c = d = e = 0$. This yields

$$G_{V^4_{14}}(t) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{l+m}t^{2l+2m}\frac{\left((l+m)!\right)^4}{(l!)^6(m!)^6} \times (1-6(m-l)H_m).$$

6.1.8. V^4_{16}. [regularized quantum period p. 16, operator p. 30]

This is the subvariety of $\mathrm{Gr}(3,6)$ cut out by a regular section of $\wedge^2 S^\star \oplus (\det S^\star)^{\oplus 2}$, where S is the universal bundle of subspaces on $\mathrm{Gr}(3,6)$. We apply Theorem F.1 in [Coates et al. 16] with $a = 2$, $b = c = d = 0$, and $e = 1$. This shows that the quantum period $G_{V^4_{16}}(t)$ is the coefficient of $(p_2-p_1)(p_3-p_1)(p_3-p_2)$ in the expression:

$$\sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty}\sum_{l_3=0}^{\infty} t^{2l_1+2l_2+2l_3} \times \frac{\prod_{k=1}^{l_1+l_2+l_3}(p_1+p_2+p_3+k)^2}{\prod_{j=1}^{j=3}\prod_{k=1}^{k=l_j}(p_j+k)^6} \prod_{1\le i<j\le 3}\prod_{k=1}^{l_i+l_j}(p_i+p_j+k) \times \prod_{1\le i<j\le 3}\left(p_j-p_i+(l_j-l_i)\right).$$

(Since this expression is totally antisymmetric in p_1, p_2, p_3, it is divisible by $(p_2-p_1)(p_3-p_1)(p_3-p_2)$.)

6.1.9. V^4_{18}. [regularized quantum period p. 16, operator p. 30]

This is the subvariety of $\mathrm{Gr}(5,7)$ cut out by a regular section of $\left(S \otimes \det S^\star\right) \oplus \det S^\star$, where S is the universal bundle of subspaces on $\mathrm{Gr}(5,7)$. We apply Theorem F.1 in [Coates et al. 16] with $a = d = 1$ and $b = c = e = 0$. This shows that the quantum period $G_{V^4_{18}}(t)$ is the coefficient of $\prod_{1\le i<j\le 5}(p_j-p_i)$ in the expression:

$$\sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty}\sum_{l_3=0}^{\infty}\sum_{l_4=0}^{\infty}\sum_{l_5=0}^{\infty} t^{2|l|}\frac{\prod_{k=1}^{k=|l|}(p_1+p_2+\cdots+p_5+k)}{\prod_{j=1}^{j=5}\prod_{k=1}^{k=l_j}(p_j+k)^7} \times \prod_{j=1}^{j=5}\prod_{k=1}^{|l|-l_j}(p_1+p_2+\cdots+p_5-p_j+k) \times \prod_{1\le i<j\le 5}\left(p_j-p_i+(l_j-l_i)\right),$$

where $|l| = l_1+l_2+\cdots+l_5$. (As above, antisymmetry implies that the long formula here is divisible by $\prod_{1\le i<j\le 5}(p_j-p_i)$.)

6.2. The case $\rho > 1$

Four-dimensional Fano manifolds with $\rho > 1$ and $r = 2$ have been classified by Mukai [Mukai 89, Mukai 88] and Wiśniewski [Wiśniewski 90]. There are 18 deformation families, as follows. We denote the kth such deformation family, as given in [Iskovskikh and Prokhorov 99, Table 12.7], by MW^4_k.

6.2.1. MW^4_1. [regularized quantum period p. 16, operator p. 31]

This is the product $\mathbb{P}^1 \times B^3_1$. Combining Theorem 2.4 with [Coates et al. 16, Example G.1] and [Coates et al. 16, Section 3] yields

$$G_{\mathrm{MW}^4_1}(t) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{(6m)!}{(l!)^2(m!)^3(2m)!(3m)!}t^{2l+2m}.$$

6.2.2. MW^4_2. [regularized quantum period p. 16, operator p. 32]

This is the product $\mathbb{P}^1 \times B^3_2$. Combining Theorem 2.4 with [Coates et al. 16, Example G.1] and [Coates et al. 16, Section 4] yields

$$G_{\mathrm{MW}^4_2}(t) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{(4m)!}{(l!)^2(m!)^4(2m)!}t^{2l+2m}.$$

6.2.3. MW^4_3. [regularized quantum period p. 16, operator p. 33]

This is the product $\mathbb{P}^1 \times B^3_3$. Combining Theorem 2.4 with [Coates et al. 16, Example G.1] and [Coates et al. 16, Section 5] yields

$$G_{\mathrm{MW}^4_3}(t)=\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{(3m)!}{(l!)^2(m!)^5}t^{2l+2m}.$$

6.2.4. MW^4_4. [regularized quantum period p. 16, operator p. 34]

This is a double cover of $\mathbb{P}^2\times\mathbb{P}^2$, branched over a divisor of bidegree $(2,2)$. Consider the toric variety F with weight data:

x_0	x_1	x_2	y_0	y_1	y_2	w	
1	1	1	0	0	0	1	L
0	0	0	1	1	1	1	M

and $\overline{\mathrm{Amp}}F=\langle L, L+M\rangle$. Let X be a member of the linear system $|2L+2M|$ defined by the equation $w^2=f_{2,2}$, where $f_{2,2}$ is a bihomogenous polynomial of degrees 2 in x_0, x_1, x_2 and 2 in y_0, y_1, y_2. Let $p: F \dashrightarrow \mathbb{P}^2\times\mathbb{P}^2$ be the rational map which sends (contravariantly) the homogenous co-ordinate functions $[x_0,x_1,x_2,y_0,y_1,y_2]$ on $\mathbb{P}^2_{x_0,x_1,x_2}\times\mathbb{P}^2_{y_0,y_1,y_2}$ to $[x_0,x_1,x_2,y_0,y_1,y_2]$. The restriction of p to X is a morphism, which exhibits X as a double cover of $\mathbb{P}^2\times\mathbb{P}^2$ branched over the locus $(f_{2,2}=0)\subset\mathbb{P}^2_{x_0,x_1,x_2}\times\mathbb{P}^2_{y_0,y_1,y_2}$. Thus $X=\mathrm{MW}^4_4$.

Recall the definition of the J-function $J_X(t,z)$ from [Coates and Givental 07, equation 11]. Recall from [Coates 14] that there is a Lagrangian cone $\mathcal{L}_X\subset H^\bullet(X;\Lambda_X)\otimes\mathbb{C}((z^{-1}))$ that encodes all genus-zero Gromov–Witten invariants of X, and a Lagrangian cone $\mathcal{L}_{\boldsymbol{e}}\subset H^\bullet(F;\Lambda_F)\otimes\mathbb{C}((z^{-1}))\otimes\mathbb{C}(\lambda)$ that encodes all genus-zero $(\boldsymbol{e}, 2L+2M)$-twisted Gromov–Witten invariants of F. Here Λ_X and Λ_F are certain Novikov rings and $\boldsymbol{e}$ is the total Chern class with parameter λ (or, equivalently, $\boldsymbol{e}$ is the S^1-equivariant Euler class with respect to an action of S^1 described in [Coates 14]; in this case one should regard λ as the standard generator for the S^1-equivariant cohomology algebra of a point). The J-function J_X is characterized by the fact that $J_X(t,-z)$ is the unique point on $\mathcal{L}_X$ of the form $-z+t+O(z^{-1})$.

Let $p_1, p_2\in H^2(F;\mathbb{Q})$ denote the first Chern class of $L, L+M$, respectively, and let $P_1, P_2\in H^2(X;\mathbb{Q})$ denote the pullbacks of p_1, p_2 along the inclusion map $i: X\to F$. Let Q_1, Q_2 denote the elements of the Novikov ring Λ_X that are dual, respectively, to P_1, P_2, and note that Λ_X and Λ_F are canonically isomorphic (via $i_\star$). Theorem 22 in [Coates et al. 14] implies that:

$$I(t_1,t_2,\lambda,z)=ze^{t_1p_1/z}e^{t_2p_2/z}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\times\frac{Q_1^lQ_2^me^{lt_1}e^{mt_2}\prod_{k=1}^{k=2m}(\lambda+2p_2+kz)}{\prod_{k=1}^{k=l}(p_1+kz)^3\prod_{k=1}^{k=m}(p_2+kz)}\times\frac{\prod_{k=-\infty}^{k=0}(p_2-p_1+kz)^3}{\prod_{k=-\infty}^{k=m-l}(p_2-p_1+kz)^3}$$

satisfies $I(t_1,t_2,\lambda,-z)\in\mathcal{L}_{\boldsymbol{e}}$. Theorem 1.1 in [Coates 14] gives that $i^\star\mathcal{L}_{\boldsymbol{e}}\big|_{\lambda=0}\subset\mathcal{L}_X$, and therefore that:

$$i^\star I(t_1,t_2,0,-z)\in\mathcal{L}_X.$$

Since the hypersurface X misses the locus $y_1=y_2=y_3=0$ in F, we have that $i^\star(p_2-p_1)^3=0$. Thus:

$$i^\star I(t_1,t_2,0,z)=ze^{t_1P_1/z}e^{t_2P_2/z}\sum_{l=0}^{\infty}\sum_{m=l}^{\infty}\times\frac{Q_1^lQ_2^me^{lt_1}e^{mt_2}\prod_{k=1}^{k=2m}(2P_2+kz)}{\prod_{k=1}^{k=l}(P_1+kz)^3\prod_{k=1}^{k=m}(P_2+kz)}\times\frac{1}{\prod_{k=1}^{k=m-l}(P_2-P_1+kz)^3}.$$

In particular, $i^\star I(t_1,t_2,0,-z)$ has the form $-z+t_1P_1+t_2P_2+O(z^{-1})$ and, from the characterization of J_X discussed above, we conclude that $J_X(t_1P_1+t_2P_2,-z)=i^\star I(t_1,t_2,0,-z)$.

To extract the quantum period G_X from the J-function $J_X(t_1P_1+t_2P_2,z)$, we take the component along the unit class $1\in H^\bullet(X;\mathbb{Q})$, set $z=1$, set $t_1=t_2=0$, and set $Q_1=1$, $Q_2=t^2$, obtaining:

$$G_{\mathrm{MW}^4_4}(t)=\sum_{l=0}^{\infty}\sum_{m=l}^{\infty}\frac{(2m)!}{(l!)^3m!((m-l)!)^3}t^{2m}.$$

6.2.5. MW^4_5. [regularized quantum period p. 16, operator p. 34]

This is a divisor on $\mathbb{P}^2\times\mathbb{P}^3$ of bidegree $(1,2)$. Theorem 2.2 yields

$$G_{\mathrm{MW}^4_5}(t)=\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{(l+2m)!}{(l!)^3(m!)^4}t^{2l+2m}.$$

6.2.6. MW^4_6. [regularized quantum period p. 16, operator p. 35]

This is the product $\mathbb{P}^1\times B^3_4$. Combining Theorem 2.4 with [Coates et al. 16, Example G.1] and [Coates et al. 16, Section 6] yields

$$G_{\mathrm{MW}^4_6}(t)=\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{(2m)!(2m)!}{(l!)^2(m!)^6}t^{2l+2m}.$$

6.2.7. MW^4_7. [regularized quantum period p. 16, operator p. 36]

This is a complete intersection of two divisors in $\mathbb{P}^3 \times \mathbb{P}^3$, each of bidegree $(1, 1)$. Theorem 2.2 yields

$$G_{\mathrm{MW}^4_7}(t) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{(l+m)!(l+m)!}{(l!)^4(m!)^4} t^{2l+2m}.$$

6.2.8. MW^4_8. [regularized quantum period p. 16, operator p. 36]

This is a divisor on $\mathbb{P}^2 \times Q^3$ of bidegree $(1, 1)$. Theorem 2.2 yields

$$G_{\mathrm{MW}^4_8}(t) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{(l+m)!(2m)!}{(l!)^3(m!)^5} t^{2l+2m}.$$

6.2.9. MW^4_9. [regularized quantum period p. 16, operator p. 37]

This is the product $\mathbb{P}^1 \times B^3_5$. Combining Theorem 2.4 with [Coates et al. 16, Example G.1] and [Coates et al. 16, Section 7] yields

$$G_{\mathrm{MW}^4_9}(t) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (-1)^{m+n} t^{2l+2m+2n} \times \frac{((m+n)!)^3}{(l!)^2(m!)^5(n!)^5}(1 - 5(n-m)H_n).$$

6.2.10. MW^4_{10}. [regularized quantum period p. 16, operator p. 38]

This is the blow-up of the quadric Q^4 along a conic that is not contained in a plane lying in Q^4. Consider the toric variety F with weight data:

s_0	s_1	s_2	x	x_3	x_4	x_5	
1	1	1	−1	0	0	0	L
0	0	0	1	1	1	1	M

and $\overline{\mathrm{Amp}}F = \langle L, M\rangle$. The morphism $F \to \mathbb{P}^5$ that sends (contravariantly) the homogenous co-ordinate functions $[x_0, x_1, \ldots, x_5]$ to $[xs_0, xs_1, xs_2, x_3, x_4, x_5]$ blows up the plane $\Pi = (x_0 = x_1 = x_2 = 0)$ in $\mathbb{P}^5$. Thus, a general member of $|2M|$ on F is the blow-up of Q^4 with center a conic on Π. In other words, a general member of $|2M|$ on F is MW^4_{10}. We have

- $-K_F = 2L + 4M$ is ample, so that F is a Fano variety;
- $\mathrm{MW}^4_{10} \sim 2M$ is ample;
- $-(K_F + 2M) \sim 2L + 2M$ is ample.

Theorem 2.2 yields

$$G_{\mathrm{MW}^4_{10}}(t) = \sum_{l=0}^{\infty}\sum_{m=l}^{\infty} \frac{(2m)!}{(l!)^3(m-l)!(m!)^3} t^{2l+2m}.$$

6.2.11. MW^4_{11}. [regularized quantum period p. 16, operator p. 38]

This is the projective bundle $\mathbb{P}_{\mathbb{P}^3}(\mathcal{E}^\vee)$, where $\mathcal{E} \to \mathbb{P}^3$ is the null-correlation bundle of Szurek–Wiśniewski [Szurek and Wiśniewski 90].

Remark 6.1. For us $\mathbb{P}(E)$ denotes the projective bundle of lines in E, whereas in Szurek–Wiśniewski and Iskovskikh–Prokhorov, $\mathbb{P}(E)$ denotes the projective bundle of one-dimensional quotients. With our conventions, if $\pi : \mathbb{P}(E) \to X$ is a projective bundle then $E^\star = \pi_\star \mathcal{O}_{\mathbb{P}(E)}(1)$, and so a regular section $s \in \Gamma(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$ vanishes on $\mathbb{P}(F^\star) \subset \mathbb{P}(E)$, where the vector bundle $F \to X$ is the cokernel of $s : \mathcal{O}_{\mathbb{P}(E)} \to E^\star$.

Proposition 6.2. *Let $V = \mathbb{C}^4$, so that $\mathbb{P}(V) = \mathbb{P}^3$. Consider the partial flag manifold $\mathrm{Fl}_{1,2}(V)$ and the natural projections*

$$\mathbb{P}(V) \xleftarrow{\;p_1\;} \mathrm{Fl}_{1,2}(V) \xrightarrow{\;p_2\;} \mathrm{Gr}(2, V) \qquad (6\text{–}4)$$

Let $|L|$ denote the linear system defined by $\mathcal{O}(1)$ for the projective bundle p_1. Then a general element of $|L|$ is $\mathbb{P}(\mathcal{E}^\vee)$, where $\mathcal{E} \to \mathbb{P}(V)$ is the null-correlation bundle.

Proof. The null-correlation bundle has rank 2, and so the perfect pairing $\mathcal{E} \otimes \mathcal{E} \to \det \mathcal{E}$ gives canonical isomorphisms $\mathcal{E}^\vee \cong \mathcal{E} \otimes (\det \mathcal{E})^{-1}$ and $\mathbb{P}(\mathcal{E}^\vee) \cong \mathbb{P}(\mathcal{E})$. There is an exact sequence:

$$0 \longrightarrow \mathcal{E}(-1) \longrightarrow T_{\mathbb{P}(V)}(-2) \longrightarrow \mathcal{O}_{\mathbb{P}(V)} \longrightarrow 0$$

and the map $s^\star : T_{\mathbb{P}(V)}(-2) \to \mathcal{O}_{\mathbb{P}(V)}$ therein defines a section $s \in \Gamma(\mathbb{P}(T_{\mathbb{P}(V)}(-2)), \mathcal{O}_{\mathbb{P}(T_{\mathbb{P}(V)}(-2))}(1))$. The construction in Remark 6.1 now exhibits $\mathbb{P}(\mathcal{E}(-1)) \cong \mathbb{P}(\mathcal{E})$ as the locus $(s = 0)$ in $\mathbb{P}(T_{\mathbb{P}(V)}(-2))$. We will identify $\mathbb{P}(T_{\mathbb{P}(V)}(-2))$ with the partial flag manifold $\mathrm{Fl}_{1,2}(V)$.

For a vector bundle $\mathcal{F} \to X$ of rank 3, the perfect pairing $\mathcal{F} \otimes \wedge^2\mathcal{F} \to \det \mathcal{F}$ gives a canonical isomorphism $\mathcal{F}^\star \cong (\wedge^2\mathcal{F}) \otimes (\det \mathcal{F})^{-1}$. Applying this with $\mathcal{F} \to X$ equal to $\Omega_{\mathbb{P}(V)}(2) \to \mathbb{P}(V)$ gives

$$T_{\mathbb{P}(V)}(-2) \cong \Omega^2_{\mathbb{P}(V)}(2),$$

where $\Omega^2_{\mathbb{P}(V)} := \wedge^2\Omega_{\mathbb{P}(V)}$. We thus need to identify $\mathbb{P}(\Omega^2_{\mathbb{P}(V)}(2))$ with $\mathrm{Fl}_{1,2}(V)$.

