

Quantum Probabilities and Violation of CHSH-Inequality from Classical Random Signals and Threshold Type Detection Scheme

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We present a purely wave model (based on classical random field) which reproduces quantum probabilities (given by the fundamental law of quantum mechanics, Born's rule) including probabilities for joint detection of a pair of quantum observables (e.g., spin or polarization projections). The crucial point of our approach is that the presence of detector's threshold and calibration procedure have to be treated not as simply experimental technicalities, but as the basic counterparts of the theoretical model. The presence of the background field (vacuum fluctuations) is also the key-element of our prequantum model. It is of the classical signal type and the methods of classical signal theory (including statistical radiophysics) are used for its development. We stress that our prequantum model is not objective, i.e., the values of observables (clicks of detectors) cannot be assigned in advance, i.e., before measurement. Hence, the dilemma, nonobjectivity or nonlocality, is resolved in favor of nonobjectivity (our model is local of the classical field type). In particular, we reproduce the probabilities for the EPR-experiment for photon polarization and, hence, violate CHSH inequality for classical random signals (measured by the threshold type and properly calibrated detectors acting in the presence of the background field).

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§1. Introduction

Recently the interest to representation of quantum mechanics (QM) as emergent theory essentially increased [see, e.g., 't Hooft,^{1)–3)} Ojima,^{4)–7)} and Else.^{8)–10)}] These studies support the original Einstein position that QM is incomplete theory.¹¹⁾ However, as is commonly accepted, violation of Bell's inequality¹²⁾ showed that if QM is incomplete, then subquantum reality is nonlocal. Is it possible, nevertheless, to combine consideration of QM as emergent theory with locality? The answer is positive and in this paper we present a very natural model of the classical field type which reproduces probabilistic predictions of QM; in particular, Bell's type inequalities are violated. The price of the classical description is contextuality of quantum observables. They cannot be considered as properties of subquantum systems, but their values depend on the measurement context (in complete agreement with Bohr's views), cf. Ozawa,¹³⁾ and Ozawa and Kitajima.¹⁴⁾

The cornerstone of our classical prequantum model is the observation of Ohya and Watanabe¹⁵⁾ that the covariance operator of a classical random (Gaussian) signal normalized by its trace can be interpreted as a density operator, i.e., a quantum state. These authors used this observation to embed classical signal theory into quantum information theory and to solve some problems related to definition of

entropy for Gaussian channels on the infinite dimensional Hilbert space.¹⁶⁾ I use the same observation the other way around: to embed QM into classical signal theory.

We stress that Bell's inequality¹²⁾ plays a crucial role in modern quantum mechanics and, especially, quantum information (QI). Its violation has not only theoretical consequences (nonlocality, nonobjectivity of quantum observables), but also applications, e.g., to quantum cryptography.¹⁷⁾ Its violation was experimentally confirmed^{18),19)} (although there are still loopholes [see, e.g., Refs. 17), 20)–22) for discussions]). There are no doubts in results of experiments. However, a proper interpretation of these results is still the subject of intensive debates [see, e.g., Refs. 22), 23), 21). By the commonly accepted interpretation it was proved that quantum observables are either nonlocal or/and nonobjective. Although Bell's test does not provide a possibility to distinguish nonlocality from nonobjectivity, the majority of QI-people made (intuitively) their choice in favour of *nonlocality*. This viewpoint has been criticized by several authors, e.g., Refs. 24)–26), and see also Ref. 22) for extended bibliography. The majority of authors criticizing the conventional interpretation of violation of Bell's inequality tried to save *both locality and objectivity* (a possibility to assign to a system the values of physical observables before measurement). This is not my approach. I agree (although this contradict to my own "old papers"²⁷⁾) that it is impossible to combine locality and realism and reproduce quantum probabilities for entangled systems; in particular, to violate Bell's type inequalities, e.g., the CHSH inequality. In this paper I present a local, but *nonobjective* classical model violating the Bell's type inequality for *probabilities of joint detections*, namely, CHSH-inequality.

Nonobjectivity of observables is typically considered as an intrinsically quantum feature. Bohr emphasized the role of measurement context in quantum measurements. At the same time classical physics is often associated with one special model, *classical statistical mechanics*, which is definitely objective. It is forgotten that, besides classical statistical mechanics, there exists another important classical model — *classical field theory*. In this paper we show that the usage of the threshold type detectors operating with (classical) random signals makes observables for classical signals nonobjective. Hence, Bohr was right, the experimental context plays a crucial role in QM. However, it also plays a similar role in some classical models of the wave-type. We call our model *threshold signal detection model*, TSD.

TSD definitely has important consequences for quantum foundations: QM can be treated as a part of classical signal theory. Hence, opposite to Bohr's claim, QM may be incomplete; opposite to Bell's claim, it may be local; opposite to Einstein's claim, it need not be objective. Of course, in physics the creation of a theoretical model, in our case TSD, is not the end of the story. The final word always should be said by experimenters. To confirm TSD experimentally, experimenters have to be able to measure components of (classical) fields corresponding to quantum particles at so to say "prequantum level" (for example, electric and magnetic components of the photon).

The impact of TSD to QI is a more complicated problem. Since QM can be embedded in classical signal theory, it seems that QI can be considered as a part of classical information theory. Surprisingly this is not the case. QI was elaborated

to operate with incomplete information^{*)} provided by quantum observables. Its operations and consequences cannot be directly derived from classical signal theory. Nevertheless, it is clear that after creation of TSD Bell's test cannot be considered guarantying 100% security of the basic quantum cryptographic protocols.

We list the basic assumptions of TSD:

(a) prequantum signals have a special temporal structure of correlations given by (5·3)–(5·5);

(b) detectors are of the threshold type;

(c) detectors are properly calibrated to eliminate the contribution of the random background field;

(d) instances of clicks of detectors for measurements on correlated signals match each other;

(e) stochastic processes inducing quantum probabilities and correlations are Gaussian.

Thus the temporal structure plays an important role in our treatment of Bell's inequality, cf. Refs. 25), 26), 22).

The usage of the threshold type detectors ruins objectivity of quantum observables. It is possible to determine only instances of detectors' clicks; in TSD we are not able to represent quantum observables in Bell's form:¹²⁾

$$a = a(\lambda), \quad (1.1)$$

where λ is a so-called hidden variable.

The calibration of detectors is not a technicality. This is a basic element of TSD; quantum correlations are obtained through discarding the contribution of the random background field. This field is fundamental and it is impossible to distil it from the quantum signal (quantum system). We are only able to eliminate it through the measurement procedure, via calibration.

It is well known that in real EPR-Bohm experiments clicks of detectors for channels corresponding to entangled photons have to match each other. In practice, this is done with the aid of time window.^{**)} Typically this matching is considered as an experimental technicality. However, as it was shown in a series of papers^{28)–30)} [see Appendix A], this is a foundational question related to the projection postulate in QM (the difference between Lüders postulate and the original von Neumann postulates for measurements on composite quantum systems). In TSD the condition (d) is also fundamental.

TSD can be considered as measurement theory for recently developed *prequantum classical statistical field theory*, PCSFT.^{31)–36)} The latter reproduced all quantum averages and correlations including correlations for entangled quantum states. In particular, PCSFT correlations violated Bell's inequality.

^{*)} This is the interpretation of QM and QI based on TSD. It differs crucially from the orthodox Copenhagen interpretation.

^{**)} A possibility to violate Bell's inequality for a classical corpuscular model by using the time window was explored in Ref. 26).

The message of PCSFT in a nutshell is that (i) quantum systems may be mapped on classical stochastic systems even if they are capable of nontrivial quantum manifestations, and that (ii) this shows that the aforesaid phenomena should be regarded more classical than it is commonly believed.

Examples of mappings with the stated properties are well known: the Q -representation for linear bosonic systems, and the so-called positive- P representation for nonlinear ones. The Q -function of an electromagnetic field in a quantum state is positive, which does not preclude such field from showing violations of Bell's inequalities in A. Aspect's experiment. The main problem for matching of PCSFT and conventional QM was that PCSFT (nor other aforementioned models) was not able to describe probabilities of discrete clicks of detectors. In particular, PCSFT is theory of correlations of *continuous signals*. "Prequantum observables" are given by quadratic forms of signals. These forms are unbounded and this is not surprising that correlations of such observables can violate Bell's type inequalities [see Ref. 22] for discussion and an elementary example]. (The condition of coincidence of ranges of values of quantum observables and corresponding "prequantum variables" plays a crucial role in Bell's argument.) TSD solved the measurement problem of PCSFT. In the same way as in Bell's consideration, TSD operates with discrete observables. In particular, in the case of photon polarization (its projection to a fixed axis) TSD operates with dichotomous variables taking values ± 1 .

§2. Resolution of dilemma: nonlocality or nonobjectivity?

This section is devoted to the general discussion on Bell's inequality, nonlocality, nonobjectivity, and contextuality. As was emphasized in the introduction, our classical field type model, PCSFT, endowed with the corresponding measurement model TSD is a local, but nonobjective. Hence, the dilemma "nonlocality or nonobjectivity?" is resolved in favor of nonobjectivity. (In our framework one cannot use the functional representation of quantum observables (1.1) and, hence, it is not surprising that Bell's inequality can be violated.) In this section we couple this (yet purely theoretical) prediction with experimental studies in quantum foundations, namely, experiments on quantum contextuality.^{37),38)} Although these experiments have no direct relation to PCSFT/TSD, their results might be interpreted as supporting nonobjective "prequantum" models.

We state again the basic assumptions of Bell's argument:

(R) **Realism:** A possibility to assign to a quantum system the values of observables before measurement.

From the philosophical viewpoint this is not precisely the definition of realism (objectivity). To be real (objective), it is enough to exist, without any relation with experiment. Such "ontic realism" is formalized through the principle of value definiteness:

(VD) *All observables defined for a QM system have definite values at all times.*

However, Bell used "measurement realism" which we presented in (R). If the values of physical observables were existing, but not coinciding with results of mea-

surement, then Bell's consideration would not imply Bell's inequality [see Ref. 22] for analysis and examples]. In philosophic literature (R) is often referred as a *principle of faithful measurement* (FM).³⁹⁾

(L) **Locality:** No action at the distance.

Therefore every one (who accepts that experiments are strong signs that local realism has to be rejected) has to make the choice between:

(NONL) Realism, but nonlocality (the original Bell's position).

(NR) No realism (nonobjectivity) and locality (the original Bohr's position).

(NONL+NR) Nonlocality + nonobjectivity.

The last possibility, (NONL+NR), seems to be too complex to happen in nature. Of course, one cannot completely reject that nature is so exotic. However, to resolve all problems one need not make this assumption, either nonlocality or nonobjectivity is enough. The (NONL+NR)-interpretation of experimental results is definitely non-minimalistic and it can be rejected, e.g., by the [Occam's razor]-reason.

Hence, one has to make his choice: either nonlocality or nonobjectivity; either De Broglie-Bohm-Bell or Bohr-Heisenberg-Pauli position. We state again that the Copenhagen interpretation of quantum mechanics had nothing to do with nonlocality. Bohr advertised the position that the values of quantum observables are "created" in the process of interaction of quantum systems with measurement devices. Hence, the main point was nonobjectivity.

