# Quantum Pseudo-Telepathy 

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#### Abstract

Quantum information processing is at the crossroads of physics, mathematics and computer science. It is concerned with that we can and cannot do with quantum information that goes beyond the abilities of classical information processing devices. Communication complexity is an area of classical computer science that aims at quantifying the amount of communication necessary to solve distributed computational problems. Quantum communication complexity uses quantum mechanics to reduce the amount of communication that would be classically required.

Pseudo-telepathy is a surprising application of quantum information processing to communication complexity. Thanks to entanglement, perhaps the most nonclassical manifestation of quantum mechanics, two or more quantum players can accomplish a distributed task with no need for communication whatsoever, which would be an impossible feat for classical players.

After a detailed overview of the principle and purpose of pseudo-telepathy, we present a survey of recent and no-so-recent work on the subject. In particular, we describe and analyse all the pseudo-telepathy games currently known to the authors.


Keywords: Entanglement, Nonlocality, Bell's theorem, Quantum information processing, Quantum communication complexity, Pseudo-telepathy.

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## 1 Introduction

### 1.1 Quantum Nonlocality

Albert Einstein was convinced that the physical world is local and realistic. By local, he meant that no action performed at some location $A$ could have an effect at some other remote location $B$ in a time faster than that required by light to travel from $A$ to $B$. In particular, instantaneous action at a distance would be impossible. By realistic, he meant that measurements could only reveal "elements of reality" that were already present in the system being measured. He attributed Heisenberg's uncertainty relations to the fact that any measurement of a particle's position, no matter how subtle, would disturb its momentum, and vice versa, and therefore one could never know both position and momentum simultaneously with arbitrary precision.

But Niels Bohr said no. In his interpretation of quantum mechanics, Heisenberg's uncertainty relations are unavoidable because it is impossible for a particle to have both its position and momentum simultaneously defined with arbitrary precision. It is the act of measuring one or the other that makes it come to existence. How could you disturb that which does not even exist in the first place? You can't deny Einstein's realism more strongly than this!

In an attempt to "prove" his point, Einstein wrote a famous paper in 1935 with Boris Podolsky and Nathan Rosen [1]. In that paper, they introduced the notion of quantum entanglement, according to which it would be possible in some cases to determine the position or the momentum of a particle without in any way interacting with it. Based on the locality assumption, they concluded that both position and momentum had to be simultaneously well-defined for that particle, contradicting Bohr's view.

In a curious turn of history, John Bell proved three decades later that the entanglement phenomenon predicted by Einstein, Podolsky and Rosen entails statistics that would be impossible to explain by any local realistic theory whatsoever [2]. In other words, Einstein's argument had backfired, with entanglement becoming the most convincing argument in favour of Bohr's view! Despite appearances, quantum mechanics does not really contradict the locality assumption of Einstein: It is still the case that no action performed at A can have an instantaneous observable effect at B. But it certainly does contradict realism.

It remained to perform actual experiments, which was done in the 1980s by Alain Aspect et al, and sure enough the predictions of quantum mechanics come true $[3,4,5]$. Subsequent experiments were conclusively conducted by Nicolas Gisin and collabora-
tors over more than 10 kilometres of distance between points $A$ and $B[6]$, and even in relativistic settings in which both detectors, each in its own inertial reference frame, are first to do the measurement [7]! To get a convincing experimental demonstration that the physical world is not local realistic, we must make a large number of careful experiments and collect significant statistical evidence that such data would be overwhelmingly unlikely in a classical local realistic world. Moreover, the apparatus must be sufficiently accurate to rule out a variety of loopholes that would make a classical "explanation" possible again-more on this issue in Sections 1.3 and 1.4.

Despite David Mermin's heroic effort at explaining Bell's theorem [8], it is not so easy to convey the significance of these experiments to a non-scientist because nontrivial probabilistic arguments are involved. It would be much nicer to exhibit an observable behaviour that is obviously impossible in any classical world so dear to Einstein ${ }^{1}$. A significant step in the right direction was taken by Daniel Greenberger, Michael Horne and Anton Zeilinger [9], and later by Lucien Hardy [10], when they introduced quantum phenomena that can be shown to be classically impossible without resorting to probabilities: A classical contradiction results from what is and what is not possible in the observed behaviour, not from the specific probabilities of various events. These phenomena have been called by such names as "Bell's theorem without inequalities". It is our belief that pseudo-telepathy - which is the topic of this survey - takes another step in this direction, even though the Greenberger-Horne-Zeilinger conundrum (but not Hardy's) can be interpreted as an early instance of pseudo-telepathy.