The Plücker embedding $\mathrm{Gr}(2, V) \to \mathbb{P}(\wedge^2 V)$ maps a subspace $W \in \mathrm{Gr}(2, V)$ to the antisymmetric linear map $L_W : V^\star \to V$, well-defined up to scale, given by

$$L_W(f) = f(w_1)w_2 - f(w_2)w_1,$$

where $\{w_1, w_2\}$ is a basis for W. The kernel of L_W is the annihilator $W^\perp \subset V^\star$. If $f \notin W^\perp$ then $\langle L_W(f)\rangle =$

$\ker f \cap W$; this implies in particular that $\mathrm{rk} L_W = 2$. Thus, the image of the Plücker embedding consists of (the lines spanned by) antisymmetric linear maps $L_W : V^\star \to V$ of rank 2, and one can recover $W \in \mathrm{Gr}(2, V)$ from its image $\langle L_W \rangle$ by taking the annihilator of the kernel:

$$W = (\ker L_W)^\perp.$$

There is a canonical isomorphism $\mathrm{Ann} : \mathrm{Gr}(2, V) \to \mathrm{Gr}(2, V^\star)$ which maps $W \in \mathrm{Gr}(2, V)$ to $W^\perp$.

Recall that our goal is to identify $\mathbb{P}(\Omega^2_{\mathbb{P}(V)}(2))$ with $\mathrm{Fl}_{1,2}(V)$. Let $q_1 : \mathbb{P}(\Omega^2_{\mathbb{P}(V)}(2)) \to \mathbb{P}(V)$ denote the projection. The Euler sequence:

$$0 \longrightarrow \Omega_{\mathbb{P}(V)} \longrightarrow \pi^\star V^\star(-1) \longrightarrow \mathcal{O}_{\mathbb{P}(V)} \longrightarrow 0$$

gives, via [Hartshorne 77, II, Exercise 5.16]:

$$0 \longrightarrow \Omega^2_{\mathbb{P}(V)} \longrightarrow \pi^\star\big(\wedge^2 V^\star\big)(-2) \longrightarrow \Omega_{\mathbb{P}(V)} \longrightarrow 0$$

and thus:

$$0 \longrightarrow \Omega^2_{\mathbb{P}(V)}(2) \longrightarrow \pi^\star\big(\wedge^2 V^\star\big) \longrightarrow \pi^\star V^\star(1) \longrightarrow \mathcal{O}_{\mathbb{P}(V)}(2) \longrightarrow 0 \qquad (6\text{–}5)$$

This defines a map $f : \mathbb{P}(\Omega^2_{\mathbb{P}(V)}(2)) \to \mathbb{P}(\wedge^2 V^\star)$. Consider the fiber of the sequence (6–5) over $[v] \in \mathbb{P}(V)$. The map $\pi^\star(\wedge^2 V^\star) \to \pi^\star V^\star(1)$ here is given by contraction with v, and so non-zero elements of the kernel are antisymmetric linear maps $V \to V^\star$ of rank 2. (They are antisymmetric, hence have rank 0, 2, or 4; they are non-zero, hence are not of rank 0; and they have the non-zero element v in their kernel, hence are not of rank 4.) In particular, we see that the image of f lies in $\mathrm{Gr}(2, V^\star) \subset \mathbb{P}(\wedge^2 V^\star)$. Given $[x] \in \mathbb{P}(\Omega^2_{\mathbb{P}(V)}(2))$, write $W_{[x]} \subset V^\star$ for the linear subspace defined by $f([x])$. Suppose that $[x] \in \mathbb{P}(\Omega^2_{\mathbb{P}(V)}(2))$ lies over $[v] \in \mathbb{P}(V)$. Then, applying the discussion in the previous paragraph but with V there replaced by $V^\star$, we see that $v \in W^\perp_{[x]}$. Thus, writing $q_2 : \mathbb{P}(\Omega^2_{\mathbb{P}(V)}(2)) \to \mathrm{Gr}(2, V)$ for the composition

$$\mathbb{P}\big(\Omega^2_{\mathbb{P}(V)}(2)\big) \xrightarrow{\ f\ } \mathrm{Gr}(2, V^\star) \xrightarrow{\ \mathrm{Ann}\ } \mathrm{Gr}(2, V)$$

we have that $q_1([x]) \subset q_2([x])$, i.e., that the diagram:

$$\begin{array}{ccc} & \mathbb{P}\big(\Omega^2_{\mathbb{P}(V)}(2)\big) & \\ {}^{q_1}\swarrow & & \searrow^{q_2} \\ \mathbb{P}(V) & & \mathrm{Gr}(2, V) \end{array}$$

coincides with the diagram (6–4). This identifies $\mathbb{P}(\Omega^2_{\mathbb{P}(V)}(2))$ with the partial flag manifold $\mathrm{Fl}_{1,2}(V)$, and exhibits $\mathbb{P}(\mathcal{E}^\vee)$ as an element of the linear system $|L|$ as claimed. □

Abelianization:

To compute the quantum period, we use the Abelian/non-Abelian Correspondence of Ciocan-Fontanine–Kim–Sabbah, as in [Coates et al. 16, Section 39]. Consider the situation as in Section 3.1 of [Ciocan-Fontanine et al. 08] with:

- $X = \mathbb{C}^{10}$, regarded as the space of pairs:

$\{(v, w) : v \in \mathbb{C}^2$ is a row vector, w is a 2×4 complex matrix$\}$

- $G = \mathbb{C}^\times \times \mathrm{GL}_2(\mathbb{C})$, acting on X as:
$$(\lambda, g) : (v, w) \mapsto (\lambda v g^{-1}, g w)$$
- $T = (\mathbb{C}^\times)^3$, the diagonal subtorus in G;
- the group that is denoted by S in [Ciocan-Fontanine et al. 08] set equal to the trivial group;
- $\mathcal{V}$ equal to the representation of G given by the determinant of the standard representation of the second factor $\mathrm{GL}_2(\mathbb{C})$.

Then $X /\!\!/ G$ is the partial flag manifold $\mathrm{Fl} = \mathrm{Fl}_{1,2}(\mathbb{C}^4)$, whereas $X /\!\!/ T$ is the toric variety with weight data:

1	1	1	1	0	0	0	0	−1	0	L_1
0	0	0	0	1	1	1	1	0	−1	L_2
0	0	0	0	0	0	0	0	1	1	H

and $\overline{\mathrm{Amp}} = \langle L_1, L_2, H \rangle$, that is, $X /\!\!/ T$ is the projective bundle $\mathbb{P}(\mathcal{O}(-1, 0) \oplus \mathcal{O}(0, -1))$ over $\mathbb{P}^3 \times \mathbb{P}^3$. The non-trivial element of the Weyl group $W = \mathbb{Z}/2\mathbb{Z}$ exchanges the two factors of $\mathbb{P}^3 \times \mathbb{P}^3$. The representation $\mathcal{V}$ induces the line bundle $\mathcal{V}_G = L$ over $X /\!\!/ G = \mathrm{Fl}$, where L was defined in the statement of Proposition 6.2, whereas the representation $\mathcal{V}$ induces the line bundle $\mathcal{V}_T = L_1 + L_2$ over $X /\!\!/ T$.

The Abelian/non-Abelian correspondence:

Let p_1, p_2, and $p_3 \in H^2(X /\!\!/ T; \mathbb{Q})$ denote the first Chern classes of the line bundles L_1, L_2, and H, respectively. We fix a lift of $H^\bullet(X /\!\!/ G; \mathbb{Q})$ to $H^\bullet(X /\!\!/ T, \mathbb{Q})^W$ in the sense of [Ciocan-Fontanine et al. 08, Section 3]; there are many possible choices for such a lift, and the precise choice made will be unimportant in what follows. The lift allows us to regard $H^\bullet(X /\!\!/ G; \mathbb{Q})$ as a subspace of $H^\bullet(X /\!\!/ T, \mathbb{Q})^W$, which maps isomorphically to the Weyl-anti-invariant part $H^\bullet(X /\!\!/ T, \mathbb{Q})^a$ of $H^\bullet(X /\!\!/ T, \mathbb{Q})$ via

$$H^\bullet(X /\!\!/ T, \mathbb{Q})^W \xrightarrow{\ \cup (p_2 - p_1)\ } H^\bullet(X /\!\!/ T, \mathbb{Q})^a$$

We compute the quantum period of $\mathrm{MW}^4_{11} \subset X /\!\!/ G$ by computing the J-function of $\mathrm{Fl} = X /\!\!/ G$ twisted, in the sense of [Coates and Givental 07], by the Euler class and the bundle $\mathcal{V}_G$, using the Abelian/non-Abelian Correspondence.

Our first step is to compute the J-function of $X /\!\!/ T$ twisted by the Euler class and the bundle $\mathcal{V}_T$. As in [Coates et al. 16, Section D.1] and as in [Ciocan-Fontanine et al. 08], consider the bundles $\mathcal{V}_T$ and $\mathcal{V}_G$ equipped with the canonical $\mathbb{C}^\times$-action that rotates fibers and acts trivially on the base. Recall the definition of the twisted J-function $J_{\boldsymbol{e},\mathcal{V}_T}$ of $X/\!\!/T$ from [Coates et al. 16, Section D.1]. We will compute $J_{\boldsymbol{e},\mathcal{V}_T}$ using the Quantum Lefschetz theorem; $J_{\boldsymbol{e},\mathcal{V}_T}$ is the restriction to the locus $\tau \in H^0(X/\!\!/T) \oplus H^2(X/\!\!/T)$ of what was denoted by $J^{S\times\mathbb{C}^\times}_{\mathcal{V}_T}(\tau)$ in [Ciocan-Fontanine et al. 08]. The toric variety $X/\!\!/T$ is Fano, so Theorem C.1 in [Coates et al. 16] gives

$$J_{X/\!\!/T}(\tau) = e^{\tau/z} \sum_{l,m,n\geq 0} \frac{Q_1^l Q_2^m Q_3^n e^{l\tau_1} e^{m\tau_2} e^{m\tau_3}}{\prod_{k=1}^{k=l}(p_1+kz)^4 \prod_{k=1}^{k=m}(p_2+kz)^4} \times \frac{\prod_{k=-\infty}^{k=0} p_3-p_1+kz}{\prod_{k=-\infty}^{k=n-l} p_3-p_1+kz} \frac{\prod_{k=-\infty}^{k=0} p_3-p_2+kz}{\prod_{k=-\infty}^{k=n-m} p_3-p_2+kz},$$

where $\tau = \tau_1 p_1 + \tau_2 p_2 + \tau_3 p_3$ and we have identified the group ring $\mathbb{Q}[H_2(X/\!\!/T;\mathbb{Z})]$ with $\mathbb{Q}[Q_1, Q_2, Q_3]$ via the $\mathbb{Q}$-linear map that sends Q^β to $Q_1^{\langle\beta,p_1\rangle} Q_2^{\langle\beta,p_2\rangle} Q_3^{\langle\beta,p_3\rangle}$. The line bundles L_1, L_2, and H are nef, and $c_1(X/\!\!/T) - c_1(\mathcal{V}_T)$ is ample, so Theorem D.3 in [Coates et al. 16] gives

$$J_{\boldsymbol{e},\mathcal{V}_T}(\tau) = e^{\tau/z} \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} Q_1^l Q_2^m Q_3^n e^{l\tau_1} e^{m\tau_2} e^{m\tau_3} \times \frac{\prod_{k=1}^{k=l+m}(\lambda + p_1 + p_2 + kz)}{\prod_{k=1}^{k=l}(p_1+kz)^4 \prod_{k=1}^{k=m}(p_2+kz)^4} \times \frac{\prod_{k=-\infty}^{k=0} p_3-p_1+kz}{\prod_{k=-\infty}^{k=n-l} p_3-p_1+kz} \frac{\prod_{k=-\infty}^{k=0} p_3-p_2+kz}{\prod_{k=-\infty}^{k=n-m} p_3-p_2+kz}.$$

Consider now $\mathrm{Fl} = X/\!\!/G$ and a point $t \in H^\bullet(\mathrm{Fl})$. Recall that $\mathrm{Fl} = \mathbb{P}(S)$ is the projectivization of the universal bundle S of subspaces on $\mathrm{Gr} := \mathrm{Gr}(2,4)$. Let $\epsilon_1 \in H^2(\mathrm{Fl};\mathbb{Q})$ be the pullback to Fl (under the projection map $p_2 : \mathrm{Fl} \to \mathrm{Gr}$) of the ample generator of $H^2(\mathrm{Gr})$, and let $\epsilon_2 \in H^2(\mathrm{Fl};\mathbb{Q})$ be the first Chern class of $\mathcal{O}_{\mathbb{P}(S)}(1)$. Identify the group ring $\mathbb{Q}[H_2(\mathrm{Fl};\mathbb{Z})]$ with $\mathbb{Q}[q_1, q_2]$ via the $\mathbb{Q}$-linear map which sends Q^β to $q_1^{\langle\beta,\epsilon_1\rangle} q_2^{\langle\beta,\epsilon_2\rangle}$. In [Ciocan-Fontanine et al. 08, Section 6.1], the authors consider the lift $\tilde{J}^{S\times\mathbb{C}^\times}_{\mathcal{V}_G}(t)$ of their twisted J-function $J^{S\times\mathbb{C}^\times}_{\mathcal{V}_G}(t)$ determined by a choice of lift $H^\bullet(X/\!\!/G;\mathbb{Q}) \to H^\bullet(X/\!\!/T,\mathbb{Q})^W$. We restrict to the locus $t \in H^0(X/\!\!/G;\mathbb{Q}) \oplus H^2(X/\!\!/G;\mathbb{Q})$, considering the lift:

$$\tilde{J}_{\boldsymbol{e},\mathcal{V}_G}(t) := \tilde{J}^{S\times\mathbb{C}^\times}_{\mathcal{V}_G}(t) \quad t \in H^0(X/\!\!/G;\mathbb{Q}) \oplus H^2(X/\!\!/G;\mathbb{Q})$$

of our twisted J-function $J_{\boldsymbol{e},\mathcal{V}_G}$ determined by our choice of lift $H^\bullet(X/\!\!/G;\mathbb{Q}) \to H^\bullet(X/\!\!/T,\mathbb{Q})^W$. Theorems 4.1.1 and 6.1.2 in [Ciocan-Fontanine et al. 08] imply that

$$\tilde{J}_{\boldsymbol{e},\mathcal{V}_G}(\varphi(t)) \cup (p_2 - p_1) = \left[\left(z\frac{\partial}{\partial\tau_2} - z\frac{\partial}{\partial\tau_1}\right) J_{\boldsymbol{e},\mathcal{V}_T}(\tau)\right]_{\tau=t,\, Q_1=Q_2=-q_1,\, Q_3=q_2}$$

for some function $\varphi : H^2(X/\!\!/G;\mathbb{Q}) \to H^\bullet(X/\!\!/G;\Lambda_G)$. Setting $t = 0$ gives

$$\tilde{J}_{\boldsymbol{e},\mathcal{V}_G}(\varphi(0)) \cup (p_2 - p_1) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (-1)^{l+m} q_1^{l+m} q_2^n \times \frac{\prod_{k=1}^{k=l+m}(\lambda + p_1 + p_2 + kz)}{\prod_{k=1}^{k=l}(p_1+kz)^4 \prod_{k=1}^{k=m}(p_2+kz)^4} \times \frac{\prod_{k=-\infty}^{k=0} p_3-p_1+kz}{\prod_{k=-\infty}^{k=n-l} p_3-p_1+kz} \frac{\prod_{k=-\infty}^{k=0} p_3-p_2+kz}{\prod_{k=-\infty}^{k=n-m} p_3-p_2+kz} \times (p_2 - p_1 + (m-l)z).$$

For symmetry reasons, the right-hand side here is divisible by $p_2 - p_1$; it takes the form:

$$(p_2 - p_1)(1 + O(z^{-2}))$$

whereas:

$$\tilde{J}_{\boldsymbol{e},\mathcal{V}_G}(\varphi(0)) \cup (p_2 - p_1) = (p_2 - p_1)(1 + \varphi(0)z^{-1} + O(z^{-2})).$$

We conclude that $\varphi(0) = 0$. Thus,

$$\tilde{J}_{\boldsymbol{e},\mathcal{V}_G}(0) \cup (p_2 - p_1) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (-1)^{l+m} q_1^{l+m} q_2^n \times \frac{\prod_{k=1}^{k=l+m}(\lambda + p_1 + p_2 + kz)}{\prod_{k=1}^{k=l}(p_1+kz)^4 \prod_{k=1}^{k=m}(p_2+kz)^4} \times \frac{\prod_{k=-\infty}^{k=0} p_3-p_1+kz}{\prod_{k=-\infty}^{k=n-l} p_3-p_1+kz} \frac{\prod_{k=-\infty}^{k=0} p_3-p_2+kz}{\prod_{k=-\infty}^{k=n-m} p_3-p_2+kz} \times (p_2 - p_1 + (m-l)z). \tag{6–6}$$

To extract the quantum period $G_{\mathrm{MW}^4_{11}}$ from the twisted J-function $J_{\boldsymbol{e},\mathcal{V}_G}(0)$, we proceed as in [Coates et al. 16, Example D.8]: we take the non-equivariant limit, extract the component along the unit class $1 \in H^\bullet(X/\!\!/G;\mathbb{Q})$, set $z = 1$, and set $Q^\beta = t^{\langle\beta,-K\rangle}$ where $K = K_{\mathrm{MW}^4_{11}}$. Thus, we consider the right-hand side of (6–6), take the non-equivariant limit, extract the coefficient of $p_2 - p_1$, set $z = 1$, and set $q_1 = q_2 = 2t$, obtaining:

$$G_{\mathrm{MW}^4_{11}}(t) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=\max(l,m)}^{\infty} (-1)^{l+m} t^{2l+2m+2n} \times \frac{(l+m)!}{(l!)^4(m!)^4(n-l)!(n-m)!} \times (1 + (m-l)(H_{n-m} - 4H_m)) + \sum_{l=0}^{\infty}\sum_{m=l+1}^{\infty}\sum_{n=l}^{m-1} (-1)^{l+n} t^{2l+2m+2n} \times \frac{(l+m)!(m-n-1)!}{(l!)^4(m!)^4(n-l)!}(m-l).$$

Remark 6.3. The quantum period of $\mathbb{P}(\Omega^2_{\mathbb{P}(V)}(2))$ can also be computed using Strangeway's reconstruction theorem for the quantum cohomology of Fano bundles [Strangeway 15, Theorem 1]. Thus, the quantum period of MW^4_{11} can be derived from this result together with the Quantum Lefschetz theorem. The Gromov–Witten invariants required as input to the reconstruction theorem can be computed via [Strangeway 15, Lemma 1], using Schubert calculus on $\mathrm{Gr}(2,4)$ and intersection numbers in $\mathbb{P}^3$.