It is typically assumed that the present experimental situation does not provide us a possibility to make the choice. And this is correct if one explores only experiments of the EPR-Bohm type in which realism and locality are mixed.

However, recently exciting experiments testing *quantum contextuality* were performed:^{37),38)} they supported the thesis that quantum mechanics is contextual.

We point that *contextuality implies nonobjectivity!*^{*)} In the contextual situation it is impossible to assign values of physical observables before measurement. Therefore the experiments^{37),38)} can be considered as supporting nonobjectivity. This experiment is about nonobjectivity of results of measurements for a single particle.

Now I present the following considerations which seem to be logically justified. If already a single particle exhibits lack of objectivity, then it is reasonable to assume that the situation cannot be improved by consideration of a pair of particles. Hence, it is reasonable to assume nonobjectivity in the EPR-Bohm experiment. This implies that among two alternatives, (NONL) or (NR), the latter is essentially more justified than the former.

We can summarize the arguments presented in this session:

Recent experiments on quantum foundations can be considered as supporting the original Bohr's position — quantum observables are nonobjective, their values cannot

*) We state again that we understood objectivity (realism) as "measurement objectivity" (realism) — the discussion after the definition of (R). Contextuality does not imply the violation of the principle of value definiteness (VD).

be assigned before measurement. The assumption of nonlocality has to be rejected, since there is no direct experimental evidence of nonlocality (similar to the tests of nonobjectivity presented in articles^{37),38}) and since in the EPR-Bohm experiment it is unnecessary — under the assumption of nonobjectivity.

We shall come back to the problem of nonobjectivity of quantum measurements and threshold detection measurements of classical waves in Appendix A.

§3. From time-correlations in prequantum random signals to quantum probabilities

3.1. Random signals

Random signals considered in signal's theory, e.g., in radio-physics, are mathematically defined as stochastic processes. A stochastic process is a function of two variables $\phi(s, \omega)$, where $s \in [0, +\infty)$ is the time parameter and ω is a random parameter. In modern theory of stochastic processes⁴⁰⁾ random parameters are invented in the abstract framework.

Starting with Kolmogorov (1933),⁴¹⁾ the probability space is defined as a triple (Ω, \mathcal{F}, P) , where Ω is a set (of any nature), \mathcal{F} is a σ -algebra of its subsets (a set system which contains Ω and the empty set, it is closed with respect to countable unions and intersections and the operation of complement), and P is a probability measure defined on \mathcal{F} . In the Kolmogorovian framework a *random variable* $\xi : \Omega \rightarrow \mathbf{C}$ is defined as a measurable function, i.e., for any Borel subset O of the set of complex numbers \mathbf{C} its preimage $\xi^{-1}(O)$ belongs to \mathcal{F} . The mean value of ξ is denoted by $E\xi$.

A function of two variables $\phi(s, \omega)$ is a stochastic process if for any instant of time s , the map $\omega \rightarrow \phi(s, \omega)$ is a random variable.

Although in formal calculations the physical nature of the random parameter ω does not play any role, in a real physical model it is important to know the origins of randomness. In classical signal theory, randomness is generated by noises in sources and transmission media.

3.2. The scheme of threshold detection

Let us consider a complex valued stochastic process (random signal) $\phi(s) = \phi(s, \omega)$ with zero average, $E\phi(s) = 0$ for any s . The quantity

$$\mathcal{E}(s, \omega) = |\phi(s, \omega)|^2 \quad (3.1)$$

is the signal energy at the instant of time s . If signals corresponding to quantum systems were smooth enough, then the detection procedure under consideration would be reduced to the condition of the energy level approaching the detection threshold, say $\mathcal{E}_d > 0$.

The instant of time τ corresponding to the signal detection (“click”) is determined as the first instant of time when the signal's energy exceeds the threshold:

$$\mathcal{E}(\tau, \omega) \geq \mathcal{E}_d. \quad (3.2)$$

In the rigorous mathematical framework this is nothing else than the *hitting time*

(stopping time)⁴⁰⁾

$$\tau \equiv \tau_{\mathcal{E}_d}(\omega) = \inf\{s \geq 0 : \mathcal{E}(s, \omega) \geq \mathcal{E}_d\}. \tag{3.3}$$

This is formalization of the basic condition (b), §1.

Mathematically our aim is to find average of the instance of detection, $\bar{\tau} = E\tau$. The quantity $1/\bar{\tau}$ will be used to find the probability of detection, “how often the detector produces clicks” [see §5].

However, classical random signals corresponding to quantum states are very singular (because of the contribution of the background field of the white noise type) and the value of a signal at the fixed instance of time is not defined (at least we cannot be sure that it is defined for almost all ω). Therefore, instead of the signal’s energy value at the fixed instance of time (3.1), we shall use the analog of the threshold approaching condition for a properly smoothed signal. We shall consider smoothing in the L_2 -space. This smoothing matches the real detection procedure. In reality, a detector cannot determine the signal’s energy at the fixed instance of time. Any detector is based on the integration of signals.

Suppose that the detection procedure is based on the *integration window* given by the step-function

$$g(s) \equiv g^\kappa(s) = \begin{cases} 1/\sqrt{\kappa}, & s \in [0, \kappa], \\ 0, & s \notin [0, \kappa], \end{cases} \tag{3.4}$$

where $\kappa > 0$ is a small parameter (of the detector). We remark that $\|g\| = 1$ (the L_2 -norm).

Mathematically the detection procedure is described in the following way. Consider the κ -smoothed signal

$$\phi^\kappa(u, \omega) = \int_{-\infty}^{+\infty} \phi(s, \omega)g(u - s)ds = \frac{1}{\sqrt{\kappa}} \int_{u-\kappa}^u \phi(s, \omega)ds \tag{3.5}$$

and its energy

$$\mathcal{E}(u, \omega; \kappa) = |\phi^\kappa(u, \omega)|^2. \tag{3.6}$$

Now the instant of time of signal’s detection τ is determined as the first instant of time when the signal’s energy exceeds the threshold:

$$\mathcal{E}(\tau, \omega; \kappa) \geq \mathcal{E}_d. \tag{3.7}$$

In our model the smoothed (through the integration over the time window) signal ϕ^κ has a.e. continuous trajectories. Therefore⁴⁰⁾ the click time can be represented as the first instant of time when signal’s energy approaches the threshold:

$$\tau(\omega) \equiv \tau_{\mathcal{E}_d}(\omega) = \inf\{s \geq \kappa : \mathcal{E}(s, \omega; \kappa) = \mathcal{E}_d\}. \tag{3.8}$$

(This τ also depends on the integration window of the detector.) Hence,

$$\mathcal{E}(\tau(\omega), \omega; \kappa) = \mathcal{E}_d. \tag{3.9}$$

We consider the special class of random signals having zero averages ($E\phi(s) = 0$ for any s). Suppose that the covariance function of $\phi(s)$ has the following form (this is concretization of the assumption (a) of §1):

$$E\phi(s_1)\overline{\phi(s_2)} = \sigma^2\delta(s_1 - s_2)\sqrt{|s_1s_2|}. \quad (3.10)$$

(The role of the parameter σ^2 will become clear in §5, Remark 3.) We find the average of the energy $\mathcal{E}(\tau, \omega; \kappa)$ of this signal. We have

$$\begin{aligned} E\mathcal{E}(\tau, \omega; \kappa) &= \frac{1}{\kappa} E \left| \int_{\tau-\kappa}^{\tau} ds \phi(s, \omega) \right|^2 \\ &= \frac{1}{\kappa} \int_{\tau-\kappa}^{\tau} \int_{\tau-\kappa}^{\tau} E\phi(s_1, \omega)\overline{\phi(s_2, \omega)} ds_1 ds_2 \\ &= \frac{\sigma^2}{\kappa} \int_{\tau-\kappa}^{\tau} \int_{\tau-\kappa}^{\tau} \delta(s_1 - s_2)\sqrt{|s_1s_2|} ds_1 ds_2 \\ &= \frac{\sigma^2}{\kappa} \int_{\tau-\kappa}^{\tau} s ds = \sigma^2(\tau - \kappa/2). \end{aligned}$$

We shall use this quantity a bit later. Now we proceed to calculation of the average detection time $\bar{\tau}$. We take the average of the equality (3.9) and obtain

$$E\mathcal{E}(\tau(\omega), \omega; \kappa) = \mathcal{E}_d. \quad (3.11)$$

(We recall that the instant of detection $\tau = \tau(\omega)$ is a random variable.) To find the quantity on the left-hand side of this equality, we use the formula of total probability

$$E\mathcal{E}(\tau, \omega; \kappa) = \int_0^{+\infty} E[\mathcal{E}(\tau(\omega), \omega; \kappa) | \tau(\omega) = \tau] P(\tau(\omega) = \tau) d\tau,$$

where $E[\mathcal{E} | \tau(\omega) = \tau]$ is the conditional expectation of the quantity \mathcal{E} under the condition $\tau(\omega) = \tau$. The conditional expectation has already been found

$$E[\mathcal{E}(\tau(\omega), \omega; \kappa) | \tau(\omega) = \tau] = \sigma^2(\tau - \kappa/2). \quad (3.12)$$

Hence,

$$\begin{aligned} E\mathcal{E}(\tau(\omega), \omega; \kappa) &= \sigma^2 \int_0^{+\infty} (\tau - \kappa/2) P(\tau(\omega) = \tau) d\tau = \sigma^2(\bar{\tau} - \kappa/2) \\ &= \bar{\tau}\sigma^2(1 + O(\kappa/\bar{\tau})), \quad \kappa/\bar{\tau} \rightarrow 0. \end{aligned} \quad (3.13)$$

Finally, the averaged condition of detection (3.11) takes the form:

$$\bar{\tau}\sigma^2(1 + O(\kappa/\bar{\tau})) = \mathcal{E}_d \quad (3.14)$$

or

$$\frac{1}{\bar{\tau}} \approx \frac{\sigma^2}{\mathcal{E}_d}, \quad \kappa/\bar{\tau} \rightarrow 0. \quad (3.15)$$

3.3. Probabilities of clicks in detection channels

Hence, during a long period of time T such a detector clicks N_{click} -times, where

$$N_{\text{click}} \approx \frac{T}{\bar{\tau}} \approx \frac{\sigma^2 T}{\mathcal{E}_d}, \quad \kappa/\bar{\tau} \rightarrow 0. \tag{3.16}$$

To find the probability of detection and match the real detection scheme which is used in quantum experiments¹⁸⁾ we have to use a proper normalization of N_{click} . This is an important point of our considerations. (The normalization problem is typically ignored in standard books on quantum foundations, cf., however, Ref. 22.) In QM-experiments probabilities are obtained through normalization corresponding to the sum of clicks in all detectors involved in the experiment, e.g., spin up and spin down detectors.

