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Quantum phase transitions in holographic models of magnetism and superconductorsNabil Iqbal,¹ Hong Liu,¹ Márk Mezei,¹ and Qimiao Si²¹*Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*²*Department of Physics and Astronomy, Rice University, Houston, Texas 77005, USA*

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We study a holographic model realizing an “antiferromagnetic” phase in which a global $SU(2)$ symmetry representing spin is broken down to a $U(1)$ by the presence of a finite electric charge density. This involves the condensation of a neutral scalar field in a charged anti-de Sitter black hole. We observe that the phase transition for both neutral and charged (as in the standard holographic superconductor) order parameters can be driven to zero temperature by a tuning of the UV conformal dimension of the order parameter, resulting in a quantum phase transition of the Berezinskii-Kosterlitz-Thouless-type. We also characterize the antiferromagnetic phase and an externally forced ferromagnetic phase by showing that they contain the expected spin waves with linear and quadratic dispersions, respectively.

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I. INTRODUCTION

Quantum phase transitions naturally occur in strongly correlated many-body systems, which often contain competing interactions and the concomitant competing orders. When the transition is continuous, or first order with weak discontinuities, it gives rise to fluctuations that are both quantum and collective [1–3]. Such quantum criticality serves as a mechanism for some of the most interesting phenomena in condensed matter physics, especially in itinerant electronic systems [4,5]. Among these are the breakdown of Fermi liquid theory and the emergence of unconventional superconductivity.

Quantum criticality is traditionally formulated within the Landau paradigm of phase transitions. The critical theory expresses the fluctuations of the order parameter, a coarse-grained classical variable manifesting the breaking of a global symmetry, in $d + z$ dimensions [1]; here d is the spatial dimension and z the dynamic exponent. More recent developments [6,7], however, have pointed to new types of quantum critical points. New modes, which are inherently quantum and are beyond order-parameter fluctuations, emerge as part of the quantum critical excitations. Quantum criticality is hence considerably richer and more delicate than its thermal classical counterpart. In turn, new methods are needed to search for, study, and characterize strongly coupled quantum critical systems.

A. AdS_2 and the emergent IR CFT

A promising new route has come from an unexpected source, the AdS/CFT duality [8], which equates a gravity theory in a weakly curved $(d + 1)$ -dimensional anti-de Sitter (AdS_{d+1}) spacetime with a strongly coupled d -dimensional field theory living on its boundary. The gravity is classical when the corresponding field theory takes a large N limit. This maps questions about strongly coupled many-body phenomena to solvable single- or few-

body classical problems in a curved geometry, often that of a black hole.

For considering a boundary theory at a finite density, a particularly simple gravity setup is a charged black hole in AdS [9] which describes [10] a boundary conformal field theory at a finite chemical potential for a $U(1)$ charge. A rich body of phenomena has been found in this relatively simple context including novel transport (see [11] for a review), holographic superconductors [12,13]¹ (see [14–16] for reviews), and non-Fermi liquids [17–20] (see also [21–29]). In particular, at low energies the system has an infrared fixed point described by a $(0 + 1)$ -dimensional conformal field theory (CFT) [20]. The CFT has nontrivial scaling behavior only in the temporal direction and is represented on the gravity side by a near-horizon geometry $AdS_2 \times \mathbb{R}^{d-1}$. This realization also offers a unified way, from the gravitational perspective, to understand both the onset of the superconducting instability and the emergence of non-Fermi liquids.

One finds that each operator in the UV theory develops an anomalous IR scaling dimension δ . For example, if a scalar operator of charge q and UV dimension Δ is dual to a minimally coupled scalar field in the bulk, its IR dimension is given by [see, e.g., Eq. (43) of [20]; in this paper we have set g_F to 1]

$$\delta = \frac{1}{2} + \sqrt{\frac{\Delta(\Delta - d)}{d(d - 1)} - \frac{q^2}{2d(d - 1)} + \frac{1}{4}}. \quad (1)$$

This determines the onset of an instability: if δ becomes complex, the system is unstable.² Because of the term

¹Technically these should be called charged superfluids, as the symmetry that is broken is global and is not gauged within the holographic framework.

²In the bulk, a complex δ is a violation of the AdS_2 BF bound and corresponds to an infinitely oscillating bulk field near the horizon and hence an instability, as pointed out in [30].

proportional to $-q^2$ in (1), even an operator which is irrelevant in the UV can be unstable in the IR if q is sufficiently large. This gives rise to interesting examples of the phenomenon of dangerously irrelevant operators and possibly suggests a novel type of pairing instability driven by an IR fixed point.³

A similar formula exists for fermions. Here there are no instabilities, but the IR dimension δ controls the nature of small excitations around a Fermi surface [20] and, in particular, whether there exist stable quasiparticles. It also naturally yields fermion self-energies that are singular only in the temporal direction, properties that are characteristic of the electron self-energy in models for non-Fermi liquids [1,31,32] and the spin self-energy of quantum critical heavy fermions [6].

In this paper we extend this development to model itinerant quantum magnetic systems, condensed matter systems in which the quantum critical phenomenon is of great current interest [4,5,33].

B. Gravity formulation of a magnetic system

In the low energy limit of an electronic system, spin-orbit couplings become suppressed and spin rotations are decoupled from spacetime rotations. Thus, spin rotations remain a symmetry even though the rotational symmetries may be broken by a lattice or other effects. The low energy theory is then characterized by an $SU(2)$ global symmetry which describes the spin rotation and a $U(1)$ symmetry describing the charge. This $SU(2)$ symmetry is of course only approximate and will be broken at high energies. However, if one is only interested in universality classes describing only low energy behavior (which does not involve spin-orbit couplings), this high energy breaking will be irrelevant and one might as well replace it by a UV completion in which the dynamics have an $SU(2)$ symmetry that is exact at all energies.

Using the standard AdS/CFT dictionary, the conserved currents j_μ^a , $a = 1, 2, 3$ for $SU(2)$ spin symmetry and J^μ for $U(1)$ charge should be dual to bulk gauge fields making up a $SU(2)_{\text{spin}} \times U(1)_{\text{charge}}$ gauge group in the bulk. For simplicity, in our bulk description we will consider a theory in which this $SU(2) \times U(1)$ symmetry is exact to all energies. Modeling spin-orbit couplings and understanding how to take into account the relation with spacetime symmetries are interesting questions which will be left for future study (see also recent discussion in [29]).

We will be interested in studying the gravity dual of an “antiferromagnetic” (AFM) phase in a continuum limit. In such a limit the background value of the spin density is zero, but there exists a staggered spin order parameter Φ^a ,

³Note that here a clear understanding of the underlying physical mechanism is somewhat hindered by the fact that at this moment the gravity model provides only a macroscopic description of the condensation.

$a = 1, 2, 3$ which transforms as a triplet under spin rotations. Its background value spontaneously breaks $SU(2)$ to the $U(1)$ subgroup corresponding to rotations about a single axis. This leads us to introduce a real scalar field ϕ^a transforming as a triplet under $SU(2)_{\text{spin}}$ [and neutral under $U(1)_{\text{charge}}$] as the corresponding bulk field. A phase with vanishing $SU(2)$ gauge fields but with a normalizable $\phi^a \neq 0$ in one direction can then be interpreted as an antiferromagnetic phase, or a spin density wave phase in an itinerant-electron context. A “ferromagnet” would have nonzero $SU(2)$ gauge fields, corresponding to a nonzero background spin density in the field theory; we also study this by applying a source analogous to an external magnetic field.

Note that the spirit of this discussion is parallel to that of holographic superconductors in that the gravity description captures the macroscopic dynamics of the order parameter and the symmetry breaking pattern, but does not explain its microscopic origin.

C. Quantum phase transitions in holographic models of symmetry breaking

To describe the AFM order and its transition, we proceed in a way analogous to the superconducting case. The magnetic case considered here involves a neutral order parameter. Indeed, with the exception of superconductivity, most ordering phenomena in condensed matter systems involve a neutral order parameter; other examples include charge density wave order and Pomeranchuk instability. We are therefore led to consider holographic phase transitions involving condensation of a neutral scalar field in a finite density system. Here we again consider a charged black hole which is dual to a boundary conformal theory at a finite chemical potential μ for a $U(1)$ charge.

Setting $q = 0$ in (1) one finds that δ becomes complex if

$$\max\left(\frac{d-2}{2}, d - \Delta_c\right) < \Delta < \Delta_c \equiv \frac{d + \sqrt{d}}{2}, \quad (2)$$

where $\frac{d-2}{2}$ corresponds to the unitarity bound on a scalar operator in d spacetime dimensions. On the gravity side this regime is where a scalar field ϕ satisfies the Breitenlohner-Freedman (BF) bound [34] of AdS_{d+1} but violates the BF bound of the near-horizon AdS_2 region, as was recognized first in [35].

In this paper we construct the phase diagram for the condensation of such a neutral scalar field in the $T - \Delta$ [equivalently, the $T - m^2$, see Eq. (18)] plane. For all Δ in the range (2) there exists a critical temperature T_c below which the neutral scalar field condenses. The phase transition is second order with mean field exponents. For the standard quantization⁴ in AdS_{d+1} , we find that T_c decreases with an increasing Δ and approaches zero at Δ_c .

⁴The story for alternative quantization is more involved, see the discussion in Sec. II for details.

Thus, if we allow ourselves to vary Δ through Δ_c , we find a quantum phase transition, as was previously noticed in [30,35] in the charged case. Within the context of a specific boundary field theory varying the UV conformal dimension, Δ seems somewhat artificial: however, the physics here is actually controlled by the effective mass of the scalar in the near-horizon AdS_2 region, and we later discuss several concrete ways to achieve this. Interestingly, the quantum phase transition at Δ_c is not described by mean field exponents, but is instead of the Berezinskii-Kosterlitz-Thouless (BKT) type with an exponentially generated scale. More explicitly, for $\Delta \sim \Delta_c$, at $T = 0$, there exists an IR scale

$$\Lambda_{\text{IR}} \sim \mu \exp\left(-\frac{C}{\sqrt{\Delta_c - \Delta}}\right), \quad C = \pi \sqrt{\frac{d(d-1)}{2\Delta_c - d}} \quad (3)$$

below which new physics appears (or in other words, the condensate becomes significant). From the point of view of the near-horizon geometry AdS_2 , this behavior can be understood from the recent discussion in [23,36], which argues that for a scalar field in AdS with a mass square below the BF bound, such an exponential scale should generally be generated.⁵ An explicit holographic model realizing this phenomenon was also recently constructed in [37] and a different quantum phase transition which occurs outside the BF boundary (i.e., the BF bound of AdS_2 is not violated) has been constructed in [38].

Below the IR scale Λ_{IR} we show that the system in fact flows to a new IR fixed point which is controlled by a $(0 + 1)$ -dimensional CFT, dual to an AdS_2 with a different cosmological constant determined by the condensed vacuum of the neutral scalar field. In Fig. 1 we give a cartoon picture of this flow.

The nature of the ordered phase can also be characterized in terms of its collective modes. The condensate breaks the global $SU(2) \rightarrow U(1)$: we show that the gapless spin waves expected from such a breaking arise naturally in the gravity description, and that they obey the proper dispersion relations.

The main focus of this current paper is on the condensation of a neutral scalar; however it is clear that one can immediately apply the above discussion to condensation of charged scalar operators which give rise to the well-studied holographic superconductors. Equation (1) implies that dialing Δ one finds a quantum critical point at Δ_c given by the larger root of (for standard quantization)

$$\frac{\Delta_c(\Delta_c - d)}{d(d-1)} - \frac{q^2}{2d(d-1)} + \frac{1}{4} = 0. \quad (4)$$

⁵At the BF bound an IR and UV fixed point, which corresponds to standard and alternative quantization, respectively, collide and move to the complex plane, in analogue with the BKT transition. Note that in our case the standard and alternative quantization described here refer to those in AdS_2 .

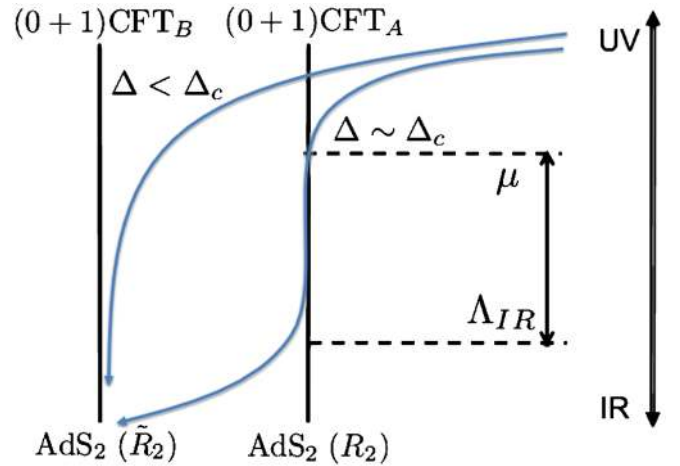


FIG. 1 (color online). A cartoon picture for the flow of the system induced by the condensation of a neutral scalar field. The CFT_A refers to the $(0 + 1)$ -dimensional IR CFT of the uncondensed system, described geometrically by an AdS_2 factor with radius R_2 . When the dimension Δ of the operator is close to the quantum critical value Δ_c , the system stays near this IR CFT for an exponentially long scale, before flowing to the new fixed point, $(0 + 1)$ CFT_B , described by an AdS_2 factor with a different radius \tilde{R}_2 .

Again for Δ close to Δ_c the system will linger around the IR fixed point of the uncondensed phase for an exponentially large scale (3) before settling into new fixed points. In this case the gravity description of the IR fixed points have been worked out before for $d = 3$ in [39–42] (see also [43]). The structure of the phase diagram will, in general, depend on the detailed form of the action, and nonminimal couplings such as those typically found in stringy embeddings (e.g., see [41,42,44,45]) will change the results. It was found in [39] that for minimally coupled scalar actions and potentials similar to ours and for minimally coupled scalar actions and potentials similar to ours and for the charge of the scalar q small the system flows to a Lifshitz fixed point with a dynamic exponent $z \sim \frac{1}{q^2}$, while for larger values of q the system flows to another AdS_4 with a different cosmological constant. Combined with the discussion above for the condensation of a neutral scalar field we thus find a unifying picture for the quantum phase transitions for both charged and neutral order parameters as presented in Fig. 2. Note that AdS_2 can be considered as describing a d -dimensional theory with a dynamic exponent $z = \infty$, while AdS_4 describes a theory with $z = 1$.