6.2.12. MW^4_{12}. *[regularized quantum period p. 16, operator p. 39]*

This is the blow-up of the quadric Q^4 along a line. Consider the toric variety F with weight data:

s_0	s_1	s_2	s_3	x	x_4	x_5	
1	1	1	1	−1	0	0	L
0	0	0	0	1	1	1	M

and $\overline{\mathrm{Amp}}F = \langle L, M\rangle$. The morphism $F \to \mathbb{P}^5$ that sends (contravariantly) the homogenous co-ordinate functions $[x_0, x_1, \ldots, x_5]$ to $[xs_0, xs_1, xs_2, xs_3, x_4, x_5]$ blows up the line $(x_0 = x_1 = x_2 = x_3 = 0)$ in $\mathbb{P}^5$, and MW^4_{12} is the proper transform of a quadric containing this line. Thus MW^4_{12} is a member of $|L+M|$ in the toric variety F. We have:

- $-K_F = 3L + 3M$ is ample, so that F is a Fano variety;
- $\mathrm{MW}^4_{12} \sim L + M$ is ample;
- $-(K_F + L + M) \sim 2L + 2M$ is ample.

Theorem 2.2 yields

$$G_{\mathrm{MW}^4_{12}}(t) = \sum_{l=0}^{\infty}\sum_{m=l}^{\infty} \frac{(l+m)!}{(l!)^4(m-l)!(m!)^2} t^{2l+2m}.$$

6.2.13. MW^4_{13}. *[regularized quantum period p. 16, operator p. 40]*

This is the projective bundle $\mathbb{P}_{Q^3}(\mathcal{O}(1) \oplus \mathcal{O})$ or, equivalently, a member of $|2L|$ in the toric variety F with weight data:

x_0	x_1	x_2	x_3	x_4	u	v	
1	1	1	1	1	0	−1	L
0	0	0	0	0	1	1	M

and $\overline{\mathrm{Amp}}F = \langle L, M\rangle$. We have

- $-K_F = 4L + 2M$ is ample, that is, F is a Fano variety;
- $\mathrm{MW}^4_{13} \sim 2L$ is nef;
- $-(K_F + 2L) \sim 2L + 2M$ is ample.

The projection $[x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : u : v] \mapsto [x_0 : x_1 : x_2 : x_3 : x_4 : x_5]$ exhibits F as the scroll $\mathbb{P}_{\mathbb{P}^4}(\mathcal{O}(1) \oplus \mathcal{O})$ over $\mathbb{P}^4$, and passing to a member of $|2L|$ restricts this scroll to $Q^3 \subset \mathbb{P}^4$. Theorem 2.2 yields

$$G_{\mathrm{MW}^4_{13}}(t) = \sum_{l=0}^{\infty}\sum_{m=l}^{\infty} \frac{(2l)!}{(l!)^5 m!(m-l)!} t^{2l+2m.}$$

6.2.14. MW^4_{14}. *[regularized quantum period p. 16, operator p. 40]*

This is the product $\mathbb{P}^1 \times \mathbb{P}^3$. Combining Theorem 2.4 with [Coates et al. 16, Example G.1] and [Coates et al. 16, Section 1] yields

$$G_{\mathrm{MW}^4_{14}}(t) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{t^{2l+4m}}{(l!)^2(m!)^4}.$$

6.2.15. MW^4_{15}. *[regularized quantum period p. 16, operator p. 41]*

This is the projective bundle $\mathbb{P}_{\mathbb{P}^3}(\mathcal{O}(1) \oplus \mathcal{O}(-1))$, or in other words, the toric variety with weight data:

x_0	x_1	x_2	x_3	u	v	
1	1	1	1	0	−2	L
0	0	0	0	1	1	M

and $\overline{\mathrm{Amp}}F = \langle L, M\rangle$. Theorem 2.1 yields

$$G_{\mathrm{MW}^4_{15}}(t) = \sum_{l=0}^{\infty}\sum_{m=2l}^{\infty} \frac{t^{2l+2m}}{(l!)^4 m!(m-2l)!}.$$

6.2.16. MW^4_{16}. *[regularized quantum period p. 16, operator p. 42]*

This is the product $\mathbb{P}^1 \times W^3$, where $W^3 \subset \mathbb{P}^2 \times \mathbb{P}^2$ is a divisor of bidegree $(1,1)$. Theorem 2.2 yields

$$G_{\mathrm{MW}^4_{16}}(t) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(m+n)!}{(l!)^2(m!)^3(n!)^3} t^{2l+2m+2n}.$$

6.2.17. MW^4_{17}. *[regularized quantum period p. 16, operator p. 43]*

This is the product $\mathbb{P}^1 \times B^3_7$, where B^3_7 is the blow-up of $\mathbb{P}^3$ at a point. Note that B^3_7 is the projective bundle $\mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(-1))$. It follows that MW^4_{17} is the toric variety with weight data:

x_0	x_1	y_0	y_1	y_2	u	v	
1	1	0	0	0	0	0	L
0	0	1	1	1	0	−1	M
0	0	0	0	0	1	1	N

and $\overline{\text{Amp}}F = \langle L, M, N\rangle$. Theorem 2.1 yields

$$G_{\text{MW}^4_{17}}(t) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=m}^{\infty} \frac{t^{2l+2m+2n}}{(l!)^2(m!)^3 n!(n-m)!}.$$

6.2.18. MW^4_{18}. *[regularized quantum period p. 16, operator p. 44]*

This is the product $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Combining Theorem 2.4 with [Coates et al. 16, Example G.1] yields

$$G_{\text{MW}^4_{18}}(t) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{t^{2k+2l+2m+2n}}{(k!)^2(l!)^2(m!)^2(n!)^2}.$$

7. Four-dimensional Fano toric manifolds

Four-dimensional Fano toric manifolds were classified by Batyrev [Batyrev 99] and Sato [Sato 00]. Øbro classified Fano toric manifolds in dimensions 2–8 [Øbro 07] and, to standardize notation, we will write BØS^4_k for the kth four-dimensional Fano toric manifold in Øbro's list. BØS^4_k is the $(23+k)$th Fano toric manifold in the Graded Ring Database [Brown and Kasprzyk], as the list there is the concatenation of Øbro's lists in dimensions 2–8. We can compute the quantum periods of the BØS^4_k using Theorem 2.1; the first few Taylor coefficients of their regularized quantum periods can be found in the tables in the Appendix.

8. Product manifolds and other index 1 examples

Quantum periods for one-, two- and three-dimensional Fano manifolds were computed in [Coates et al. 16]. Combining these results with Theorem 2.4 allows us to compute the quantum period of any four-dimensional Fano manifold that is a product of lower-dimensional manifolds. Many of these examples have Fano index $r = 1$.

In his thesis [Strangeway 14], Strangeway determined the quantum periods of two four-dimensional Fano manifolds of index $r = 1$ that have not yet been discussed. These manifolds arise as complete intersections in the nine-dimensional projective bundle $F = \mathbb{P}(\Omega^2_{\mathbb{P}^4}(2))$. Let $\pi : F \to \mathbb{P}^4$ denote the canonical projection, let $p \in H^2(F)$ be the first Chern class of $\pi^\star\mathcal{O}_{\mathbb{P}^4}(1)$, and let $\xi \in H^2(F)$ be the first Chern class of the tautological bundle $\mathcal{O}_F(1)$. The manifold F is Fano of Picard rank 2, with nef cone generated by $\{\xi, p\}$ and $-K_F = 6\xi + 2p$. Let:

$\text{Str}_1 \subset F$ denote a complete intersection of five divisors of type ξ

$\text{Str}_2 \subset F$ denote a complete intersection of four divisors of type ξ and a divisor of type p

We consider also:

$\text{Str}_3 \subset F$, a complete intersection of four divisors of type ξ and a divisor of type $\xi + p$

which was unaccountably omitted from [Strangeway 14].

The manifolds Str_k, $k \in \{1, 2, 3\}$, each have Picard rank two. To see this, observe that the ambient manifold F is the blow-up of $\mathbb{P}^9$ along $\text{Gr}(2, 5)$, where $\text{Gr}(2, 5) \to \mathbb{P}^9$ is the Plücker embedding [Strangeway 15]; the blow-up $F \to \mathbb{P}^9$ and the projection $\pi : F \to \mathbb{P}^4$ are the extremal contractions corresponding to the two extremal rays in $\overline{\text{NE}}(F)$. Thus, Str_1 is the blow-up of $\mathbb{P}^4$ along an elliptic curve $E_5 \subset \mathbb{P}^4$ of degree 5. Consider the five-dimensional Fano manifold F_5 given by the complete intersection of four divisors of type ξ in F. Then F_5 is the blow-up of $\mathbb{P}^5$ along a del Pezzo surface S_5 of degree 5; in particular, F_5 has Picard rank two. Str_3 is an ample divisor (of type $\xi + p$) in F_5, so the Picard rank of Str_3 is also two. The manifold Str_2 is a divisor in F_5 of type p, and F_5 arises as the closure of the graph of the map $\mathbb{P}^5 \to \mathbb{P}^4$ given by the five-dimensional linear system of quadrics passing through S_5. This exhibits Str_2 as the blow-up of a smooth four-dimensional quadric Q^4 along S_5, which implies that the Picard rank of Str_2 is two.

We can compute the quantum periods of Str_k, $k \in \{1, 2, 3\}$, by observing that a complete intersection in F of five divisors of type ξ and one divisor of type p is a three-dimensional Fano manifold $\text{MM}^3_{2\text{–}17}$, "unsectioning" to compute the quantum period for F, and then applying the quantum Lefschetz theorem to compute the quantum periods for Str_1, Str_2, and Str_3. Recall the definition of the J-function $J_X(t, z)$ from [Coates and Givental 07, equation 11]. The identity component of the J-function of $\text{MM}^3_{2\text{–}17}$ is

$$\begin{aligned} e^{-q_1-q_2} &\sum_{l_1,l_2,l_3\geq 0} (-q_1)^{l_1+l_2} q_2^{l_3} \\ &\times \frac{(l_1+l_2)!(l_1+l_3)!(l_2+l_3)!(l_1+l_2+l_3)!}{(l_1!)^4(l_2!)^4(l_3!)^4 z^{l_1+l_2+l_3}} \\ &\times (1 + (l_2 - l_1)(H_{l_2+l_3} - 4H_{l_2})), \end{aligned} \tag{8–7}$$

where q_1, q_2 are generators of the Novikov ring for $\text{MM}^3_{2\text{–}17}$ dual, respectively, to ξ and p; see [Coates et al. 16, Section 34]. The identity component of the J-function of F takes the form:

$$\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} c_{l,m} z^{-6l-2m} q_1^l q_2^m$$

for some coefficients $c_{l,m} \in \mathbb{Q}$. The Quantum Lefschetz theorem implies (see [Coates et al. 16, Section D.1]) that the identity component of the J-function of $\text{MM}^3_{2\text{–}17}$ is equal to

$$e^{-c_{1,0}q_1-c_{0,1}q_2}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}(l!)^5 m! c_{l,m} z^{-l-m} q_1^l q_2^m \qquad (8\text{–}8)$$

and it is known that $c_{1,0}=1$ and $c_{0,1}=0$ [Strangeway 15, Section 5.1]. Equating (8–7) and (8–8) determines the $c_{l,m}$:

$$c_{l,m}=\sum_{i=0}^{l}\sum_{j=0}^{m}(-1)^{j+l} \times \frac{(m+l-i-j)!(i+m-j)!(m+l-j)!}{((l-i)!)^4(i!)^4((m-j)!)^4 j! m! (l!)^4} \times (1+(2i-l)(H_{i+m-j}-4H_i)).$$

The Quantum Lefschetz theorem now gives that

$$G_{\mathrm{Str}_1}(t)=e^{-t}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}(l!)^5 c_{l,m} t^{l+2m}$$

$$G_{\mathrm{Str}_2}(t)=\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}(l!)^4 m!\, c_{l,m} t^{2l+m}$$

$$G_{\mathrm{Str}_3}(t)=e^{-t}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}(l!)^4 (l+m)!\, c_{l,m} t^{l+m}.$$

9. Numerical calculations of quantum differential operators

As discussed in Section 2, the regularized quantum period $\widehat{G}_X(t)$ of a Fano manifold X satisfies a differential equation:

$$L_X\widehat{G}_X \equiv 0 \qquad L_X=\sum_{k=0}^{k=N} p_k(t) D^k \qquad (9\text{–}9)$$

called the regularized quantum differential equation. Here the p_m are polynomials and $D=t\frac{d}{dt}$. The regularized quantum differential equation for X coincides with the (unregularized) quantum differential equation for an anticanonical Calabi–Yau manifold $Y\subset X$; the study of the regularized quantum period from this point of view was pioneered by Batyrev–Ciocan-Fontanine–Kim–van Straten [Batyrev et al. 98, Batyrev et al. 00]. The differential equation (9–9) is expected to be Fuchsian, and the local system of solutions to $L_X f \equiv 0$ is expected to be of *low ramification* in the following sense.

Definition 9.1 **([Coates et al. 12]).** Let $S\subset \mathbb{P}^1$ a finite set, and $\mathbb{V}\to \mathbb{P}^1\setminus S$ a local system. Fix a basepoint $x\in \mathbb{P}^1\setminus S$. For $s\in S$, choose a small loop that winds once anticlockwise around s and connect it to x via a path, thereby making a loop γ_s about s based at x. Let $T_s:\mathbb{V}_x\to\mathbb{V}_x$ denote the monodromy of $\mathbb{V}$ along γ_s. The *ramification* of $\mathbb{V}$ is

$$\mathrm{rf}(\mathbb{V}) := \sum_{s\in S}\dim(\mathbb{V}_x/\mathbb{V}_x^{T_s}).$$

The ramification defect of $\mathbb{V}$ is the quantity $\mathrm{rf}(\mathbb{V})-2\mathrm{rk}(\mathbb{V})$. Non-trivial irreducible local systems $\mathbb{V}\to\mathbb{P}^1\setminus S$ have non-negative ramification defect; this gives a lower bound for the ramification of $\mathbb{V}$. A local system of ramification defect zero is called extremal.

Definition 9.2. The ramification (respectively, ramification defect) of a differential operator L_X is the ramification (respectively, ramification defect) of the local system of solutions $L_X f\equiv 0$.

Definition 9.3. The *quantum differential operator* for a Fano manifold X is the operator $L_X\in\mathbb{Q}[t]\langle D\rangle$ such that $L_X\widehat{G}_X\equiv 0$ which is of lowest order in D and, among all such operators of this order, is of lowest degree in t. (This defines L_X only up to an overall scalar factor, but this suffices for our purposes.)

Suppose that each of the polynomials $p_0,\ldots,p_N$ are of degree at most r, and write:

$$L_X=\sum_{k=0}^{k=N}p_k(t)D^k \qquad p_k(t)=\sum_{l=0}^{r}a_{kl}t^l.$$

The differential equation $L_X\widehat{G}_X\equiv 0$ gives a system of linear equations for the coefficients a_{kl} which, given sufficiently many terms of the Taylor expansion of $\widehat{G}_X$, becomes over-determined. Given *a priori* bounds on N and r, therefore, we could compute the quantum differential operator L_X by calculating sufficiently many terms in the Taylor expansion. In general, we do not have such bounds, but nonetheless by ensuring the linear system for (a_{kl}) is highly over-determined we can be reasonably confident that the operator L_X which we compute is correct. In addition, since L_X is expected to correspond under mirror symmetry to a Picard–Fuchs differential equation for the mirror family, L_X is expected to be of Fuchsian type. This is an extremely delicate condition on the coefficients (a_{kl}), and it can be checked by exact computation.

We computed candidate quantum differential operators L_X for all four-dimensional Fano manifolds of Fano index $r>1$, and checked the Fuchsian condition in each case. The operators L_X, together with their ramification defects and the log-monodromy data $\{\log T_s : s\in S\}$ in Jordan normal form, can be found in Appendix B. In 24 cases, the local system of solutions to the regularized quantum differential equation is extremal, and in the remaining 11 cases it is of ramification defect 1.

To compute the ramification of L_X, we follow Kedlaya [Kedlaya 10, Section 7.3]. This involves only linear algebra over a splitting field for $p_N(t)$—recall that every singular point of L_X occurs at a root of $p_N(t)$—and thus can be implemented using exact (not numerical) computer algebra. For this we use Steel's symbolic implementation of $\overline{\mathbb{Q}}$ in the computational algebra system Magma [Bosma et al. 97, Steel 10].