It is important to mention a common misconception in reasoning about experimental realizations of Bell's theorem without inequalities. Too often, we read that they allow to rule out local realism in a single run: "The quantum nonlocality can thus, in principle, be manifest in a single run of a certain measurement" [11]. The fallacy here is that if the players (classical or quantum) win for a single run, we cannot conclude anything: they could have been lucky at guessing. As Asher Peres wrote, about those who made this mistake: "The list of authors is too long to give explicitly, and it would be unfair to give only a partial list" [12].

### 1.2 Pseudo-Telepathy

Consider two parties, whom we shall call Alice and Bob. They pretend to be endowed with telepathic powers. To convince scientists Xavier and Yolande, they agree to play the following game. First, Xavier and Yolande secretly decide on two lists of animals,

[^1]Table 1: Example of a game.

| Round | Xavier | Yolande |
| :---: | :---: | :---: |
| 1 | lion | giraffe |
| 2 | tiger | tiger |
| 3 | hyena | elephant |
| 4 | crocodile | alligator |
| 5 | platypus | platypus |
| $\ldots$ | $\ldots$ | $\ldots$ |

such as that shown in Table 1. Then, Xavier takes Alice far away from Bob and Yolande. At predetermined times, Xavier and Yolande name animals from their lists to Alice and Bob. For example, Xavier and Yolande simultaneously say "lion" and "giraffe", respectively. Without consulting each other, Alice and Bob must immediately decide whether or not they were presented with the same animal; in this case, they should both answer "no!". If Alice and Bob succeed systematically in a sufficiently long sequence of trials, Xavier and Yolande will conclude that Alice and Bob are able to communicate somehow. But if communication is classically impossible because Alice and Bob are sufficiently far apart that a signal from Alice going at the speed of light would not reach Bob in time to influence his answer (and vice versa from Bob to Alice), then Xavier and Yolande would be forced into believing that Alice and Bob are able to communicate in a way unknown to (classical) physics. In this case, telepathy would not seem to be worse than any other esoteric "explanation", would it?

It turns out that quantum mechanics cannot help Alice and Bob win this animalguessing game. (Otherwise, a solution to this game could be harnessed to provide faster-than-light signalling.) But there are other similar games that are equally impossible for classical players, yet they can be won systematically, without any communication, provided Alice and Bob share prior entanglement. This is the phenomenon we call "pseudo-telepathy" because it would appear as magical as "true" telepathy to a classical physicist, yet it has a perfectly scientific explanation: quantum mechanics.

There are several reasons to be interested in pseudo-telepathy games.
$\diamond$ Some of these games are conceptually very simple. Their classical impossibility can be explained in a few minutes to any reasonably intelligent person. This makes their quantum possibility all the more impressive. The best examples in
this category are the Mermin-GHZ game and the magic square game, described in Sections 3 and 5, respectively.
$\diamond$ If implemented successfully, any one of these games would provide extremely convincing evidence that classical physics does not rule the world for anyone who is already convinced that faster-than-light signalling is impossible (and therefore that it is not because Alice and Bob are able to communicate, however covertly, that they win the game systematically).
$\diamond$ The theoretical and experimental study of pseudo-telepathy has the potential to provide the first loophole-free demonstration that the physical world cannot be local realistic.
$\diamond$ In introducing the broader field of quantum communication complexity in 1993, Andrew Yao asked if quantum mechanics could be used to reduce the amount of communication required to solve some distributed computational tasks, but he could not answer the question [13]. Surprisingly, a pseudo-telepathic version of this question had already been solved ten years previously [14] by Peter Heywood and Michael Redhead ${ }^{2}$ - see Section 2. Please consult [15] for a survey of quantum communication complexity and its interplay with pseudo-telepathy.

We are now ready to give a formal definition of pseudo-telepathy. For the sake of simplicity, we define it in the context of two players, Alice and Bob. The generalization to multi-party settings is obvious.

A two-party game is defined as a sextuple $G=\langle X, Y, A, B, P, W\rangle$, where $X, Y, A$ and $B$ are sets, $P \subseteq X \times Y$ and $W \subseteq X \times Y \times A \times B$. It is convenient to think of $X$ and $Y$ as the input sets, $A$ and $B$ as the output sets, $P$ as a predicate on $X \times Y$ known as the promise, and $W$ as the winning condition, which is a relation between inputs and outputs that has to be satisfied by Alice and Bob whenever the promise is fulfilled. [In the generalization to 3 parties, we use $X, Y, Z$ as input sets and $A$, $B, C$ (for Charlie) as output sets; $P$ and $W$ are changed accordingly. In the $n$-party generalization, we use $X=X_{1} \times X_{2} \times \cdots \times X_{n}$ and $A=A_{1} \times A_{2} \times \cdots \times A_{n}$ as input and output sets, respectively, $P$ becomes a predicate on $X$ and $W$ becomes a relation on $X \times A$.]