Note that in our above discussion we have imagined the existence of an “experimental knob” which can be used to adjust the UV scaling dimension Δ of an operator. It is important to note that we do this purely for convenience, and the nature of our discussion is completely insensitive to the precise realization of such a knob. The most useful knob will likely depend on the UV geometry into which this AdS_2 is embedded. For example, if one is studying a

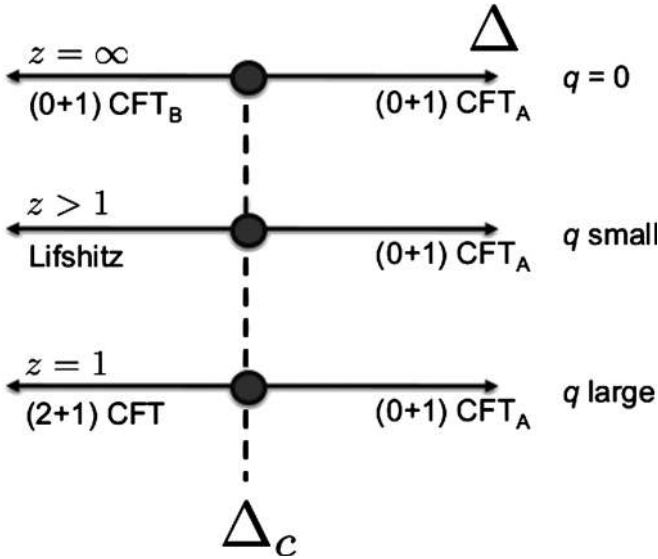


FIG. 2. Tuning the UV dimension of the order parameter we find quantum phase transitions between a (0 + 1)-dimensional IR CFT corresponding to AdS₂ in the unbroken phase and various types of symmetry breaking phases. The type of symmetry-broken phase depends on the charge q of the order parameter.

holographic superconductor an external magnetic field will also allow one to tune the IR scaling dimension [46]. A similar UV realization is provided in a $D3/D5$ brane construction in [37], where a precisely analogous transition is studied. For convenience in the remainder of this paper we will simply imagine that we are free to tune the bulk mass and thus, directly, the UV dimension of the scalar. In Sec. IV A we give a simple model illustrating how this may be achieved.

A rough plan of the paper is as follows. In the next section we describe a holographic phase transition corresponding to the condensation of a neutral scalar order parameter in a finite density system in a probe approximation. In Sec. III we discuss some qualitative features of the effects of backreaction and the zero temperature limit of our solution. In Sec. IV we discuss the quantum phase transition that we find by tuning the UV conformal dimension. In Sec. V we embed the scalar solution discussed in Sec. II into an $SU(2)$ system describing the spin rotational symmetry. We proceed to study the spin waves from spontaneous breaking of the spin symmetry in the antiferromagnetic phase. In a probe limit we are able to isolate the spin wave excitations directly in the bulk and find their dispersion relations. We do not construct a spontaneous ferromagnet; however in Sec. VI we do show that if one considers aligning the “spins” with an external magnetic field then techniques similar to those in Sec. V can be used to find a spin wave which has a quadratic dispersion relation in line with field theoretical expectations. We conclude in Sec. VII with a discussion of further directions.

II. CONDENSATION OF A NEUTRAL ORDER PARAMETER AT A FINITE DENSITY

In this section we provide a gravity dual description of the condensation of a neutral scalar field in a charged AdS black hole geometry which describes the onset of a real order parameter in the boundary theory at a finite density. We begin our analysis by studying the transition at a finite temperature. While our discussion applies to any space-time dimension, for definiteness we will consider a (2 + 1)-dimensional boundary theory.

A. Setup

To put the system at a finite density we turn on a chemical potential μ for the $U(1)_{\text{charge}}$ in the boundary. This is described on the gravity side by a charged black hole with a nonzero electric field for the corresponding $U(1)$ gauge field B_M . The action for B_M coupled to AdS gravity can be written as

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[\mathcal{R} + \frac{6}{R^2} - R^2 G_{MN} G^{MN} \right], \quad (5)$$

with $G_{MN} = \partial_M B_N - \partial_N B_M$ and R is the curvature radius of AdS. The equations of motion following from (5) are solved by the geometry of a charged black hole [9,10],

$$\frac{ds^2}{R^2} \equiv g_{MN} dx^M dx^N = r^2 (-f dt^2 + d\vec{x}^2) + \frac{1}{r^2} \frac{dr^2}{f}, \quad (6)$$

with

$$f = 1 + \frac{3\eta}{r^4} - \frac{1+3\eta}{r^3}, \quad B_t = \mu \left(1 - \frac{1}{r} \right), \quad (7)$$

where we have rescaled the coordinates so that the horizon is at $r = 1$ and all coordinates are dimensionless (see Appendix A 1 for more details). The chemical potential and temperature are given by

$$\mu \equiv \sqrt{3}\eta^{(1/2)}, \quad T = \frac{3}{4\pi}(1 - \eta). \quad (8)$$

η is a parameter between 0 and 1, where $\eta = 1$ corresponds to the extremal black hole with $T = 0$ and $\eta = 0$ corresponds to a finite temperature system with zero chemical potential (and charge density).⁶ In the zero temperature limit, the near-horizon geometry reduces to $\text{AdS}_2 \times \mathbb{R}^2$ with the curvature radius of the AdS_2 region related to that of the UV AdS_4 by

$$R_2 = \frac{R}{\sqrt{6}}. \quad (9)$$

As discussed in [20,35], a neutral scalar field χ can develop an instability if the mass square of the scalar

⁶Note that since we are considering a conformal theory, only the dimensionless ratio $\frac{\mu}{T}$ is physically relevant, and it is clear that as η is varied from 0 to 1, $\frac{\mu}{T}$ takes all values from 0 to ∞ .

violates the near-horizon AdS₂ BF bound, while still satisfying the AdS₄ BF bound, i.e.,⁷

$$-\frac{9}{4} < m^2 R^2 < -\frac{3}{2}, \quad (11)$$

where the lower limit is the BF bound in AdS₄, the upper limit is the BF bound for the near-horizon AdS₂ region, and we have used (9) to convert from AdS₂ to AdS₄ radii.

Once this condition is met, the scalar will want to condense near the horizon but will be stable at infinity. The condensed solution will involve a nontrivial radial profile for the scalar; we will see that at low temperatures the scalar will probe the extreme values of its potential and nonlinearities in the potential will be important. We choose to study a nonlinear Mexican hat potential with the Lagrangian for χ given by

$$\mathcal{L}_\chi = \frac{1}{2\kappa^2\lambda} \left[-\frac{1}{2}(\partial\chi)^2 - V(\chi) \right], \quad (12)$$

with

$$V(\chi) = \frac{1}{4R^2}(\chi^2 + m^2 R^2)^2 - \frac{m^4 R^2}{4}. \quad (13)$$

Here m^2 is the effective mass near the point $\chi = 0$, and should be chosen to satisfy the condition (11) (in particular, it is *negative*). λ is a coupling constant and we have chosen the constant in (12) so that at $\chi = 0$, there is no net contribution to the cosmological constant. The precise form of the potential in (13) is not important for our discussion below, provided that it does have a minimum and satisfies the condition (11).⁸

At zero temperature, we expect the scalar to condense until the value at the horizon reaches some point near the bottom of the Mexican hat, at which point the “effective AdS₂ mass” will again satisfy the AdS₂ BF bound and condensation will halt. At finite temperature, we expect a phase transition at some temperature T_c below which χ condenses. Note that at the classical level the coefficient of the χ^4 term is arbitrary, as it can be absorbed into λ in (12) via a rescaling of χ .

B. Phase diagram

We now seek the endpoint of the instability, i.e., a nontrivial scalar profile for χ . We first consider the finite

⁷For general d boundary theory dimensions, Eq. (10) becomes

$$-\frac{d^2}{4} < m^2 R^2 < -\frac{d(d-1)}{4}. \quad (10)$$

⁸The condensed phase $q = 0$ solutions without the stabilizing χ^4 term in the potential were studied in [35,40]; at nonzero temperature these are similar to ours, but in the low temperature limit the structure described in Sec. III—which depends critically on the existence of a minimum to the potential—is not shared by those examples.

temperature case and take λ to be parametrically large so that we can ignore the backreaction of χ to the background geometry. We will discuss the backreaction in Sec. III and there we argue that this approximation is good even at zero temperature.

The equation of motion for $\chi(r)$ is given by

$$\frac{1}{r^2} \partial_r(r^4 f \partial_r \chi) - \chi(\chi^2 + m^2 R^2) = 0. \quad (14)$$

We are interested in a solution which is regular at the horizon and normalizable at the boundary, which can be found numerically.

We first consider the asymptotic behavior for χ near the horizon and the boundary. We require the solution to be regular at the horizon, i.e., to have the expansion

$$\chi(r) = \chi_h + \chi'_h(r-1) + \dots, \quad r \approx 1. \quad (15)$$

At finite temperature the factor f in (14) has a first order zero at the horizon, i.e., $f(r) = 4\pi T(r-1) + \dots$. Demanding that (14) be nonsingular then leads to a condition linking the near-horizon value of χ to its derivative:

$$\chi'_h = \frac{1}{4\pi T} \chi_h(\chi_h^2 + m^2 R^2). \quad (16)$$

A choice of χ_h fixes also χ'_h and thus completely specifies the solution.

Near the boundary $r \rightarrow \infty$, the linearized equation of (14) gives the standard asymptotic behavior

$$\chi(r) \approx Ar^{\Delta-3} + Br^{-\Delta}, \quad r \rightarrow \infty, \quad (17)$$

with Δ given by

$$\Delta = \frac{3}{2} + \sqrt{m^2 R^2 + \frac{9}{4}}. \quad (18)$$

In the standard quantization A has the interpretation of the source⁹ while B gives the response $\langle \Phi \rangle_A$ of the order parameter Φ dual to χ in the presence of source A . Note that the mass range (11) lies within the range $-\frac{9}{4} < m^2 R^2 < -\frac{3}{4}$ for which an alternative quantization exists, in which the roles of A and B are exchanged [47]. The conformal dimension of Φ (i.e., the “dimension” of fluctuations about the point $\chi = 0$) is given by Δ (standard quantization) and $3 - \Delta$ (alternative quantization), respectively.

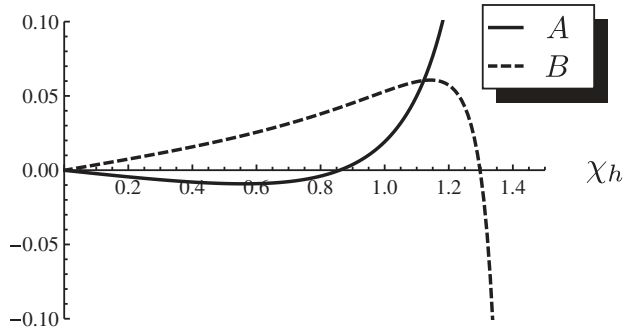
The condensed phase for standard quantization is characterized by a normalizable nontrivial solution with $A = 0$, and then B gives the expectation value of the order parameter Φ . For the alternative quantization we look instead

⁹For example, in Sec. V we interpret χ as corresponding to the staggered magnetization, in which case A can be interpreted as the staggered magnetic field.

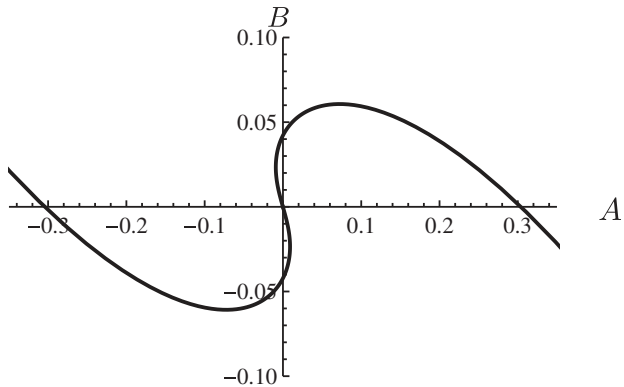
for a solution with $B = 0$, and A then gives the expectation value. The task before us now is to pick a value for χ_h , and then numerically integrate the radial evolution equation to the boundary. As expected for a sufficiently low temperature, we find a nontrivial scalar hair solution, which means that there will exist a χ_h for which A or B vanishes, as shown in Fig. 3.

For the standard quantization we find a continuous phase transition for all values of m^2 falling into the range (11); there is a nontrivial profile for χ for T smaller than some temperature T_c and none for T greater. In particular, as we increase m^2 , the critical temperature T_c decreases to zero as the upper bound in (11) is approached. Precisely at the critical value $m_c^2 R^2 = -\frac{3}{2}$, we find a quantum phase transition: the physics in the vicinity of this point is discussed in Sec. IV. The phase diagram for standard quantization is plotted in Fig. 4.

For the alternative quantization, the phase structure in the vicinity of the quantum critical point at $m^2 = m_c^2$ is the



(a) A and B as a function of the horizon value of the scalar field χ_h .



(b) B as a function of A : different points on the curve correspond to different values of χ_h .

FIG. 3. The constants A and B determine the behavior of the scalar profile asymptotically. This is a representative plot where we scan the case $m^2 R^2 = -2.1$ and $T = 0.00024$ (with $T/T_c = 0.22$) by varying χ_h . There is symmetry breaking if $A = 0, B \neq 0$ in the normal quantization or if $A \neq 0, B = 0$ in the alternative quantization.

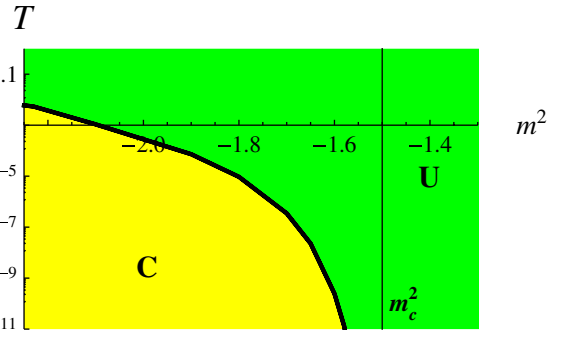


FIG. 4 (color online). Phase diagram for the standard quantization. Note logarithmic scale for T . C denotes the condensed phase and U the uncondensed phase. $T_c \rightarrow 0$ as $m^2 \rightarrow m_c^2$, leading to a quantum critical point.

same as in the standard quantization. However the global structure of the phase diagram is somewhat different; in particular, the specific value of the mass $m^2 R^2 = -\frac{27}{16}$ plays an important role. This value of the mass corresponds to the UV scaling dimension for which the first nonlinear (i.e., arising from the ϕ^4 term in the potential) correction to the near-boundary asymptotics becomes degenerate with the term proportional to B . For $m^2 R^2 < -\frac{27}{16}$ the phase structure is as described above, but for $m^2 R^2 > -\frac{27}{16}$ one finds a new condensed phase in the *high-temperature* regime. In particular, the critical temperature T_{c2} appears to increase with m^2 . These solutions appear to be closely related to the thermodynamically unstable scalar hair solutions constructed in [48,49], which studied uncharged black holes and thus correspond to the high-temperature limit of our construction. As these new phases appear to involve UV physics and are not related to the low temperature quantum critical behavior that is the focus of this work, we defer an in-depth study of these phases and the critical value $m^2 R^2 = -\frac{27}{16}$ to later work.