The situation in lower dimensions

The classification of three-dimensional Fano manifolds is known [Iskovskih 77, Iskovskih 78, Iskovskih 79, Mori and Mukai 82, Mori and Mukai 83, Mori and Mukai 86, Mori and Mukai 03, Mori and Mukai 04], and the quantum periods of all three-dimensional Fano manifolds have been computed [Coates et al. 16]. In each case, the regularized quantum differential equation (9–9) is extremal, and coincides with the Picard–Fuchs differential equation associated with a Laurent polynomial $f : (\mathbb{C}^\times)^3 \to \mathbb{C}$ in three variables.[1] This latter phenomenon is a manifestation of Mirror Symmetry, and when it occurs then we say that the Laurent polynomial f is a mirror to the corresponding Fano manifold X. The classification of two-dimensional Fano manifolds, which are called del Pezzo surfaces, is classical, and it was proved in [Akhtar et al. 16] that del Pezzo surfaces correspond under Mirror Symmetry to a distinguished family of Laurent polynomials in two variables called *maximally mutable* Laurent polynomials. This mirror correspondence extends to two-dimensional Fano orbifolds too [Akhtar et al. 16, Oneto and Petracci, Cavey and Prince 17], and this inspired the classification of del Pezzo surfaces with $\frac{1}{3}(1, 1)$ singularities by Corti–Heuberger [Corti and Heuberger 17, Kasprzyk et al. 17]. Kasprzyk and Tveiten have defined what it means for a Laurent polynomial in any number of variables to be maximally mutable [Kasprzyk and Tveiten], and each of the three-dimensional Fano manifolds corresponds under mirror symmetry to a maximally mutable Laurent polynomial in three variables. It would be very interesting to find out whether the correspondence between maximally mutable Laurent polynomials and Fano manifolds (or, more precisely, Fano varieties with an appropriate class of mild singularities) persists to higher dimensions. Furthermore, Golyshev has observed a connection, which holds in low dimensions, between the ramification defect of the regularized quantum differential operator L_X of a Fano manifold X and the dimension of the primitive part of the middle-dimensional cohomology of X. The regularized quantum differential operators for three-dimensional Fano manifolds are all extremal, which is consistent with the fact that this primitive part automatically vanishes in odd dimensions. It will be interesting to see how much of this picture persists beyond dimension 3.

[1] The proof of extremality involves applying the generalized Griffiths–Dwork algorithm of Lairez [Lairez 16] to f.

Source code

This paper is accompanied by full source code, written in Magma. See the included file README.txt for usage instructions. The source code, but not the text of this paper, is released under a Creative Commons CC0 license [CCO]: see the included file COPYING.txt for details. If you make use of the source code in an academic or commercial context, please acknowledge this by including a reference or citation to this paper.

Acknowledgments

We thank Alessio Corti for a number of very useful conversations.

Funding

This research was supported by a Royal Society University Research Fellowship (TC); ERC Starting Investigator Grant number 240123; the Leverhulme Trust; grant MK-1297.2014.1; AG Laboratory NRU-HSE, RF government grant ag. 11.G34.31.0023; Grant of Leading Scientific Schools (N.Sh. 2998.2014.1); EPSRC grant EP/I008128/1, and EPSRC Fellowship EP/N022513/1.

ORCID

Sergey Galkin http://orcid.org/0000-0003-3175-2670

References

[Akhtar et al. 16] M. Akhtar, T. Coates, A. Corti, L. Heuberger, A. Kasprzyk, A. Oneto, A. Petracci, T. Prince, and K. Tveiten. "Mirror Symmetry and the Classification of Orbifold del Pezzo Surfaces," *Proc. Amer. Math. Soc.* 144:2 (2016), 513–527.

[Batyrev 99] V. V. Batyrev. "On the Classification of Toric Fano 4-Folds," *J. Math. Sci. (New York)* 94:1 (1999), 1021–1050.

[Batyrev et al. 98] V. V. Batyrev, I. Ciocan-Fontanine, B. Kim, and D. van Straten. "Conifold Transitions and Mirror Symmetry for Calabi-Yau Complete Intersections in Grassmannians," *Nucl. Phys. B* 514:3 (1998), 640–666.

[Batyrev et al. 00] V. V. Batyrev, I. Ciocan-Fontanine, B. Kim, and D. van Straten. "Mirror Symmetry and Toric Degenerations of Partial Flag Manifolds," *Acta Math.* 184:1 (2000), 1–39.

[Bosma et al. 97] W. Bosma, J. Cannon, and C. Playoust. "The Magma Algebra System. I. The User Language," *J. Symb. Comput.* 24:3–4 (1997), 235–265.

[Brown and Kasprzyk] G. Brown and A. Kasprzyk. "The Graded Ring Database," Available online http://www.grdb.co.uk/.

[Cavey and Prince 17] D. Cavey and T. Prince. "Del Pezzo Surfaces with a Single $\frac{1}{k}(1,1)$ Singularity," arXiv:1707.09213 [math.AG], 2017.

[CC0] CC0, "Creative Commons CC0 License," Available online https://creativecommons.org/publicdomain/zero/1.0/ and https://creativecommons.org/publicdomain/zero/1.0/legalcode.

[Ciocan-Fontanine et al. 08] I. Ciocan-Fontanine, B. Kim, and C. Sabbah. "The Abelian/nonabelian Correspondence and Frobenius Manifolds," *Invent. Math.* 171:2 (2008), 301–343.

[Coates 14] T. Coates. "The Quantum Lefschetz Principle for Vector Bundles as a Map Between Givental Cones," arXiv:1405.2893 [math.AG], 2014.

[Coates et al. 12] T. Coates, A. Corti, S. Galkin, V. Golyshev, and A. Kasprzyk. "Mirror Symmetry and Fano Manifolds." *European Congress of Mathematics, Kraków*, July 2–7, 2012, 2014, pp. 285–300.

[Coates et al. 16] T. Coates, A. Corti, S. Galkin, and A. Kasprzyk. "Quantum Periods for 3-dimensional Fano Manifolds," *Geom. Topol.* 20:1 (2016), 103–256.

[Coates et al. 14] T. Coates, A. Corti, H. Iritani, and H.-H. Tseng. "Some Applications of the Mirror theorem for Toric Stacks," arXiv:1401:2611 [math.AG], 2014.

[Coates and Givental 07] T. Coates and A. Givental. "Quantum Riemann-Roch, Lefschetz and Serre," *Ann. Math.* (2) 165:1 (2007), 15–53.

[Corti and Heuberger 17] A. Corti and L. Heuberger. "Del Pezzo Surfaces with $\frac{1}{3}(1,1)$ Points," *Manuscr. Math.* 153:1–2 (2017), 71–118.

[Fujita 80] T. Fujita. "On the Structure of Polarized Manifolds with Total Deficiency One. I," *J. Math. Soc. Japan* 32:4 (1980), 709–725.

[Fujita 81] T. Fujita. "On the Structure of Polarized Manifolds with Total Deficiency One. II," *J. Math. Soc. Japan* 33:3 (1981), 415–434.

[Fujita 84] T. Fujita. "On the Structure of Polarized Manifolds with Total Deficiency One. III, *J. Math. Soc. Japan* 36:1 (1984), 75–89.

[Fujita 90] T. Fujita, Classification Theories of Polarized Varieties. In *London Mathematical Society Lecture Note Series*, vol. 155. Cambridge: Cambridge University Press, 1990.

[Givental 96] A. Givental. "A Mirror Theorem for Toric Complete Intersections." In *Topological Field Theory, Primitive Forms and Related Topics (Kyoto, 1996)*, pp. 141–175, Progr. Math., vol. 160. Boston, MA: Birkhäuser, 1998.

[Hartshorne 77] R. Hartshorne. "Algebraic Geometry." In *Graduate Texts in Mathematics*, vol. 52. New York-Heidelberg: Springer-Verlag, 1977.

[Iskovskih 77] V. A. Iskovskih. "Fano Threefolds. I," *Izv. Akad. Nauk SSSR Ser. Mat.* 41:3 (1977), 516–562, 717.

[Iskovskih 78] V. A. Iskovskih. "Fano Threefolds. II," *Izv. Akad. Nauk SSSR Ser. Mat.* 42:3 (1978), 506–549.

[Iskovskih 79] V. A. Iskovskih. "Anticanonical Models of Three-dimensional Algebraic Varieties." In *Current Problems in Mathematics*, vol. 12 (Russian), pp. 59–157, Moscow: VINITI, 1979; 239 (loose errata).

[Iskovskikh and Prokhorov 99] V. A. Iskovskikh and Yu. G. Prokhorov. "Fano Varieties, Algebraic Geometry, V." In *Encyclopaedia Math. Sci.*, vol. 47, pp. 1–247, Berlin: Springer, 1999.

[Kasprzyk et al. 17] A. Kasprzyk, B. Nill, and T. Prince. "Minimality and Mutation-Equivalence of Polygons," *Forum Math. Sigma* 5:e18 (2017), 48.

[Kasprzyk and Tveiten] A. Kasprzyk and K. Tveiten. "Maximally Mutable Laurent Polynomials," in preparation.

[Kedlaya 10] K. S. Kedlaya. "*p*-adic Differential Equations." In *Cambridge Studies in Advanced Mathematics*, vol. 125. Cambridge: Cambridge University Press, 2010.

[Kobayashi and Ochiai 73] S. Kobayashi and T. Ochiai. "Characterizations of Complex Projective Spaces and Hyperquadrics," *J. Math. Kyoto Univ.* 13 (1973), 31–47.

[Kollár 81] Y. Kollár. "Higher-dimensional Fano Varieties of Large Index," *Vestnik Moskov. Univ. Ser. I Mat. Mekh.* no. 3 (1981), 31–34, 80–81.

[Lairez 16] P. Lairez. "Computing Periods of Rational Integrals, *Math. Comp.* 85:300 (2016), 1719–1752.

[Mori and Mukai 82] S. Mori and S. Mukai. "Classification of Fano 3-folds with $B_2 \geq 2$," *Manusc. Math.* 36:2 (1981/82), 147–162.

[Mori and Mukai 83] S. Mori and S. Mukai. "On Fano 3-Folds with $B_2 \geq 2$." In *Algebraic Varieties and Analytic Varieties (Tokyo, 1981)*, pp. 101–129, Adv. Stud. Pure Math., vol. 1. Amsterdam: North-Holland, 1983.

[Mori and Mukai 86] S. Mori and S. Mukai. "Classification of Fano 3-Folds with $B_2 \geq 2$. I." *Algebraic and Topological Theories (Kinosaki, 1984)*, pp. 496–545, Tokyo: Kinokuniya, 1986.

[Mori and Mukai 03] S. Mori and S. Mukai. "Erratum: "Classification of Fano 3-folds with $B_2 \geq 2$ [Manuscripta Math. 36 (1981/82), no. 2, 147–162]," *Manusc. Math.* 110:3 (2003), 407.

[Mori and Mukai 04] S. Mori and S. Mukai. "Extremal Rays and Fano 3-Folds." The Fano Conference, Univ. Torino, Turin, 2004, pp. 37–50.

[Mukai 88] S. Mukai. "Curves, $K3$ Surfaces and Fano 3-folds of Genus ≤ 10." In *Algebraic Geometry and Commutative Algebra*, vol. I, pp. 357–377, Tokyo: Kinokuniya, 1988.

[Mukai 89] S. Mukai. "Biregular Classification of Fano 3-folds and Fano Manifolds of Coindex 3," *Proc. Nat. Acad. Sci. U.S.A.* 86:9 (1989), 3000–3002.

[Øbro 07] M. Øbro. "An Algorithm for the Classification of Smooth Fano Polytopes," arXiv:0704.0049 [math.CO], 2007.

[Oneto and Petracci] A. Oneto and A. Petracci. "On the Quantum Periods of del Pezzo Surfaces with $\frac{1}{3}(1,1)$ Singularities," *to appear in Adv. Geom.* arXiv:1507.08589 [math.AG].

[Sato 00] H. Sato. "Toward the Classification of Higher-dimensional Toric Fano Varieties," *Tohoku Math. J. (2)* 52:3 (2000), 383–413.

[Serpico 80] M. E. Serpico. "Fano Varieties of Dimensions $n \geq 4$ and of Index $r \geq n-1$," *Rend. Sem. Mat. Univ. Padova* 62 (1980), 295–308.

[Shokurov 85] V. V. Shokurov. "A Nonvanishing Theorem," *Izv. Akad. Nauk SSSR Ser. Mat.* 49:3 (1985), 635–651.

[Steel 10] A. K. Steel. "Computing with Algebraically Closed Fields," *J. Symb. Comput.* 45:3 (2010), 342–372.

[Strangeway 14] A. Strangeway. "Quantum Reconstruction for Fano Bundles," PhD diss., Imperial College London, 2014.

[Strangeway 15] A. Strangeway. "Quantum Reconstruction for Fano Bundles on Projective Space," *Nagoya Math. J.* 218 (2015), 1–28.

[Szurek and Wiśniewski 90] M. Szurek and J. A. Wiśniewski. “Fano Bundles Over $\mathbf{P}^3$ and Q_3,” *Pac. J. Math.* 141:1 (1990), 197–208.

[Wilson 87] P. M. H. Wilson. “Fano Fourfolds of Index Greater than One,” *J. Reine Angew. Math.* 379 (1987), 172–181.

[Wiśniewski 90] J. Wiśniewski. “Fano 4-folds of Index 2 with $b_2 \geq 2$. A Contribution to Mukai Classification,” *Bull. Polish Acad. Sci. Math.* 38:1–12 (1990), 173–184.

Appendix A. Regularized quantum period sequences

In this appendix, we record the description, degree, and Picard rank ρ_X for each of the four-dimensional Fano manifolds X considered in this paper, together with the first few Taylor coefficients α_d of the regularized quantum period:

$$\widehat{G}_X(t) = \sum_{d=0}^{\infty} \alpha_d t^d.$$

The tables are divided by Fano index r. We include only coefficients α_d with $d \equiv 0 \bmod r$, since coefficients α_d with $d \not\equiv 0 \bmod r$ are zero. Notation is as follows:

- $\mathbb{P}^n$ denotes n-dimensional complex projective space;
- Q^n denotes a quadric hypersurface in $\mathbb{P}^{n+1}$;
- FI^4_k is as in Section 5 above;
- V^4_k is as in Section 6.1 above;
- MW^4_k is as in Section 6.2 above;
- $\mathrm{BØS}^4_k$ is as in Section 7 above;
- Str_k is as in Section 8 above;
- S^2_k denotes the del Pezzo surface of degree k;
- F_1 denotes the Hirzebruch surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1})$;
- V^3_k denotes the three-dimensional Fano manifold of Picard rank 1, Fano index 1, and degree k;
- B^3_k denotes the three-dimensional Fano manifold of Picard rank 1, Fano index 2, and degree $8k$;
- $\mathrm{MM}^3_{\rho-k}$ denotes the kth entry in the Mori–Mukai list of three-dimensional Fano manifolds of Picard rank ρ [Mori and Mukai 82, Mori and Mukai 83, Mori and Mukai 86, Mori and Mukai 03, Mori and Mukai 04]. We use the ordering as in [Coates et al. 16], which agrees with the original papers of Mori–Mukai except when $\rho = 4$.

We prefer to express manifolds as products of lower-dimensional manifolds where possible, so, for example, $\mathrm{BØS}^4_{122}$ is the product $\mathbb{P}^1 \times \mathbb{P}^3$, but we refer to this space as $\mathbb{P}^1 \times \mathbb{P}^3$ rather than $\mathrm{BØS}^4_{122}$. The tables for Fano index r with $r \in \{2, 3, 4, 5\}$ are complete. The table for $r = 1$ is very far from complete.

It appears from Table A.5 as if the regularized quantum period might coincide for the pairs $\{\mathrm{BØS}^4_6, \mathrm{BØS}^4_{41}\}$

Table A.1. Four-dimensional Fano manifolds with Fano index $r = 5$.

X	$(-K_X)^4$	ρ_X	α_0	α_5	α_{10}	α_{15}	α_{20}
$\mathbb{P}^4$	625	1	1	120	113400	168168000	305540235000

Table A.2. Four-dimensional Fano manifolds with Fano index $r = 4$.

X	$(-K_X)^4$	ρ_X	α_0	α_4	α_8	α_{12}	α_{16}
Q^4	512	1	1	48	15120	7392000	4414410000

Table A.3. Four-dimensional Fano manifolds with Fano index $r = 3$.

X	$(-K_X)^4$	ρ_X	α_0	α_3	α_6	α_9	α_{12}
$\mathbb{P}^2 \times \mathbb{P}^2$	486	2	1	12	900	94080	11988900
FI^4_5	405	1	1	18	1710	246960	43347150
FI^4_4	324	1	1	24	3240	672000	169785000
FI^4_3	243	1	1	36	8100	2822400	1200622500
FI^4_2	162	1	1	72	37800	31046400	31216185000
FI^4_1	81	1	1	360	1247400	6861254400	46381007673000

Table A.4. Four-dimensional Fano manifolds with Fano index $r = 2$.

X	$(-K_X)^4$	ρ_X	α_0	α_2	α_4	α_6	α_8
MW^4_{15}	640	2	1	2	6	380	6790
$\mathbb{P}^1 \times \mathbb{P}^3$	512	2	1	2	30	740	12670
MW^4_{13}	480	2	1	2	54	740	21910
MW^4_{12}	416	2	1	2	54	1100	28630
$\mathbb{P}^1 \times \mathrm{MM}^3_{2-35}$	448	3	1	4	60	1480	41020
MW^4_{11}	384	2	1	4	84	2200	70420
MW^4_{10}	352	2	1	4	84	2560	87220
MW^4_7	320	2	1	4	108	3280	126700
$\mathbb{P}^1 \times \mathrm{MM}^3_{2-32}$	384	3	1	6	114	3300	114450
MW^4_8	320	2	1	6	138	4740	194250
V^4_{18}	288	1	1	6	162	6180	284130
MW^4_5	256	2	1	6	186	7980	410970
$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	384	4	1	8	168	5120	190120
$\mathbb{P}^1 \times B^3_5$	320	2	1	8	192	6920	303520
V^4_{16}	256	1	1	8	240	10880	597520
V^4_{14}	224	1	1	8	288	15200	968800
MW^4_4	192	2	1	8	360	22400	1695400
$\mathbb{P}^1 \times B^3_4$	256	2	1	10	318	15220	886270
V^4_{12}	192	1	1	10	438	28900	2310070
V^4_{10}	160	1	1	12	684	58800	6129900
$\mathbb{P}^1 \times B^3_3$	192	2	1	14	690	50900	4540690
V^4_8	128	1	1	16	1296	160000	24010000
V^4_6	96	1	1	24	3240	672000	169785000
$\mathbb{P}^1 \times B^3_2$	128	2	1	26	2814	447380	84832510
V^4_4	64	1	1	48	15120	7392000	4414410000
$\mathbb{P}^1 \times B^3_1$	64	2	1	122	84606	84187220	98308169470
V^4_2	32	1	1	240	498960	1633632000	6558930378000

Table A.5. Certain four-dimensional Fano manifolds with Fano index $r = 1$.