At the outset of the game, Alice and Bob are allowed to discuss strategy and exchange any amount of classical information, including the value of random variables. This is carried out in secret of Xavier and Yolande. If Alice and Bob are quantum

[^2]players, they are also allowed to share unlimited amounts of entanglement. Xavier and Yolande are also allowed to discuss strategy in secret of Alice and Bob. Then, Alice and Bob are separated and they will not be allowed to communicate any further until the game is over. In each round of the game, Alice and Bob are presented by Xavier and Yolande with inputs $x \in X$ and $y \in Y$, respectively. The task of Alice and Bob is to produce outputs $a \in A$ and $b \in B$, respectively. We say that Alice and Bob win the round if either $(x, y) \notin P$ (the promise does not hold) or $(x, y, a, b) \in W$. The pairs $(x, y)$ and $(a, b)$ are called the question and the answer for this round, respectively. Any question $(x, y)$ is said to be legitimate if it belongs to $P$, and any answer $(a, b)$ is said to be appropriate if $(x, y, a, b) \in W$. We say that Alice and Bob win the game if they keep winning round after round. Finally, we say that Alice and Bob have a winning strategy if they are mathematically certain to win the game as long as they have not exhausted all the classical information and quantum entanglement (if any) shared at the outset of the game. Please note the difference between winning the game and having a winning strategy: no matter how many rounds Alice and Bob have won, only statistical evidence can be gathered by Xavier and Yolande towards the hypothesis that they indeed have a winning strategy, but a single lost round suffices to discredit any claim to a winning strategy. (See Section 1.3 for practical considerations.)

To continue our earlier example from Table 1, $X$ and $Y$ would be nontrivial sets of animal names, $A=B=\{$ yes, no $\}, P=X \times Y$ (which is to say that all questions are legitimate) and ( $x, y, a, b) \in W$ if and only if either $x=y$ and $a=b=$ yes or $x \neq y$ and $a=b=$ no. In the first round, for instance, Xavier and Yolande ask the question (lion, giraffe) and Alice and Bob should provide the answer (no, no). As we said before, this game does not admit a winning strategy, be it classical or quantum, but Alice and Bob could still win by being exponentially lucky in the number of rounds.

Finally, we say that a game exhibits pseudo-telepathy if it does not admit a winning strategy whenever Alice and Bob are restricted to being classical players, yet it does admit a winning strategy provided quantum players Alice and Bob share a sufficient supply of entanglement.

We can quantify the quantumness of a pseudo-telepathy game by how bad classical players would be at it. This is interesting because the harder a pseudo-telepathy game is for classical players, the more convincing it is when quantum players win it round after round. A more subtle reason to be interested in this issue is that quantum players will not be perfect in real life: it is unavoidable that they will lose a few rounds now and then. But if they are still better at the game than any classical player could be, the main purpose of proving that the world cannot be local realistic will still be fulfilled.

This is how we hope to use pseudo-telepathy games to circumvent the various loopholes that have plagued until now every attempt at reaching this holy Grail [16, 17, 18].

A classical strategy is deterministic if there are two functions $f: X \rightarrow A$ and $g: Y \rightarrow B$ such that Alice and Bob systematically output $f(x)$ and $g(y)$ on inputs $x$ and $y$, respectively. The success of a deterministic strategy is defined by the proportion of legitimate questions on which it provides an appropriate answer. Given a game $G$, we let $\widetilde{\omega}(G)$ stand for the maximum success proportion, over all possible deterministic strategies, for classical players that play game $G$. Formally:

$$
\begin{equation*}
\widetilde{\omega}(G)=\max _{f, g} \frac{\#\{(x, y) \in P \mid(x, y, f(x), g(y)) \in W\}}{\# P} \tag{1.1}
\end{equation*}
$$

Classical players can implement probabilistic strategies when they are allowed to toss local coins and share random variables. In that case, we are interested in the probability that the strategy succeeds on given questions. We shall judge the merit of a probabilistic strategy by its performance on the worst possible legitimate question. More precisely, we say that a probabilistic strategy is successful with probability $p$ if it produces an appropriate answer with probability at least $p$ on all legitimate questions. Note that any deterministic strategy that does not win the game with certainty is "successful" with probability zero since there is at least one legitimate question on which it fails systematically. Given a game $G$, we let $\omega(G)$ stand for the maximum success probability, over all possible probabilistic strategies, for classical players that play game $G$. Formally, given a strategy $s$ and a legitimate question $(x, y)$, let $\operatorname{Pr}_{s}($ win $\mid(x, y))$ denote the probability that strategy $s$ provides an appropriate answer on that question. Then

$$
\begin{equation*}
\omega(G)=\max _{s} \min _{(x, y) \in P} \operatorname{Pr}_{s}(\text { win } \mid(x, y)) \tag{1.2}
\end{equation*}
$$

The next theorem provides a simple yet important relation between the maximum success proportion for deterministic strategies and the maximum probability of success for probabilistic strategies. But first we need a technical proposition.