In QM such a collection of detectors is symbolically represented as quantum observable, say C . In the mathematical formalism observable C is represented by the Hermitian operator \widehat{C} . In the case of purely discrete (nondegenerate) spectrum, the QM-probabilities of detection are determined by the basis of eigenvectors $\{e_j\}$ of the operator \widehat{C} through Born's rule:

$$P_j = |\langle \Psi, e_j \rangle|^2 \tag{3.17}$$

for quantum systems in the pure state Ψ or more generally, for quantum systems in the mixed state ρ , we have:

$$P_j = \text{Tr} \rho C_j, \tag{3.18}$$

where \widehat{C}_j is the projector onto the vector e_j , i.e., $\widehat{C}_j = |e_j\rangle\langle e_j|$. In QM Born's rule (3.18) is postulated.²²⁾

To reproduce the QM-scheme (in the model in which the spatial degrees of freedom are still absent), we consider a family of stochastic processes $\phi(i, s)$, $i = 1, 2, \dots, m$. The signal ϕ is split into a family of disjoint channels coupled to detectors $D(i) : \phi(s) = (\phi(i, s))_{i=1}^m$.^{*)} Thus we have the vector valued random signal $\phi(s)$. Suppose that the covariance function of $\phi(s)$ has the following form:

$$E\phi(i, s_1)\overline{\phi(j, s_2)} = \delta(s_1 - s_2)\sqrt{|s_1 s_2|}b(ij). \tag{3.19}$$

(This is simply generalization of (3.10) to the vector valued process; this is further concretization of the assumption (a) [see §1].) Hence, its covariance function can be represented as

$$B(s_1, s_2) = \delta(s_1 - s_2)\sqrt{|s_1 s_2|}B, \tag{3.20}$$

$B = (b(ij))$. The matrix B does not depend on temporal correlations; it represents only correlations of internal degrees of freedom (such as e.g. spin or polarization). We set

$$b(ii) = \sigma_i^2 \quad \text{and} \quad \Sigma^2 = \sum_i \sigma_i^2 = \text{Tr} B.$$

^{*)} We consider only the detection scheme for discrete observables, e.g., spin, $i = +1$, spin up, and $i = -1$, spin down.

We repeat the previous detection scheme (based on threshold detectors) for each of these processes, so m detectors are involved; the only assumption is that all these detectors have the same detection threshold $\mathcal{E}_b > 0$. We obtain [see (3.16)]

$$N_{\text{click}}(i) \approx \frac{T}{\bar{\tau}_i} \approx \frac{\sigma_i^2 T}{\mathcal{E}_d}, \quad \kappa/\bar{\tau}_i \rightarrow 0. \quad (3.21)$$

Hence, the total number of clicks:

$$N = \sum_i N_{\text{click}}(i) \approx \frac{T \Sigma^2}{\mathcal{E}_d}, \quad (3.22)$$

The probability of detection for the j th detector is given by

$$P(j) = N_{\text{click}}(j)/N \approx \frac{\sigma_j^2}{\Sigma^2}. \quad (3.23)$$

In fact, this is Born's rule. Consider the matrix

$$\rho = B/\text{Tr}B = (b(ij))/\Sigma^2. \quad (3.24)$$

This is the Hermitian positive trace one matrix; so formally it has all properties of the *density matrix* used in QM. In \mathbf{C}^n take the canonical basis $e_j = (0\dots 1\dots 0)$; set $\widehat{C}_j = |e_j\rangle\langle e_j|$. Then the equality for the probability of detection (3.23) can be written as

$$P(j) = \text{Tr} \rho \widehat{C}_j. \quad (3.25)$$

This is the QM-rule for calculation of probabilities of detection.

In the quantum formalism for a given state ρ , density operator, we are able to determine probabilities of detection in corresponding channels not only for one fixed observable, the fixed family of disjoint channels, but for any observable, any family of disjoint channels. The same feature has our model. We have a stochastic process $\phi(s)$ valued in the m -dimensional complex Hilbert space H ; denote its covariance function by $B(s_1, s_2)$. Suppose that it has the form (3.20) where $B : H \rightarrow H$ is Hermitian positive operator (in general $\text{Tr}B \neq 1$). This operator describes correlations of internal degrees of freedom in the signal ϕ .

Suppose now that *all measurement procedures under consideration have the form of projections of the signal $\phi(s)$ onto some orthogonal directions $\{e_j\}$ and the threshold type measurements for components $\phi_j(s) = \langle \phi(s), e_j \rangle$* . Hence, selection of each measurement of this type is equivalent to decomposition of the random signal $\phi(s)$ into orthogonal components.*) Set $b(ij) = \langle e_i | B | e_i \rangle$ and repeat the previous considerations; we obtain (3.25) for the "density operator" $\rho = B/\text{Tr}B$. Opposite to the canonical scheme of QM, this operator has a natural interpretation in theory of

*) In the QM-formalism such a decomposition of a signal corresponds to the measurement scheme based on the projection postulate. Formally the latter works very well, but its origin cannot be explained in physical terms. This brings a bit of mystery to QM-measurement theory: collapse of the wave function and so on. In our model the split of a physical signal into a family of signals is the standard operation of the classical signal theory, in particular, in classical optics.

classical stochastic processes (classical signal theory) — the normalized covariance operator of the internal degrees of freedom of a signal.

Summary. We considered stochastic processes (with temporal correlations of the special type). They can be used to model (classical) random signals with finite-dimensional state space representing non-temporal degrees of freedom, “internal degree of freedom”. The covariance operator for the internal degrees of freedom normalized by its trace can be formally treated as a density operator, so to say, quantum state. By splitting the random signal into its components corresponding to projections onto vectors of an orthogonal basis in the space of internal degrees of freedom we reproduce the detection scheme of QM.

Remark 1. We stress that the presented derivation was done under the assumption

$$\kappa/\bar{\tau}_i \rightarrow 0. \tag{3.26}$$

Hence, the integration window κ has to be essentially smaller than the average time between clicks. This is a natural physical assumption.

Remark 2. We remark that the detection threshold \mathcal{E}_d disappeared from the final formula for the probability of detection. However, the average time between clicks depends linearly on the threshold [see (3.14)].

Remark 3. (Dimension analysis) The squared-signal $|\phi(s)|^2$ has the dimension of energy. From the equality (3.10) we obtain that $\sigma^2 \times \text{time} \sim \text{energy}$. Hence, $\sigma^2 \sim \frac{\text{energy}}{\text{time}} \sim \text{power}$. The detection threshold $\mathcal{E}_d \sim \text{energy}$. We now comment the equality (3.16) from the dimensional viewpoint. The number of clicks of a detector, N_+ , is proportional to signal’s power σ^2 and the duration of the experiment run and inverse proportional to the detection threshold. Hence, *signal’s power (and not its total energy) is crucial for detection.*

§4. Threshold/calibration detection scheme for classical signals representing entangled quantum systems

The detection scheme presented in this section describes detection of internal degrees of freedom, e.g., spin components, for pairs of correlated quantum particles.

Consider a *Gaussian** signal with two correlated components (bi-signal) $\phi(s) = (\phi_1(s), \phi_2(s))$. We proceed under the following assumptions (see the basic assumption (a), §1) on averages and correlations ($k = 1, 2$):

$$E\phi_k(s) = 0, \tag{4.1}$$

$$E\phi_k(s_1)\overline{\phi_k(s_2)} = \sigma_k^2\delta(s_1 - s_2)\sqrt{|s_1s_2|} + \mathcal{E}_0\delta(s_1 - s_2), \quad \mathcal{E}_0 > 0, \tag{4.2}$$

$$E\phi_1(s_1)\overline{\phi_2(s_2)} = 2\sqrt{\mathcal{E}_0}\sigma_{12}\delta(s_1 - s_2)|s_1s_2|^{1/4}, \quad \sigma_{12} \in \mathbf{C}, \tag{4.3}$$

$$\sigma_1^2 = \sigma_2^2 = |\sigma_{12}|^2 \equiv \sigma^2. \tag{4.4}$$

*) We proceed with only Gaussian signals. It may be possible to use non-Gaussian signals. However, mathematics is essentially more complicated.

Remark 4. We stress the appearance of the additional term in (4.2) comparing with (3.10). Physically this is the contribution (to correlations) of the background field of the white noise type. We shall see that, in fact, \mathcal{E}_0 is the mean energy of this field. (It does not depend on s). Its necessity was not evident in the case of the one-component signal (corresponding to a single quantum particle), so in §3 we ignored the contribution of the background field. However, in the case of bi-signals (corresponding to composite two particle systems) one cannot proceed classically without the background component. Surprisingly the presence of the background field started to play a role only in joint detection, or other way around: the presence of the background can be detected only through joint measurement of correlated signals. We shall see that probabilities of joint detection predicted by QM (and tested experimentally) correspond to the well-defined classical stochastic process only if the presence of the background field is taken into account. This is a tricky situation. The contribution of the background field is not directly present in quantum probabilities. It is eliminated through calibration of detectors [see (4.10)]. However, in the absence of this field “prequantum stochastic process” is not well defined. (Of course, one may simply deny the existence of the prequantum classical process.)

Remark 5. (Dimension analysis) Here, cf. Remark 3, $\sigma_k^2 \sim \text{power}$. The detection threshold $\mathcal{E}_d \sim \text{energy}$. From (4.3) we have that $E\phi_1(s_1)\overline{\phi_2(s_2)} = k\delta(s_1 - s_2)|s_1s_2|^{1/4}$, where $k^2 \times \text{time} \sim \text{energy}^2$, i.e., $k^2 \sim \frac{\text{energy}^2}{\text{time}} \sim \text{energy} \times \text{power}$. Hence, it is natural to represent $k = k_0 \times \sigma_{12}$, where $|\sigma_{12}|^2 \sim \text{power}$ and $k_0^2 \sim \text{energy}$. We can select $k_0^2 = \mathcal{E}_0$, the energy of vacuum fluctuations. The equality (4.4) encodes matching of statistics of measurements on each of components $\phi_j(s)$, $j = 1, 2$, and joint measurement of these components. Hence, we consider a very special class of signals.

First, we show that this stochastic process is well defined. Consider the covariance function of this process

$$\begin{aligned} D(s_1, s_2) &= \begin{pmatrix} D_{11}(s_1, s_2) & D_{12}(s_1, s_2) \\ D_{21}(s_1, s_2) & D_{22}(s_1, s_2) \end{pmatrix} \\ &= \delta(s_1 - s_2) \begin{pmatrix} \sigma^2\sqrt{|s_1s_2|} + \mathcal{E}_0 & 2\sqrt{\mathcal{E}_0}\sigma_{12}|s_1s_2|^{1/4} \\ 2\sqrt{\mathcal{E}_0}\bar{\sigma}_{12}|s_1s_2|^{1/4} & \sigma^2\sqrt{|s_1s_2|} + \mathcal{E}_0 \end{pmatrix}. \end{aligned} \quad (4.5)$$

We now prove that the operator \widehat{D} defined by the kernel (4.5) is positively defined. Take two L_2 -functions, $y_1(s), y_2(s)$. We have

$$\begin{aligned} \langle \widehat{D}y_1, y_2 \rangle &= \int (\sigma^2|s| + \mathcal{E}_0)(|y_1(s)|^2 + |y_2(s)|^2)ds \\ &\quad + 2\sqrt{\mathcal{E}_0} \int \sqrt{|s|}(\sigma_{12}y_2(s)\bar{y}_1(s) + \bar{\sigma}_{12}y_1(s)\bar{y}_2(s))ds = I_1 + I_2. \end{aligned}$$

We have

$$I_2 \geq -4\sqrt{\mathcal{E}_0}|\sigma_{12}| \int |y_1(s)||y_2(s)|ds.$$

Hence, $I_1 + I_2 \geq$

$$\int (\sigma\sqrt{|s|}|y_1(s)| - \sqrt{\mathcal{E}_0}|y_2(s)|)^2 + (\sigma\sqrt{|s|}|y_2(s)| - \sqrt{\mathcal{E}_0}|y_1(s)|)^2 ds \geq 0.$$

For each component of the bi-signal $\phi = (\phi_1, \phi_2)$, we consider the smoothed signal corresponding the integration window κ , $\phi^\kappa = (\phi_1^\kappa, \phi_2^\kappa)$ [see (3.5)]. Denote by $\mathcal{E}_k(s, \omega; \kappa)$ the energy of the k th component of the κ -smoothed signal, i.e., $\mathcal{E}_k(s, \omega; \kappa) = |\phi_k^\kappa(s, \omega)|^2$, $k = 1, 2$.