X	$(-K_X)^4$	ρ_X	α_0	α_1	α_2	α_3	α_4	α_5	α_6	α_7
BØS$^4_{115}$	512	2	1	0	0	0	24	120	0	0
BØS$^4_{21}$	594	2	1	0	0	6	0	0	90	1260
BØS$^4_{118}$	513	2	1	0	0	6	0	120	90	0
BØS$^4_{17}$	450	3	1	0	0	6	0	120	90	1260
BØS$^4_{47}$	513	2	1	0	0	6	24	0	90	2520
BØS$^4_{94}$	459	3	1	0	0	6	24	120	90	1260
BØS$^4_{37}$	417	3	1	0	0	6	24	120	90	2520
BØS$^4_{74}$	486	3	1	0	0	6	48	0	90	2520
BØS$^4_{86}$	405	3	1	0	0	6	48	120	90	2520
BØS$^4_{114}$	401	3	1	0	0	12	0	120	900	0
BØS$^4_{46}$	406	3	1	0	0	12	24	0	900	3780
BØS$^4_{87}$	364	4	1	0	0	12	24	120	900	3780
BØS$^4_{32}$	322	4	1	0	0	12	24	240	900	5040
BØS$^4_{30}$	327	4	1	0	0	12	48	120	900	7560
BØS$^4_{31}$	249	5	1	0	0	18	72	360	2430	18900
Str$_1$	225	2	1	0	0	30	120	240	5850	50400
BØS$^4_{2}$	800	2	1	0	2	0	6	0	20	840
BØS$^4_{1}$	605	3	1	0	2	0	6	0	380	840
BØS$^4_{12}$	560	3	1	0	2	0	6	60	380	840
BØS$^4_{121}$	544	2	1	0	2	0	6	120	20	2520
BØS$^4_{105}$	489	3	1	0	2	0	6	120	380	2520
BØS$^4_{18}$	529	3	1	0	2	0	30	60	380	840
BØS$^4_{10}$	496	4	1	0	2	0	30	60	740	840
BØS$^4_{109}$	464	3	1	0	2	0	30	120	380	2520
BØS$^4_{104}$	431	3	1	0	2	0	30	120	740	2520
BØS$^4_{15}$	433	4	1	0	2	0	30	180	380	3360
BØS$^4_{11}$	415	4	1	0	2	0	30	180	740	3360
BØS$^4_{8}$	576	3	1	0	2	6	6	60	110	1680
BØS$^4_{26}$	560	3	1	0	2	6	6	60	470	420
BØS$^4_{7}$	592	3	1	0	2	6	6	120	110	1260
BØS$^4_{20}$	400	3	1	0	2	6	6	120	830	2520
BØS$^4_{111}$	480	3	1	0	2	6	6	180	110	2940
BØS$^4_{24}$	442	4	1	0	2	6	6	180	470	2940
BØS$^4_{106}$	496	3	1	0	2	6	30	60	470	2940
BØS$^4_{45}$	432	3	1	0	2	6	30	60	830	2940
BØS$^4_{41}$	433	3	1	0	2	6	30	120	470	3780
BØS$^4_{6}$	463	4	1	0	2	6	30	120	470	3780
$\mathbb{P}^1 \times \mathrm{MM}^3_{2\text{–}33}$	432	3	1	0	2	6	30	120	830	2520
BØS$^4_{82}$	432	4	1	0	2	6	30	180	470	4200
BØS$^4_{113}$	400	3	1	0	2	6	30	180	470	5460
BØS$^4_{92}$	384	4	1	0	2	6	30	180	830	5460
BØS$^4_{70}$	411	4	1	0	2	6	30	240	470	5040
BØS$^4_{16}$	337	4	1	0	2	6	30	240	1190	7560
BØS$^4_{52}$	464	4	1	0	2	6	54	60	830	2940
BØS$^4_{71}$	390	4	1	0	2	6	54	120	1190	3780
BØS$^4_{91}$	384	4	1	0	2	6	54	180	830	5460
BØS$^4_{13}$	368	4	1	0	2	6	54	180	830	5880
BØS$^4_{81}$	357	4	1	0	2	6	54	240	1190	6300
$\mathbb{P}^2 \times F_1$	432	3	1	0	2	12	6	180	920	1680
$\mathbb{P}^1 \times Q^3$	432	2	1	0	2	12	6	240	560	2520
BØS$^4_{27}$	417	4	1	0	2	12	6	240	560	3360
BØS$^4_{60}$	448	4	1	0	2	12	30	120	920	4620
BØS$^4_{88}$	389	4	1	0	2	12	30	180	1280	5460

(*Continued on next page*)

Table A.5. (Continued).

X	$(-K_X)^4$	ρ_X	α_0	α_1	α_2	α_3	α_4	α_5	α_6	α_7
$\mathrm{BØS}^4_{35}$	369	4	1	0	2	12	30	180	1280	5460
$\mathbb{P}^1 \times \mathrm{MM}^3_{2-30}$	368	3	1	0	2	12	30	240	1280	5040
$\mathrm{BØS}^4_{93}$	347	4	1	0	2	12	30	300	1280	7980
$\mathrm{BØS}^4_{85}$	352	4	1	0	2	12	54	240	1280	9660
$\mathrm{BØS}^4_{42}$	326	4	1	0	2	12	54	240	1640	10080
$\mathrm{BØS}^4_{51}$	480	4	1	0	2	18	6	180	1370	1260
$\mathbb{P}^1 \times \mathrm{MM}^3_{2-28}$	320	3	1	0	2	18	30	360	2090	7560
$\mathrm{BØS}^4_{73}$	352	4	1	0	2	18	54	180	2090	11340
Str_2	240	2	1	0	2	30	54	600	6590	26040
$\mathbb{P}^1 \times \mathrm{MM}^3_{2-36}$	496	3	1	0	4	0	36	60	400	3360
$\mathrm{BØS}^4_{43}$	464	3	1	0	4	0	36	120	400	5040
$\mathbb{P}^1 \times \mathrm{MM}^3_{3-29}$	400	4	1	0	4	0	60	60	1480	3360
$\mathrm{BØS}^4_{36}$	384	4	1	0	4	0	60	120	1480	5040
$\mathrm{BØS}^4_{3}$	558	4	1	0	4	6	36	120	490	3360
$\mathrm{BØS}^4_{22}$	505	4	1	0	4	6	36	120	850	2100
$\mathrm{BØS}^4_{5}$	478	4	1	0	4	6	36	180	490	5460
$\mathrm{BØS}^4_{9}$	382	4	1	0	4	6	36	180	1210	6720
$\mathrm{BØS}^4_{95}$	447	4	1	0	4	6	36	240	490	7140
$\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$	432	3	1	0	4	6	36	240	490	7560
$\mathrm{BØS}^4_{25}$	409	4	1	0	4	6	36	240	850	7140
$\mathrm{BØS}^4_{100}$	415	4	1	0	4	6	60	120	1570	4620
$\mathbb{P}^1 \times \mathrm{MM}^3_{3-30}$	400	4	1	0	4	6	60	180	1570	5460
$\mathrm{BØS}^4_{34}$	369	4	1	0	4	6	60	180	1570	6720
$\mathrm{BØS}^4_{56}$	405	5	1	0	4	6	60	240	1210	8400
$\mathbb{P}^1 \times \mathrm{MM}^3_{3-26}$	368	4	1	0	4	6	60	240	1570	8820
$\mathrm{BØS}^4_{102}$	367	4	1	0	4	6	60	240	1570	9660
$\mathrm{BØS}^4_{44}$	351	4	1	0	4	6	60	240	1930	9660
$\mathrm{BØS}^4_{48}$	442	5	1	0	4	6	84	120	1930	4620
$\mathbb{P}^1 \times \mathrm{MM}^3_{3-22}$	320	4	1	0	4	6	84	300	2650	13440
$\mathrm{BØS}^4_{29}$	310	5	1	0	4	6	84	360	2650	15120
$\mathbb{P}^1 \times \mathrm{MM}^3_{3-31}$	416	4	1	0	4	12	36	360	940	8400
$F_1 \times F_1$	384	4	1	0	4	12	36	360	1300	8400
$S^2_7 \times \mathbb{P}^2$	378	4	1	0	4	12	36	360	1300	9660
$\mathbb{P}^1 \times \mathrm{MM}^3_{2-31}$	368	3	1	0	4	12	36	420	940	11760
$\mathrm{BØS}^4_{54}$	405	5	1	0	4	12	60	300	1660	10080
$\mathrm{BØS}^4_{58}$	373	5	1	0	4	12	60	300	2020	10080
$\mathbb{P}^1 \times \mathrm{MM}^3_{3-25}$	352	4	1	0	4	12	60	360	2020	10920
$\mathrm{BØS}^4_{66}$	332	5	1	0	4	12	60	360	2020	13440
$\mathbb{P}^1 \times \mathrm{MM}^3_{3-23}$	336	4	1	0	4	12	60	420	2020	14280
$\mathrm{BØS}^4_{28}$	321	5	1	0	4	12	60	420	2020	16380
$\mathrm{BØS}^4_{65}$	331	5	1	0	4	12	84	420	2380	17640
$\mathrm{BØS}^4_{80}$	325	5	1	0	4	12	84	420	2740	17640
$\mathbb{P}^1 \times \mathrm{MM}^3_{3-19}$	304	4	1	0	4	12	84	480	3100	20160
$\mathrm{BØS}^4_{50}$	394	5	1	0	4	18	36	480	1750	10500
$\mathrm{BØS}^4_{68}$	363	5	1	0	4	18	36	480	2110	10500
$\mathrm{BØS}^4_{59}$	341	5	1	0	4	18	60	480	2830	15540
$\mathbb{P}^1 \times \mathrm{MM}^3_{2-27}$	304	3	1	0	4	18	60	600	2830	19740
$\mathrm{BØS}^4_{53}$	330	5	1	0	4	18	84	480	3190	20580
$\mathrm{BØS}^4_{69}$	310	5	1	0	4	18	84	480	3550	20580
$\mathrm{BØS}^4_{84}$	299	5	1	0	4	18	84	600	3550	25620
$\mathbb{P}^1 \times \mathrm{MM}^3_{3-14}$	256	4	1	0	4	18	132	780	6070	42420
$\mathbb{P}^1 \times \mathrm{MM}^3_{3-9}$	208	4	1	0	4	36	228	1560	15340	122640
$\mathrm{BØS}^4_{38}$	385	4	1	0	6	0	90	120	1860	7560
$\mathbb{P}^1 \times \mathbb{P}^1 \times F_1$	384	4	1	0	6	6	90	300	1950	13020

(*Continued on next page*)

Table A.5. (Continued).

X	$(-K_X)^4$	ρ_X	α_0	α_1	α_2	α_3	α_4	α_5	α_6	α_7
$\mathbb{P}^1 \times MM^3_{4-13}$	368	5	1	0	6	6	114	240	3390	9660
$\mathbb{P}^1 \times MM^3_{3-24}$	336	4	1	0	6	6	114	300	3390	14280
$BØS^4_{33}$	305	5	1	0	6	6	114	360	3750	18480
$BØS^4_{4}$	364	5	1	0	6	12	90	420	2760	17220
$BØS^4_{23}$	354	5	1	0	6	12	90	480	2760	20160
$\mathbb{P}^1 \times MM^3_{4-12}$	352	5	1	0	6	12	90	540	2400	21420
$S^2_7 \times F_1$	336	5	1	0	6	12	90	540	2760	21420
$\mathbb{P}^1 \times MM^3_{2-29}$	320	3	1	0	6	12	90	600	2400	26040
$BØS^4_{96}$	334	5	1	0	6	12	114	480	3840	22680
$\mathbb{P}^1 \times MM^3_{4-10}$	320	5	1	0	6	12	114	540	3840	23940
$\mathbb{P}^1 \times MM^3_{3-20}$	304	4	1	0	6	12	114	600	3840	28560
$\mathbb{P}^1 \times MM^3_{3-17}$	288	4	1	0	6	12	138	600	5280	31080
$S^2_6 \times \mathbb{P}^2$	324	5	1	0	6	18	90	720	3570	28980
$\mathbb{P}^1 \times MM^3_{3-18}$	288	4	1	0	6	18	114	840	4650	38220
$BØS^4_{57}$	298	6	1	0	6	18	138	780	5730	39480
$\mathbb{P}^1 \times MM^3_{3-16}$	272	4	1	0	6	18	138	900	6090	46620
$\mathbb{P}^1 \times MM^3_{2-25}$	256	3	1	0	6	24	114	1200	5820	57120
$BØS^4_{49}$	308	6	1	0	6	24	138	960	6180	46200
$BØS^4_{55}$	298	6	1	0	6	24	138	960	6540	46200
$BØS^4_{63}$	278	6	1	0	6	24	138	1080	6540	53760
$BØS^4_{64}$	268	6	1	0	6	24	162	960	7980	53760
$\mathbb{P}^1 \times MM^3_{2-24}$	240	3	1	0	6	24	186	1260	10140	78120
$BØS^4_{39}$	307	5	1	0	8	0	168	120	5120	10080
$S^2_7 \times \mathbb{P}^1 \times \mathbb{P}^1$	336	5	1	0	8	6	168	360	5210	19740
$\mathbb{P}^1 \times MM^3_{3-21}$	304	4	1	0	8	6	192	360	7010	21000
$\mathbb{P}^1 \times MM^3_{4-9}$	304	5	1	0	8	12	168	720	5660	39480
$S^2_7 \times S^2_7$	294	6	1	0	8	12	168	720	6020	39480
$\mathbb{P}^1 \times MM^3_{4-8}$	288	5	1	0	8	12	192	720	7460	42000
$\mathbb{P}^1 \times MM^3_{2-26}$	272	3	1	0	8	12	192	780	7460	47880
$S^2_6 \times F_1$	288	6	1	0	8	18	168	1020	6830	54600
$\mathbb{P}^1 \times MM^3_{5-2}$	288	6	1	0	8	18	192	1020	7910	57120
$\mathbb{P}^1 \times MM^3_{4-7}$	272	5	1	0	8	18	192	1080	8270	63000
$\mathbb{P}^1 \times MM^3_{3-15}$	256	4	1	0	8	18	216	1140	10070	72660
$\mathbb{P}^1 \times MM^3_{4-5}$	256	5	1	0	8	24	216	1440	10880	89040
$\mathbb{P}^1 \times MM^3_{2-22}$	240	3	1	0	8	24	216	1560	11240	100800
$\mathbb{P}^1 \times MM^3_{3-13}$	240	4	1	0	8	24	240	1560	13040	105840
$\mathbb{P}^1 \times MM^3_{3-11}$	224	4	1	0	8	30	264	1980	16370	142800
$\mathbb{P}^1 \times MM^3_{2-18}$	192	3	1	0	8	48	360	3360	31040	295680
$BØS^4_{40}$	230	6	1	0	10	0	270	240	10900	25200
$S^2_6 \times \mathbb{P}^1 \times \mathbb{P}^1$	288	6	1	0	10	12	270	840	11080	55440
$\mathbb{P}^1 \times MM^3_{2-23}$	240	3	1	0	10	12	318	960	15760	74760
$S^2_6 \times S^2_7$	252	7	1	0	10	18	270	1320	12610	91560
$\mathbb{P}^1 \times MM^3_{4-6}$	256	5	1	0	10	18	294	1320	14050	94080
$\mathbb{P}^1 \times MM^3_{4-4}$	240	5	1	0	10	24	318	1800	17380	135240
$\mathbb{P}^1 \times MM^3_{2-21}$	224	3	1	0	10	24	342	1920	19900	154560
$\mathbb{P}^1 \times MM^3_{3-12}$	224	4	1	0	10	30	342	2340	21070	186060
$\mathbb{P}^1 \times MM^3_{2-19}$	208	3	1	0	10	30	342	2520	21430	208740
$\mathbb{P}^2 \times S^2_5$	270	6	1	0	10	36	270	2160	15040	134400
$\mathbb{P}^1 \times MM^3_{2-20}$	208	3	1	0	10	36	390	2940	27640	255360
$S^2_6 \times S^2_6$	216	8	1	0	12	24	396	2160	23160	186480
$\mathbb{P}^1 \times MM^3_{4-3}$	224	5	1	0	12	24	444	2160	26760	191520
$F_1 \times S^2_5$	240	7	1	0	12	36	396	2820	24060	219240
$\mathbb{P}^1 \times MM^3_{3-10}$	208	4	1	0	12	36	492	3360	35220	319200
$\mathbb{P}^1 \times MM^3_{5-1}$	224	6	1	0	12	42	468	3480	32430	300300
$\mathbb{P}^1 \times MM^3_{2-17}$	192	3	1	0	12	42	540	4140	43230	423360

(*Continued on next page*)

Table A.5. (Continued).