C. Critical exponents

A continuous phase transition can be characterized by various critical exponents: in the normal quantization with A the source and $B \sim \langle \Phi \rangle$ the expectation value we have the following behavior close to T_c :

- (1) The expectation value $\langle \Phi \rangle \propto (T_c - T)^\beta$ with mean field value $\beta = \frac{1}{2}$.
- (2) The specific heat: $C \propto |T_c - T|^{-\alpha}$ with mean field value $\alpha = 0$.
- (3) Zero field susceptibility: $\frac{\partial \langle \Phi \rangle}{\partial A} |_{A=0} \propto |T - T_c|^{-\gamma}$ with mean field value $\gamma = 1$.
- (4) Precisely at $T = T_c$, $\langle \Phi \rangle \sim A^{(1/\delta)}$ with mean field value $\delta = 3$

There are also other exponents associated with correlation functions at finite spatial or time separation which we will leave for future study. For our phase transition we find that

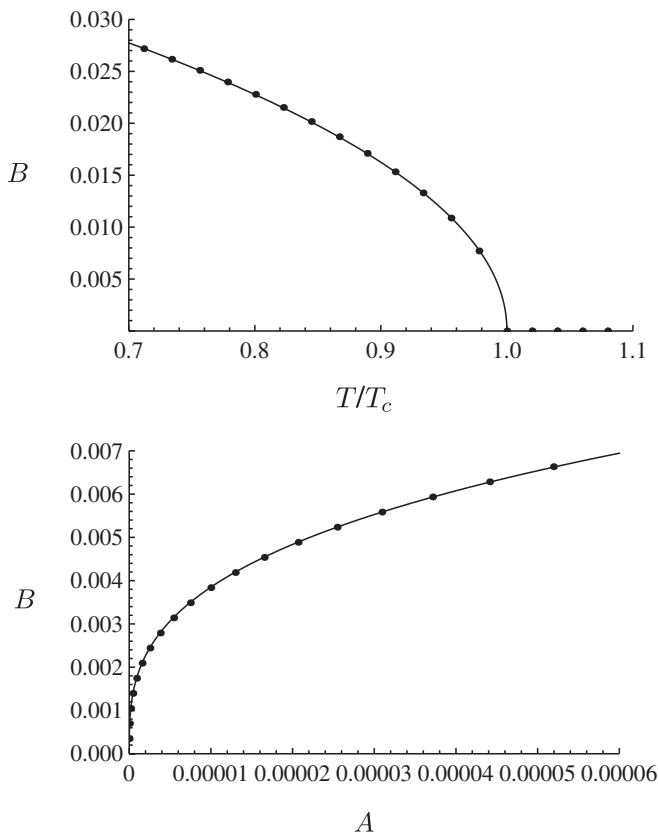


FIG. 5. *Top*: Plot for exponent β , defined as $B \sim (T_c - T)^\beta$. $\beta = 0.49 \pm 0.03$ from numerical fit, compared with $\beta_{\text{mean field}} = \frac{1}{2}$. *Bottom*: Plot for exponent δ , defined as $B \sim A^{(1/\delta)}$ at T_c . $\delta = 3.03 \pm 0.05$ from numerical fit, compared with $\delta_{\text{mean field}} = 3$.

all of the above exponents are precisely those of mean field theory. See Fig. 5 for numerical fits of exponent β and δ .

The existence of mean field exponents in this classical gravity analysis is not surprising¹⁰ and can be translated into the statement that A and B are analytic functions of the horizon field χ_h . This follows from the fact that χ is regular at the horizon and the black hole geometry is smooth; we will find a rather different situation at $T = 0$ in the next section.

We now sketch the relevant arguments for finite T . As argued above, for small χ_h we can expand A as

$$A \approx a_1(T)\chi_h + a_2(T)\chi_h^3 + \dots, \quad (19)$$

and similarly for B . Note that both A and B must both be odd functions of χ_h , due to the $\chi \rightarrow -\chi$ symmetry. At T_c , a new zero of A should be generated at $\chi_h = 0$ and move to a finite value of χ_h as T is further lowered.¹¹ Thus near T_c

¹⁰Mean field exponents also appear in the phase transition associated with a charged scalar field in the holographic superconductor story [13,50].

¹¹In fact given $\chi \rightarrow -\chi$ symmetry, a pair of zeros are generated and move to opposite directions.

we should have $a_1(T) = a(T - T_c)$ and $a_2(T) = b + \dots$ with a and b having the same sign for $T > T_c$. For $T < T_c$, then A has a zero for χ_h taking a value χ_A given by

$$\chi_A = \sqrt{-\frac{a_1(T)}{a_2(T)}} \propto (T_c - T)^{(1/2)} \rightarrow 0. \quad (20)$$

Using χ_A as the initial value at the horizon gives the solution for the condensed phase. Assuming that B is still linear in χ_h near $\chi_h = 0$, we thus find that

$$B(\chi_A) \propto \chi_A \propto (T_c - T)^{(1/2)}, \quad (21)$$

which gives us the critical exponent $\beta = \frac{1}{2}$. Turning now to the susceptibility, we find

$$\left. \frac{dB}{dA} \right|_{A=0} = \left. \frac{dB}{d\chi_h} \frac{d\chi_h}{dA} \right|_{A=0} \sim (T - T_c)^{-1}, \quad (22)$$

leading to $\gamma = 1$. Furthermore, precisely at $T = T_c$, $A \sim a\chi_h^3$ and B stays linear near $\chi_h = 0$. Thus, we find that at $T = T_c$,

$$B(\chi_h) \propto \chi_h \propto A^{(1/3)}, \quad (23)$$

which gives the critical exponent $\delta = 3$. One can also compute the free energy using holographic renormalization and indeed find the mean field exponent $F \propto -(T_c - T)^2$ for $T < T_c$. In particular, the analytic expansion (19) for A and B in terms of χ_h guarantees that F has the Landau-Ginsburg form.

Note that the gravity analysis can be extended to all spacetime dimensions with the same mean field scalings. Presumably this has to do with the fact that we are working in the large N limit, which suppresses fluctuations. This is also consistent with picture obtained in [20] in which the scalar instability essentially follows from a RPA type analysis. We expect that a $1/N$ computation involving quantum corrections in the bulk will reveal corrections to these mean field exponents.

III. BACKREACTION AND THE ZERO TEMPERATURE LIMIT

In the usual studies of holographic superconductors, the scalar is charged under a $U(1)$ and so the usual probe approximation involves treating the scalar and $U(1)$ gauge field as negligible perturbations to the background metric. In these models lowering the temperature essentially means increasing the ratio μ/T ; thus at sufficiently low temperatures the gauge field (and thus also the condensed scalar) will necessarily backreact strongly on the geometry, causing a breakdown of the probe approximation. The zero temperature limit of the backreacted geometries have been constructed [39–42] and typically depend on the details of the couplings and charge of the scalar. One recurring theme is that at zero temperature the charges that were previously carried by the black hole are now completely sucked out of the hole and into its scalar hair. Once the black hole

horizon is relieved of the burden of carrying a large charge, it is usually replaced by a degenerate horizon with vanishing entropy.

In our model the situation is different. We always include the backreaction of the gauge field on the metric, as we start from the beginning with the charged black hole solution; it is consistent to solve for a scalar profile on this fixed background only because the scalar is uncharged and so its contribution to the backreaction can be cleanly suppressed by taking λ large.

In this section we will discuss the backreacted solution in the IR at zero temperature and argue that even if backreaction is included its effects are rather benign and do not change any qualitative conclusions. This is essentially because all of the charge must stay in the black hole itself, greatly constraining its near-horizon form.

First, we note that Gauss's law states that

$$\partial_r(\sqrt{-g}g^{rr}g^{tt}\partial_r A_t) = 0 \rightarrow g_{xx}\sqrt{g^{rr}g^{tt}}\partial_r A_t = \text{const.} \quad (24)$$

Thus, if the electric field is nonsingular at the horizon, g_{xx} must be finite there: this is simply saying that the \mathbb{R}^2 at the horizon cannot degenerate as it has a nonzero electric field flux through it. We then expect that the near-horizon geometry factorizes into the form $\mathcal{M}_2 \times \mathbb{R}^2$, where \mathcal{M}_2 is some 2d manifold involving (t, r) . Now consider the trace of the Einstein equation arising from the variation of (5) plus (12); the $U(1)$ field strength does not contribute to this equation as it is classically scale invariant, and we find

$$\mathcal{R} + \frac{12}{R^2} = \frac{1}{2\lambda}[(\nabla\chi)^2 + 4V(\chi)], \quad (25)$$

where \mathcal{R} is the Ricci scalar of the geometry and comes purely from the \mathcal{M}_2 factor. Now let us further assume that in the near-horizon region χ asymptotes to some constant value χ_h (we will show this to be consistent shortly). We then find that \mathcal{M}_2 has constant negative curvature, and thus must be AdS_2^{12} with radius \tilde{R}_2 satisfying

$$\frac{1}{\tilde{R}_2^2} = \frac{1}{R_2^2} - \frac{1}{\lambda}V(\chi), \quad (26)$$

where R_2 is the curvature radius of the original AdS_2 near-horizon geometry of the extremal charged black hole. Note that since $V(\chi_h) < 0$, $\tilde{R}_2 < R_2$. Thus, we see that the backreacted IR geometry is very similar to the unperturbed geometry, except that its AdS_2 factor has a radius that is corrected by the presence of the near-horizon scalar potential.

Let us now study the scalar equation of motion (14) near the backreacted AdS_2 horizon:

$$\frac{1}{r^2}\partial_r(r^4\tilde{f}\partial_r\chi) = R^2\frac{dV}{d\chi}, \quad (27)$$

where \tilde{f} is the warp factor for the backreacted geometry and at zero temperature has a double zero at the horizon r_0 ; expanding near the horizon we see that regularity at the horizon requires

$$\frac{dV}{d\chi}(\chi(r=r_0)) = 0. \quad (28)$$

Thus, we see that at the horizon χ will sit at the bottom of its potential. To understand how this AdS_2 region matches onto the asymptotic geometry, we expand $\chi = \chi_h + \delta(r)$ where χ_h is the bottom of the potential $V(\chi)$. Now working in the AdS_2 region and linearizing (27) near χ_h we find that δ obeys the standard AdS_2 wave equation:

$$\partial_r((r-r_0)^2\partial_r\delta) - \tilde{R}_2^2V''(\chi_h)\delta = 0, \quad (29)$$

whose solutions are

$$\delta = \alpha(r-r_0)^{-(1/2)+\nu} + \beta(r-r_0)^{-(1/2)-\nu}, \quad (30)$$

with

$$\nu = \sqrt{\frac{1}{4} + \tilde{R}_2^2V''(\chi_h)}. \quad (31)$$

Now we would like χ to approach χ_h as we approach the horizon; this means that we should not take the β solution above, as it invariably blows up as $r \rightarrow r_0$. Note that for the α solution to also not blow up, we need $-\frac{1}{2} + \nu > 0$ and thus $V''(\chi_h) > 0$. To stabilize the AdS_2 region we must truly be sitting at a minimum of the potential at the horizon. In this case the solution

$$\chi(r) = \chi_h + \alpha(r-r_0)^{-(1/2)+\nu} \quad (32)$$

can be interpreted as an irrelevant deformation of the new AdS_2 IR CFT that we are flowing to.¹³ The value of the coefficient α is not fixed at this linearized level and must be determined by matching to the UV solution.

We have not solved for the fully backreacted geometry but have used this method to find a normalizable scalar solution on the $T=0$ charged black hole geometry using the above exponents. Our results match smoothly onto the $T \rightarrow 0$ limit of the profiles calculated using the finite temperature matching procedure discussed in Sec. II. When λ is large we do not expect the inclusion of the backreaction to qualitatively change any of these results given the change of the cosmological constant is small.

It is instructive to compare this to the situation for charged holographic superconductors [39]. In this case for large charge it is found that the IR geometry flows to an AdS_4 , while for sufficiently small charge it flows to a

¹²More precisely, it could also be an AdS_2 black hole; the true zero temperature solution corresponds however to pure AdS_2 .

¹³Similar considerations are used in [39] to determine when an emergent AdS_4 can exist.

Lifshitz geometry, with exponent z satisfying

$$q^2 \sim \frac{1}{z}, \quad (33)$$

at very small q [see Eq. (81) in [39]]. We see that our AdS₂ solution corresponds to $z = \infty$ at $q = 0$; increasing the charge we have a Lifshitz solution with finite z , and finally increasing further we find $z = 1$ for AdS₄.

IV. A QUANTUM PHASE TRANSITION FROM CLASSICAL GRAVITY

We recall from the discussion in Sec. II B that when $m_c^2 R^2 \equiv -\frac{3}{2}$, the critical temperature approaches zero: thus we should obtain a “quantum phase transition” at zero temperature as m^2 is varied from above m_c^2 to below. In this section we consider the behavior near the quantum critical point from the condensed side.

The accessibility of this quantum critical point depends on having the ability to tune the IR dimension of the field χ in some way. The simplest possible way is by directly tuning the mass, and before proceeding we pause briefly to explain how this may be possible in a simple bulk realization. A similar mechanism is discussed for uncharged black holes in AdS in [51].