In the absence of the background field, we would have the threshold approaching detection conditions

$$\mathcal{E}_k(\tau_k(\omega), \omega; \kappa) = \mathcal{E}_d, \quad k = 1, 2, \tag{4.6}$$

for each component, ϕ_k , $k = 1, 2$, where the random instant of time $\tau_k(\omega)$ is defined as

$$\tau_k(\omega) = \inf\{s \geq \kappa : \mathcal{E}_k(s, \omega; \kappa) \geq \mathcal{E}_d\} = \inf\{s \geq \kappa : \mathcal{E}_k(s, \omega; \kappa) = \mathcal{E}_d\}.$$

The latter equality is a consequence of continuity of (almost all) trajectories of signals smoothed through integration in detectors. (We assume that both detectors have the same detection threshold.)

However, in the present model our signals are mixed with the background field. Denote the latter by $\eta(s) \equiv \eta(s, \omega)$. Moreover, this field cannot be distilled from signals. There is no filter removing the background field. Its contribution can be strong enough to play an important role in production of clicks. We only can make cutoff in detectors by their calibration — subtraction the energy of the background field. This field is very singular, so its energy for a fixed instance of time is not well defined. However, this problem is solved through using detectors with the integration windows given by functions of g^κ type. They measure the energy of the smoothed η :

$$\eta^\kappa(u) = \int \eta(s)g(u - s)ds = \langle g_u, \eta \rangle,$$

where $g_u(s) = g(u - s)$. For such a detector, the energy contribution of the background field is given by

$$\mathcal{E}_0(u, \omega; \kappa) = |\eta^\kappa(u)|^2 = |\langle g_u, \eta \rangle|^2. \tag{4.7}$$

Hence, the detection condition for each component of the bi-signal can be modified from (4.6) to

$$\mathcal{E}_k(\tau_k(\omega), \omega; \kappa) - \mathcal{E}_0(\tau_k(\omega), \omega; \kappa) = \mathcal{E}_d, \tag{4.8}$$

where

$$\begin{aligned} \tau_k(\omega) &= \inf\{s \geq \kappa : \mathcal{E}_k(s, \omega; \kappa) - \mathcal{E}_0(s, \omega; \kappa) \geq \mathcal{E}_d\} \\ &= \inf\{s \geq \kappa : \mathcal{E}_k(s, \omega; \kappa) - \mathcal{E}_0(s, \omega; \kappa) = \mathcal{E}_d\}. \end{aligned}$$

The latter equality is a consequence of continuity of (almost all) trajectories of signals and the background noise smoothed through integration in detectors. Thus,

the detection conditions (4.6) have to be modified by taking into account the presence of the background field:

$$\mathcal{E}_k(\tau_k(\omega), \omega; \kappa) = \mathcal{E}'_d, \quad (4.9)$$

where $\mathcal{E}'_d = \mathcal{E}_d + \mathcal{E}_0(\tau_k(\omega), \omega; \kappa)$ is the calibrated threshold. However, the threshold \mathcal{E}'_d is random. And what is even worse, it depends on the detection moment. So, it is not useful for the practical purpose. The (random) contribution of the background is unknown. Therefore in practice the detection condition (4.9) is changed to coarser condition with *calibration by the mean value of the detected energy of the background field* [see the basic assumption (c), §1].

First we find this mean value for the fixed (i.e., nonrandom) τ . We use the general result on quadratic forms of Gaussian random variables valued in Hilbert spaces³⁶ [see Eq. (B.2) in Appendix B]. (Thus here we use the basic assumption (d), §1.) Consider in L_2 the operator $\widehat{A} \equiv \widehat{A}_{\tau; \kappa} = |g_\tau\rangle\langle g_\tau|$, where, as always, $g_\tau(s) = g(\tau - s)$ and the function g was defined in (3.4). Set $f_A(y) = \langle \widehat{A}y, y \rangle, y \in L_2$, the quadratic form corresponding to the operator \widehat{A} . By (B.2) we obtain

$$E\mathcal{E}_0(\tau, \omega; \kappa) = Ef_A(\eta^\kappa) = \mathcal{E}_0 \text{Tr} \widehat{A} = \mathcal{E}_0 \|g_\tau\|^2 = \mathcal{E}_0.$$

This quantity does not depend on τ and this is not surprising, since the background field is translation invariant. If τ is random (as it is in (4.7)), then we can use the formula of total probability:

$$E\mathcal{E}_0(\tau(\omega), \omega; \kappa) = \int_0^\infty E[\mathcal{E}_0(\tau(\omega), \omega; \kappa) | \tau(\omega) = \tau] P(\tau(\omega) = \tau) d\tau = \mathcal{E}_0.$$

Now we modify the detection condition (4.9) and proceed with conditions ($k = 1, 2$)

$$\mathcal{E}_k(\tau_k(\omega), \omega; \kappa) - \mathcal{E}_0 = \mathcal{E}_d, \quad (4.10)$$

where, for

$$\mathcal{E}'_d = \mathcal{E}_0 + \mathcal{E}_d, \quad (4.11)$$

the random instant of time $\tau_k(\omega)$ is defined as

$$\tau_k(\omega) = \inf\{s \geq \kappa : \mathcal{E}_k(s, \omega; \kappa) \geq \mathcal{E}'_d\} = \inf\{s \geq \kappa : \mathcal{E}_k(s, \omega; \kappa) = \mathcal{E}'_d\}.$$

The latter equality is a consequence of continuity of (almost all) trajectories of signals smoothed through integration. Hence, the detection conditions have the form:

$$\mathcal{E}_k(\tau_k(\omega), \omega; \kappa) = \mathcal{E}'_d, k = 1, 2. \quad (4.12)$$

For each component of the bi-signal, we repeat the scheme of §§3 and 5, but with the new threshold given by (4.11).

The only difference is that the process $\phi_k(s)$ has the covariance operator $\widehat{D}_{kk} = \widehat{D}_{kk}^{(0)} + \mathcal{E}_0 I$, where $\widehat{D}_{kk}^{(0)}$ is the covariance operator of the process which was considered in §3. We have [see Appendix B]

$$E\mathcal{E}_k(\tau, \omega; \kappa) = \text{Tr} \widehat{D}_{kk} \widehat{A} = \text{Tr} \widehat{D}_{kk}^{(0)} \widehat{A} + \mathcal{E}_0 \text{Tr} \widehat{A} = \text{Tr} \widehat{D}_{kk}^{(0)} \widehat{A} + \mathcal{E}_0. \quad (4.13)$$

Therefore the detection condition (4.12) with (4.11) implies (after averaging and the use of the formula of total probability) the same condition as in §3

$$\text{Tr} \widehat{D}_{kk}^{(0)} \widehat{A} = \bar{\tau} \sigma^2 (1 + O(\kappa/\bar{\tau})) = \mathcal{E}_d. \quad (4.14)$$

Thus the contribution of the background field was completely excluded — through the proper calibration of detectors. (We stress again that this can be done only “afterward”, i.e., on the level of detectors and not fields; this is a crucial point of our approach to QM, as theory of measurements with threshold’s type and properly calibrated detectors.)

Now we consider the joint clicks in detectors corresponding to the components of the bi-signal. Thus (4.12) holds for both k s and moreover the instances of detection for corresponding detectors, $\tau_k = \tau_k(\omega)$, are constrained by the equality [see the basic assumption (d), §1]:

$$\tau(\omega) = \tau_1(\omega) = \tau_2(\omega). \quad (4.15)$$

Remark 6. Of course, in the real experiment we cannot proceed with the precise coincidence of instances of detection. One has to use the joint detection time window, say v , and proceed under the condition

$$|\tau_1 - \tau_2| \leq v. \quad (4.16)$$

In our ideal model we ignore this experimental technicality. Opposite to the model from works,²⁶⁾ the presence of the joint detection time window $v \neq 0$ in real experiments does not play a crucial role in our model; i.e., we can obtain quantum correlations even for $v = 0$.

We now find average of the joint detection time τ . The system of equalities (4.10), $k = 1, 2$, and (4.15) imply

$$(\mathcal{E}_1(\tau(\omega), \omega; \kappa) - \mathcal{E}_0)(\mathcal{E}_2(\tau(\omega), \omega; \kappa) - \mathcal{E}_0) = \mathcal{E}_d^2. \quad (4.17)$$

We take the average of both the sides

$$E(\mathcal{E}_1(\tau(\omega), \omega; \kappa) - \mathcal{E}_0)(\mathcal{E}_2(\tau(\omega), \omega; \kappa) - \mathcal{E}_0) = \mathcal{E}_d^2 \quad (4.18)$$

or

$$E\mathcal{E}_1(\tau(\omega), \omega; \kappa)\mathcal{E}_2(\tau(\omega), \omega; \kappa) - \mathcal{E}_0(E\mathcal{E}_1(\tau(\omega), \omega; \kappa) + E\mathcal{E}_2(\tau(\omega), \omega; \kappa)) + \mathcal{E}_0^2 = \mathcal{E}_d^2. \quad (4.19)$$

We start with the first term on the left-hand side of this equality. We shall again use the formula of total probability

$$\begin{aligned} & E\mathcal{E}_1(\tau(\omega), \omega; \kappa)\mathcal{E}_2(\tau(\omega), \omega; \kappa) \\ &= \int_0^\infty E[\mathcal{E}_1(\tau(\omega), \omega; \kappa)\mathcal{E}_2(\tau(\omega), \omega; \kappa)|\tau(\omega) = \tau]P(\tau(\omega) = \tau)d\tau. \end{aligned} \quad (4.20)$$