X	$(-K_X)^4$	ρ_X	α_0	α_1	α_2	α_3	α_4	α_5	α_6	α_7
$\mathbb{P}^1 \times \mathrm{MM}^3_{3\text{–}7}$	192	4	1	0	12	48	564	4680	48000	486360
$\mathbb{P}^1 \times \mathrm{MM}^3_{2\text{–}16}$	176	3	1	0	12	60	636	6120	63300	693000
$\mathbb{P}^1 \times \mathbb{P}^1 \times S^2_5$	240	7	1	0	14	30	546	2760	33350	246540
$S^2_7 \times S^2_5$	210	8	1	0	14	36	546	3480	37040	330540
$\mathbb{P}^1 \times \mathrm{MM}^3_{2\text{–}15}$	176	3	1	0	14	36	714	4320	59720	519120
$\mathbb{P}^1 \times \mathrm{MM}^3_{4\text{–}2}$	208	5	1	0	14	42	618	4200	46490	425880
$\mathbb{P}^1 \times \mathrm{MM}^3_{4\text{–}1}$	192	5	1	0	14	48	690	5280	59540	594720
$\mathbb{P}^1 \times \mathrm{MM}^3_{3\text{–}8}$	192	4	1	0	14	54	690	5700	61070	631260
$\mathbb{P}^1 \times V^3_{22}$	176	2	1	0	14	60	786	6960	78800	859320
$S^2_6 \times S^2_5$	180	9	1	0	16	42	720	4920	58390	567840
$\mathbb{P}^1 \times \mathrm{MM}^3_{3\text{–}6}$	176	4	1	0	16	66	936	8280	97630	1086540
$\mathbb{P}^1 \times \mathrm{MM}^3_{2\text{–}12}$	160	3	1	0	16	72	1056	9840	122920	1428000
$\mathbb{P}^1 \times \mathrm{MM}^3_{2\text{–}13}$	160	3	1	0	16	84	1104	11400	137860	1685040
$\mathbb{P}^1 \times \mathrm{MM}^3_{2\text{–}11}$	144	3	1	0	16	108	1248	15600	188260	2538480
$\mathbb{P}^1 \times \mathrm{MM}^3_{2\text{–}14}$	160	3	1	0	18	90	1302	13260	168570	2089080
$S^2_5 \times S^2_5$	150	10	1	0	20	60	1140	9120	121700	1377600
$S^2_4 \times \mathbb{P}^2$	216	7	1	0	20	102	1188	11760	123050	1391880
$\mathbb{P}^1 \times V^3_{18}$	144	2	1	0	20	120	1788	20760	285680	3926160
Str_3	86	2	1	0	20	156	2700	41040	697700	12503400
$S^2_4 \times F_1$	192	8	1	0	22	102	1434	13740	160510	1881180
$\mathbb{P}^1 \times \mathrm{MM}^3_{3\text{–}3}$	144	4	1	0	22	132	2058	24360	345280	4867800
$S^2_4 \times \mathbb{P}^1 \times \mathbb{P}^1$	192	8	1	0	24	96	1704	14400	193920	2150400
$S^2_4 \times S^2_7$	168	9	1	0	24	102	1704	15720	205530	2452380
$\mathbb{P}^1 \times \mathrm{MM}^3_{3\text{–}5}$	160	4	1	0	24	126	1992	21300	290130	3813600
$\mathbb{P}^1 \times \mathrm{MM}^3_{2\text{–}9}$	128	3	1	0	24	174	2784	37680	578490	9059820
$S^2_4 \times S^2_6$	144	10	1	0	26	108	1998	19080	270440	3435600
$\mathbb{P}^1 \times \mathrm{MM}^3_{3\text{–}4}$	144	4	1	0	26	156	2574	31080	457640	6657840
$\mathbb{P}^1 \times V^3_{16}$	128	2	1	0	26	192	3198	44160	700820	11249280
$\mathbb{P}^1 \times \mathrm{MM}^3_{2\text{–}8}$	112	3	1	0	28	216	3900	58800	984520	17334240
$S^2_4 \times S^2_5$	120	11	1	0	30	126	2658	27720	439590	6247500
$\mathbb{P}^1 \times \mathrm{MM}^3_{2\text{–}10}$	128	3	1	0	30	216	3858	54000	891660	14726880
$\mathbb{P}^1 \times V^3_{14}$	112	2	1	0	34	312	5910	97920	1820140	34520640
$\mathbb{P}^1 \times \mathrm{MM}^3_{2\text{–}7}$	112	3	1	0	38	348	6954	117840	2268560	44336040
$S^2_4 \times S^2_4$	96	12	1	0	40	192	4776	59520	1120000	19138560
$\mathbb{P}^1 \times \mathrm{MM}^3_{2\text{–}6}$	96	3	1	0	46	528	11826	238560	5341780	122340960
$\mathbb{P}^1 \times V^3_{12}$	96	2	1	0	50	600	13758	288480	6659420	157802400
$S^2_3 \times \mathbb{P}^2$	162	8	1	0	54	498	9882	162000	2938770	54057780
$S^2_3 \times F_1$	144	9	1	0	56	498	10536	171900	3240110	60897480
$\mathbb{P}^1 \times \mathrm{MM}^3_{3\text{–}1}$	96	4	1	0	56	672	16296	350400	8393600	205470720
$S^2_3 \times \mathbb{P}^1 \times \mathbb{P}^1$	144	9	1	0	58	492	11214	178440	3502120	65938320
$S^2_3 \times S^2_7$	126	10	1	0	58	498	11214	181800	3561250	68151720
$S^2_3 \times S^2_6$	108	11	1	0	60	504	11916	195120	3962040	78104880
$\mathbb{P}^1 \times \mathrm{MM}^3_{3\text{–}2}$	112	4	1	0	60	600	13884	259440	5613000	122354400
$S^2_3 \times S^2_5$	90	12	1	0	64	522	13392	225720	4887190	102194400
$\mathbb{P}^1 \times \mathrm{MM}^3_{2\text{–}5}$	96	3	1	0	68	816	21012	465960	11662880	297392760
$S^2_3 \times S^2_4$	72	13	1	0	74	588	17550	319560	7862600	185440080
$\mathbb{P}^1 \times V^3_{10}$	80	2	1	0	80	1320	38688	1078320	32604200	1016215200
$\mathbb{P}^1 \times \mathrm{MM}^3_{2\text{–}4}$	80	3	1	0	92	1518	47172	1357680	42774050	1385508600
$S^2_3 \times S^2_3$	54	14	1	0	108	984	37260	848880	26609400	804368880
$\mathbb{P}^1 \times V^3_8$	64	2	1	0	154	3840	159486	6504960	284808340	12889551360
$S^2_2 \times \mathbb{P}^2$	108	9	1	0	276	6822	314532	12870000	570227370	25599296520
$S^2_2 \times F_1$	96	10	1	0	278	6822	317850	13006380	579688190	26140920540
$S^2_2 \times \mathbb{P}^1 \times \mathbb{P}^1$	96	10	1	0	280	6816	321192	13126080	588430720	26621521920
$S^2_2 \times S^2_7$	84	11	1	0	280	6822	321192	13142760	589248730	26688271260
$S^2_2 \times S^2_6$	72	12	1	0	282	6828	324558	13295880	599727720	27308448720

(*Continued on next page*)

Table A.5. (Continued).

X	$(-K_X)^4$	ρ_X	α_0	α_1	α_2	α_3	α_4	α_5	α_6	α_7
$S^2_2 \times S^2_5$	60	13	1	0	286	6846	331362	13619400	621807910	28636256460
$S^2_2 \times S^2_4$	48	14	1	0	296	6912	348840	14492160	681885440	32334274560
$\mathbb{P}^1 \times MM^3_{2-3}$	64	3	1	0	302	8472	442194	21352560	1128405740	61403700960
$S^2_2 \times S^2_3$	36	15	1	0	330	7308	413838	18050760	935040840	48854892240
$\mathbb{P}^1 \times V^3_6$	48	2	1	0	398	17616	1221810	85572960	6386359700	493612489440
$\mathbb{P}^1 \times MM^3_{2-2}$	48	3	1	0	472	21216	1568424	115141440	9050108800	736102993920
$S^2_2 \times S^2_2$	24	16	1	0	552	13632	1086120	63331200	4672300800	350133073920
$\mathbb{P}^1 \times V^3_4$	32	2	1	0	1946	215808	35318526	5981882880	1074550170260	200205416839680
$S^2_1 \times \mathbb{P}^2$	54	10	1	0	10260	2021286	618874020	184451042160	57876574021290	18570362883899400
$S^2_1 \times F_1$	48	11	1	0	10262	2021286	618997146	184491467820	57895141165310	18578110239211740
$S^2_1 \times \mathbb{P}^1 \times \mathbb{P}^1$	48	11	1	0	10264	2021280	619120296	184531277760	57913469449600	18585729302999040
$S^2_1 \times S^2_7$	42	12	1	0	10264	2021286	619120296	184531893480	57913712003290	18585859292400540
$S^2_1 \times S^2_6$	36	13	1	0	10266	2021292	619243470	184572934920	57932529089640	18593740045797840
$S^2_1 \times S^2_5$	30	14	1	0	10270	2021310	619489890	184655634120	57970416902950	18609636764945100
$S^2_1 \times S^2_4$	24	15	1	0	10280	2021376	620106408	184864542720	58066057475840	18649867837440000
$S^2_1 \times S^2_3$	18	16	1	0	10314	2021772	622208142	185592555720	58399032538440	18790790073224400
$\mathbb{P}^1 \times MM^3_{2-1}$	32	3	1	0	10382	2082840	650724306	199392674160	64624270834220	21530238491351520
$S^2_1 \times S^2_2$	12	17	1	0	10536	2028096	636179112	190741334400	60762986684160	19811992617768960
$S^2_1 \times S^2_1$	6	18	1	0	20520	4042560	1869353640	783667509120	387953543059200	204188081194137600
$\mathbb{P}^1 \times V^3_2$	16	2	1	0	68762	55200000	61055606526	71592493125120	88810659628444820	114429017109750013440

and $\{\text{BØS}^4_{35}, \text{BØS}^4_{88}\}$. This is not the case. The coefficients α_8, α_9 in these cases are:

X	α_8	α_9
BØS^4_6	14350	87360
BØS^4_{35}	32830	227640
BØS^4_{41}	10990	102480
BØS^4_{88}	32830	212520

Thus 10 terms of the Taylor expansion of the regularized quantum period suffice to distinguish all of the four-dimensional Fano manifolds considered in this paper.

Appendix B. Quantum differential operators for four-dimensional Fano manifolds of index $r > 1$: Numerical results

In this appendix, we record the quantum differential operators for all four-dimensional Fano manifolds of Fano index $r > 1$. These were computed numerically, as described in Section 9, from 500 terms of the Taylor expansion of the quantum period. They pass a number of strong consistency checks, and so we are reasonably confident that they are correct, but this has not been rigorously proven. We record also the local log-monodromies and ramification defect for the quantum local system, that is, for the local system of solutions to the regularized quantum differential equation. These are derived using exact computer algebra from the (numerically computed) operators L_X, as described in Section 9.

B1. $\mathbb{P}^4$. *[description p. 2, regularized quantum period p. 15]*

The quantum differential operator is

$$(5t-1)(625t^4+125t^3+25t^2+5t+1)D^4+31250t^5D^3 \\ +109375t^5D^2+156250t^5D+75000t^5.$$

The local log-monodromies for the quantum local system:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t=0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t=\frac{1}{5}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at the roots of } 625t^4+125t^3 +25t^2+5t+1=0.$$

The operator L_X is extremal.

B2. Q^4. *[description p. 2, regularized quantum period p. 15]*

The quantum differential operator is

$$(32t^2-1)(32t^2+1)D^4+8192t^4D^3+23552t^4D^2 \\ +28672t^4D+12288t^4.$$

The local log-monodromies for the quantum local system:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at the roots of } 32t^2 - 1 = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at the roots of } 32t^2 + 1 = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = \infty.$$

The operator L_X is extremal.

B3. FI^4_1. [description p. 3, regularized quantum period p. 15]

The quantum differential operator is

$$(11664t^3 - 1)D^4 + 69984t^3D^3 + 142884t^3D^2 + 113724t^3D + 29160t^3.$$

The local log-monodromies for the quantum local system:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at the roots of } 11664t^3 - 1 = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \quad \text{at } t = \infty.$$

The operator L_X is extremal.

B4. FI^4_2. [description p. 3, regularized quantum period p. 15]

The quantum differential operator is

$$(12t - 1)(144t^2 + 12t + 1)D^4 + 10368t^3D^3 + 21924t^3D^2 + 19116t^3D + 5832t^3.$$

The local log-monodromies for the quantum local system:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = \frac{1}{12}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at the roots of } 144t^2 + 12t + 1 = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} \quad \text{at } t = \infty.$$

The operator L_X is extremal.

B5. FI^4_3. [description p. 3, regularized quantum period p. 15]

The quantum differential operator is

$$(9t - 1)(81t^2 + 9t + 1)D^4 + 4374t^3D^3 + 9477t^3D^2 + 8748t^3D + 2916t^3.$$

The local log-monodromies for the quantum local system:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = \frac{1}{9}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at the roots of } 81t^2 + 9t + 1 = 0$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = \infty.$$

The operator L_X is extremal.

B6. FI^4_4. [description p. 3, regularized quantum period p. 15]

The quantum differential operator is

$$(432t^3 - 1)D^4 + 2592t^3D^3 + 5724t^3D^2 + 5508t^3D + 1944t^3.$$

The local log-monodromies for the quantum local system:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at the roots of } 432t^3 - 1 = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \quad \text{at } t = \infty.$$

The operator L_X is extremal.

B7. FI^4_5. [description p. 3, regularized quantum period p. 15]

The quantum differential operator is

$$(729t^6 + 297t^3 - 1)D^4 + 162t^3(54t^3 + 11)D^3 + 27t^3(1323t^3 + 148)D^2 + 81t^3(702t^3 + 49)D + 1458t^3(20t^3 + 1).$$

The local log-monodromies for the quantum local system:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at the roots of } 729t^6 + 297t^3 - 1 = 0.$$

The ramification defect of L_X is 1.

B8. $\mathbb{P}^2 \times \mathbb{P}^2$. [description p. 3, regularized quantum period p. 15]

The quantum differential operator is

$$(3t + 1)(6t - 1)(9t^2 - 3t + 1)(36t^2 + 6t + 1)D^4 + 162t^3(432t^3 + 7)D^3 + 27t^3(10584t^3 + 95)D^2 + 1296t^3(351t^3 + 2)D + 972t^3(240t^3 + 1).$$

The local log-monodromies for the quantum local system:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = \frac{1}{6}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = -\frac{1}{3}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at the roots of } 9t^2 - 3t + 1 = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at the roots of } 36t^2 + 6t + 1 = 0.$$

The ramification defect of L_X is 1.

B9. V^4_2. [description p. 3, regularized quantum period p. 16]

The quantum differential operator is

$$(6912t^2 - 1)D^4 + 27648t^2D^3 + 38400t^2D^2 + 21504t^2D + 3840t^2.$$

The local log-monodromies for the quantum local system:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at the roots of } 6912t^2 - 1 = 0$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix} \quad \text{at } t = \infty.$$

The operator L_X is extremal.

B10. V^4_4. [description p. 3, regularized quantum period p. 16]

The quantum differential operator is

$$(32t - 1)(32t + 1)D^4 + 4096t^2D^3 + 5888t^2D^2 + 3584t^2D + 768t^2.$$

The local log-monodromies for the quantum local system:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = \frac{1}{32}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = -\frac{1}{32}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \quad \text{at } t = \infty.$$

The operator L_X is extremal.

B11. V_6^4. [description p. 4, regularized quantum period p. 16]

The quantum differential operator is

$$(432t^2 - 1)D^4 + 1728t^2D^3 + 2544t^2D^2 + 1632t^2D + 384t^2.$$

The local log-monodromies for the quantum local system:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at the roots of } 432t^2 - 1 = 0$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix} \quad \text{at } t = \infty.$$

The operator L_X is extremal.

B12. V_8^4. [description p. 4, regularized quantum period p. 16]

The quantum differential operator is

$$(16t - 1)(16t + 1)D^4 + 1024t^2D^3 + 1536t^2D^2 + 1024t^2D + 256t^2.$$

The local log-monodromies for the quantum local system:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = \frac{1}{16}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = -\frac{1}{16}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = \infty.$$

The operator L_X is extremal.

B13. V_{10}^4. [description p. 4, regularized quantum period p. 16]

The quantum differential operator is

$$(256t^4 + 176t^2 - 1)D^4 + 64t^2(32t^2 + 11)D^3 + 16t^2(352t^2 + 67)D^2 + 32t^2(192t^2 + 23)D + 192t^2(12t^2 + 1).$$

The local log-monodromies for the quantum local system:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at the roots of } 256t^4 + 176t^2 - 1 = 0$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = \infty.$$

The ramification defect of L_X is 1.

B14. V_{12}^4. [description p. 4, regularized quantum period p. 16]

The quantum differential operator is

$$(4t^2 - 12t + 1)(4t^2 + 12t + 1)D^4 + 32t^2(4t^2 - 17)D^3 + 8t^2(46t^2 - 105)D^2 + 16t^2(28t^2 - 37)D + 32t^2(6t^2 - 5).$$

The local log-monodromies for the quantum local system:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at the roots of } 4t^2 - 12t + 1 = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at the roots of } 4t^2 + 12t + 1 = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = \infty.$$

The operator L_X is extremal.

B15. V^4_{14}. [description p. 4, regularized quantum period p. 16]

The quantum differential operator is

$$
\begin{aligned}
&(4t^2+1)(108t^2-1)D^4 + 32t^2(108t^2+13)D^3\\
&\quad +24t^2(406t^2+27)D^2 + 16t^2(708t^2+29)D\\
&\quad +128t^2(36t^2+1).
\end{aligned}
$$

The local log-monodromies for the quantum local system:

$$\begin{pmatrix} 0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0 \end{pmatrix} \quad \text{at } t=0$$

$$\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix} \quad \text{at the roots of } 108t^2-1=0$$

$$\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix} \quad \text{at the roots of } 4t^2+1=0$$

$$\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&\frac{2}{3}&0\\0&0&0&\frac{1}{3} \end{pmatrix} \quad \text{at } t=\infty.$$

The ramification defect of L_X is 1.

B16. V^4_{16}. [description p. 4, regularized quantum period p. 16]

The quantum differential operator is

$$
\begin{aligned}
&(16t^2-8t-1)(16t^2+8t-1)D^4 + 128t^2(16t^2-3)D^3\\
&\quad +32t^2(184t^2-19)D^2\\
&\quad +448t^2(4t-1)(4t+1)D + 128t^2(24t^2-1).
\end{aligned}
$$

The local log-monodromies for the quantum local system:

$$\begin{pmatrix} 0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0 \end{pmatrix} \quad \text{at } t=0$$

$$\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix} \quad \text{at the roots of } 16t^2-8t-1=0$$

$$\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix} \quad \text{at the roots of } 16t^2+8t-1=0$$

$$\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix} \quad \text{at } t=\infty.$$

The operator L_X is extremal.

B17. V^4_{18}. [description p. 4, regularized quantum period p. 16]

The quantum differential operator is

$$
\begin{aligned}
&(432t^4+72t^2-1)D^4 + 288t^2(12t^2+1)D^3\\
&\quad +24t^2(414t^2+19)D^2 + 336t^2(36t^2+1)D\\
&\quad +96t^2(54t^2+1).
\end{aligned}
$$

The local log-monodromies for the quantum local system:

$$\begin{pmatrix} 0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0 \end{pmatrix} \quad \text{at } t=0$$

$$\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix} \quad \text{at the roots of } 432t^4+72t^2-1=0$$

$$\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix} \quad \text{at } t=\infty.$$

The operator L_X is extremal.