Proposition 1 Let $G$ be a game. Consider any probabilistic strategy s. If the questions are asked uniformly at random among all legitimate questions, the probability that the players win using s is $\widetilde{\omega}(G)$ at best.

Proof. We consider a general probabilistic strategy $s$, which is a probability distribution over a finite set of deterministic strategies, say $\left\{s_{1}, s_{2}, \ldots s_{m}\right\}$. Let $\operatorname{Pr}\left(s_{i}\right)$ be the probability that strategy $s_{i}$ be chosen, and let $p_{i} \leq \widetilde{\omega}(G)$ be the success proportion of strategy $s_{i}$. The probability that the players win the game according to strategy $s$ is:

$$
\begin{equation*}
\sum_{i=1}^{m} p_{i} \operatorname{Pr}\left(s_{i}\right) \leq \widetilde{\omega}(G) \sum_{i=1}^{m} \operatorname{Pr}\left(s_{i}\right)=\widetilde{\omega}(G) \tag{1.3}
\end{equation*}
$$

Theorem 2 For any game $G, \omega(G) \leq \widetilde{\omega}(G)$.

Proof. Consider any probabilistic strategy $s$ that is successful with maximal probability $\omega(G)$. By definition of $\omega(G)$, given any legitimate question ( $x, y$ ), $\operatorname{Pr}_{s}(\operatorname{win} \mid(x, y)) \geq \omega(G)$. If the question is chosen uniformly at random, the probability $q$ of winning the game using strategy $s$ is

$$
\begin{equation*}
q=\sum_{(x, y) \in P} \frac{\operatorname{Pr}_{s}(\operatorname{win} \mid(x, y))}{\# P} \geq \sum_{(x, y) \in P} \frac{\omega(G)}{\# P}=\omega(G) \tag{1.4}
\end{equation*}
$$

By proposition $1, \widetilde{\omega}(G) \geq q$, and since $q \geq \omega(G)$, then $\widetilde{\omega}(G) \geq \omega(G)$.
The next observation is useful to determine values and bounds on $\widetilde{\omega}(G)$ and $\omega(G)$.

Lemma 3 Let $G$ be a game whose set of legitimate questions is P. If it does not admit a classical winning strategy, then

$$
\begin{equation*}
\omega(G) \leq \widetilde{\omega}(G) \leq \frac{\# P-1}{\# P} \tag{1.5}
\end{equation*}
$$

Proof. Since $\widetilde{\omega}(G)$ is the maximum success proportion, over all possible deterministic strategies, for classical players that play game $G$, it is the ratio of some integer to the total number $P$ of legitimate questions. Since the game does not admit a classical winning strategy, this ratio is not 1 . The next best alternative, which may or may not be achievable by some strategy, would be that $\widetilde{\omega}(G)=(\# P-1) / \# P$. The Lemma follows from Theorem 2.

### 1.3 How Convincing Can Actual Experiments Be?

A major motivation for studying pseudo-telepathy games is that their physical implementation could provide increasingly convincing and loophole-free demonstrations that the physical world is not local realistic. After establishing that a given game is indeed a pseudo-telepathy game, the ideal experiment would consist in implementing the quantum winning strategy and running it on a significant number of rounds until either:
$\diamond$ the players lose even a single round, in which case the predictions of quantum mechanics have been shown to fail, demonstrating that quantum mechanics must be wrong, and it's back to the drawing board; or
$\diamond$ the players win systematically enough rounds to rule out (with high confidence) classical local realistic theories because classical strategies would be overwhelmingly unlikely to perform flawlessly.

Of course, physical realizations of quantum winning strategies for pseudo-telepathy games will never be perfect. Disavowing one hundred years of quantum mechanics on a single failure - as we light-heartedly suggested above - would be foolish! We should be content as long as real-life experiments allow Alice and Bob to win substantially more often than any classical players could hope to do, except with overwhelmingly small probability. We expound on this issue in the next section.