A. Tuning across the quantum critical point

Imagine that the scalar χ contains a coupling to another scalar field ψ ,

$$S_\chi \supset \int d^{d+1}x \sqrt{-g} F(\psi) \chi^2, \quad (34)$$

where $F(\psi)$ is some function whose detailed form will not be important for us. Now consider turning on a constant source h for ψ at the boundary, i.e., we require the boundary value of ψ to approach h . In the boundary theory language this corresponds to adding a term

$$\int d^d x h \mathcal{O}(x) \quad (35)$$

to the action with \mathcal{O} the boundary operator dual to ψ . The source induces a nontrivial bulk solution for ψ , which in turn contributes to the action of χ as a mass term, shifting the effective m^2 . However the new effective mass will typically be radially dependent; we require it to be nonzero at the AdS₂ horizon. If we write the IR dimension (1) of ψ as

$$\delta_\psi = \frac{1}{2} + \nu_\psi \quad (36)$$

then the asymptotics of ψ in the near-horizon AdS₂ region are

$$\psi \sim (r - r_0)^{-(1/2) \pm \nu_\psi}. \quad (37)$$

For ψ to be regular at the horizon we need to choose the + sign in the exponent. So if $\nu_\psi = \frac{1}{2}$ then ψ approaches a

constant at the horizon, and turning on a source h for ψ will change the effective IR mass of the field χ . From (1), ν_ψ is $\frac{1}{2}$ if ψ is dual to an operator with UV dimension $\Delta = d$, i.e., it is massless in the bulk. Thus, whenever the UV CFT has such an operator with a nontrivial operator product expansion with the order-parameter operator Φ , we expect to be able to tune the system through the quantum critical point by varying h in (35), i.e., there exists a critical value h_c of h at which we expect a quantum phase transition. It would be interesting to understand this mechanism further. Also note that for type II theories in an asymptotic AdS geometry, a natural candidate for ψ is the dilaton.

B. BKT scaling behavior and Efimov states

For $m^2 < m_c^2$, the BF bound of the near-horizon region AdS₂ region is violated. It has been argued in [36] that for a general AdS gravity dual with mass slightly below the BF bound conformality is lost and an IR scale Λ_{IR} is generated exponentially as

$$\Lambda_{\text{IR}} \sim \mu \exp\left(-\frac{\pi}{\sqrt{m_c^2 R_2^2 - m^2 R_2^2}}\right), \quad (38)$$

where μ represents some UV scale that in our case is the chemical potential. This exponential behavior is characteristic of the BKT phase transition. One expects that this IR scale controls the physics near the quantum phase transition. In particular, the critical temperature T_c and the value of the condensate $\langle \Phi \rangle$ should be related to this scale. Here we present a reformulation of the arguments of [36] that demonstrates the behavior of T_c and $\langle \Phi \rangle$ explicitly.

We parametrize the AdS₂ region of the uncondensed geometry as

$$\frac{ds_2^2}{R_2^2} = \frac{-dt^2 + dz^2}{z^2}. \quad (39)$$

The linearized equation for the scalar about the point $\chi = 0$ is then

$$-\frac{d^2}{dz^2} \chi + \frac{R_2^2(m^2 - m_c^2) - 1/4}{z^2} \chi = \omega^2 \chi, \quad (40)$$

where we have assumed time dependence $e^{-i\omega t}$. This is essentially a Schrödinger equation with energy ω^2 : the existence of a negative energy bound state indicates a mode growing exponentially in time and hence an instability, and it is a well-known fact that if z can take values from 0 to ∞ then such negative energy bound states exist when $m^2 < m_c^2$, leading to the BF bound.

In our problem, however, z does not take values on the whole half line. At some small UV value z_{UV} we should match onto the asymptotic UV geometry. We will now show that if we also choose an appropriate infrared cutoff z_{IR} then even when m^2 is below the BF bound (40) will

have no bound states. To understand this, let us assume for simplicity that χ satisfies Dirichlet boundary conditions $\chi(z_{\text{IR,UV}}) = 0$ ¹⁴ and consider the zero energy $\omega = 0$ solutions to (40). We find that the solution is oscillatory (see also discussion in [23])

$$\chi(z) = \sqrt{z} \sin \left[R_2 \sqrt{m_c^2 - m^2} \log \frac{z}{z_{\text{UV}}} \right], \quad (41)$$

and importantly, the zero energy solution satisfies the boundary condition only when

$$\log \frac{z_{\text{IR}}}{z_{\text{UV}}} = \frac{\pi}{R_2 \sqrt{m_c^2 - m^2}}. \quad (42)$$

Essentially we must fit a single half-period of the oscillatory wave function inside. This wave function has no nodes; thus if z_{IR} satisfies this condition, then the lowest energy state has zero energy and there is no instability. Decreasing the distance between z_{IR} and z_{UV} will only increase the energy of the ground state. On the other hand, if we increase this distance-if z_{IR} is too high-then the ground state will have negative energy, indicating an instability.

Now in our problem z_{IR} can be provided by a small finite temperature, which ends the geometry with a horizon at some large value z_h . We thus find that if $z_h > z_{\text{IR}}$ as defined above, there will be an instability that can be resolved only by condensation of the scalar. The critical temperature T_c is thus given by

$$T_c \sim \frac{1}{z_h} \sim \mu \exp \left[- \frac{\pi}{R_2 \sqrt{m_c^2 - m^2}} \right], \quad (43)$$

where $\mu \sim \exp[z_{\text{UV}}]$ is some UV scale. We have been able to confirm this numerically, including the prefactor in the exponent (see Fig. 6).

What if $T < T_c$? Now the IR cutoff cannot be provided by the horizon, and must instead be provided by nonlinearities associated with the scalar potential, which will become important near z_{IR} . We then expect that the condensate $\langle \Phi \rangle$ should itself exhibit similar exponential scaling as (43), as was found explicitly in a similar context in [37]. We present here a simple way to understand this result. Note that for m^2 only slightly below m_c^2 the IR theory is basically the original conformal IR CFT for an exponential hierarchy of scales. In this IR CFT, χ is unstable and has dimension given by

$$\delta = \frac{1}{2} + i \sqrt{m_c^2 R_2^2 - m^2 R_2^2} \approx \frac{1}{2}. \quad (44)$$

Since χ has dimension $\frac{1}{2}$, it should transform under a scaling transformation as

$$\chi(\lambda z) = \lambda^{(1/2)} \chi(z). \quad (45)$$

¹⁴Note that more general boundary conditions simply result in slight changes in the value of z_{IR} and z_{UV}

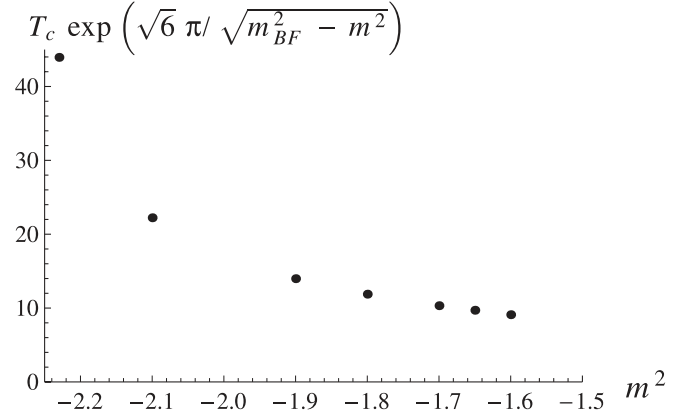


FIG. 6. The exponential dependence of T_c as a function of $\sqrt{m_c^2 - m^2} R_2$. Here we have compensated for the expected theoretical behavior of T_c and plotted the resulting function versus m^2 ; we see that as $m^2 \rightarrow m_c^2$ the values approach a constant, verifying our prediction.

We expect that $\langle \Phi \rangle$ should be simply related to the value of $\chi(z)$ at the UV matching value z_{UV} , as essentially only linear UV evolution will be required to relate them, and thus

$$\langle \Phi \rangle \sim \chi(z_{\text{UV}}). \quad (46)$$

Similarly, the value of ϕ in the deep IR will be determined by nonlinearities, which we expect to result in

$$\chi(z_{\text{IR}}) \sim O(1). \quad (47)$$

We thus find from (45) that

$$\chi(z_{\text{IR}}) = \sqrt{\frac{z_{\text{IR}}}{z_{\text{UV}}}} \chi(z_{\text{UV}}), \quad (48)$$

which results upon evolution through the UV region to the AdS₄ boundary in

$$\langle \Phi \rangle \sim \sqrt{\frac{z_{\text{UV}}}{z_{\text{IR}}}} \sim \Lambda_{\text{IR}} \exp \left[- \frac{\pi}{2 R_2 \sqrt{m_c^2 - m^2}} \right]. \quad (49)$$

Note that the exponent for $\langle \Phi \rangle$ is half that obtained for T_c . In the intermediate conformal regime χ scales as a dimension 1/2 operator and the temperature scales as dimension 1: this is the origin of the factor of 2 difference in the exponents above. Eventually in the far IR nonlinearities become important and a scale is generated, but the theory is not gapped; provided that the scalar potential has a minimum, as described in Sec. III, it instead flows to a different AdS₂ fixed point, and the IR scale manifests itself only as an irrelevant perturbation (32) along which we flow to this new fixed point.

It is possible to perform a more explicit calculation of (49), following the techniques used in [37] where similar scaling is obtained in an AdS₂ that arises from a D3/D5 brane construction. From such a treatment it is clear that there should exist an infinite tower of ‘‘Efimov states’’ that correspond to allowing oscillatory solutions such as (41) to

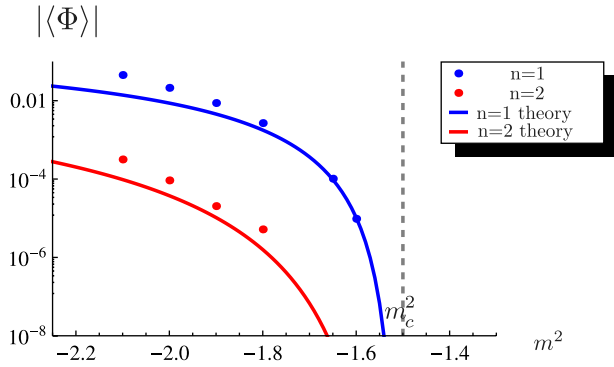


FIG. 7 (color online). Efimov states: for comparison we plot expected results from the analytic arguments described in the text. Agreement can be seen at this level but numerical difficulties prevent us from probing this region more carefully.

move through more periods before matching to the IR solution. These have $\langle\Phi\rangle \sim \exp\left[-\frac{n\pi}{2R_2\sqrt{m_c^2-m^2}}\right]$ with n a positive integer. We have been able to find the first two of these states numerically (see Fig. 7); as explained in [37] the relevant wave functions have more nodes with increasing n and so the $n = 1$ state is energetically favored.

To conclude this subsection we note that turning on any finite temperature the phase transition becomes that of the mean field, since the physics depended in a smooth way on the horizon value of the scalar χ_h . At zero temperature since the horizon is degenerate we expect the boundary values of χ to depend on the initial value α at the horizon in Eq. (32). It would be interesting to understand this better.

C. Quantum critical points for holographic superconductors

Finally, note that the scaling behavior (43) and (49) also applies to the condensate of a charged scalar, which results in the well-studied holographic superconductor.¹⁵ In such a case the coupling to the background electric field is important; also one now has the possibility of supplying an external magnetic field H for $U(1)_{\text{charge}}$ ¹⁶ [46]. The effect of the electric field on the conformal dimension is shown in (1), and the generalization of this formula to include the effect of the magnetic field is (see Appendix A for a derivation)

$$\delta = \frac{1}{2} + \sqrt{\frac{m^2 R^2}{6} + (6|bq| - q^2) \frac{\sqrt{1 + 12b^2} - 1}{72b^2}} + \frac{1}{4}, \quad (50)$$

¹⁵Similar exponential dependence of the critical temperature in such a case has been observed by Faulkner and Roberts.

¹⁶Note that this magnetic field is an external field strength for $U(1)_{\text{charge}}$ and has absolutely nothing to do with the “magnetic” field associated with the antiferromagnetic ordering discussed later in this paper, which would correspond to a chemical potential for the $SU(2)$ gauge field A_7^a .

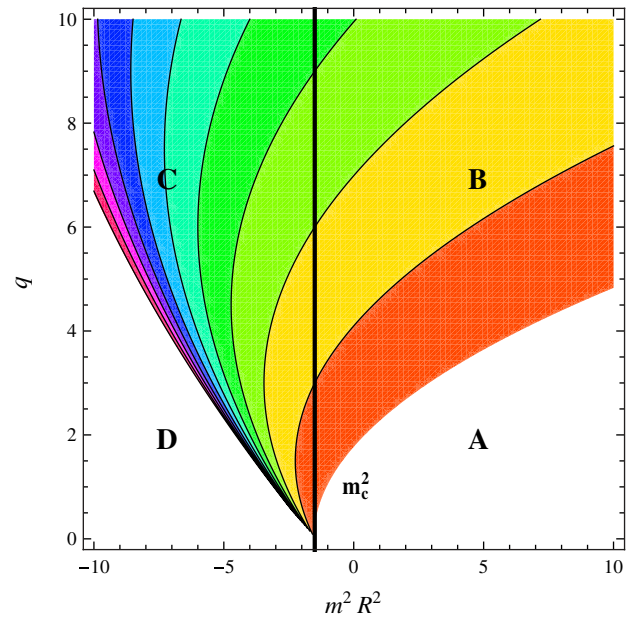


FIG. 8 (color online). Contour plot for the critical magnetic field in the m^2, q plane. The critical field H_c increases as we move to the left. In region A the IR dimension, even with $H = 0$, is real and there is no condensate. In regions B and C there exists a finite magnetic field H_c above which the condensate is destroyed. However, H_c diverges as we move to the left and is infinite on the boundary between regions C and D. In region D even an infinitely strong magnetic field will not stop the condensation. This is understandable: everywhere to the left of the line $m^2 = m_c^2$ the scalar has a sufficiently negative mass that it would have condensed even if it was uncharged and so it is perhaps not unexpected that a magnetic field cannot halt this condensation. In region C again the scalar would have condensed even if it was uncharged, but here the charge is high enough that a sufficiently strong magnetic field can stop the condensation.

where $b \equiv \frac{H}{\mu_B^2}$ is the dimensionless ratio between the boundary magnetic field H and chemical potential.¹⁷ Again the quantum phase transition will happen when δ becomes complex. It is clear from here that there is a different critical mass m_c^2 than in the neutral case; alternatively, one can now imagine tuning the magnetic field through a critical value H_c , which is again found by setting the expression inside the square root of (50) to zero. For $H > H_c$ there will be no condensate, and for H slightly less than H_c one will find similar nonanalytic behavior as above as a function of the deviation of H from H_c . The explicit expression for H_c is rather complicated and is given in (A28).