B18. MW^4_1. [description p. 5, regularized quantum period p. 16]

The quantum differential operator is

$$
\begin{aligned}
&(2t-1)^2(2t+1)^2(1724t^2-4t-1)(1724t^2+4t-1)\\
&\quad (1384128950480t^6-34997928616t^4-263676995t^2+9409)D^6\\
&\quad +2(2t-1)(2t+1)(1810104932909611157120t^{12}-17468834144875533568t^{10}+37442934049578483 2t^8\\
&\quad +3959280486757728t^6-1227737299988t^4-725619617t^2+18818)D^5\\
&\quad +4\left(3126544884116601804800t^{14}-7255579544869796106 24t^{12}+31166631689741025792t^{10}\right.\\
&\quad \left.-440963660134839040t^8-7399870298607304t^6-348759582360t^4-504354223t^2+9409\right)D^4\\
&\quad +8t^2(6746754749935824947200t^{12}-1095289161198143939072t^{10}+33852557194447324800t^8\\
&\quad -138056179574882528t^6-6267098983057824t^4-620133376448t^2-10735669)D^3\\
&\quad +32t^2(3803277296533945221760t^{12}-460852243532660846400t^{10}+13408867107650109352t^8\\
&\quad +72880113460188392t^6-1247120973283936t^4-144892007990t^2-573949)D^2
\end{aligned}
$$

$$
\begin{aligned}
&+256t^4(524004808703094940640t^{10} - 510439696703766668752t^8 + 157380143707792310 2t^6 \\
&+16338545311012128t^4 - 54334824441981t^2 - 5942952755)D \\
&+1536t^4(359964049158161392 00t^{10} - 302551142001992096 0t^8 \\
&+99559010182515260t^6 + 1234528802429310t^4 - 1125770982819t^2 - 89024854)
\end{aligned}
$$

The local log-monodromies for the quantum local system:

$$
\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = 0
$$

$$
\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{5}{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \end{pmatrix} \quad \text{at } t = \frac{1}{2}
$$

$$
\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{5}{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \end{pmatrix} \quad \text{at } t = -\frac{1}{2}
$$

$$
\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at the roots of } 1724t^2 - 4t - 1 = 0
$$

$$
\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at the roots of } 1724t^2 + 4t - 1 = 0.
$$

The operator L_X is extremal.

B19. MW^4_2. [description p. 5, regularized quantum period p. 16]

The quantum differential operator is

$$
\begin{aligned}
&(2t-1)^2(2t+1)^2(14t-1)(14t+1)(18t-1)(18t+1)(7544656t^6 - 3112t^4 - 6667t^2 + 1)D^6 \\
&\quad +2(2t-1)(2t+1) \\
&\quad (21081096723456t^{12} - 1556683668736t^{10} - 25006224512t^8 + 2282941792t^6 - 4014452t^4 - 18937t^2 + 2)D^5 \\
&\quad +4\left(364128034314240t^{14} - 74616093755392t^{12} + 1020882362880t^{10}\right. \\
&\quad \left.+ 104092887296t^8 - 4150928136t^6 - 1064664t^4 - 12743t^2 + 1\right) D^4 \\
&\quad +8t^2(785749968783360t^{12} - 103037075309056t^{10} - 578957540736t^8 \\
&\quad +167061307168t^6 - 3800819360t^4 - 2728128t^2 - 205)D^3 \\
&\quad +32t^2(442942589109888t^{12} - 38313217780544t^{10} - 591376611224t^8 \\
&\quad +79457857384t^6 - 864235264t^4 - 697078t^2 - 13)D^2 \\
&\quad +256t^4(61027379435232t^{10} - 3662113129808t^8 - 80739092050t^6 + 9991256448t^4 - 46797421t^2 - 33179)D \\
&\quad +7680t^4(838452710592t^{10} - 37503518528t^8 - 1048500436t^6 + 130762470t^4 - 249799t^2 - 110)
\end{aligned}
$$

The local log-monodromies for the quantum local system:

$$\begin{pmatrix} 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix} \quad \text{at } t=0$$

$$\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix} \quad \text{at } t=\frac{1}{2}$$

$$\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix} \quad \text{at } t=\frac{1}{14}$$

$$\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix} \quad \text{at } t=\frac{1}{18}$$

$$\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix} \quad \text{at } t=-\frac{1}{18}$$

$$\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix} \quad \text{at } t=-\frac{1}{14}$$

$$\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix} \quad \text{at } t=-\frac{1}{2}.$$

The operator L_X is extremal.

B20. MW_3^4. [description p. 5, regularized quantum period p. 16]

The quantum differential operator is

$$\begin{aligned}
&(2t-1)^2(2t+1)^2(104t^2-4t-1)(104t^2+4t-1)(124883200t^6+3445552t^4-190621t^2+50)D^6\\
&\quad +2(2t-1)(2t+1)(59432414412800t^{12}-3046657163264t^{10}-307452667136t^8\\
&\quad +12555781056t^6-36422116t^4-548263t^2+100)D^5\\
&\quad +4\left(1026559885312000t^{14}-181195540611072t^{12}-5355282845184t^{10}+794966941312t^8\right.\\
&\quad \left.-20516326820t^6-1544700t^4-364427t^2+50\right)D^4\\
&\quad +8t^2(2215208173568000t^{12}-216948202172416t^{10}-15687219459072t^8\\
&\quad +1005364344896t^6-20392735992t^4-36932632t^2-5075)D^3\\
&\quad +64t^2(624378035507200t^{12}-30899527376640t^{10}-4029754891664t^8
\end{aligned}$$

$$+205143060452t^6 - 2542262335t^4 - 5022410t^2 - 175)D^2$$
$$+256t^4(172050086041600t^{10} - 3462364820096t^8$$
$$-968115899896t^6 + 47585432988t^4 - 312087909t^2 - 517438)D$$
$$+1536t^4(11818946048000t^{10} - 33971342080t^8 - 59641497680t^6 + 2994403590t^4 - 9592347t^2 - 9140)$$

The local log-monodromies for the quantum local system:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{5}{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \end{pmatrix} \quad \text{at } t = \frac{1}{2}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{5}{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \end{pmatrix} \quad \text{at } t = -\frac{1}{2}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at the roots of } 104t^2 - 4t - 1 = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at the roots of } 104t^2 + 4t - 1 = 0.$$

The operator L_X is extremal.

B21. MW^4_4. [description p. 5, regularized quantum period p. 16]

The quantum differential operator is

$$(16t^2 + 1)(128t^2 - 1)D^4 + 64t^2(256t^2 + 7)D^3 + 16t^2(2816t^2 + 43)D^2 + 96t^2(512t^2 + 5)D + 128t^2(144t^2 + 1).$$

The local log-monodromies for the quantum local system:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at the roots of } 128t^2 - 1 = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at the roots of } 16t^2 + 1 = 0$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = \infty.$$

The ramification defect of L_X is 1.

B22. MW^4_5. [description p. 6, regularized quantum period p. 16]

The quantum differential operator is

$$\begin{aligned}
&(32t^4 - 144t^3 + 40t^2 - 12t + 1)(32t^4 + 144t^3 + 40t^2 + 12t + 1)\\
&\quad (1379024896t^8 - 181690112t^6 + 32203856t^4 + 160775t^2 + 136)D^6\\
&\quad +2\left(14121214935040t^{16} - 152437496479744t^{14} + 18570092085248t^{12} - 3527276827648t^{10}\right.\\
&\quad \left. - 277037824256t^8 - 556812800t^6 - 140567296t^4 - 499733t^2 - 272\right) D^5\\
&\quad +4\left(54366677499904t^{16} - 340827602288640t^{14} + 64054093561856t^{12} - 13315483720192t^{10}\right.\\
&\quad \left. -406200526464t^8 - 986296160t^6 + 122964452t^4 + 370629t^2 + 136\right) D^4\\
&\quad +8t^2(103084869025792t^{14} - 369261049675776t^{12} + 91720177627136t^{10} - 21127821675520t^8\\
&\quad -305989253824t^6 - 1259677440t^4 - 1614768t^2 - 5831)D^3\\
&\quad +128t^2(12720125640704t^{14} - 26591028115456t^{12} + 8287153333632t^{10}\\
&\quad -2016286904136t^8 - 15422020074t^6 - 337096385t^4 - 533452t^2 - 102)D^2\\
&\quad +256t^4(6200095932416t^{12} - 7989995554816t^{10} + 3020985441920t^8\\
&\quad -739274260600t^6 - 4109085596t^4 - 179641501t^2 - 331640)D\\
&\quad +1536t^4(386126970880t^{12} - 333456404480t^{10} + 147420386560t^8\\
&\quad -34919816144t^6 - 204213358t^4 - 10473275t^2 - 20944).
\end{aligned}$$

The local log-monodromies for the quantum local system:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at the roots of } 32t^4 - 144t^3 + 40t^2 - 12t + 1 = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at the roots of } 32t^4 + 144t^3 + 40t^2 + 12t + 1 = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = \infty.$$

The ramification defect of L_X is 1.

B23. MW^4_6. [description p. 6, regularized quantum period p. 16]

The quantum differential operator is

$$
\begin{aligned}
&(2t-1)^2(2t+1)^2(6t-1)(6t+1)(10t-1)(10t+1)(1433040t^6+80728t^4-2579t^2+1)D^6\\
&\quad +2(2t-1)(2t+1)(226993536000t^{12}-6984049920t^{10}\\
&\quad -2174297216t^8+67707232t^6-106452t^4-7441t^2+2)D^5\\
&\quad +4\left(3920797440000t^{14}-584516290560t^{12}-52264472064t^{10}\right.\\
&\quad \left.+4812821760t^8-86750536t^6+175464t^4-4911t^2+1\right)D^4\\
&\quad +8t^2(8460668160000t^{12}-550314631680t^{10}-112753881472t^8\\
&\quad +5245181728t^6-97715360t^4-329920t^2-69)D^3\\
&\quad +32t^2(4769443728000t^{12}-59034734400t^{10}-53369502424t^8\\
&\quad +1881440232t^6-26802560t^4-94678t^2-5)D^2\\
&\quad +768t^4(219040164000t^{10}+4402781840t^8-2035415446t^6+67181120t^4-620103t^2-1737)D\\
&\quad +4608t^4(15046920000t^{10}+593723200t^8-120630700t^6+4065994t^4-22137t^2-34).
\end{aligned}
$$

The local log-monodromies for the quantum local system:

$$
\begin{pmatrix} 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix} \quad \text{at } t=0
$$

$$
\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&\frac{1}{2}&1\\ 0&0&0&0&0&\frac{1}{2} \end{pmatrix} \quad \text{at } t=\frac{1}{2}
$$

$$
\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix} \quad \text{at } t=\frac{1}{6}
$$

$$
\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix} \quad \text{at } t=\frac{1}{10}
$$

$$
\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix} \quad \text{at } t=-\frac{1}{10}
$$

$$\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix} \quad \text{at } t=-\frac{1}{6}$$

$$\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&\frac{1}{2}&1\\ 0&0&0&0&0&\frac{1}{2} \end{pmatrix} \quad \text{at } t=-\frac{1}{2}.$$

The operator L_X is extremal.

B24. MW^4_7. *[description p. 6, regularized quantum period p. 16]*

The quantum differential operator is

$$(8t-1)(8t+1)(16t^2+1)D^4+64t^2(128t^2+3)D^3+16t^2(1456t^2+19)D^2+32t^2(864t^2+7)D+64t^2(180t^2+1).$$

The local log-monodromies for the quantum local system:

$$\begin{pmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix} \quad \text{at } t=0$$

$$\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix} \quad \text{at } t=\frac{1}{8}$$

$$\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix} \quad \text{at } t=-\frac{1}{8}$$

$$\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix} \quad \text{at the roots of } 16t^2+1=0$$

$$\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&\frac{1}{2}&0\\ 0&0&0&\frac{1}{2} \end{pmatrix} \quad \text{at } t=\infty.$$

The ramification defect of L_X is 1.

B25. MW^4_8. *[description p. 6, regularized quantum period p. 16]*

The quantum differential operator is

$$\begin{aligned}
&(2t-1)(2t+1)(8000t^6+528t^4+60t^2-1)(64404500t^8-1791160t^6+729408t^4+2144t^2+5)D^6\\
&\quad +4\big(10304720000000t^{16}-1452885248000t^{14}+158410065280t^{12}\\
&\quad -11837396096t^{10}-726357052t^8+2787020t^6-1531744t^4-3536t^2-5\big)D^5\\
&\quad +4\big(79346344000000t^{16}-7715142848000t^{14}+1509213811040t^{12}\\
&\quad -96608493568t^{10}-1083561820t^8+48291628t^6+2808508t^4+5602t^2+5\big)D^4\\
&\quad +32t^2(37612228000000t^{14}-2748389976000t^{12}+827208001440t^{10}\\
&\quad -40569448408t^8-339211968t^6+5857560t^4-32366t^2-51)D^3\\
&\quad +32t^2(74258388500000t^{14}-4572988268000t^{12}+1872888803480t^{10}\\
&\quad -62657669272t^8-239219588t^6-13755238t^4-70290t^2-15)D^2
\end{aligned}$$

$$+512t^4(4524416125000t^{12} - 258077892500t^{10} + 128903007200t^8$$
$$-2846763420t^6 - 5734916t^4 - 1401952t^2 - 4225)D$$
$$+3072t^4(281769687500t^{12} - 15710668750t^{10} + 8827474000t^8 - 131446365t^6 - 200152t^4 - 95918t^2 - 245).$$

The local log-monodromies for the quantum local system:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = \frac{1}{2}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = -\frac{1}{2}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at the roots of } 8000t^6 + 528t^4 + 60t^2 - 1 = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = \infty.$$

The ramification defect of L_X is 1.

B26. MW^4_9. *[description p. 6, regularized quantum period p. 16]*

The quantum differential operator is

$$(176t^4 - 144t^3 + 20t^2 + 8t - 1)(176t^4 + 144t^3 + 20t^2 - 8t - 1)(2109888t^6 + 174528t^4 - 2941t^2 + 2)D^6$$
$$+2\left(718914797568t^{14} - 137405030400t^{12} - 5419619584t^{10}\right.$$
$$\left.+2224774528t^8 - 45409520t^6 - 364024t^4 + 8407t^2 - 4\right)D^5$$
$$+4\left(3104404807680t^{14} - 245927227392t^{12} - 51116043392t^{10}\right.$$
$$\left.+3834427616t^8 - 24121292t^6 + 553996t^4 - 5471t^2 + 2\right)D^4$$
$$+8t^2(6698978795520t^{12} - 43313504256t^{10} - 101803764992t^8$$
$$+3311154624t^6 - 46433864t^4 - 300504t^2 - 107)D^3$$
$$+64t^2(1888172529408t^{12} + 75804905088t^{10} - 23177683476t^8 + 452012856t^6 - 8043103t^4 - 46340t^2 - 4)D^2$$
$$+4608t^4(28905231168t^{10} + 2010341344t^8 - 284249759t^6 + 4423780t^4 - 79284t^2 - 316)D$$
$$+55296t^4(992819520t^{10} + 86769760t^8 - 8133271t^6 + 123438t^4 - 1829t^2 - 4).$$

The local log-monodromies for the quantum local system:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at the roots of } 176t^4 - 144t^3 + 20t^2 + 8t - 1 = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at the roots of } 176t^4 + 144t^3 + 20t^2 - 8t - 1 = 0.$$

The operator L_X is extremal.

B27. MW^4_{10}. [description p. 6, regularized quantum period p. 16]

The quantum differential operator is

$$\begin{aligned}
&(4t^2+1)(16384t^6+512t^4+44t^2-1)(1473757184t^8+13746176t^6+7592448t^4+46808t^2+55)D^6 \\
&\quad +4\left(482920754053120t^{16}+86447658893312t^{14}+4886210281472t^{12}+534951231488t^{10}\right. \\
&\quad \left.+18501230592t^8+119944000t^6+16234576t^4+72412t^2+55\right)D^5 \\
&\quad +4\left(3718489806209024t^{16}+396693178679296t^{14}+31496439267328t^{12}+3577124749312t^{10}\right. \\
&\quad \left.+37524877312t^8-202496928t^6-28001956t^4-105906t^2-55\right)D^4 \\
&\quad +16t^2(3525321504587776t^{14}+219939516448768t^{12}+31919813689344t^{10} \\
&\quad +2870602727424t^8+30098031616t^6+29514768t^4+38264t^2+755)D^3 \\
&\quad +64t^2(1740023841947648t^{14}+63218454626304t^{12}+17856939016192t^{10} \\
&\quad +1163732019200t^8+10148827776t^6+77992960t^4+145739t^2+55)D^2 \\
&\quad +256t^4(424064787152896t^{12}+8915881033728t^{10}+4878831042560t^8 \\
&\quad +233990370304t^6+2025100864t^4+24310608t^2+53911)D \\
&\quad +30720t^4(1320486436864t^{12}+15835594752t^{10}+16557547520t^8+617585152t^6+5801632t^4+78192t^2+187).
\end{aligned}$$

The local log-monodromies for the quantum local system:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at the roots of } 4t^2 + 1 = 0$$

$$\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix} \quad \text{at the roots of } 16384t^6+512t^4+44t^2-1=0$$

$$\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix} \quad \text{at } t=\infty.$$

The ramification defect of L_X is 1.