But even an unfailing sequence of wins would not be entirely convincing if the experiment is not performed with extreme care. There are many ways in which unscrupulous Alice and Bob-or an incredibly devious Nature out to fool us into believing in quantum mechanics - could provide classical (local, realistic) players with a winning strategy despite the mathematical proofs that this is impossible! Before they can be used to rule out classical theories beyond any reasonable doubt, pseudo-telepathy experiments must fulfil the following conditions simultaneously.
$\diamond$ It must be physically impossible for Alice and Bob to communicate between the time they receive the question and the time they have to produce the answer. No amount of shielding can achieve this requirement. For instance, it would be unscientific to claim it impossible for Alice to have a means to modulate neutrinos and for Bob to detect them, however unlikely. The only known law of physics that can be used to enforce this restriction is the fact that information cannot be signalled faster than the speed of light. Hence, Alice and Bob must be sufficiently far apart, their questions must be asked with near-perfect simultaneity, and they must be requested to answer very quickly.
$\diamond$ It must be impossible for Alice and Bob to know ahead of time what the questions will be. We must first come to grip with the fact that there is no way to enforce this condition if the universe is purely deterministic. In the view of Laplace, nothing prevents Alice and Bob from analysing the entire state of the universe with enough precision to be able to predict with certainty what questions Xavier and Yolande will ask. This would allow Alice and Bob to conspire together and decide on their answers before they are separated, and Xavier and Yolande would be none the wiser. Unfortunately, there is no resolution to this problem: no pseudo-telepathy experiment will ever convince a determinism-addict that classical physics is wrong.
$\diamond$ Following up on the previous point, the questions must be determined at random by Xavier and Yolande after they have separated to go with Alice and Bob, respectively. Indeed, we said earlier that "Xavier and Yolande are also allowed to discuss strategy in secret of Alice and Bob". But no laws of physics can guarantee them unconditional secrecy (not even quantum cryptography!) if they choose their questions ahead of time. Therefore, each round must begin with Xavier and Yolande choosing their questions purely at random, independently from one another. The entire round, including registering Alice's and Bob's answers, must take place in less time than it takes light to travel between Alice and Bob. This brings up a subtle point: given that Xavier and Yolande cannot prepare their questions together, there can be no guarantee that the promise will be fulfilled. After they meet again, Xavier and Yolande will have to apply post-selection to their data to keep only the answers to legitimate questions. It follows that it is desirable for pseudo-telepathy games to have a high density of legitimate questions.
$\diamond$ And finally, the last condition to rule out classical local realistic theories is that Xavier and Yolande's post-selected data must show that appropriate answers were provided on legitimate questions significantly more often by Alice and Bob than could be expected by any possible classical process.

### 1.4 Imperfect Detectors

Several reasons could explain an incorrect answer provided by the physical implementation of a mathematically-proven quantum winning strategy: (1) quantum mechanics is incorrect, (2) Alice and Bob are unable to keep high-grade entanglement for a macroscopic amount of time, (3) the unitary quantum process to which they subject their entanglement after receiving their questions (see all the following sections) is imperfect, and/or (4) the final measurement that should reveal the classical answer goes wrong.

Here we concentrate on the last possibility. We ask how bad can the detectors be to allow nevertheless for an outcome that would be classically impossible (in probability), provided everything else is perfect in the experiment. For this purpose, we must distinguish between two different types of imperfections in detectors: noise and inefficiency.
$\diamond$ A noisy detector may register the wrong result. This causes Alice or Bob to output a bit that is the complement of what she or he should have produced according to the protocol ${ }^{3}$. In most pseudo-telepathy games, a single error of this sort will result in an inappropriate answer to a legitimate question. But it is often the case that two wrongs make a right.
$\diamond$ An inefficient detector may fail to register an outcome at all. When this happens, the qubit that was supposed to be measured is usually lost forever, and there is nothing the unlucky party can do to recover the information (without communicating with the other party). But at least, contrary to the case of noisy detectors, that party knows that there is a problem.
$\diamond$ Of course, real detectors can be both noisy and inefficient.

In each case, the relevant issue is to determine the level of imperfection detectors can have, and still allow Alice and Bob to accomplish a task that would exceed the capabilities of classical players.

### 1.4.1 Noisy Detectors

For simplicity, let us assume that all the detectors are independent and that they all have the same error probability. Let $p$ stand for the probability for each detector to perform correctly, and therefore each bit of the answer is flipped with probability $1-p$, independently from one another. For any given pseudo-telepathy game, this entails a probability for quantum players to provide inappropriate answers to legitimate questions. If that probability exceeds the error probability that can be reached by some classical strategy, the quantum implementation becomes worthless. This gives rise to a threshold that depends on the pseudo-telepathy game being played.

[^3]Definition 1 Consider a pseudo-telepathy game $G$. We define $p_{*}(G)$ as the maximum value of $p$ for which a classical strategy can succeed as well as an implementation of the best possible quantum strategy that has to deal with detectors that give the correct answer with probability $p$.

Unfortunately, the exact value of $p_{*}(G)$ is unknown for most games.