We plot the critical values H_c in the $q - m^2$ plane in Fig. 8. Note, in particular, that the expression in (50)

¹⁷ μ_B is the dimensional version of the dimensionless chemical potential μ we were using earlier; see Appendix A 1 for further explanation.

containing b saturates at a finite value $\frac{|q|}{2\sqrt{3}}$ as $b \rightarrow \infty$. Thus if

$$m^2 R^2 + \sqrt{3}|q| < -\frac{3}{2} \quad (51)$$

no matter how large the magnetic field is, a condensate cannot be prevented. This is region D in Fig. 8. This is understandable: everywhere in this region the scalar mass is below the neutral AdS₂ BF bound $m^2 R^2 = -\frac{3}{2}$ and so it would have condensed even if it was uncharged. As the electric field is in some sense not responsible for the instability, it may make sense that a magnetic field cannot halt it. It will be interesting to find field theoretical models with this feature.

V. ANTIFERROMAGNETISM AND SPIN WAVES

We have understood in detail the mechanism by which a neutral scalar field can condense in a finite density geometry. We would now like to employ this new understanding to holographically model symmetry breaking with such a neutral order parameter. One immediate example is antiferromagnetic order: as explained in the Introduction, in the low energy limit of interest to us spin rotations may be thought of as a global $SU(2)$ symmetry. Antiferromagnetic ordering then corresponds to a spontaneous breaking $SU(2) \rightarrow U(1)$, where the unbroken direction corresponds to rotations about a single axis. In such a system the background value of the spin density remains zero, but there is a neutral order parameter Φ^a that corresponds to the staggered magnetism.

In this section we construct the bulk dual of the operator Φ^a : we embed the neutral scalar field χ discussed in Sec. II, III, and IV into part of an $SU(2)$ triplet scalar field ϕ^a charged under the $SU(2)$ bulk gauge field corresponding to the global spin $SU(2)$ symmetry of a boundary theory. The phase transition discussed earlier then becomes a transition to an AFM phase.

Note that while we use the word antiferromagnetic there is no sense in which the microscopic degrees of freedom of our system consist of spins that are antialigned on a bipartite lattice. We use the term to describe only the symmetry breaking pattern described above, realized in a manner that manifestly requires a finite density for symmetry breaking. We do feel that if indeed gravity duals could be constructed top-down for such systems they would likely contain ingredients similar to those in our description.

A. Embedding of χ

More explicitly, we can consider the following action:

$$S = S_{\text{grav}} + S_{\text{matter}}, \quad (52)$$

with

$$S_{\text{grav}} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left(\mathcal{R} + \frac{6}{R^2} \right), \quad (53)$$

where R is the curvature radius of AdS₄. The relevant part of the bulk Lagrangian for matter fields is then given by

$$\begin{aligned} \frac{2\kappa^2}{R^2} \mathcal{L}_{\text{matter}} = & -\frac{1}{4g_A^2} F_{MN}^a F^{MNa} - G_{MN} G^{MN} \\ & - \frac{1}{\lambda} \left(\frac{1}{2} (D\phi^a)^2 - V(\phi^a) \right), \end{aligned} \quad (54)$$

with

$$\begin{aligned} F_{MN}^a &= \partial_M A_N^a - \partial_N A_M^a + \epsilon^{abc} A_M^b A_N^c, \\ G_{MN} &= \partial_M B_N - \partial_N B_M, \end{aligned} \quad (55)$$

and

$$D_M \phi^a = \partial_M \phi^a + \epsilon^{abc} A_M^b \phi^c. \quad (56)$$

We will take the potential V to have the double well form of (12)

$$V(\phi^a) = \frac{1}{2} m^2 \vec{\phi} \cdot \vec{\phi} + \frac{1}{4} (\vec{\phi} \cdot \vec{\phi})^2. \quad (57)$$

We again put the system at a finite density by turning on a chemical potential for the $U(1)_{\text{charge}}$, with a background metric given by (6). We note that in this background all $SU(2)$ gauge fields are inactive. This highlights an important difference between this setup and the usual $U(1)$ holographic superconductor; in the Abelian case there is a background chemical potential μ for the $U(1)$ charge; this does not break the $U(1)$ but does interact via the bulk equations of motion with the charged scalar order parameter, causing it to condense for a suitable choice of mass and charge. This is distinctly different from the $SU(2)$ case studied here; specifying a chemical potential for the $SU(2)$ would involve picking a direction μ^a in the $SU(2)$ space, corresponding to an explicit breaking of the symmetry. This is analogous to applying an external magnetic field. We study this explicit breaking in Sec. VI and for now focus on the spontaneous breaking of $SU(2)$.

When m^2 falls in the range (11), the background (6) becomes unstable toward the condensation of the scalar ϕ^a , which will then spontaneously break the $SU(2)$ to a $U(1)$ subgroup. More explicitly, consider the following ansatz:

$$\phi_0^a = \left(0, 0, \frac{\chi(r)}{R} \right), \quad A_M^a = 0, \quad (58)$$

which can be readily checked by examining the equations of motion as being self-consistent, if we ignore the back-reaction to the background geometry. We note, in particular, that one can consistently set the $SU(2)$ gauge field to 0. This is because of the non-Abelian nature of the interactions of the gauge field and scalar, e.g., $f^{abc} \phi_b A_c$ etc., which clearly vanish if all objects point only in one direction in the $SU(2)$ space.

Plugging (58) into the equations of motion following from (54) we precisely find (14) and our previous discus-

sion of the phase transition in Secs. II, III, and IV can be taken over completely. Note that if we consider a finite λ and turn on backreaction, the profile for the $U(1)$ gauge field in (6) and the background metric will be modified, but one can still set the $SU(2)$ gauge field to zero.

B. Spin waves

To characterize the ordered phase, we study in this section perturbations around the symmetry breaking solution (58) with a nonzero order parameter ϕ^a but zero background gauge field A_M^a . We show that the system has two linearly dispersing gapless modes, corresponding in the field theory to the two Goldstone modes arising from the spontaneous breaking of the global $SU(2)$ symmetry, and that their velocity is given by the expression expected from the standard field theory for a quantum antiferromagnet. From the bulk point of view, low-frequency rotations of the order parameter in the broken symmetry directions will source bulk gauge field fluctuations, which we will find to be normalizable at the AdS boundary if they obey a specific dispersion.¹⁸

Our analysis for the remainder of this section will not depend at all on the details of the metric or scalar profile (except that it is normalizable). We will work in a general boundary spacetime with a bulk black brane metric given by

$$ds^2 \equiv g_{MN} dx^M dx^N = g_{tt} dt^2 + g_{rr} dr^2 + g_{xx} d\vec{x}^2. \quad (59)$$

We work at finite temperature with g_{tt} (g_{rr}) having a first order zero (pole) at the horizon $r = r_0$.

Let us begin by understanding *why* from the gravity point of view there should exist a gapless mode. Consider a global $SU(2)$ gauge rotation of the background order parameter $\phi_0^a(r)$, at $\omega = 0$ and $k = 0$;

$$\delta\phi_i^a(r) = \epsilon\tau_{ab}^i \phi_0^b(r), \quad (60)$$

where τ^i , $i = 1, 2$ is a generator along one of the broken directions so that $\delta\phi^a$ is nonzero and ϵ is a constant. This small perturbation will obviously be a solution to the bulk gravity equations of motion; it is in fact a normalizable solution at the AdS boundary, precisely because ϕ_0^a is normalizable. This appears somewhat trivial—but note that this is exactly a gapless mode, as it is a normalizable solution that exists in the limit $\omega \rightarrow 0$. Now consider a *local* $SU(2)$ gauge rotation with the gauge parameter ϵ having a small frequency and momentum in the x direction, i.e., $\epsilon(t, x) \propto e^{-i\omega t + ikx}$. The perturbation to the scalar takes the same form (60), but now we have extra perturbations to the gauge fields

$$\delta A_t^i = -i\omega\epsilon, \quad \delta A_x^i = ik\epsilon. \quad (61)$$

¹⁸Methods similar to those used here may be used to obtain analytic control over the hydrodynamics of the $U(1)$ holographic superconductor [52].

These are not normalizable at infinity, and thus this pure gauge transform is no longer a normalizable solution. Thus, at any nonzero ω and k to find a normalizable mode we must move off of the pure gauge solution and actually solve the dynamical equations of motion. The existence of the global solution (60) guarantees that we as we take $\omega, k \rightarrow 0$ we will find a gapless solution, but the dynamical bulk equations of motion will show that it will be normalizable only when a certain dispersion relation $\omega(k)$ is satisfied.

1. Pions in the bulk

The previous discussion has convinced us that the existence of the gapless mode is intimately related to our ability to perform global rotations on the order-parameter, i.e., to bulk Goldstone modes. Let us thus parametrize fluctuations of ϕ in terms of bulk Goldstone fields (or “pions”) $\pi^i(r, t, x)$ as follows:

$$\phi(r, x) = \exp(i\pi^i(r, t, x)\tau^i)\phi_0(r), \quad (62)$$

where $\phi_0(r)$ is the background solution and as before τ^i are the broken symmetry generators. We now work out the bulk quadratic action of the π^i to be¹⁹

$$S(\pi) = -\frac{R^2}{2\kappa^2} \int d^4x \frac{1}{2\lambda} \sqrt{-g} g^{MN} (\partial_M \pi^i - A_M^i) \times (\partial_N \pi^j - A_N^j) h_{ij}. \quad (63)$$

Here h_{ij} can be viewed as a metric on the Goldstone boson space, given by

$$h_{ij}(r) = \frac{1}{2} \phi_0^\dagger(r) \{\tau_i, \tau_j\} \phi_0(r) = \frac{\chi^2}{R^2} \delta_{ij}. \quad (64)$$

Thus, the two Goldstone modes decouple. Below we will focus on one of them and drop the index i for notational simplicity.

The equation of motion for π is then

$$\partial_M (\chi^2 \sqrt{-g} g^{MN} (\partial_N \pi - A_N)) = 0. \quad (65)$$

It is clear that in the limit $\omega, k \rightarrow 0$, a constant pion profile $\pi(r) = \pi_0$ and vanishing gauge field $A = 0$ provide a solution to the equations of motion. This is the gauge transform discussed earlier.

Let us now briefly detour to discuss the asymptotic behavior of the pion field. Carrying out the standard analysis and using the fact that the background solution is normalizable²⁰: $\chi(r) \sim r^{-\Delta}$ for large r , we find that as $r \rightarrow \infty$;

¹⁹In writing this expression we have assumed that the background gauge field is zero; thus we have neglected intrinsically non-Abelian terms of the form $f^{abc} \pi_b A_c$ which all arise at higher order.

²⁰We restrict ourselves to the ordinary quantization for this section.

$$\pi(r) \approx B + Ar^{-d+2\Delta}, \quad r \rightarrow \infty. \quad (66)$$

To understand this it is useful to remember that fluctuations in the original field $\delta\phi(r)$ are related to π as $\delta\phi(r) = \pi(r)^i \tau^i \phi_0(r) \sim \pi(r)r^{-\Delta}$. Comparing this to the asymptotic behavior of the scalar (17) we see that the constant piece B of π at infinity is actually normalizable, while the piece A is not normalizable and should be viewed as the source for the pion field.²¹

2. Yang-Mills equations

Fluctuations of π will excite the non-Abelian gauge field $A_M^a(r, t, x)$, whose linearized equations are

$$\frac{1}{\tilde{g}_A^2} \nabla_M F^{MN} + \chi^2 g^{NP} (\partial_P \pi - A_P) = 0. \quad (67)$$

Again we have suppressed the index i which labels the broken generators, since at the linearized level different directions decouple. We have absorbed various factors into a rescaled gauge coupling

$$\frac{1}{\tilde{g}_A^2} = \frac{R^2 \lambda}{g_A^2}, \quad (68)$$

which controls the strength of the effects of the scalar sector on the gauge field. We could, of course, always choose a unitary gauge to get rid of the Goldstone boson π —but to make the connection to boundary Goldstone modes more obvious we will not do this and instead choose the gauge $A_r = 0$. It is convenient, as in [53], to work with the momenta J^μ conjugate to the bulk gauge field A_μ , defined as²²

$$J^\mu \equiv \frac{1}{\tilde{g}_A^2} \sqrt{-g} F^{\mu r} = -\frac{1}{\tilde{g}_A^2} \sqrt{-g} g^{rr} g^{\mu\nu} \partial_r A_\nu. \quad (69)$$

The value of the bulk field J^μ at the AdS boundary $r \rightarrow \infty$ equal to the expectation value of the current in the quantum field theory $\langle \mathcal{J}^\mu \rangle_{\text{QFT}}$.

The $N = r$ component of the Maxwell equations (67) can now be written

$$\partial_\mu J^\mu = -\sqrt{-g} g^{rr} \chi^2(r) \partial_r \pi. \quad (70)$$

If the symmetry is unbroken then $\chi = 0$ and this is nothing but the conservation of current. Let us now consider evaluating this expression at the AdS boundary. Comparing (70) to the asymptotic behavior of π in (66) we see that the entire right-hand side of this expression is proportional (with no extra factors of r) to the coefficient A in the near-boundary expansion of π , i.e., to the source for the

²¹It is amusing to note that computing the momentum conjugate to π precisely extracts out the source in this case, in exactly the same way that it extracts out the vacuum expectation value in the more familiar case of a standard massless scalar.

²²For convenience of discussion here we have chosen a different normalization for the currents below.

Goldstone boson field. This is simply the usual field theory Ward identity, which says that in the absence of a Goldstone source the field theory current is conserved.

The other nontrivial equations from (67) are those in t and x directions with nonzero $A_x(r, t, x)$ and $A_t(r, t, x)$ only, given by

$$-\partial_r J^t + \frac{1}{\tilde{g}_A^2} \nabla_x F^{xt} + \sqrt{-g} g^{tt} \chi^2 (\partial_t \pi - A_t) = 0, \quad (71)$$

$$-\partial_r J^x + \frac{1}{\tilde{g}_A^2} \nabla_t F^{tx} + \sqrt{-g} g^{xx} \chi^2 (\partial_x \pi - A_x) = 0. \quad (72)$$

Equations (65) and (70)–(72) are the full set of equations for this system.