B28. MW^4_{11}. *[description p. 7, regularized quantum period p. 16]*

The quantum differential operator is

$$\begin{aligned}
&(176t^4-96t^3+8t^2-4t-1)(176t^4+96t^3+8t^2+4t-1)(46963840t^6-6320080t^4+10817t^2-60)D^6\\
&\quad+2\big(16002270986240t^{14}-4453229608960t^{12}+179162526976t^{10}+27549781760t^8\\
&\quad-288092960t^6+25314336t^4-28611t^2+120\big)D^5\\
&\quad+4\big(69100715622400t^{14}-16714357862400t^{12}+879319798400t^{10}\\
&\quad+38172435520t^8-54159304t^6-23633284t^4+6819t^2-60\big)D^4\\
&\quad+8t^2(149112070553600t^{12}-33265162726400t^{10}+1526941953280t^8\\
&\quad+17140032640t^6-188273072t^4+3160464t^2+1615)D^3\\
&\quad+128t^2(21014345918720t^{12}-4511525020640t^{10}+157602014826t^8\\
&\quad-362882280t^6+22427697t^4+282960t^2+30)D^2\\
&\quad+512t^4(5790594508160t^{10}-1225795140240t^8+31283552519t^6-265518840t^4+10109387t^2+49308)D\\
&\quad+15360t^4(79556744960t^{10}-16784238240t^8+319629734t^6-3585072t^4+146945t^2+456).
\end{aligned}$$

The local log-monodromies for the quantum local system:

$$\begin{pmatrix} 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix} \quad \text{at } t=0$$

$$\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix} \quad \text{at the roots of } 176t^4-96t^3+8t^2-4t-1=0$$

$$\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix} \quad \text{at the roots of } 176t^4+96t^3+8t^2+4t-1=0.$$

The operator L_X is extremal.

B29. MW^4_{12}. *[description p. 10, regularized quantum period p. 16]*

The quantum differential operator is

$$\begin{aligned}
&(16t^4+44t^2-1)(432t^4+36t^2+1)(314924112t^8-117964512t^6+14238144t^4+164850t^2+221)D^6\\
&\quad+4\big(10883777310720t^{16}+14017882540032t^{14}-7127443839744t^{12}
\end{aligned}$$

$$
\begin{aligned}
&+913572411264t^{10} + 49215528432t^{8} + 208091040t^{6} + 29910072t^{4} + 249043t^{2} + 221\big) D^{5} \\
&+4\big(83805085292544t^{16} + 42638961696768t^{14} - 38573795723328t^{12} + 6607519967520t^{10} \\
&+197257285008t^{8} + 2651761872t^{6} - 52981980t^{4} - 386208t^{2} - 221\big) D^{4} \\
&+16t^{2}(79451574368256t^{14} + 656487804672t^{12} - 24209536131072t^{10} \\
&+5352527374416t^{8} + 119397810240t^{6} + 1063433832t^{4} + 1841528t^{2} + 1513)D^{3} \\
&+32t^{2}(78431220245376t^{14} - 24146078481216t^{12} - 14886119891520t^{10} \\
&+4353594087744t^{8} + 89900843112t^{6} + 822085824t^{4} + 1734954t^{2} + 221)D^{2} \\
&+2304t^{4}(1061924105664t^{12} - 538846899984t^{10} - 114223084416t^{8} \\
&+48172072716t^{6} + 1084223724t^{4} + 9219514t^{2} + 22737)D \\
&+27648t^{4}(33067031760t^{12} - 20911507560t^{10} - 1683208512t^{8} + 1243973394t^{6} + 32605206t^{4} + 248744t^{2} + 663).
\end{aligned}
$$

The local log-monodromies for the quantum local system:

$$
\begin{pmatrix} 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix} \quad \text{at } t = 0
$$

$$
\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix} \quad \text{at the roots of } 432t^{4} + 36t^{2} + 1 = 0
$$

$$
\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix} \quad \text{at the roots of } 16t^{4} + 44t^{2} - 1 = 0
$$

$$
\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix} \quad \text{at } t = \infty.
$$

The ramification defect of L_X is 1.

B30. MW^4_{13}. [description p. 10, regularized quantum period p. 16]

The quantum differential operator is

$$
\begin{aligned}
&(2t-1)(2t+1)(136t^{3} - 20t^{2} + 6t - 1)(136t^{3} + 20t^{2} + 6t + 1)(66683996t^{6} - 1058780t^{4} - 19394t^{2} + 15)D^{6} \\
&\quad +4\big(27134518180352t^{14} - 3636686279168t^{12} - 76696829376t^{10} \\
&\quad +3834603264t^{8} - 107218310t^{6} + 2061400t^{4} + 29091t^{2} - 15\big) D^{5} \\
&\quad +4\big(234343566103040t^{14} - 21804386465792t^{12} + 102596245216t^{10} \\
&\quad +15956069184t^{8} + 542705432t^{6} - 1694520t^{4} - 36762t^{2} + 15\big) D^{4} \\
&\quad +32t^{2}(126422186976640t^{12} - 8667906294144t^{10} + 129634375864t^{8} \\
&\quad +2690207848t^{6} + 17651600t^{4} - 204240t^{2} - 49)D^{3} \\
&\quad +32t^{2}(285066614292448t^{12} - 15534154687040t^{10} + 251203935368t^{8} \\
&\quad -291393828t^{6} - 1136928t^{4} - 258660t^{2} - 15)D^{2} \\
&\quad +128t^{4}(78551346664144t^{10} - 3663416463632t^{8} + 48257529676t^{6} - 820898270t^{4} - 4423658t^{2} - 37545)D \\
&\quad +3072t^{4}(1349017239080t^{10} - 57143052340t^{8} + 533807468t^{6} - 19445050t^{4} - 103234t^{2} - 345).
\end{aligned}
$$

The local log-monodromies for the quantum local system:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = \frac{1}{2}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = -\frac{1}{2}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at the roots of } 136t^3 - 20t^2 + 6t - 1 = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at the roots of } 136t^3 + 20t^2 + 6t + 1 = 0.$$

The operator L_X is extremal.

B31. MW^4_{14}. [description p. 10, regularized quantum period p. 16]

The quantum differential operator is

$$\begin{aligned}
&(2t-1)(2t+1)(6t-1)(6t+1)(20t^2-4t+1)(20t^2+4t+1)(127920t^6-45016t^4-293t^2-1)D^6 \\
&\quad +2\left(81050112000t^{14}-42347458560t^{12}+4138403072t^{10}\right. \\
&\quad \left.+109225472t^8+95280t^6+187248t^4+911t^2+2\right)D^5 \\
&\quad +4\left(349989120000t^{14}-177028331520t^{12}+12857605632t^{10}\right. \\
&\quad \left.+227619584t^8+1578760t^6-171496t^4-769t^2-1\right)D^4 \\
&\quad +8t^2(755239680000t^{12}-383806594560t^{10}+19129972096t^8 \\
&\quad +278237408t^6+1853984t^4+10304t^2+13)D^3 \\
&\quad +32t^2(425743344000t^{12}-221217604800t^{10}+7159648792t^8+109530520t^6+1224240t^4+4582t^2+1)D^2 \\
&\quad +768t^4(19552572000t^{10}-10422296720t^8+211812198t^6+4163952t^4+55827t^2+173)D \\
&\quad +23040t^4(268632000t^{10}-146189120t^8+1864348t^6+52366t^4+729t^2+2).
\end{aligned}$$

The local log-monodromies for the quantum local system:

$$\begin{pmatrix} 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix} \quad \text{at } t=0$$

$$\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix} \quad \text{at } t=\frac{1}{2}$$

$$\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix} \quad \text{at } t=\frac{1}{6}$$

$$\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix} \quad \text{at } t=-\frac{1}{6}$$

$$\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix} \quad \text{at } t=-\frac{1}{2}$$

$$\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix} \quad \text{at the roots of } 20t^2-4t+1=0$$

$$\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix} \quad \text{at the roots of } 20t^2+4t+1=0.$$

The operator L_X is extremal.

B32. MW^4_{15}. [description p. 10, regularized quantum period p. 16]

The quantum differential operator is

$$\begin{aligned}
&(416t^4-144t^3+8t^2+4t-1)(416t^4+144t^3+8t^2-4t-1)(119179008t^6+10942640t^4+192779t^2+980)D^6\\
&\quad+2\left(226871066492928t^{14}+10977991065600t^{12}-661584438272t^{10}\right.\\
&\quad\left.-1433003264t^8-363743360t^6-47681568t^4-641057t^2-1960\right)D^5\\
&\quad+4\left(979670514401280t^{14}+74598187089920t^{12}-1635236114944t^{10}\right.\\
&\quad\left.-46841533568t^8+1268915136t^6+55009348t^4+503977t^2+980\right)D^4\\
&\quad+8t^2(2114025846865920t^{12}+203938713538560t^{10}-1101057117184t^8\\
&\quad-117410965440t^6-475968000t^4+3889200t^2-12299)D^3
\end{aligned}$$

$$
\begin{aligned}
&+128t^2(297929404790784t^{12}+33127871570560t^{10}+111493204952t^8\\
&-15417741170t^6-129552291t^4+148450t^2-245)D^2\\
&+256t^4(164191489173504t^{10}+19958501546880t^8+167108388648t^6-7614297740t^4-82910507t^2-24740)D\\
&+1536t^4(11279101317120t^{10}+1447829510400t^8+16271407536t^6-474232070t^4-5907869t^2-4760).
\end{aligned}
$$

The local log-monodromies for the quantum local system:

$$
\begin{pmatrix}0&1&0&0&0&0\\0&0&0&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\0&0&0&0&0&0\end{pmatrix}\quad \text{at } t=0
$$

$$
\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&0\end{pmatrix}\quad \text{at the roots of } 416t^4-144t^3+8t^2+4t-1=0
$$

$$
\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&0\end{pmatrix}\quad \text{at the roots of } 416t^4+144t^3+8t^2-4t-1=0.
$$

The operator L_X is extremal.

B33. MW^4_{16}. *[description p. 11, regularized quantum period p. 16]*

The quantum differential operator is

$$
\begin{aligned}
&(8t^2-4t+1)(8t^2+4t+1)(28t^2-4t-1)(28t^2+4t-1)(1245440t^6+159472t^4-353t^2+2)D^6\\
&\quad+2\left(687403171840t^{14}+55847092224t^{12}-2940634112t^{10}\right.\\
&\quad\left.+646645248t^8-7816080t^6-602296t^4+771t^2-4\right)D^5\\
&\quad+4\left(2968331878400t^{14}+342433251328t^{12}-16569335296t^{10}\right.\\
&\quad\left.+962246144t^8+20691276t^6+609636t^4-327t^2+2\right)D^4\\
&\quad+8t^2(6405347737600t^{12}+905030160384t^{10}-30840349696t^8+529133120t^6+4111656t^4-69016t^2-79)D^3\\
&\quad+64t^2(1805409751040t^{12}+290148947200t^{10}-6893123664t^8+13678020t^6-1333787t^4-13834t^2-3)D^2\\
&\quad+256t^4(497488517120t^{10}+86925519744t^8-1508603672t^6-6132868t^4-554137t^2-2494)D\\
&\quad+1536t^4(34174873600t^{10}+6291237120t^8-85724560t^6-530810t^4-38831t^2-116).
\end{aligned}
$$

The local log-monodromies for the quantum local system:

$$
\begin{pmatrix}0&1&0&0&0&0\\0&0&0&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\0&0&0&0&0&0\end{pmatrix}\quad \text{at } t=0
$$

$$
\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&0\end{pmatrix}\quad \text{at the roots of } 8t^2-4t+1=0
$$

$$\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix} \quad \text{at the roots of } 28t^2-4t-1=0$$

$$\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix} \quad \text{at the roots of } 28t^2+4t-1=0$$

$$\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix} \quad \text{at the roots of } 8t^2+4t+1=0.$$

The operator L_X is extremal.

B34. MW^4_{17}. [description p. 11, regularized quantum period p. 16]

The quantum differential operator is

$$\begin{aligned}
&(2752t^6-1152t^5+224t^4+96t^3-52t^2+12t-1)(2752t^6+1152t^5+224t^4-96t^3-52t^2-12t-1)\\
&\quad(4516691026120601600t^{18}+1089417553578401952 0t^{16}+161057014788668272t^{14}+2231864238469018 25t^{12}\\
&\quad+1489656860655194t^{10}-37076036387883t^8-944190030122t^6+1145046509t^4+34077463t^2+3136)D^8\\
&\quad+2\left(71835072861485809470013440 0t^{30}+1808773177510341584824565760t^{28}\right.\\
&\quad+1102994647458607838324326 4t^{26}+385512856215387015817134 08t^{24}-573876948723528931270656t^{22}\\
&\quad-554109078152771599052 80t^{20}-108686191812366553256 96t^{18}-2057179881006753699 84t^{16}\\
&\quad+5469013305971207356 8t^{14}-2288987306843185052t^{12}-22309755134355486t^{10}\\
&\quad\left.+253350406694334t^8+8580745232122t^6-7798176720t^4-238542241t^2-18816\right)D^7\\
&\quad+4\left(6234258109050375607576166400t^{30}+16563148489700147386089144320t^{28}\right.\\
&\quad+15388402567597881479353139 2t^{26}+396376397592542781690715136t^{24}-2440863110736050596925 44t^{22}\\
&\quad-293272503356977770941696t^{20}-34041797107502106577312t^{18}+652775043341067168068t^{16}\\
&\quad-449173288488497578 00t^{14}+1157406156124045856 4t^{12}+618056086449925 88t^{10}\\
&\quad\left.-1336778298658238t^8-25795200795638t^6+35399715688t^4+581693399t^2+40768\right)D^6\\
&\quad+8\left(2906754912573693647483043840 0t^{30}+8185187030120176976408838144 0t^{28}\right.\\
&\quad+939457832323447863369383936t^{26}+2148139133199504182578666496t^{24}+537102363857931753762508 8t^{22}\\
&\quad-957869720387013465562880t^{20}-502027855580520126838 4t^{18}-17918825365022233354 04t^{16}\\
&\quad-8565071302823242976 0t^{14}-2084328316325865994 6t^{12}-1012807214484074 30t^{10}\\
&\quad\left.+1741410204251200t^8+30185303650830t^6-49184868070t^4-583650151t^2-37632\right)D^5\\
&\quad+16\left(7923023705874412539134607360 0t^{30}+2369910572455188878638719795 20t^{28}\right.\\
&\quad+306475600992798048521861939 2t^{26}+672555815401257404235198233 6t^{24}+257145153803159692577569 28t^{22}\\
&\quad-221458718333665760927651 2t^{20}+14905942833336792786992t^{18}+242529639655820577809 2t^{16}\\
&\quad+7003090664826524140 0t^{14}+1303966269637213352 3t^{12}+68594134702584874t^{10}\\
&\quad\left.-642583292426727t^8-12336212241784t^6+23535298811t^4+206616410t^2+12544\right)D^4\\
&\quad+64t^2(6424428636759576879967764480 0t^{28}+20383520225153903007867666432 0t^{26}\\
&\quad+282639675724523456352549068 8t^{24}+617605839465917509424647705 6t^{22}+300609790056751486720224 00t^{20}\\
&\quad-17508481065520436663874 24t^{18}-3168072610345148806422 4t^{16}+3289191995937725203 66t^{14}\\
&\quad+3305279728049457466 0t^{12}-311682535251628245t^{10}-3943292702734802t^8\\
&\quad-5555362326911t^6+105428531260t^4-716272453t^2-28448)D^3
\end{aligned}$$

$$
\begin{aligned}
&+512t^2(15009134071247813042455347200t^{28} + 50222389883894861028705538560t^{26}\\
&+7253350719573480165383100l6t^{24} + 1603230224901344022068592328t^{22} + 915489389738265863481035 2t^{20}\\
&-428192374996045407448412t^{18} - 155195232490019542067 92t^{16} + 5278180644461079504 2t^{14}\\
&-127646208056482029 8t^{12} - 107386844952242335t^{10} - 792211501649686t^{8}\\
&-5580848553730t^6 + 16358540366t^4 - 35218463t^2 - 1568)D^2\\
&+6144t^6(12168602116468301625577472 00t^{24} + 425413834485292365055166592 0t^{22}\\
&+628793956213672622681130 88t^{20} + 14120355547354651582831934 8t^{18} + 892698338486015205162136t^{16}\\
&-368713413364399788991 76t^{14} - 163721855337978967092 8t^{12} + 2207989224928555682t^{10}\\
&-562698415042789318t^8 - 9517978064339709t^6 - 42261757561740t^4 - 328350172920t^2 + 368309382)D\\
&+15482880t^6(1837299575605338318848 00t^{24} + 66326524726825082023488 0t^{22}\\
&+99293417132710413021 12t^{20} + 22607862359961003907882t^{18} + 1519630333596783623 64t^{16}\\
&-584635999194169424 9t^{14} - 27615245351651082 2t^{12} + 64100253358393t^{10}\\
&-116439515097332t^8 - 1405235953421t^6 - 3886412020t^4 - 21911230t^2 - 41272).
\end{aligned}
$$

The local log-monodromies for the quantum local system:

$$
\begin{pmatrix}
0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0
\end{pmatrix} \quad \text{at } t = 0
$$

$$
\begin{pmatrix}
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0
\end{pmatrix} \quad \text{at the roots of } 2752t^6 - 1152t^5 + 224t^4 + 96t^3 - 52t^2 + 12t - 1 = 0
$$

$$
\begin{pmatrix}
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0
\end{pmatrix} \quad \text{at the roots of } 2752t^6 + 1152t^5 + 224t^4 - 96t^3 - 52t^2 - 12t - 1 = 0.
$$

The ramification defect of L_X is 1.

B35. MW^4_{18}. [description p. 11, regularized quantum period p. 16]

The quantum differential operator is

$$
\begin{aligned}
&(4t-1)(4t+1)(8t-1)(8t+1)D^4 + 64t^2(128t^2-5)D^3 + 16t^2(1472t^2-33)D^2\\
&\quad +32t^2(896t^2-13)D + 128t^2(96t^2-1).
\end{aligned}
$$

The local log-monodromies for the quantum local system:

$$
\begin{pmatrix}
0&1&0&0\\
0&0&1&0\\
0&0&0&1\\
0&0&0&0
\end{pmatrix} \quad \text{at } t = 0
$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = \frac{1}{4}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = \frac{1}{8}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = -\frac{1}{8}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = -\frac{1}{4}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{at } t = \infty.$$

The operator L_X is extremal.