### 1.4.2 Inefficient Detectors

If a detector fails to respond to Alice or Bob, one approach would be to ignore the problem and output a random bit. This would be a way to turn inefficient detectors into noisy ones. But a much better solution is to allow the parties to admit ignorance. When a party's detector fails to register the result of a measurement, this party simply outputs $\perp$. In this context, we must redefine slightly the notion of a game. The output sets $A$ and $B$ are augmented with this new symbol $\perp$. We say that the players win if the question is not legitimate or if their answer is appropriate, it's a draw if at least one player outputs $\perp$, and they lose otherwise. The key observation is that if detector inefficiency is the only imperfection in the quantum players' implementation, they will either win or draw, but they will never lose a round.

Of course, to be fair, we must allow classical players to output $\perp$ as well. This makes it possible for classical players to have an error-free strategy, in which they never lose, at the expense of occasional draws. Implementations of quantum strategies are interesting only if the occurrence of $\perp$ is smaller than with the best error-free classical strategy. Whether or not this is possible depends on the efficiency $\eta$ of the detectors, which is assumed to be the same, independently, for each detector.

Definition 2 Consider a pseudo-telepathy game $G$. We define $\eta_{*}(G)$ as the maximum value of $\eta$ for which an error-free classical strategy can win at least as often as an implementation of the best quantum strategy that uses detectors of efficiency $\eta$.

Parameter $\eta_{*}(G)$ has been analysed for more games than $p_{*}(G)$ because it bears direct relevance to the widely studied "detection loophole" [17, 19], but nevertheless its exact value is also mostly unknown.

### 1.5 Outline of the Paper

After this Introduction, the rest of the paper provides a systematic survey of all the pseudo-telepathy games known to the authors. We assume that the reader is already familiar with the basic principles of quantum information processing [20]. The games are presented in chronological order of discovery. Each game is defined by the task that the players must accomplish in order to win, then a quantum winning strategy is described, and the impossibility of a classical winning strategy is discussed. Additional information is provided for some games. Before we conclude with open problems in Section 9, we have a quick look in Section 8 at the smallest possible games that exhibit pseudo-telepathy.

## 2 Impossible Colouring Games

In the 1960s, Bell was not alone in proving that Einstein's realism is incompatible with other well-accepted principles [2]. Recall that Bell's argument was to reach a contradiction by accepting also Einstein's nonlocality, as well as the predictions of quantum mechanics. At about the same time, Simon Kochen and Ernst Specker proved that realism is also incompatible with another principle: non-contextuality [21]. Briefly stated, non-contextuality is the principle according to which the probability of a given outcome in a projective measurement does not depend on the choice of the other orthogonal outcomes used to define that measurement. (To be historically precise, it should be noted that Bell himself published a similar idea [22] before Kochen and Specker [21], but Specker by himself presented an earlier proof as early as 1960 [23], and a similar idea goes back to Gleason in 1957 [24].)

It has been argued that the Kochen-Specker approach is less useful than Bell's because it in inherently counterfactual, and therefore it seems impossible to subject it to actual experiments. However, when Kochen-Specker is used together with entanglement, Peter Heywood and Michael Redhead [14] realized that it can be turned into a testable scenario. A little imagination suffices to turn their argument into a pseudotelepathy game, which is probably the first such game ever. However, that paper was overlooked until recently, which explains why the first pseudo-telepathy game is often credited to Greenberger, Horne and Zeilinger [9] or to Mermin [25]—see Section 3.

Theorem 4 (Kochen-Specker Theorem) There exists an explicit, finite set of vectors in $\mathbb{R}^{3}$ that cannot be $\{0,1\}$-coloured so that both of the following conditions hold simultaneously:

1. For every orthogonal pair of vectors, at most one of them is coloured 1.
2. For every mutually orthogonal triple of vectors, at least one of them-and therefore exactly one -is coloured 1.

We say of any such set of vectors that is has the Kochen-Specker property.

Originally, Theorem 4 was proved using 117 vectors [21], but this has been reduced to 31 (with 17 orthogonal triples) by Conway and Kochen [26]. A Kochen-Specker construction, similar to that of Theorem 4 , can be obtained in any dimension $d \geq 3$, either by a geometric argument, or by extending a construction in dimension $d$ to dimension $d+1$ [26]. Following the approach of Richard Cleve, Peter Høyer, Benjamin Toner and John Watrous [27], we show below that any Kochen-Specker construction in any dimension gives rise to a two-party pseudo-telepathy game. Renato Renner and Stefan Wolf have proved a weak converse of this result: Any two-party pseudo-telepathy game can be turned into a Kochen-Specker construction [28], but under two conditions: Alice and Bob must share a maximally entangled state (of any dimension) and they are restricted to making projective measurements only (not POVMs) on their shared entanglement, without the addition of an ancillary system.