For the fixed τ , we have to find the correlation of two quadratic forms of the component of the Gaussian bi-signal satisfying the aforementioned assumptions. We again

use the general result on quadratic forms of Gaussian random variables valued in Hilbert spaces^{31)–36)} [see Eq. (B·3) in Appendix B]. Consider again the operator $\widehat{A} = |g_\tau\rangle\langle g_\tau|$ and its quadratic form $f_A(y)$. Then by (B·3) we have

$$\begin{aligned} E\mathcal{E}_1(\tau, \omega; \kappa)\mathcal{E}_2(\tau, \omega; \kappa) &= Ef_A(\phi_1)f_A(\phi_2) \\ &= \text{Tr}\widehat{D}_{11}\widehat{A} \text{Tr}\widehat{D}_{22}\widehat{A} + \langle \widehat{A} \otimes \widehat{A} D_{12}, D_{12} \rangle = J_1 + J_2, \end{aligned} \quad (4.21)$$

where

$$\widehat{D} = \begin{pmatrix} \widehat{D}_{11} & \widehat{D}_{12} \\ \widehat{D}_{21} & \widehat{D}_{22} \end{pmatrix} \quad (4.22)$$

is the covariance operator corresponding to the kernel $D(s_1, s_2)$. We start with the last term. It is determined by the off-diagonal term $D_{12}(s_1, s_2)$ of the covariance function $D(s_1, s_2)$:

$$\begin{aligned} J_2 &= \left| \int \int g_\tau(s_1)g_\tau(s_2)D_{12}(s_1, s_2)ds_1ds_2 \right|^2 = \left| \frac{2\sqrt{\mathcal{E}_0}\sigma_{12}}{\kappa} \int_{\tau-\kappa}^\tau \sqrt{|s|}ds \right|^2 \\ &= \left| \frac{4\sqrt{\mathcal{E}_0}\sigma_{12}}{3\kappa} [\tau^{3/2} - (\tau - \kappa)^{3/2}] \right|^2 = 4\mathcal{E}_0\sigma^2\tau(1 + O(\kappa/\tau))^2 \\ &= 4\mathcal{E}_0\sigma^2\tau(1 + O(\kappa/\tau)), \quad \kappa/\tau \rightarrow 0. \end{aligned}$$

Now we consider

$$J_1 = (\text{Tr}\widehat{D}_{11}\widehat{A})(\text{Tr}\widehat{D}_{22}\widehat{A}) = (\text{Tr}\widehat{D}_{11}\widehat{A})^2 = \langle \widehat{D}_{11}g_\tau, g_\tau \rangle^2.$$

We have

$$\langle \widehat{D}_{11}g_\tau, g_\tau \rangle = \int (\sigma^2|s_1| + \mathcal{E}_0)g_\tau^2(s)ds = \frac{1}{\kappa} \int_{\tau-\kappa}^\tau (\sigma^2s + \mathcal{E}_0)ds = \sigma^2\tau(1 + O(\kappa/\tau)) + \mathcal{E}_0.$$

Hence,

$$J_1 = (\sigma^4\tau^2 + 2\sigma^2\tau\mathcal{E}_0 + \mathcal{E}_0^2)(1 + O(\kappa/\tau))$$

and

$$\begin{aligned} E\mathcal{E}_1(\tau, \omega; \kappa)\mathcal{E}_2(\tau, \omega; \kappa) &= [4\mathcal{E}_0\sigma^2\tau + (\sigma^4\tau^2 + 2\sigma^2\tau\mathcal{E}_0 + \mathcal{E}_0^2)](1 + O(\kappa/\tau)) \\ &\approx 4\mathcal{E}_0\sigma^2\tau + (\sigma^4\tau^2 + 2\sigma^2\tau\mathcal{E}_0 + \mathcal{E}_0^2). \quad \kappa/\tau \rightarrow 0. \end{aligned}$$

We now turn to the formula of total probability (4·20) and we obtain

$$\begin{aligned} E\mathcal{E}_1(\tau(\omega), \omega; \kappa)\mathcal{E}_2(\tau(\omega), \omega; \kappa) &\approx \int_0^\infty [\sigma^2\tau + (\sigma^4\tau^2 + 2\sigma^2\tau\mathcal{E}_0 + \mathcal{E}_0^2)]P(\tau(\omega) = \tau)d\tau \\ &= 4\mathcal{E}_0\sigma^2\bar{\tau} + (\sigma^4\bar{\tau}^2 + 2\sigma^2\bar{\tau}\mathcal{E}_0 + \mathcal{E}_0^2), \quad \kappa/\tau \rightarrow 0. \end{aligned} \quad (4.23)$$

Finally, turn to the basic detection condition (4·19):

$$4\mathcal{E}_0\sigma^2\bar{\tau} + (\sigma^4\bar{\tau}^2 + 2\sigma^2\bar{\tau}\mathcal{E}_0 + \mathcal{E}_0^2) - 2\mathcal{E}_0(\sigma^2\bar{\tau} + \mathcal{E}_0) + \mathcal{E}_0^2 \approx \mathcal{E}_d^2$$

or

$$4\mathcal{E}_0\sigma^2\bar{\tau} + \sigma^4\bar{\tau}^2 \approx \mathcal{E}_d^2. \quad (4.24)$$

Suppose now that

$$\sigma^4 \bar{\tau}^2 \ll \mathcal{E}_0 \sigma^2 \bar{\tau}. \quad (4.25)$$

Thus the second term on the left-hand side of the equality (4.24) is essentially less than the second term. Hence, we have

$$4\mathcal{E}_0 \sigma^2 \bar{\tau} \approx \mathcal{E}_d^2. \quad (4.26)$$

Now we analyze the condition (4.25). It can be written as

$$\sigma^2 \ll \frac{\mathcal{E}_0 \bar{\tau}}{\tau^2} \quad (4.27)$$

or

$$\sigma^2 \ll \frac{\mathcal{E}_0 \bar{\tau}^2}{\bar{\tau} \tau^2}. \quad (4.28)$$

By the Cauchy-Bunyakovsky inequality $\bar{\tau}^2 \leq \tau^2$. Hence, we have

$$\sigma^2 \ll \frac{\mathcal{E}_0}{\bar{\tau}}. \quad (4.29)$$

We state again that the quantity σ^2 has the dimension of *signal's power*. Hence, the condition (4.27) is a constraint to signal's power. The quantity $\frac{\mathcal{E}_0}{\bar{\tau}}$ is average power of the background field (vacuum fluctuations) during the period of detection ("click's production"). Hence, our approach is about detection of *weak signals on the strong random background*.

§5. Probability of coincidence of clicks

Hence, during a long period of time T a pair of detectors clicks jointly N_{click} -times, where

$$N_{\text{click}} \approx \frac{T}{\bar{\tau}} \approx \frac{4\mathcal{E}_0 \sigma^2 T}{\mathcal{E}_d^2}, \quad (5.1)$$

where $\kappa/\bar{\tau} \rightarrow 0$ and the condition (4.29) holds. To find the probability of detection and match the real detection scheme which is used quantum experiments,¹⁸⁾ we have to use a proper normalization. This is again an important point of our considerations, cf. §5. In QM-experiments with composite systems probabilities are obtained through normalization corresponding to the sum of joint clicks in all pairs of detectors involved in the experiment. For example, for measurement of spin projections for a pair of electrons (e.g., entangled) to some axes a and b , we use two pairs of detectors: D_{1+}, D_{1-} , spin up and spin down for the first electron, and D_{2+}, D_{2-} , spin up and spin down for the second electron. We collect the numbers of clicks for the pairs of detectors: $N_{\text{click}(++)}$ for D_{1+}, D_{2+} , ..., $N_{\text{click}(--)}$ for D_{1-}, D_{2-} . Then we compute the total sum of clicks $N = N_{\text{click}(++)} + N_{\text{click}(+-)} + N_{\text{click}(-+)} + N_{\text{click}(--)}$ and it is used as the normalization factor for computing of probabilities, e.g., $P(++) = \frac{N_{\text{click}(++)}}{N}$. We repeat this scheme in the general case of detection of observable with discrete spectrum. Suppose that each component of a random bi-signal

$\phi(s) = (\phi_1(s), \phi_2(s))$ is complex vector $\phi_1(s) = (\phi_1(i, s))_{i=1}^m$, $\phi_2(s) = (\phi_2(i, s))_{i=1}^m$. Consider a *Gaussian* bi-signal. Assumptions (4.1)–(4.4) are modified ($i, j = 1, \dots, m$):

$$E\phi_k(i, s) = 0, \quad (5.2)$$

$$E\phi_k(i, s_1)\overline{\phi_k(j, s_2)} = \sigma_k^2(ij)\delta(s_1 - s_2)\sqrt{|s_1s_2|} + \mathcal{E}_0\delta(s_1 - s_2), \mathcal{E}_0 > 0, \quad (5.3)$$

$$E\phi_1(i, s_1)\overline{\phi_2(j, s_2)} = 2\sqrt{\mathcal{E}_0}\sigma_{12}(ij)\delta(s_1 - s_2)|s_1s_2|^{1/4}, \sigma_{12}(ij) \in \mathbf{C}. \quad (5.4)$$

To match completely the QM-theory, the condition (4.4), coupling between powers of signal's components σ_k^2 and “power of correlations between component” σ_{12} , has to be generalized to the case of vector processes in a rather tricky way [see (6.1)]. To clarify the main points of derivation of probabilities for coincidences, we start with a simpler stochastic model which will reproduce probabilities for coincidences, but not yet probabilities for measurements on each fixed component.*) In this section we proceed with stochastic processes satisfying Eqs. (5.2)–(5.4) and

$$\sigma_1^2(ij) = \sigma_2^2(ij) = |\sigma_{12}(ij)|^2 \equiv \sigma^2(ij). \quad (5.5)$$

We have

$$N_{\text{click}}(ij) \approx \frac{T}{\bar{\tau}_{ij}} \approx \frac{4\mathcal{E}_0\sigma^2(ij)T}{\mathcal{E}_d^2}. \quad (5.6)$$

The total number of clicks

$$N_{\text{click}} = \sum_{ij} N_{\text{click}}(ij) \approx \frac{4\mathcal{E}_0\Sigma^2T}{\mathcal{E}_d^2}, \quad (5.7)$$

where $\Sigma^2 = \sum_{ij} \sigma^2(ij)$ is the total averaged power of the bi-signal. The probability of detection for the pair of detectors D_{1i} and D_{2j} is given by

$$P_{\text{click}}(ij) = \frac{N_{\text{click}}(ij)}{N_{\text{click}}} \approx \frac{\sigma^2(ij)}{\Sigma^2}. \quad (5.8)$$

In fact, this is Born's rule. Consider the complex vector

$$\psi = (\sigma_{12}(ij)). \quad (5.9)$$

(Its dimension is m^2 , i.e., the squared dimension of the state space of components ϕ_k .) We remark that it is not normalized by one. Its squared norm is $\|\psi\|^2 = \Sigma^2$. We normalize this vector:

$$\Psi = \frac{\psi}{\|\psi\|}. \quad (5.10)$$

By our interpretation of QM this is a state vector. (So, the quantum state vector of a composite system is constructed from correlations between components of the

*) Of course, by knowing probabilities for coincidences we can derive probabilities for measurements on signals ϕ_k , $k = 1, 2$, by using the laws of (classical) probability theory. However, we want to introduce such a random bi-signal $\phi = (\phi_1, \phi_2)$ that by performing measurement (with detectors of the threshold type and the background field calibration) only on ϕ_k we shall obtain the corresponding quantum probability.

“prequantum stochastic process”; the quantum system is its symbolic representation in the operational formalism called QM.) In such notation we have

$$P_{\text{click}}(ij) = |\Psi(ij)|^2. \quad (5.11)$$

In our approach the QM-formalism is the operational formalism in which connection of the quantum state vector with correlations inside “prequantum random signals” is ignored. In QM the Ψ -state is invented formally; then it is used to find correlations. In our approach the Ψ -state is nothing else than the symbolic representation of correlations in the classical signals.