3. Boundary Goldstone modes and their spin wave velocity

We will now solve the above equations in the hydrodynamic limit to show explicitly the existence of a Goldstone mode, i.e., a normalizable solution to (65) and (70)–(72) that is infalling (or regular) at the horizon and exists in the limit of small ω or k . We work in Fourier space

$$\pi = \pi(r) e^{-i\omega t + ikx}, \quad A_{t,x} = A_{t,x}(r) e^{-i\omega t + ikx}. \quad (73)$$

Recall that at precisely 0 frequency and momentum we already know a solution to these equations: a constant pion profile $\pi(r) = \pi_0$ and $A = 0$. We now expand around this solution in powers of ω and k . To lowest order we simply solve the *forced* equations for A given by (71) and (72) with the forcing term provided by the zeroth-order solution for π , meaning that

$$A_t(r) \sim \mathcal{O}(\omega \pi_0), \quad A_x(r) \sim \mathcal{O}(\pi_0 k). \quad (74)$$

The first correction to $\pi(r)$ enters through (65) and involves one extra field theory derivative, and thus we see that

$$\pi(r) = \pi_0 + \pi_1(r), \quad \pi_1(r) \sim \mathcal{O}(\pi_0 \omega^2, \pi_0 k^2). \quad (75)$$

The solution we want for A is both infalling at the black hole horizon and normalizable at the boundary; such a solution is nontrivial because of the forcing term²³ and is given by

$$A_t = -i\omega \pi_0 (1 - a_t(r)), \quad A_x = ik \pi_0 (1 - a_x(r)), \quad (76)$$

where a_t and a_x are defined to be infalling at the horizon and satisfy the homogenous part (and zero frequency) part of (71) and (72):

$$\frac{1}{\tilde{g}_A^2} \partial_r (\sqrt{-g} g^{rr} g^{tt} \partial_r a_t) - \sqrt{-g} g^{tt} \chi^2 a_t = 0, \quad (77)$$

²³Of course as the force π_0 vanishes the only solution that is infalling and normalizable becomes the trivial one $A = 0$.

$$\frac{1}{\tilde{g}_A^2} \partial_r (\sqrt{-g} g^{rr} g^{xx} \partial_r a_x) - \sqrt{-g} g^{xx} \chi^2 a_x = 0. \quad (78)$$

To ensure that A_t and A_x in (76) are normalizable we also require that

$$a_t(\infty) = 1, \quad a_x(\infty) = 1. \quad (79)$$

The corresponding equations for the pion profile are

$$\pi_1 = \omega^2 \pi_0 C^t(r) + k^2 \pi_0 C^x(r), \quad (80)$$

where the dynamical equations for C^t and C^x can be written from (65) as expressions for the radial independence of two objects α^t and α^x :

$$\partial_r \alpha^x = 0, \quad (81)$$

$$\alpha^x \equiv -\sqrt{-g} \chi^2 g^{rr} \partial_r C^x + \frac{1}{\tilde{g}_A^2} \sqrt{-g} g^{rr} g^{xx} \partial_r a_x,$$

$$\partial_r \alpha^t = 0, \quad (82)$$

$$\alpha^t \equiv \sqrt{-g} \chi^2 g^{rr} \partial_r C^t - \frac{1}{\tilde{g}_A^2} \sqrt{-g} g^{rr} g^{tt} \partial_r a_t.$$

The precise form of these equations turns out to not be important: the crucial fact is that $C^{t,x}$ are essentially driven by $a_{t,x}$, and so we can find solutions to them that are also infalling and normalizable—where normalizable in this case means that the term $\chi^2 \partial_r C^{x,t}$ vanishes at infinity [as explained in the discussion around (66)].

So far it appears that we have found a solution that is both infalling and normalizable, for all small frequency and momenta. This should not be possible, and indeed we have yet to impose the constraint (70). We find then the relation

$$\omega^2 = v_s^2 k^2, \quad v_s^2 = \frac{\alpha_x}{\alpha_t}. \quad (83)$$

This is consistent only because of the radial independence of α_x and α_t , which away from the boundary involves fluctuations of the pion field. Thus, we have shown that there exists a normalizable infalling mode provided that ω , k obey a linear relation. This is our main result.

It is convenient to evaluate (81) and (82) at $r = \infty$, where the terms depending on $C^{t,x}$ do not contribute, which leads to

$$\alpha_x = \frac{1}{\tilde{g}_A^2} \lim_{r \rightarrow \infty} \sqrt{-g} g^{rr} g^{xx} \partial_r a_x, \quad (84)$$

$$\alpha_t = -\frac{1}{\tilde{g}_A^2} \lim_{r \rightarrow \infty} \sqrt{-g} g^{rr} g^{tt} \partial_r a_t. \quad (85)$$

Note that this is equivalent to the statement that at the boundary the right-hand side of (70) vanishes—in the absence of a Goldstone boson source the current is conserved. Recall that a_t , a_x obey the zero frequency and unitary gauge infalling wave equations. It was shown in

[53] that on such a field configuration at infinity the ratio of the current $J_{x,t} \sim \partial_r a_{x,t}$ to the field $a_{x,t}$ [which are unity in this case from (79)] itself is simply the field theory Green's function for the current. We thus conclude that

$$\alpha_x = \lambda R^2 G_{xx}(\omega = 0, k = 0), \quad (86)$$

$$\alpha_t = -\lambda R^2 G_{tt}(\omega = 0, k = 0), \quad (87)$$

where

$$G_{\mu\nu}(\omega, k) = \langle j_\mu(\omega, k) j_\nu(-\omega, k) \rangle_{\text{retarded}}, \quad (88)$$

are momentum space retarded Green function of the spin current j_μ along a symmetry-broken direction. We thus find that the spin velocity v_s can be written as

$$v_s = \left(\frac{G_{xx}(\omega = 0, k = 0)}{-G_{tt}(\omega = 0, k = 0)} \right)^{1/2}. \quad (89)$$

This is the expected expression for an antiferromagnet. In particular, recognizing that [54–57]

$$\rho_s = G_{xx}(\omega = 0, k = 0), \quad (90)$$

$$\chi_\perp = -G_{tt}(\omega = 0, k = 0),$$

where ρ_s is the spin stiffness and χ_\perp is the transverse magnetic susceptibility, we recover the standard expression

$$v_s = \sqrt{\frac{\rho_s}{\chi_\perp}}. \quad (91)$$

Note that Eq. (76) has a simple boundary theory interpretation: $j_\mu \propto \partial_\mu \pi_0$, as expected for the superfluid part of the current density. In our probe analysis, there is no normal component. Also note that our analysis above gives a nice correspondence between the Higgs mechanism in the bulk and the dynamics of Goldstone modes in the boundary theory.

C. Evaluation of spin wave velocity

We now evaluate G_{xx} and G_{tt} to find the spin velocity. It is convenient to derive a flow equation for them as in [53]. We start with a_x ; examining the near-horizon behavior of Eq. (77) we see that for the solution to be nonsingular we need

$$\frac{\partial_r a_x}{a_x} = \frac{\tilde{g}_A^2 \chi^2}{4\pi T}, \quad (92)$$

where T is the temperature. Introducing

$$\mathcal{G}_x(r) = \frac{1}{\tilde{g}_A^2} \frac{\sqrt{-g} g^{rr} g^{xx} \partial_r a_x}{a_x} \quad (93)$$

from (78) we find \mathcal{G}_x satisfies the flow equation

$$\partial_r \mathcal{G}_x(r) = \sqrt{-g} g^{xx} \chi^2 - \frac{\tilde{g}_A^2 \mathcal{G}_x^2(r)}{\sqrt{-g} g^{rr} g^{xx}}. \quad (94)$$

This equation should be integrated from the horizon [where from (92) we have $\mathcal{G}_x(r_h) = 0$] to the AdS boundary, where it is equal to the Green's function $G_{xx}(\omega = 0, k = 0)$.

Similarly to above, for a_t we introduce

$$\mathcal{H}_t(r) = \frac{\tilde{g}_A^2 a_t}{\sqrt{-g} g^{rr} g^{tt} \partial_r a_t}. \quad (95)$$

This is convenient, as at the horizon $a_t = 0$. The corresponding flow equation for \mathcal{H}_t is

$$\partial_r \mathcal{H}_t(r) = \frac{\tilde{g}_A^2}{\sqrt{-g} g^{rr} g^{tt}} - \mathcal{H}_t^2(r) \sqrt{-g} g^{tt} \chi^2, \quad (96)$$

and it should be integrated from the horizon, where the relevant boundary condition is $\mathcal{H}_t(r_h) = 0$, to infinity. In terms of (93) and (95) the spin wave velocity (89) can be written as

$$v_s = (-\mathcal{G}_x(\infty) \mathcal{H}_t(\infty))^{(1/2)}. \quad (97)$$

Evaluation of this requires knowledge of the scalar profile and can only be done numerically; some representative plots are shown in Fig. 9.

It is interesting to see what happens near the phase transition, where we have some analytic control; here we have $\chi^2 \sim (T - T_c)$ from earlier results. Now, examining the flow equations we see that $\mathcal{H}_t(\infty)$ remains finite in this limit, and is in fact given [up to corrections analytic in $(T - T_c)$] by

$$-\mathcal{H}_t = - \int_{r_0}^{\infty} \frac{\tilde{g}_A^2}{\sqrt{-g} g^{rr} g^{tt}} = \frac{1}{\Xi}, \quad (98)$$

where Ξ is the spin susceptibility in the unbroken phase. The second equality in (98) follows from the $U(1)$ analysis

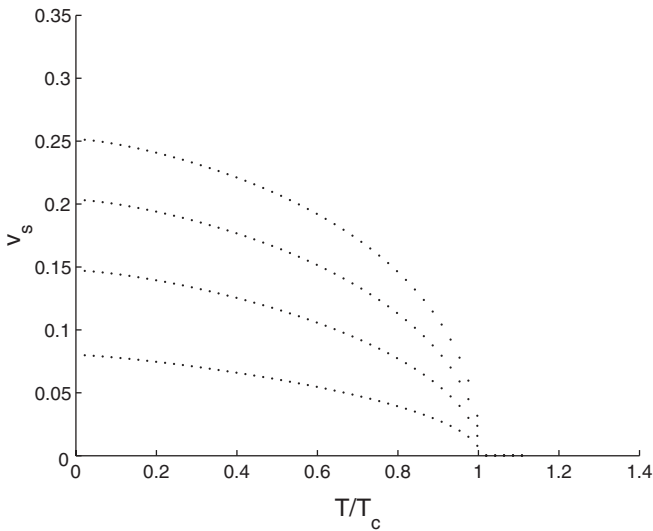


FIG. 9. Spin wave velocity as a function of T/T_c for various values of the rescaled gauge coupling \tilde{g}_A : \tilde{g}_A is varied from 1 (lowest curve) in unit increments to 4 (highest curve).

in Appendix D of [53]. On the other hand for \mathcal{G}_x we find from (94) that near the transition $\mathcal{G}_x \sim (T - T_c)$. Thus the nonlinear term in the evolution can be neglected, leading to

$$\mathcal{G}_x(\infty) = \int_{r_0}^{\infty} dr \sqrt{-g} g^{xx} \chi^2. \quad (99)$$

We find then near T_c

$$v_s^2 = \frac{1}{\Xi} \int_{r_0}^{\infty} dr \sqrt{-g} g^{xx} \chi^2 \propto (T - T_c). \quad (100)$$

Note that above T_c , $\rho_s = G_{xx}(\omega = 0, k = 0)$ becomes identically zero. It would be interesting to extend this analysis to understand what happens to the spin wave velocity at zero temperature near the quantum critical transition.

VI. EXTERNAL “MAGNETIC FIELDS” AND FORCED FERROMAGNETIC MAGNONS

We have constructed a gravity description of the spontaneous breaking of an $SU(2)$ symmetry—analogueous to the Neel phase in a spin system—and displayed the associated spin waves. In this section we turn briefly to a ferromagnet. A ferromagnet can be understood as a system undergoing spontaneous symmetry breaking in which the broken vacuum is charged under the unbroken generator. The spin waves in this case possess a quadratic dispersion $\omega \sim k^2$.

This setup is not straightforward to realize in holography—the spontaneous generation of a nonzero spin density means that essentially one now wants the unbroken non-Abelian gauge field itself to condense, without supplying any external chemical potentials. We leave this for future work.

In this section we consider something simpler—imagine applying an external magnetic field H to a sample containing spins, forcing them to align along the direction of the field. In our setup this corresponds to picking a direction in the $SU(2)$ space—we will pick the 3 direction—and supplying a chemical potential $\mu \sim H$ for the gauge field in that direction [in this section only μ refers to the chemical potential for the non-Abelian gauge field; we will set the $U(1)$ chemical potential to 0 here, as it does not play an essential role]. It is interesting to note that this model has been studied before: it is precisely the “normal phase” of the holographic p -wave superconductor studied by [58], but our interpretation is different: in particular, in those models a $U(1)$ subgroup of the $SU(2)$ is usually taken to be an electric charge, which is quite different from our approach. We will show that small fluctuations around such a configuration exhibit spin waves whose dispersion is not gapless, but whose dependence on momentum is indeed quadratic.

A. Gravity setup

We consider a generic spacetime metric (59) and a nontrivial non-Abelian gauge field profile

$$A_i^3(r) = \mu\alpha(r), \quad \alpha(r \rightarrow \infty) = 1, \quad \alpha(r_0) = 0, \quad (101)$$

with all other matter fields turned off. In particular the scalar ϕ^a is uncondensed and will play no role. We will also set the gauge coupling g_A to 1 as when the scalar is inactive g_A plays no role in the classical dynamics that follows. The profile $\alpha(r)$ solves the equation

$$\frac{1}{\sqrt{-g}} \partial_r (\sqrt{-g} g^{rr} g^{tt} \partial_r \alpha(r)) = 0, \quad (102)$$

$$\Rightarrow J^{3t} \equiv -\sqrt{-g} g^{rr} g^{tt} \partial_r A_t^3 = \text{const}, \quad (103)$$

where as above J^{3t} is the canonical momentum conjugate to A_t^3 and gives the expectation value of the spin density j^{3t} in the boundary theory as $\langle j^{3t} \rangle = J^{3t}(r = \infty)$. For simplicity, we will ignore the backreaction to the background metric, requiring that both μ and J^{3t} are small when expressed in units of the temperature. Note that μ and J^{3t} are related by the spin susceptibility Ξ

$$\langle j^{3t} \rangle = \mu \Xi, \quad (104)$$

with Ξ given by (98).