### 2.1 The Game

Consider any dimension $d \geq 3$ and let $V$ be a set of vectors in $\mathbb{R}^{d}$ with the KochenSpecker property. In the corresponding impossible colouring pseudo-telepathy game, Alice receives either an orthogonal pair of vectors $v_{1}, v_{2}$, or an orthogonal $d$-tuple of vectors $v_{1}, \ldots, v_{d}$. Bob receives a single vector $v_{\ell}$. All these vectors are taken from $V$. The promise states that $v_{\ell}$ is one of Alice's vectors.

The challenge that Alice and Bob face is that they must $\{0,1\}$-colour the vectors they have received so that (1) two orthogonal vectors are not both coloured $1,(2)$ exactly one of Alice's vectors is coloured 1 if she was given $d$ vectors, and (3) Alice and Bob assign the same colour to vector $v_{\ell}$. Note that it is not necessary for Alice to output the colour of all her vectors because she must satisfy conditions (1) and (2). In case she is given two vectors as input, she outputs a trit that indicates whether $v_{1}$ is coloured $1, v_{2}$ is coloured 1 , or neither; in case she is given $d$ vectors, she outputs the $a \in\{1,2, \ldots, d\}$ so that she assigned colour 1 to $v_{a}$. As for Bob, his one-bit output $b$ is simply the colour he assigns to vector $v_{\ell}$. In any case, the winning condition is that Alice and Bob must assign the same colour to $v_{\ell}$.

### 2.2 Quantum Winning Strategy

The players' strategy is to share entangled state $\frac{1}{\sqrt{d}} \sum_{j=0}^{d-1}|j\rangle|j\rangle$. After receiving their input, Alice and Bob do the following:

1. If Alice was given only two vectors $v_{1}$ and $v_{2}$, she chooses $d-2$ additional vectors $v_{3}, \ldots, v_{d}$, not necessarily taken from $V$, so that $v_{1}, \ldots, v_{d}$ forms an orthogonal $d$-tuple.
2. Alice performs a measurement on her share of the entangled state in basis $B_{a}=\left\{\left|v_{1}\right\rangle, \ldots,\left|v_{d}\right\rangle\right\}$ of $\mathbb{R}^{d}$ (after normalization if necessary). Let $k$ be the result of her measurement. As explained above, she produces the appropriate output $a$, corresponding to assigning colour 1 to vector $v_{k}$.
3. Bob chooses $d-1$ additional vectors $w_{1}, \ldots, w_{d-1}$, not necessarily taken from $V$, so that $v_{\ell}, w_{1}, \ldots, w_{d-1}$ forms an orthogonal $d$-tuple. He performs a measurement on his share of the entangled state in basis $B_{b}=\left\{\left|v_{\ell}\right\rangle,\left|w_{1}\right\rangle, \ldots,\left|w_{d-1}\right\rangle\right\}$ of $\mathbb{R}^{d}$ (after normalization if necessary). He outputs $b=1$ if the outcome is $v_{\ell}$; he outputs $b=$ 0 otherwise.

To show that this quantum strategy works, consider the probability that Alice measures $v_{k}(k \in\{1,2, \ldots, d\})$ and Bob measures $v_{\ell}$. Since the bases $B_{a}$ and $B_{b}$ have real coefficients,

$$
\begin{equation*}
\sum_{j=0}^{d-1}\left\langle j \mid v_{k}\right\rangle\left\langle j \mid v_{\ell}\right\rangle=\left\langle v_{k} \mid v_{\ell}\right\rangle \tag{2.1}
\end{equation*}
$$

and so this probability is

$$
\begin{align*}
\left.\left|\frac{1}{\sqrt{d}} \sum_{j=0}^{d-1}\langle j|\left\langle j \mid v_{k}\right\rangle\right| v_{\ell}\right\rangle\left.\right|^{2} & =\left|\frac{1}{\sqrt{d}} \sum_{j=0}^{d-1}\left\langle j \mid v_{k}\right\rangle\left\langle j \mid v_{\ell}\right\rangle\right|^{2}  \tag{2.2}\\
& =\left|\frac{1}{\sqrt{d}}\left\langle v_{k} \mid v_{\ell}\right\rangle\right|^{2}  \tag{2.3}\\
& = \begin{cases}\frac{1}{d}, & k=\ell \\
0, & k \neq \ell\end{cases} \tag{2.4}
\end{align*}
$$

It follows that Bob measures $v_{\ell}$ if and only if Alice measures $v_{\ell}$ as well. This proves that Alice and Bob will assign the same colour to the vector they have received in common, and therefore their answer will be appropriate.

### 2.3 Classical Players

Any classical deterministic winning strategy would correspond precisely to a $\{0,1\}$ colouring of the vectors in $V$ that can satisfy simultaneously properties (1) and (2) in the statement of the Kochen-Specker theorem. But $V$ was chosen to have the KochenSpecker property, which means that such a colouring cannot exist. It follows that $\widetilde{\omega}(G)<1$, and by Theorem $2, \omega(G)<1$ as well. Therefore, no classical winning strategy can exist for this game. The exact success proportion and probability are not known precisely, but of course they depend on the the specific Kochen-Specker construction that is used. We conjecture that $\widetilde{\omega}(G)$ and $\omega(G)$ cannot be made much smaller than 1 with this type of construction.