§6. The final stochastic model

We now consider a more tricky (classical) stochastic process. It satisfies the conditions (5.2)–(5.4) and, instead of condition (5.5), the condition:

$$\sigma_1^2(ij) = \sum_n \sigma_{12}(in)\bar{\sigma}_{12}(jn), \quad \sigma_2^2(ij) = \sum_n \sigma_{12}(ni)\bar{\sigma}_{12}(nj). \quad (6.1)$$

Later we shall write this condition in the matrix form, by using the matrix of cross-correlations $\hat{\sigma}_{12} = (\sigma_{12}(ij))$.

First we show that this process also reproduces the quantum probability for coincidence measurements on components ϕ_1 and ϕ_2 , cf. §5. We slightly modify the results of calculation in §4. We set $\mathcal{E}_k(i, s, \omega; \kappa) = |\phi_k^\kappa(i, s, \omega)|^2, k = 1, 2; i = 1, \dots, m$. Generalizing (4.13) and (4.14), we obtain

$$E\mathcal{E}_k(i, \tau, \omega; \kappa) = \tau\sigma_k^2(ii)(1 + O(\kappa/\tau)) + \mathcal{E}_0, k = 1, 2. \quad (6.2)$$

We now find $E\mathcal{E}_1(i, \tau, \omega; \kappa)\mathcal{E}_2(j, \tau, \omega; \kappa) = J_1(ij) + J_2(ij)$, cf. (4.21):

$$\begin{aligned} J_2(ij) &= 4\mathcal{E}_0|\sigma_{12}(ij)|^2\tau(1 + O(\kappa/\tau)), \quad \kappa/\tau \rightarrow 0, \\ J_1(ij) &= (\sigma_1^2(ii)\tau + \mathcal{E}_0)(\sigma_2^2(jj)\tau + \mathcal{E}_0)(1 + O(\kappa/\tau)) \\ &= [\sigma_1^2(ii)\sigma_2^2(jj)\tau^2 + \mathcal{E}_0\tau(\sigma_1^2(ii) + \sigma_2^2(ii)) + \mathcal{E}_0^2](1 + O(\kappa/\tau)), \quad \kappa/\tau \rightarrow 0. \end{aligned}$$

As always by using the formula of total probability, we obtain

$$\begin{aligned} E\mathcal{E}_1(i, \tau(\omega), \omega; \kappa)\mathcal{E}_2(j, \tau(\omega), \omega; \kappa) \\ \approx 4\mathcal{E}_0|\sigma_{12}(ij)|^2\bar{\tau} + \sigma_1^2(ii)\sigma_2^2(jj)\bar{\tau}^2 + \mathcal{E}_0\bar{\tau}(\sigma_1^2(ii) + \sigma_2^2(ii)) + \mathcal{E}_0^2, \quad \kappa/\tau \rightarrow 0. \end{aligned} \quad (6.3)$$

Bu using the condition (4.19) for i th and j th coordinates of the signals ϕ_1 and ϕ_2 we obtain

$$4\mathcal{E}_0|\sigma_{12}(ij)|^2\bar{\tau} + \sigma_1^2(ii)\sigma_2^2(jj)\bar{\tau}^2 \approx \mathcal{E}_d^2. \quad (6.4)$$

Under the assumption

$$\sigma_1^2(ii)\sigma_2^2(jj)\bar{\tau}^2 \ll \mathcal{E}_0|\sigma_{12}(ij)|^2\bar{\tau}, \quad (6.5)$$

we obtain the detection condition

$$4\mathcal{E}_0|\sigma_{12}(ij)|^2\bar{\tau} \approx \mathcal{E}_d^2, \quad (6.6)$$

which is the basic to derive detection probabilities for coincidence of clicks.

The condition (6.5) implies that

$$\sum_{ij} \sigma_1^2(ii) \sigma_2^2(jj) \ll \frac{\mathcal{E}_0}{\tau} \sum_{ij} |\sigma_{12}(ij)|^2, \quad (6.7)$$

i.e., for

$$\sigma_k^2 = \sum_i \sigma_k^2(ii), k = 1, 2, |\sigma_{12}|^2 = \sum_{ij} |\sigma_{12}(ij)|^2, \quad (6.8)$$

we have

$$\frac{\sigma_1^2 \sigma_2^2}{|\sigma_{12}|^2} \ll \frac{\mathcal{E}_0}{\tau}. \quad (6.9)$$

Quantities $\sigma_k^2, k = 1, 2$, have the meaning of average powers of signal's components ϕ_k ; the physical meaning of the quantity $|\sigma_{12}|^2$ is not straightforward. Formally, it can be considered as "power of correlations between components". By using this terminology we can say that our (coming) derivation of Born's rule is valid for signals of sufficiently low relative power (comparing with power of the background field) of the signal's components comparing with power of correlations between components. Consider again the complex vector $\psi = (\sigma_{12}(ij))$ [see (5.9)], and its normalization Ψ [see (5.10)]. Starting with detection condition (6.4) and repeating the steps of the derivation of §5, we obtain again Born's rule for detection of coincidences. Now we show that even for each single detector we obtain the quantum formula for probability.

By using the formula of total probability we obtain from (6.2) $E\mathcal{E}_k(i, \tau, \omega; \kappa) \approx \bar{\tau} \sigma_k^2(ii) + \mathcal{E}_0, k = 1, 2$. For the i th coordinate of the component ϕ_k we have the detection condition $\mathcal{E}_k(i, \tau_k(i), \omega; \kappa) = \mathcal{E}'_d$ where $\mathcal{E}'_d = \mathcal{E}_0 + \mathcal{E}_d$. Hence, $\bar{\tau} \sigma_k^2(ii) = \mathcal{E}_d$. The number of clicks is given by $N_{\text{click},k}(i) = \frac{T\sigma_k^2(ii)}{\mathcal{E}_d}$; the total number of clicks at all detectors for coordinates $\phi_k(i)$ of the component ϕ_k is given by $N_{\text{click},k} = \sum_i N_{\text{click},k}(i) = \frac{T\sigma_k^2}{\mathcal{E}_d}$ [see (6.8)]. Hence, $P_{\text{click},k}(i) = \frac{\sigma_k^2(ii)}{\sigma_k^2}$.

Now, for the vector Ψ consider the corresponding projection operator $\rho_\Psi = |\Psi\rangle\langle\Psi|$ and its partial traces $\rho_\Psi^{(k)}, k = 1, 2$. We also introduce operators $\hat{\sigma}_k^2 = (\sigma_k^2(ij))$. We have $\text{Tr}\hat{\sigma}_k^2 = \sigma_k^2$ and the equality (6.1) implies that $\rho_\Psi^{(k)} = \hat{\sigma}_k^2 / \text{Tr}\hat{\sigma}_k^2$. The final formula derived for the detection probability has the form

$$P_{\text{click},k}(i) = \text{Tr}\rho_\Psi^{(k)} \hat{C}_j,$$

where $\hat{C}_j = |e_j\rangle\langle e_j|$ is the projector onto the vector e_j corresponding to the detection in the i th channel for the ϕ_k .

We state again that each measurement under consideration corresponds to expansion of the signal's components with respect to some bases, say $\{e_{ki}\}, k = 1, 2$, in the state spaces of the signal's components ϕ_k . Detectors measure signals $\phi_k(i) = \langle\phi_k, e_{ki}\rangle, i = 1, \dots, m$.

§7. Violation of CHSH inequality

We borrow from QM the singlet state $\Psi = \frac{1}{\sqrt{2}}(|+\rangle|-\rangle - |-\rangle|+\rangle)$, where $e_{\pm} = |\pm\rangle$ is z -polarization basis. The simplest way to select the proper classical correlations is to identify ψ [see (5.9)] with $\Psi : \sigma(12) = -\sigma(21) = \frac{1}{\sqrt{2}}$. These correlations determine the classical random bi-signal $\phi(s) = (\phi_1(s), \phi_2(s))$. Each component is valued in the two dimensional complex space: $\phi_j(s) = \phi_j(+, s)e_+ + \phi_j(-, s)e_-$, $j = 1, 2$. We fix two angles θ_1, θ_2 and the corresponding bases: $e_{\pm}^{\theta_j}$. Consider expansions of the bi-signal's components: $\phi_j(s) = \phi_{\theta_j}(+, s)e_+^{\theta_j} + \phi_{\theta_j}(-, s)e_-^{\theta_j}$. Consider probabilities for joint measurements of the signals $\phi_{\theta_1}(\pm, s)$ and $\phi_{\theta_2}(\pm, s)$. Since they coincide with the corresponding quantum probabilities, these probabilities for the joint detection of classical random signals by the threshold type and properly calibrated detectors violate CHSH inequality.

The QM state Ψ determines correlations $\sigma_{12}(ij)$ up to a normalization factor. This state corresponds to a family of classical random fields. So, the correspondence between classical and quantum models is not one-to-one.

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Appendix A

— Derivation of Realism —

A.1. On Einstein-Podolsky-Rosen “derivation of realism”

In the EPR paper⁽¹¹⁾ objectivity was derived from the presence of the precise correlations (or anti-correlations). Einstein, Podolsky and Rosen claimed that they demonstrated reality of two incompatible variables, position and momentum. According to the paper⁽¹¹⁾ these variables can be jointly assigned (but not measured) to each of entangled systems. A detailed analysis of the EPR-derivation can be found in the author's works.^{(28)–(30)} It was shown that *the projection postulate plays the fundamental role* in this derivation. Then it was pointed out that Einstein, Podolsky and Rosen used, in fact, the Lüders form of this postulate. (Of course, at that time Lüders had not yet formulated his postulate in the precise form, but Einstein, Podolsky and Rosen used precisely this postulate, although in a very special situation.)

Lüders postulate. Let Ψ be a pure state and let $\hat{A} = \sum_k a_k \hat{P}_k$ be a quantum observable with the purely discrete spectrum; here (a_k) are its eigenvalues and (\hat{P}_k)

are the projections on corresponding eigenspaces; $a_k \neq a_l, k \neq l$, and in general spectrum is degenerate, i.e., the image of \hat{P}_k can have the dimension larger than 1. Suppose that a measurement of this quantum observable was done and the result a_k was obtained. Then the initial state Ψ is projected onto the state $\Psi_k = \frac{\hat{P}_k \Psi}{\|\hat{P}_k \Psi\|}$.

In particular, by the Lüders postulate, for any pure initial state, the output state is also pure. However, von Neumann motivated that such a type of the projection postulate is applicable only to observables with *nondegenerate spectra*. In the case of measurements of observables with degenerate spectra the situation is more complicated. In particular, by the von Neumann projection postulate in this case even for a pure initial state the output state is a mixture state. As was already stressed, the EPR-derivation of the presence of elements of reality corresponding to incompatible observables was fundamentally based on usage of the Lüders postulate. This is simply impossible to do on the basis of the von Neumann postulate. (We remark that in the EPR-measurement scheme for compound systems the operators have degenerate spectra: measurement on one of subsystems of a compound system is always degenerate, as the result of the multidimensionality of the state space of another subsystem.)