We will now slightly perturb this solution. We do not expect to find a gapless mode, as we have an ‘‘external magnetic field;’’ however at $k = 0$ we do expect to find a normalizable mode with $\omega = \mu$. To see the existence of this mode consider performing a gauge transform on the background (101) with infinitesimal gauge parameter Λ^a :

$$\delta A_M^a = \epsilon^{abc} \Lambda^c A_M^b + \partial_M \Lambda^a. \quad (105)$$

It is convenient to work with the linear combinations

$$A^\pm = A^1 \pm iA^2, \quad (106)$$

as these are charge eigenstates under the rotations in the 3 direction for which we have supplied a chemical potential. If we assume a time dependence of the form $e^{-i\omega t}$ for Λ^+ we find the resulting A_t^+ can be written as

$$A_t^+ = \Lambda(A_t^3 - \omega)e^{-i\omega t}, \quad (107)$$

where Λ is an overall (complex) constant. This perturbation is generically not normalizable at the AdS boundary; however if we demand that A_t^+ vanish at $r \rightarrow \infty$ then this fixes the frequency and we find

$$\omega = \mu, \quad (108)$$

$$A_t^+(t, r) = \Lambda\mu(\alpha(r) - 1)e^{-i\mu t} \equiv \Lambda A_{t0}(r)e^{-i\mu t}.$$

Thus, there is a normalizable mode at $\omega = \mu$ with the specified radial profile $A_{t0}(r) = \mu(\alpha(r) - 1)$.²⁴ We will now turn on a small k to see how this mode evolves; as expected for a system with a background spin density, we

²⁴The existence of such a mode was observed before in [58] in the context of conductivity of a p -wave superconductor.

will find that the frequency has a quadratic dependence on k .

B. Dispersion relation for the ferromagnetic magnon

To do this, we will need the full bulk Yang-Mills equations:

$$\frac{1}{\sqrt{-g}} \partial_M (\sqrt{-g} F^{aMN}) + \epsilon^{abc} A_M^b F^{cMN} = 0. \quad (109)$$

The presence of the background gauge potential A_t^3 can be conveniently taken into account by defining a gauge-covariant partial time derivative d_t that acts in the \pm basis as

$$d_t A_{t,x}^+ = (\partial_t + iA_t^3) A_{t,x}^+. \quad (110)$$

We give all fields a spacetime dependence $e^{-i\mu t} e^{-i\Omega t + ikx}$. Here $\Omega \equiv \omega - \mu$ will parametrize deviation from the $\omega = \mu$ solution found above. The relevant equations of motion are those in (109) for $N = (r, t, x)$. We will work again with the bulk canonical momenta, defined as before:

$$J^{+t} \equiv -\sqrt{-g} F^{+rt} = -\sqrt{-g} g^{tt} g^{rr} \partial_r A_t^+, \quad (111)$$

$$J^{+x} \equiv -\sqrt{-g} F^{+rx} = -\sqrt{-g} g^{xx} g^{rr} \partial_r A_x^+. \quad (112)$$

We first examine the $N = r$ component of (109), which can be written as

$$d_t J^{+t} + \partial_x J^{+x} = iJ^{3t} A_t^+. \quad (113)$$

This equation is a bulk constraint that reduces on the boundary to the non-Abelian conservation of current. The remaining dynamical equations are

$$\frac{1}{\sqrt{-g}} \partial_r J^{+t} + \partial_x F^{+xt} = 0, \quad (114)$$

$$\frac{1}{\sqrt{-g}} \partial_r J^{+x} + d_t F^{+tx} = 0. \quad (115)$$

Only one of the components of the field strength tensor is affected by the gauge field background:

$$F_{xt}^+ = \partial_x A_t^+ - d_t A_x^+ = ikA_t^+ - (i\mu(\alpha - 1) - i\Omega)A_x^+. \quad (116)$$

We now want to perturb around the $\omega = \mu$ solution found above. We thus turn on a small k dependence in (108) and consider a perturbation of the form

$$A_t^+ = e^{-i\mu t} e^{-i\Omega t + ikx} (A_{t0}(r) + A_{t1}(r; \omega, k) + \dots), \quad (117)$$

where $A_{t0}(r) \sim (\alpha(r) - 1)$ is the previously found profile in (108). Recall that $\Omega \equiv \omega - \mu$ parametrizes departure from μ and will be the small parameter in the expansions that follow. A nonzero A_t^+ will also excite an A_x^+ which we take to have the form

$$A_x^+ = A_x^+(r; \Omega, k) e^{-i\mu t} e^{-i\Omega t + ikx}. \quad (118)$$

The other components of the gauge fields can be consistently set to zero except for $A_{t,x}^-$, which are related to $A_{t,x}^+$ by complex conjugation.

To obtain the magnon dispersion relation we will expand the above equations to lowest order in the Ω and k expansion and look for solutions that are both infalling (or regular) at the horizon and normalizable at the AdS boundary. These equations will have the same structure as in the antiferromagnetic case above; we will find second-order radial equations for A_x^+ and the correction A_{t1} that are forced by the known solution A_{t0} .

More explicitly, plugging (117) and (118) into (114) and (115), and expanding them in powers of Ω and k , we find that

$$A_x^+(r) \sim O(k), \quad A_{t0} \sim O(k^2). \quad (119)$$

We thus introduce the following:

$$A_x^+(r) = k \left(1 - \frac{a_x(r)}{a_x(\infty)} \right), \quad A_{t1}(r) = k^2 a_{t1}(r). \quad (120)$$

Here by construction A_x^+ is normalizable, and from (115) we find that $a_x(r)$ satisfies the homogenous equation

$$\frac{1}{\sqrt{-g}} \partial_r (\sqrt{-g} g^{rr} g^{xx} \partial_r a_x) - \mu^2 (\alpha - 1)^2 g^{tt} g^{xx} a_x = 0. \quad (121)$$

There is a forced radial equation for the correction to the A_{t1} profile $a_{t1}(r)$, but we will not need to solve it explicitly to find the dispersion, analogous to the antiferromagnetic case where the correction to the pion profile was not explicitly needed;

$$\frac{1}{\sqrt{-g}} \partial_r (\sqrt{-g} g^{rr} g^{tt} \partial_r a_{t1}) = \mu (\alpha - 1) g^{tt} g^{xx} a_x. \quad (122)$$

It is of course critical to note that an infalling and normalizable solution to this equation can always be found.

We now impose the constraint arising from the conservation of current (113). Again it is simplest to evaluate it at the AdS boundary first; here the right-hand side of (113) vanishes [as by construction we are looking at a normalizable solution with $A_t^+(\infty) = 0$]. We find the dispersion relation

$$\Omega = \omega - \mu = \left(\frac{1}{J^{3t}} \lim_{r \rightarrow \infty} \frac{\sqrt{-g} g^{rr} g^{xx} \partial_r a_x}{a_x(\infty)} \right) k^2. \quad (123)$$

This is the desired dispersion relation.

Let us briefly understand the physical origin of the differences from the linear and gapless dispersion found in the antiferromagnetic case. The dispersion is not linear because the background value of J^{3t} is finite in the $\omega \rightarrow 0$ limit, unlike in the antiferromagnetic case where it is proportional to ω : this means that we are now balancing a term of order ω against one of order k^2 , rather than a term of order ω^2 . The mode is not gapless because at infinity the

gauge-covariant derivative $d_t = \partial_t + iA_t^3 \rightarrow -i\omega + i\mu$, resulting in a shift in ω . These considerations lead us to expect that if one were able to create a situation with a nonzero J^{3t} in the absence of a background chemical potential—i.e., a true spontaneous ferromagnet—one would find precisely the gapless quadratic dispersion of the standard ferromagnetic magnon.

Note that we can interpret the expression (123) in terms of field theory quantities. Again using expressions from [53], we can rewrite the ratio of $\partial_r a_x$ to a_x in terms of a field theory correlator to find

$$\Omega = \left(\frac{G^R(\omega = \mu, k = 0)}{J_3^t} \right) k^2, \quad (124)$$

with

$$G^R(\omega, k) = \langle j_x^+(\omega, k) j_x^-(-\omega, k) \rangle_{\text{retarded}}, \quad (125)$$

i.e., G_{xx}^R is the field theory retarded correlator for j_x^+ evaluated at the nonzero frequency $\omega = \mu$. The prefactor of the quadratic dispersion is consistent with the expected result for a ferromagnet [59].

We now evaluate the constraint (113) at arbitrary r . Now the right-hand side no longer vanishes, and we find

$$\Omega = \omega - \mu = \gamma k^2, \quad (126)$$

where

$$\gamma = -\frac{1}{J^{3t}} \left(\mu (\alpha - 1) \sqrt{-g} g^{rr} g^{tt} \partial_r a_{t1}(r) - \frac{\sqrt{-g} g^{rr} g^{xx} \partial_r a_x(r)}{a_x(\infty)} \right) + a_{t1}(r). \quad (127)$$

It can readily be checked from (121) and (122) that γ is independent of r , and evaluated at $r \rightarrow \infty$ we find that γ reduces to the expression in (123). Again, even though we did not need to explicitly solve for $a_{t1}(r)$ to find the dispersion, its fluctuations are essential to make sure that the constraint from non-Abelian current conservation is upheld at all points in the bulk.

C. Evaluation of dispersion

We now turn to the evaluation of the dispersion γ . This bears a large formal similarity to the evaluation of conductivities studied in [53]. We consider the following “transport coefficient” defined at all values of r :

$$\sigma(r; \mu) = -\frac{\sqrt{-g} g^{rr} g^{xx} \partial_r a_x(r)}{i\mu a_x(r)}. \quad (128)$$

The motivation behind this name will soon be made clear. Q is simply related to the boundary value of this object:

$$\gamma = -\frac{i\sigma(r \rightarrow \infty; \mu)}{\Xi}, \quad (129)$$

where we have used (104). From (121), we find that σ obeys a simple radial evolution equation,

$$\partial_r \sigma = i\mu \sqrt{\frac{g_{rr}}{-g_{tt}}} \left(\frac{\sigma^2}{\Sigma} - \Sigma(\alpha - 1)^2 \right), \quad (130)$$

with

$$\Sigma(r) \equiv \sqrt{\frac{-g}{-g_{rr}g_{tt}}} g^{xx}(r), \quad (131)$$

and at the horizon the value of σ is fixed by the infalling boundary condition to be $\sigma(r_h; \mu) = \Sigma(r_h)$ (this is explained in detail in [53]).

Before considering general μ , let us consider the limit $\mu \rightarrow 0$, in which case $\partial_r \sigma = 0$, σ becomes constant in r , and is in fact equal to the normal (Abelian) DC conductivity σ_{DC} of a $U(1)$ current on this background. Thus we find for the ‘‘magnon’’ dispersion (123)

$$\omega = -ik^2 \frac{\sigma_{\text{DC}}}{\Xi}. \quad (132)$$

This relation is not surprising; when taking μ to 0 we are removing all non-Abelian effects and returning to the zero density system, and (132) simply describes the diffusion in the longitudinal channel of a $U(1)$ current. The diffusion constant is seen to be $\frac{\sigma_{\text{DC}}}{\Xi}$, as is required by the Einstein relation.

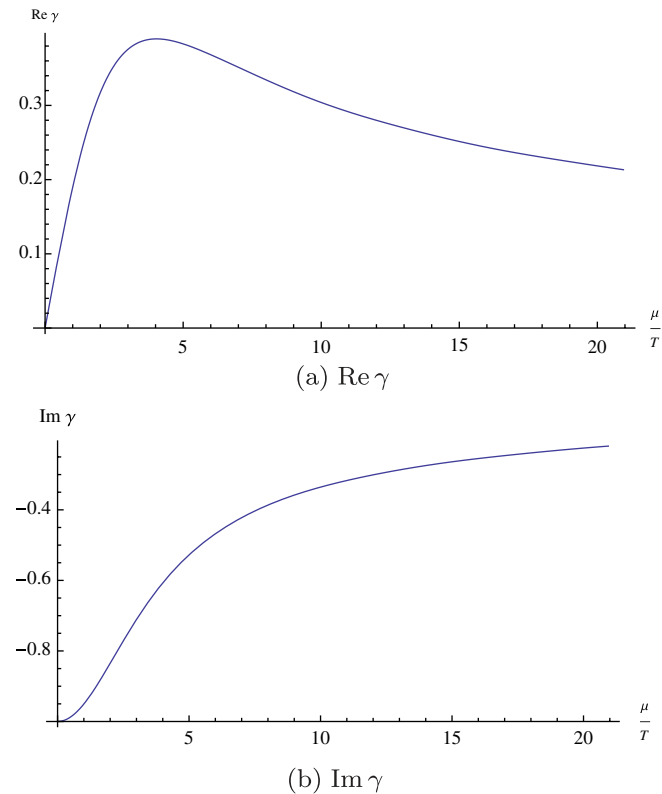


FIG. 10 (color online). The real and imaginary parts of γ are shown as a function of μ/T . Note as $\mu \rightarrow 0$, $\gamma \rightarrow -i$ as required by the Einstein relation.

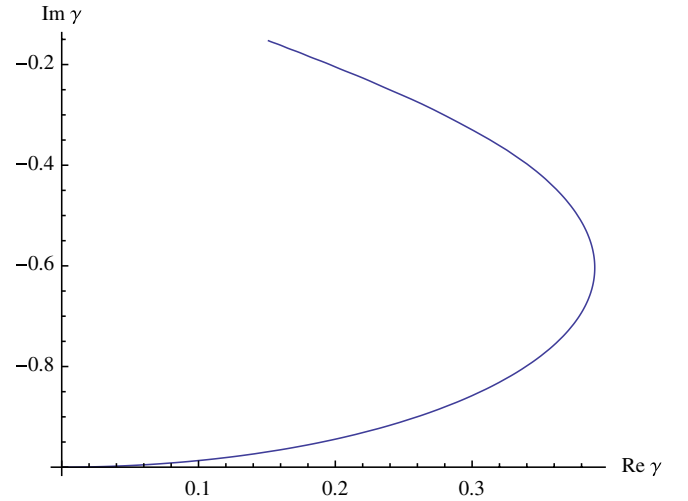


FIG. 11 (color online). Movement of γ in the complex plane as μ/T is varied from 0 to 40.

For generic μ we must evaluate this expression numerically. In this section alone for simplicity we work with the normal AdS-Schwarzschild metric in coordinates with η set to 0. We do not require the Reissner-Nordstrom metric here; even in the absence of a net $U(1)$ charge, one can imagine polarizing thermally excited spins with an external magnetic field. We do not expect inclusion of a $U(1)$ charge to qualitatively change the results. We work in units where both σ_{DC} and Ξ have been set to 1. The results of the numerical evaluation are shown in Figs. 10 and 11 as a function of μ/T .