We note that in his book⁴²⁾ von Neumann, in fact, analyzed the scheme of measurements on entangled systems. His analysis is very similar to consideration in the EPR paper. However, since von Neumann used another form of the projection postulate, he did not come to such fundamental conclusions as Einstein, Podolsky and Rosen did in Ref. 11). Moreover, von Neumann supported Born's claim about nonobjectivity of quantum observables.

Remark 7. We also remark that the position and momentum operators have continuous spectra, but in their analysis Einstein, Podolsky and Rosen proceeded by manipulating with operators having the purely discrete spectra. This is an additional problem of the EPR paper. However, this problem is not so basic as the aforementioned problem of usage of the restrictive version of the projection postulate. Von Neumann stressed⁴²⁾ that only observables represented by operators with discrete spectra can be really measured. He considered observables represented by operators with continuous spectra as mathematical idealizations; their physical meaning can be approached only via approximation by operators with discrete spectra.

Hence, if one takes seriously von Neumann's motivation of impossibility of usage of the projection postulate in "Lüders' form" for compound systems, he should accept that the analysis of Einstein, Podolsky and Rosen of the measurement scheme for entangled quantum systems is too restrictive and their "derivation" of realism was not justified.

A.2. *Von Neumann's viewpoint to measurement of compatible observables: classical and quantum nonlocality of the Bell-measurements*

We remind that Bohm elaborated a version of the EPR-argument for discrete observables such as e.g. spin or polarization. This Bohm approach (and not the original EPR-approach) was used by Bell¹²⁾ to support experimentally his argument

based on violation of Bell's inequality. Sometimes Bell's argument is identified with the ERP-argument. However, they are fundamentally different. Einstein, Podolsky and Rosen discussed determination of elements of reality for one subsystem of a compound system from measurement performed on another subsystem. To find statistical correlations, Bell was interested in *joint measurements* on both subsystems.

The Bell-scheme is based on measurement of pairs of quantum observables A_1 and A_2 represented by commutative operators, say \hat{A}_1 and \hat{A}_2 . These observables are measured on subsystems S_1 and S_2 of the compound system $S = (S_1, S_2)$. Since the first detailed analysis of joint measurement of compatible observables was performed by von Neumann⁴²⁾ (in fact, Dirac⁴³⁾ proceeded along the same lines and earlier, but he used generalized eigenvectors), this is useful to remind the von Neumann formalization of the procedure of joint measurement. By him, to measure jointly the pair (A_1, A_2) , we have to find an observable, say B (represented by \hat{B}), such that the results of measurements of A_1 and A_2 are represented as functions of the results of measurements of B : $a_{1i} = f_1(b_i)$, $a_{2i} = f_2(b_i)$. The corresponding operators are coupled by the same functional relation, $\hat{A}_1 = f_1(\hat{B})$, $\hat{A}_2 = f_2(\hat{B})$. This is evident [see also works²⁸⁾–³⁰⁾] that in the case of measurement on a compound system composed of measurements on its subsystems, the observable B has to be “nonlocal”, in the sense that the results of measurements on S_1 and S_2 have to be coupled in some way. This is not “quantum nonlocality” in Bell's sense of action at the distance, this is classical nonlocality of the measurement setup. As was pointed out in papers,²⁸⁾–³⁰⁾ in the case of the Bell-type measurements nonlocality of the measurement setup is a consequence of usage of the constraint of matching of clicks at the detectors D_{1i} , $i = 1, 2$ and D_{2i} , $i = 1, 2$ measuring observables on the subsystems S_1 and S_2 — to identify two clicks as corresponding to a pair of entangled photons. We emphasize that this condition (based on a time window) is not just an experimental technicality (as it is rather commonly believed); in our approach this is an important part of the Bell-experiment. In TSD it plays the crucial role [see (4-15)]. Otherwise TSD provides a local measurement scheme, each detector works totally locally without any action at the distance. (These detectors measure classical fields; in principle, the TSD can be experimentally realized even for macroscopic classical electromagnetic fields. And the quantum correlations will be reproduced as well. It seems that nobody would suspect that measurements of macroscopic fields with the aid of threshold-type detectors are nonlocal in the sense of action at the distance.) The correlations are created by the signal's source and the background field. These are correlations of subquantum degrees of freedom. They can be lifted to the level of quantum observables, i.e., the correlations of quantum observables are “derivatives” of the correlations of the subquantum degrees of freedom.

Remark 8. (On a role of the background field.) The background field [see (4-5)] is a random field and its contribution to correlations has no deterministic counterpart wanted by Einstein, Podolsky and Rosen. The presence of the common background field may be interpreted as a sort of nonlocality. However, this is again usual classical nonlocality. In particular, it can be imitated by using macroscopic electromagnetic fields: one source of correlated signals combined with a source of the

random background field. It is impossible to reproduce the quantum correlations for entangled states without this field. We cannot put $\mathcal{E}_0 = 0$ in (4.5).

A.3. Bell's "derivation" of realism from precise (anti)correlations

Bell's position can be characterized as *nonlocal realism*. And for him it was evident that Einstein, Podolsky, and Rosen¹¹⁾ proved the presence of elements of reality for the EPR-states. In his works¹²⁾ he did not go deeply in the EPR-derivation and, in particular, he was not aware about the role played by the projection postulate. Bell simply claimed that the presence of precise (anti)correlations evidently implied the objectivity of corresponding observables. (This is a rather common misunderstanding of the EPR paper. If it were true, the authors of Ref. 11) would simply present this viewpoint in one sentence; there would be no need for long and technical derivation of the presence of elements of reality for incompatible observables and, in particular, application of the projection postulate, its "Lüders version".) Since Bell proceeded without appealing to the quantum formalism by just playing with the EPR-argument, it is useful to reply to him in similar heuristically clear terms. Before to do this, we would like to stress again that in TSD the main *experimental "technicalities" are lifted to the level of the key elements of theory*. The detection threshold is such a fundamental technicality.

The detection threshold scheme for classical waves implies that, in fact, we not only could not assign the values of physical observables to a quantum system before measurement, but we *cannot assign one concrete value even after measurement*. The key point is that there are no "quantum systems", there are only waves. As was well known, since Bohr's analysis of the two slit experiment,^{21),47)} a classical wave should produce *double clicks* with nonzero probability. A single "photon-wave" which passes a beam splitter would produce (with some probability) clicks in detectors placed in both channels of the beam splitter. Although the quantum theory predicts that the number of such double clicks is zero, in reality it is far from to be zero. Instead of the idealized claim on the absence of double clicks, experimenters checked the value of the relative number of double clicks in the form of the coefficient of the *second order coherence* $g^{(2)}(0)$ [see works⁴⁴⁾⁻⁴⁶⁾]. In experiments with photons this coefficient is less than one; at the same time all known classical and semiclassical theories of quantum light predicted the values exceeding one.⁴⁶⁾ (This was the main reason for rejection of these theories which were very popular in the laser physics community.) In TSD the second order interference is less than one [see Ref. 48) for details], but *it is fundamentally nonzero*. So, even in pure theory we cannot claim that the double clicks can be totally excluded from consideration.

Denote the detection threshold by \mathcal{E}_d and the average energy of photons emitted by a source by \mathcal{E}_δ . For example, if we were having a source emitting photons of the fixed frequency ν , then in this experiment $\mathcal{E}_\delta = h\nu$. In reality (at least in the Bohm-Bell experiment) experimenters are not able to prepare photons of the fixed frequency; instead wave packets are emitted. Here \mathcal{E}_δ is the average energy of these wave packets. In Ref. 48) it was shown that, for any two channels, the coefficient $g^{(2)}(0) = f(\mathcal{E}_d, \mathcal{E}_\delta) \sim \left(\frac{\mathcal{E}_d}{\mathcal{E}_\delta}\right)^2$. By increasing \mathcal{E}_d experimenters can decrease the num-

ber of double clicks, but it still be nonzero. Moreover, one cannot increase \mathcal{E}_d too much. By TSD the number of clicks in a channel during a period of time T is inverse proportional to \mathcal{E}_d [see (3·16)]. So, if an experimenter decreases essentially the number of double clicks by increasing \mathcal{E}_d , he automatically decreases the size of the statistical sample. In this situation, to collect significant statistics, he should wait for a very long time. However, he could not do this, since it would be impossible to preserve stability of experimental conditions for so long a time. We summarize: *in TSD double clicks are inescapable; they are not technicalities, but exhibitions of fundamental properties of classical signals of the wave-type.*

The presence of double clicks destroys Bell's "derivation of realism" from precise (anti)correlations. Consider a pair entangled photons, $S = (S_1, S_2)$, in the Bell state, $\Psi = (|++\rangle + |--\rangle)/\sqrt{2}$, which were emitted by a source, e.g., via parametric down conversion. One may hope to assign to e.g. S_1 the polarization vector. To determine its projection on the θ -axis (we consider the two dimensional illustration), he selects the θ -orientation of the polarization beam splitter for S_2 . Then in the absence of double clicks on the S_2 -side, he could assign the result of measurement on S_2 as the element of reality for S_1 . However, if, instead of a single click in only one of two polarization channels for S_2 , he gets clicks in both channels, and then the aforementioned argument collapses. One may argue that the relative number of double clicks can be done sufficiently small even in TSD and he may hope for realism at least for the restricted sample without double clicks in polarization channels at either side.

However (this is an important point!), in TSD *the presence/absence of double clicks depends not only on the photon pair, but also on the orientation of polarization beam splitter.* Hence, for a fixed photon pair $S = (S_1, S_2)$, by making the θ -setting measurement on S_2 and even being lucky not to see the clicks in both channels, we can only assign to S_1 the observed value of the θ -projection, since we are not sure that if the θ' -setting, $\theta' \neq \theta$, were selected, we would not get double clicks on the S_2 -side.

This tricky situation at the level of "reproduction of elements of reality" via measurements reflects the random wave structure of signals described by PCSFT. For example, a photon is a pulse of random electromagnetic field, its polarization fluctuates with time. We are not yet able (and it may happen that we would be never able) to approach such a fine time scale that it would be possible to monitor the intrinsic polarization of the "photonic random wave". Quantum measurement devices realize indirect measurements of e.g. polarization. In reality we have nonzero signals in both channels of a polarization beam splitter and both branches have polarizations. (Of course, if PCSFT is correct.) By getting a click in one branch, say polarization up, (if there are no double clicks) we are only sure that the signal in this branch has sufficiently high energy to approach the detection threshold. The situation is even more complicated. By TSD a detector integrates a signal over some time window κ [see (3·4)]. And it reacts to the energy of the integrated signal. Hence, a click in one fixed channel of the polarization beam splitter only means that the energy of integrated signal was sufficiently high to approach the detection threshold.

Now let us continue to play around the TDS-detection scheme. Let the detec-

tion threshold be \mathcal{E}_d . Suppose that at the instant of time τ the click was produced by the detector D_{2+} , polarization up. And no double clicks were seen (i.e., nothing happened in D_{2-}). By Einstein-Podolsky-Rosen-Bell “an element of reality” can be assigned to S_1 . (We proceed under the assumption of locality, no action at the distance.) However, this “element of reality” depends on the selection of the detection threshold. To illustrate this statement, we have to use *counterfactual arguments* (essential in the Einstein-Podolsky-Rosen-Bell derivation of realism). Suppose that, instead of the above measurement with the threshold \mathcal{E}_d , we would perform measurement with another threshold \mathcal{E}'_d . Let us assume that: $\mathcal{E}_d < \mathcal{E}'_d$. Then it might happen that we get the click not D_{2+} , but in D_{2-} . Of course, the instant of detection (“creation of the element of reality”) τ' would be larger. At the instant of time τ the signal in the D_{2+} -channel was not strong enough to go over \mathcal{E}'_d , and a bit later the signal in D_{2-} -channel induced the click. In the absence of double clicks an element of reality, polarization down, would be assigned to S_1 . Instead of Einstein-Podolsky-Rosen-Bell “absolute elements of reality” we have context dependent elements of reality, cf. Ref. 22). This is very much in the spirit of Bohr’s reply⁴⁷⁾ to the EPR paper.¹¹⁾

Hence, my position is *a rather strange mixture of the EPR and Bohr positions. On one hand, I disagree with Bohr’s claim that quantum mechanics is complete, so I support the EPR-claim on incompleteness of quantum mechanics. On the other hand, I reject the EPR-claim on a possibility to assign (noncontextual) elements of reality to at least some quantum states and I completely agree with Bohr’s statement that the whole experimental context has to be taken into account.*

Appendix B

— Gaussian Integrals —

Let W be a real Hilbert space. Consider a σ -additive Gaussian measure p on the σ -field of Borel subsets of W . This measure is determined by its covariance operator $B : W \rightarrow W$ and mean value $m \in W$. For example, B and m determine the Fourier transform of p :

$$\tilde{p}(y) = \int_W e^{i(y,\phi)} dp(\phi) = e^{\frac{1}{2}(By,y) + i(m,y)}, y \in W.$$

(In probability theory it is called the characteristic functional of the probability distribution p .) In what follows we restrict our considerations to *Gaussian measures with zero mean value*: $(m,y) = \int_W (y,\psi) dp(\psi) = 0$ for any $y \in W$. Sometimes there will be used the symbol p_B to denote the Gaussian measure with the covariance operator B and $m = 0$. We recall that the covariance operator B is defined by its bilinear form $(By_1, y_2) = \int (y_1, \phi)(y_2, \phi) dp(\phi)$, $y_1, y_2 \in W$.

Let Q and P be two copies of a real Hilbert space. Let us consider their Cartesian product $H = Q \times P$, “phase space”, endowed with the symplectic operator $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Consider the class of Gaussian measures (with zero mean value) which are invariant with respect to the action of the operator J ; denote this class

$\mathcal{S}(H)$. It is easy to show that $p \in \mathcal{S}(H)$ if and only if its covariance operator commutes with the symplectic operator.^{31)–36)}

As always, we consider complexification of H (which will be denoted by the same symbol), $H = Q \oplus iP$. The complex scalar product is denoted by the symbol $\langle \cdot, \cdot \rangle$. The space of bounded Hermitian operators acting in H is denoted by the symbol $\mathcal{L}_s(H)$.

We introduce the complex covariance operator of a measure p on the complex Hilbert space $H : \langle Dy_1, y_2 \rangle = \int_H \langle y_1, \phi \rangle \langle \phi, y_2 \rangle dp(\phi)$. Let p be a measure on the Cartesian product $H_1 \times H_2$ of two complex Hilbert spaces. Then its covariance operator has the block structure

$$D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}, \tag{B.1}$$

where $D_{ii} : H_i \rightarrow H_i$ and $D_{ij} : H_j \rightarrow H_i$. The operator is Hermitian. Hence $D_{ii}^* = D_{ii}$, and $D_{12}^* = D_{21}$.

Let H be a complex Hilbert space and let $\hat{A} \in \mathcal{L}_s(H)$. We consider its quadratic form (which will play an important role in our further considerations) $\phi \rightarrow f_A(\phi) = \langle \hat{A}\phi, \phi \rangle$. We make a trivial, but ideologically important remark: $f_A : H \rightarrow \mathbf{R}$, is a “usual function” which is defined point wise. We use the equality [see, e.g., Ref. 36]):

$$\int_H f_A(\phi) dp_D(\phi) = \text{Tr } D\hat{A}. \tag{B.2}$$

Let p be a Gaussian measure of the class $\mathcal{S}(H_1 \times H_2)$ with the (complex) covariance operator D and let operators \hat{A}_i belong to the class $\mathcal{L}_s(H_i), i = 1, 2$. Then

$$\int_{H_1 \times H_2} f_{A_1}(\phi_1) f_{A_2}(\phi_2) dp(\phi) = \text{Tr } D_{11} \hat{A}_1 \text{Tr } D_{22} \hat{A}_2 + \text{Tr } D_{12} \hat{A}_2 D_{21} \hat{A}_1. \tag{B.3}$$

This equality is a consequence of the following general result:³⁶⁾

Let $p \in \mathcal{S}(H)$ with the (complex) covariance operator D and let $\hat{A}_i \in \mathcal{L}_s(H)$. Then

$$\int_H f_{A_1}(\phi) f_{A_2}(\phi) dp(\phi), = \text{Tr } D\hat{A}_1 \text{Tr } D\hat{A}_2 + \text{Tr } D\hat{A}_2 D\hat{A}_1. \tag{B.4}$$

References

- 1) G. 't Hooft, gr-qc/9903084.
- 2) G. 't Hooft, quant-ph/0604008.
- 3) G. 't Hooft, Herald of Russian Academy of Science **81** (2011), 907; quant-ph/0701097v1.
- 4) I. Ojima, arXiv:1112.5525.
- 5) I. Ojima, arXiv:1102.1527.
- 6) R. Harada and I. Ojima, arXiv:0810.3400v1.
- 7) I. Ojima, AIP Conf. Proc. **1327** (2011), 197.
- 8) H.-T. Elze, J. Phys.: Conf. Ser. **174** (2009), 012009.
- 9) H.-T. Elze, J. Phys.: Conf. Ser. **67** (2007), 012016.
- 10) H.-T. Elze, arXiv:0806.3408.
- 11) A. Einstein, B. Podolsky and N. Rosen, Phys. Rev. **47** (1935), 777.

- 12) J. Bell, *Speakable and Unspeakable in Quantum Mechanics* (Cambridge University Press, Cambridge, 1987).
- 13) M. Ozawa, AIP Conf. Proc. **1327** (2011), 53.
- 14) M. Ozawa and Yu. Kitajima, Found. Phys. **42** (2012), 475.
- 15) M. Ohya and N. Watanabe, Jpn. J. Industrial and Appl. Math. **3** (1986), 197.
- 16) N. Watanabe, AIP Conf. Proc. **1327** (2011), 289.
- 17) G. Jaeger, *Entanglement, Information, and the Interpretation of Quantum Mechanics* (Springer, Heidelberg-Berlin-New York, 2010).
- 18) A. Aspect, J. Dalibard and G. Roger, Phys. Rev. Lett. **49** (1982), 1804.
- 19) A. Zeilinger, *Dance of the Photons: From Einstein to Quantum Teleportation* (Farrar, Straus and Giroux, New York, 2010).
- 20) G. Adenier, J. Russian Laser Research **29** (2008), 409.
- 21) A. Plotnitsky, *Epistemology and Probability: Bohr, Heisenberg, Schrödinger, and the Nature of Quantum-Theoretical Thinking* (Springer, Heidelberg-Berlin-New York, 2009).
- 22) A. Khrennikov, *Contextual Approach to Quantum Formalism* (Springer, Heidelberg-Berlin-New York, 2009).
- 23) A. Khrennikov, *Quantum Theory: Reconsideration of Foundations-5* (AIP, Melville, NY, 2010).
- 24) A. Fine, Phys. Rev. Lett. **48** (1982), 291.
- 25) K. Hess and W. Philipp, Europys. Lett. **57** (2002), 775.
- 26) H. De Raedt, K. De Raedt and K. Michielsen, Europys. Lett. **69** (2005), 861.
K. De Raedt, K. Keimpema, H. De Raedt, K. Michielsen and S. Miyashita, Eur. Phys. J. B **53** (2006), 139.
S. Zhao, S. Yuan, H. De Raedt, K. De Raedt and K. Michielsen, Europhys. Lett. **82** (2008), 40004.
K. Hess, K. Michielsen and H. De Raedt, Europhys. Lett. **87** (2009), 60007.
- 27) A. Khrennikov, Found. Phys. **35** (2005), 1655.
- 28) A. Khrennikov, Laser Physics **19** (2009), 346.
- 29) A. Khrennikov, Et Int. J. Quantum Information (IJQI) **7** (2009), 71.
- 30) A. Khrennikov, J. Russian Laser Research **29** (2008), 296.
- 31) A. Khrennikov, J. of Phys. A **38** (2005), 9051.
- 32) A. Khrennikov, Phys. Lett. A **357** (2006), 171.
- 33) A. Khrennikov, Phys. Lett. A **372** (2008), 6588.
- 34) A. Khrennikov, Physica E **42** (2010), 287.
- 35) A. Khrennikov, Europhys. Lett. **8** (2009), 40005.
- 36) A. Khrennikov, Europhys. Lett. **90** (2010), 40004.
- 37) H. Bartosik, J. Klepp, C. Schmitzer, S. Sponar, A. Cabello, H. Rauch and Y. Hasegawa, Phys. Rev. Lett. **103** (2009), 040403.
- 38) G. Kirchmair, F. Zahringer, R. Gerritsma, M. Kleinmann, O. Gühne, A. Cabello, R. Blatt and C. F. Roos, Nature **460** (2009), 494.
- 39) H. Carsten, "The Kochen-Specker Theorem", in *The Stanford Encyclopedia of Philosophy* (Winter 2008 Edition), ed. E. N. Zalta,
<http://plato.stanford.edu/archives/win2008/entries/kochen-specker/>
- 40) I. I. Gikhman and A. V. Skorokhod, *The Theory of Stochastic Processes* (Springer, Berlin-Heidelberg-New York, 1979).
- 41) A. N. Kolmogoroff, *Grundbegriffe der Wahrscheinlichkeitsrechnung* (Springer Verlag, Berlin, 1933); English translation: A. N. Kolmogorov, *Foundations of Theory of Probability* (Chelsea Publishing Company, New York, 1956).
- 42) J. von Neuman, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, 1955).
- 43) P. A. M. Dirac, *The Principles of Quantum Mechanics* (Clarendon Press, Oxford, 1995).
- 44) P. Grangier, *Etude Expérimentale de Propriétés Non-classiques de la Lumière: Interférence à un Seul Photon* (Université de Paris-Sud, Centre D'Orsay, 1986).
- 45) P. Grangier, G. Roger and A. Aspect, Europhys. Lett. **1** (1986), 173.
- 46) M. Beck, J. Opt. Soc. Am. B **24** (2007), 2972.
- 47) N. Bohr, Phys. Rev. **48** (1935), 696.
- 48) A. Khrennikov, B. Nilsson and S. Nordebo, J. Phys.: Conf. Ser. **361** (2012), 012030.