Perhaps the most obvious feature of these diagrams is that γ has a large imaginary part, corresponding to a strong dissipation. This is due to the fact that we are evaluating our bulk equations as a perturbation about a nonzero frequency $\omega = \mu$; these waves are infalling at the black horizon and result in a large dissipation. We expect that if we were able to construct a true spontaneous ferromagnet with $\mu = 0$ then our bulk equations would be evaluated at $\omega = 0$ and γ would then be both real and gapless.

As mentioned earlier, this model is the normal phase of the holographic p -wave superconductor of [58]. According to their discussion, at sufficiently large $\frac{\mu}{T}$ (roughly $\frac{\mu}{T} \sim 15.9$ in our units) this system becomes unstable towards a condensate of the vector fields $A_{x,y}^{1,2}$. It would be interesting to understand further such a condensate—in our language, a persistent spin current—from the magnetic point of view taken in this paper.

VII. DISCUSSION AND CONCLUSIONS

In this paper we studied holographic phase transitions associated with condensation of a neutral order parameter, both at finite temperature and associated quantum critical point. We also considered the embedding of the neutral order parameter into the staggered magnetization for an

antiferromagnetic phase. We show that at the macroscopic level one recovers the expected features of the antiferromagnetic phases including existence of two gapless spin waves and their dispersion relations.²⁵ A similar discussion in a forced ferromagnetic phase reveals spin waves with quadratic dispersion relations which again agrees with field theoretical expectations.

Our discussions left some loose ends which should be studied further. The first is the phase diagram for the system in the alternative quantization, for which we only gave a qualitative picture. The phase boundary for the low temperature stable condensed phase should be mapped out more precisely. In particular it should be better understood what happens to the system at $m^2 R^2 = -\frac{27}{16}$. It seems possible that the various scaling exponents may not obey mean field behavior there. As mentioned earlier while the appearance of a special value $m^2 R^2 = -\frac{27}{16}$ has to do with the nonlinear structure of the potential, we expect this value and the qualitative behavior we find may not depend on the details of the potential as far as the next order nonlinear term is given by ϕ^4 (i.e., dictated by a Z_2 symmetry). In particular, we expect similar phenomena should also appear in the context of holographic superconductors. Another important question is to work out the solution for backreacted spacetime geometry. While we have shown that deep in the IR one should find an $\text{AdS}_2 \times \mathbb{R}^2$ with a different cosmological constant, working out the backreacted metric for all spacetime may reveal new qualitative features. Also, it would be interesting to extend our analysis for both antiferromagnetic and ferromagnetic spin waves to include their nonlinear interactions or their backreaction on the geometry [60].

The investigation of this paper suggests various other interesting directions to explore. We name a few here:

A. The nature of the quantum critical point

It would be very interesting to better understand the nature of the quantum critical point and quantum phase transitions discussed in this paper. On the gravity side, along with the example pointed out in [37], they all have to do with violation of the BF bound in the AdS_2 region, whose critical behavior thus falls into the universality class discussed in [36], where a critical line [in the sense of a $(0 + 1)$ -dimensional CFT] ends at critical point. To the left of the critical point an IR scale is generated, exhibiting the BKT scaling behavior. Below this IR scale, the system flows to different IR fixed points depending on the charge q of the condensate. For a neutral order parameter the system flows to a fixed point with $z = \infty$, while to a $(2 + 1)$ -dimensional CFT when q is sufficiently large. That such distinct physical phenomena share similar IR behavior is striking, yet viewed from a $((2 + 1)$ -dimensional perspec-

tive, they involve completely different phenomena involving complete distinct fixed points. We have not yet studied in detail the critical behavior from uncondensed side and spatial as well as temporal correlations near these critical points, which should be accessible from our gravity treatment.

Despite our gravity description, a field theoretical understanding of these critical points remains completely obscure. Can these transitions be understood as lying in the Landau paradigm? What do they suggest regarding the underlying mechanism for the condensation of charged and neutral order parameters in theories with gravity duals? Can one find explicit field theoretical models with similar features? We hope to return to these questions in the future.

B. Competition between different orders

From a more phenomenological point of view, more fields can be added to model more physics: e.g., one natural addition is the inclusion of a superconducting order parameter corresponding to a bulk field ψ charged under $U(1)_{\text{charge}}$. One could also turn on a potential $V(\phi^a, \psi)$ to model the interaction between an AFM and superconducting order parameter. For example, the simplest possibility one could consider is

$$V(\phi^a, \psi) = b \vec{\phi}^2 |\psi|^2. \quad (133)$$

Note that a negative value of b will lead to mutual enhancement of both types of instabilities, while a positive value of b will make them compete. We expect a rich phase structure to arise from such a construction, which may provide a strongly coupled example for many condensed matter problems in which such a competition exists [61].

A different observation is that in the new IR fixed point for the condensed phase for the neutral scalar field, since $\tilde{R}_2 < R_2$, the IR dimension (1) for a (different) charged scalar field appears to be smaller than that in the uncondensed phase. This appears to imply that the condensed phase or associated quantum critical point may enhance the superconducting instability, say making a previously stable mode unstable or enhancing the transition temperature to the superconducting order. A precise understanding of this will require the explicit construction of the backreacted geometry.

C. Coupling to fermions

Finally, our work raises a number of concrete questions regarding the properties of fermions on such backgrounds. It will be important to consider coupling a Fermi surface to our system [62]. In an antiferromagnetically ordered state, the nature of the Fermi surface provides yet another means to characterize the phase. In particular, the Fermi surface will reconstruct as a result of the antiferromagnetic order, in a way that is dependent on the size of the Fermi surface

²⁵As discussed earlier these features only depend on a symmetry breaking pattern and not details of the microscopic theory.

in the paramagnetic phase. For the system we have analyzed here, in which a continuum limit has already been taken, it will be instructive to see how to take into account such effects. Relatedly, the evolution of the Fermi surface across an antiferromagnetic quantum critical point has emerged as an important characterization of the nature of the quantum critical point, and it will be interesting to address this issue in our system as well.

Another issue related to fermions involves their low-energy spectral properties. Similarly to the scalar case described above, one expects that the IR scaling dimension δ of the fermion operator will be different before and after the condensation of the scalar χ . In the holographic formulation, δ controls the spectral behavior of the fermions; $\delta > 1$ is somewhat similar to a standard Fermi liquid, whereas $\delta < 1$ corresponds to non-Fermi liquid behavior [20]. Thus we find the tantalizing possibility that the condensation of the scalar and thus the existence of antiferromagnetic order could change the value of δ , perhaps driving the system between Fermi liquid and non-Fermi liquid regimes.

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APPENDIX A: EFFECT OF MAGNETIC FIELD ON IR CONFORMAL DIMENSION

In this Appendix we outline the derivation of (50) demonstrating the effect of a background magnetic field on the IR conformal dimension of a charged scalar field.

1. Dyonic black hole

We will need to consider the effect of the magnetic field on the geometry. Turning on such an external boundary magnetic field for the $U(1)$ current dual to the bulk gauge B_M in (5), the corresponding bulk geometry becomes that of a dyonic black hole with both electric and magnetic charges,

$$ds^2 \equiv g_{MN} dx^M dx^N = \frac{r^2}{R^2} (-f dt^2 + d\vec{x}^2) + \frac{R^2}{r^2} \frac{dr^2}{f}, \quad (\text{A1})$$

with

$$f = 1 + \frac{Q^2 + P^2}{r^4} - \frac{M}{r^3}, \quad (\text{A2})$$

$$B_t = \mu_B \left(1 - \frac{r_0}{r}\right), \quad B_x = -\frac{P}{R^4} y, \quad \mu_B \equiv \frac{Q}{R^2 r_0}. \quad (\text{A3})$$

r_0 is the horizon radius determined by the largest positive root of the redshift factor

$$f(r_0) = 0, \quad \rightarrow M = r_0^3 + \frac{Q^2 + P^2}{r_0}. \quad (\text{A4})$$

The geometry (A1) describes the boundary theory at a finite density with the charge density ρ , energy density ϵ , entropy density s , respectively, given by

$$\rho = 2 \frac{Q}{\kappa^2 R^2 g_F}, \quad \epsilon = \frac{M}{\kappa^2 R^4}, \quad s = \frac{2\pi}{\kappa^2} \left(\frac{r_0}{R}\right)^2. \quad (\text{A5})$$

The external magnetic field H and temperature T are

$$H = \frac{P}{R^4}, \quad T = \frac{3r_0}{4\pi R^2} \left(1 - \frac{Q^2 + P^2}{3r_0^4}\right). \quad (\text{A6})$$

μ_B in (A3) corresponds to the chemical potential of the boundary system.

It is convenient to work with dimensionless quantities by introducing

$$Q = \mu r_0^2, \quad P = h r_0^2, \quad (\text{A7})$$

and rescaling coordinates as

$$r \rightarrow r r_0, \quad (t, \vec{x}) \rightarrow \frac{R^2}{r_0} (t, \vec{x}), \quad (\text{A8})$$

after which Eq. (A1) becomes

$$\frac{ds^2}{R^2} = r^2 (-f dt^2 + d\vec{x}^2) + \frac{1}{r^2} \frac{dr^2}{f}, \quad (\text{A9})$$

with

$$f = 1 + \frac{3\eta}{r^4} - \frac{1 + 3\eta}{r^3}, \quad B_t = \mu \left(1 - \frac{1}{r}\right), \quad (\text{A10})$$

$$B_x = -h y,$$

and

$$3\eta \equiv \mu^2 + h^2. \quad (\text{A11})$$

Setting $h = 0$ above one recovers the metric (6) used in the main text.

We will be interested in the system at zero temperature, for which

$$Q^2 + P^2 = 3r_0^4 \quad \text{or} \quad \mu^2 + h^2 = 3, \quad (\text{A12})$$

and the near-horizon region becomes $\text{AdS}_2 \times \mathbb{R}^2$ with curvature radius

$$R_2 = \frac{R}{\sqrt{6}}. \quad (\text{A13})$$

Note that in the zero temperature limit, due to conformal invariance of the underlying vacuum theory the physically relevant quantity is the dimensionless ratio

$$b \equiv \frac{H}{\mu_B^2} = \frac{h}{\mu^2}. \quad (\text{A14})$$

Using the second relation in (A12) we can then express bulk quantities like h (and thus μ) in terms of b ,

$$h = \frac{\sqrt{1 + 12b^2} - 1}{2b}. \quad (\text{A15})$$

2. Scalar operator dimension in the IR

Now consider a scalar field in AdS₄ of charge q and mass m , with an action

$$S = - \int d^4x \sqrt{-g} [(D_M \phi)^* D^M \phi + m^2 \phi^* \phi], \quad (\text{A16})$$

where the gauge-covariant derivative satisfies

$$D_M \phi = (\partial_M - iqB_M) \phi. \quad (\text{A17})$$

Note that the action (A16) depends on q only through

$$\mu_q \equiv \mu q, \quad h_q \equiv h q, \quad (\text{A18})$$

which are the effective chemical potential and effective magnetic field for a field of charge q .

This problem is now similar to a Landau-level analysis from elementary quantum mechanics. After separation of variables using

$$\phi = e^{-i\omega t + ikx} Y(y) X(r), \quad (\text{A19})$$

we find that the equations of motion can be written as

$$\begin{aligned} -\frac{1}{\sqrt{-g}} \partial_r (\sqrt{-g} g^{rr} \partial_r X) + (-g^{ii} u^2 + m^2 + g^{ii} \lambda^2) X &= 0, \\ -\partial_y^2 Y + (v^2 - \lambda^2) Y &= 0, \end{aligned} \quad (\text{A20})$$

with

$$v(y) \equiv k + h_q y, \quad u(r) \equiv \sqrt{\frac{g_{ii}}{-g_{tt}}} \left(\omega + \mu_q \left(1 - \frac{1}{r} \right) \right). \quad (\text{A21})$$

One then finds that

$$Y_n(y) = e^{((- \xi^2)/2)} H_n(\xi), \quad \xi \equiv \sqrt{|h_q|} \left(y + \frac{k}{h_q} \right), \quad (\text{A22})$$

with H_n the usual Hermite polynomials. $X(r)$ is a radial

profile that satisfies the scalar wave equation with zero magnetic field $h = 0$, except that the momentum k on each constant- r slice has been discretized into Landau levels:

$$k^2 \rightarrow 2|h_q| \left(n + \frac{1}{2} \right), \quad n = 0, 1, \dots \quad (\text{A23})$$

A similar discussion can be applied to the AdS₂ region, where one finds that each Landau level has an effective mass given by

$$m_n^2 = m^2 + 2|h_q| \left(n + \frac{1}{2} \right) \frac{1}{R^2}. \quad (\text{A24})$$

The rest then follows exactly from the analysis in [20], and the IR dimension for ϕ is given by

$$\delta_n^{(B)} = \frac{1}{2} + \nu_n^{(B)}, \quad (\text{A25})$$

with

$$\nu_n^{(B)} = \sqrt{m_n^2 R^2 - \frac{\mu^2 q^2}{36} + \frac{1}{4}}. \quad (\text{A26})$$

Let us examine a bit more closely the $n = 0$ mode, which is the most likely to condense,

$$\nu_0^{(B)} = \sqrt{\frac{m^2 R^2}{6} + (6|bq| - q^2) \frac{\sqrt{1 + 12b^2} - 1}{72b^2} + \frac{1}{4}}, \quad (\text{A27})$$

where we used (A15) to express h and μ in terms of the dimensionless boundary quantity b (A14). Also note that $m^2 R^2 = \Delta(\Delta - 3)$. This is the result (50) in the main text. The critical magnetic field b_c can be found by setting the quantity inside the square root to 0, and is

$$b_c = |q| \frac{D[1 + \frac{1}{\sqrt{3}} \sqrt{q^2 - 2m^2 R^2}] - 2q^2}{D^2 - 12q^2}, \quad (\text{A28})$$

where

$$D \equiv (3 + 2m^2 R^2) \quad (\text{A29})$$

is a quantity that goes to 0 when the scalar mass is precisely at the neutral AdS₂ BF bound.

It is interesting that the expression in (A27) containing b saturates at a value $\frac{|q|}{2\sqrt{3}}$ as $b \rightarrow \infty$. Thus, if

$$m^2 R^2 + \sqrt{3}|q| < -\frac{3}{2} \quad (\text{A30})$$

no matter how large the magnetic field is, a condensate cannot be prevented. This is surprising and is discussed further in the main text.

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