### 2.4 Special Case of the Impossible Colouring Game

We have presented a family of pseudo-telepathy games based on the Kochen-Specker theorem. It is interesting to mention the particular case where $d=3$. For the quantum strategy, Alice and Bob share an entangled qutrit pair $|\psi\rangle=\frac{1}{\sqrt{3}}|00\rangle+\frac{1}{\sqrt{3}}|11\rangle+\frac{1}{\sqrt{3}}|22\rangle$. This entangled state of dimension 9 is the smallest possible state that can be used for any two-player pseudo-telepathy game [29]-more on this in Section 8.

Independently of the general approach that we described above, similar pseudotelepathy games have been obtained for a specific 4-dimensional construction in [30], and generalized to any dimension in [31]. See also [32].

## 3 Parity Games

This is a family of games for $n$ players, $n \geq 3$, with the property that the player's outputs are single bits, and the winning condition depends on their parity.

### 3.1 The Game

The task that the $n$ players face is the following: Each player $i$ receives as input a bitstring $x_{i} \in\{0,1\}^{\ell}$, which is also interpreted as an integer in binary, with the promise that $\sum_{i=1}^{n} x_{i}$ is divisible by $2^{\ell}$. The players must each output a single bit $a_{i}$ and the
winning condition is:

$$
\begin{equation*}
\sum_{i=1}^{n} a_{i} \equiv \frac{\sum_{i=1}^{n} x_{i}}{2^{\ell}} \quad(\bmod 2) \tag{3.1}
\end{equation*}
$$

When restricted to parameters $n=3$ and $\ell=1$, this is what is known as the Mermin-GHZ game. It was originally presented as a four-player game [9] and later recast with three players [33]. The game was greatly popularized by Mermin [25, 34].

When restricted to $\ell=1$, but with any number $n \geq 3$ of players, this is Mermin's parity game, introduced in [35] and recast from a computer science point of view in [36, 37].

Finally, by setting $\ell=\lceil\lg n\rceil-1$, still with any number $n \geq 3$ of players, we get the extended parity game, proposed by Harry Buhrman, Peter Høyer, Serge Massar and Hein Röhrig [38]. (We use the symbol "lg" to denote the base-two logarithm.)

### 3.2 Quantum Winning Strategy

The players' strategy for any game in this family is to share entangled state $\frac{1}{\sqrt{2}}\left|0^{n}\right\rangle+\frac{1}{\sqrt{2}}\left|1^{n}\right\rangle$. After receiving his input $x_{i}$, each player $i$ does the following:

1. apply to his share of the entangled state the unitary transformation $S$ defined by

$$
\begin{align*}
|0\rangle & \mapsto|0\rangle  \tag{3.2}\\
|1\rangle & \mapsto e^{\pi 2 x_{i} / 2^{\ell}}|1\rangle \tag{3.3}
\end{align*}
$$

where we use a dotless $\imath$ to denote $\sqrt{-1}$ in order to distinguish it from index $i$, which is used to identify a player
2. apply the Walsh-Hadamard transform, $H$, defined as usual by

$$
\begin{align*}
|0\rangle & \mapsto \frac{1}{\sqrt{2}}|0\rangle+\frac{1}{\sqrt{2}}|1\rangle  \tag{3.4}\\
|1\rangle & \mapsto \frac{1}{\sqrt{2}}|0\rangle-\frac{1}{\sqrt{2}}|1\rangle \tag{3.5}
\end{align*}
$$

3. measure the qubit in the computational basis to obtain $a_{i}$
4. output $a_{i}$

The resulting state after step 1 is

$$
\frac{1}{\sqrt{2}}\left(\left|0^{n}\right\rangle+e^{\pi \tau \frac{\sum x_{i}}{2^{\ell}}}\left|1^{n}\right\rangle\right)= \begin{cases}\frac{1}{\sqrt{2}}\left|0^{n}\right\rangle+\frac{1}{\sqrt{2}}\left|1^{n}\right\rangle & \text { if } \frac{\sum x_{i}}{2^{\ell}} \text { is even }  \tag{3.6}\\ \frac{1}{\sqrt{2}}\left|0^{n}\right\rangle-\frac{1}{\sqrt{2}}\left|1^{n}\right\rangle & \text { if } \frac{\sum x_{i}}{2^{\ell}} \text { is odd }\end{cases}
$$

We know by the promise that $\frac{\sum x_{i}}{2^{\ell}}$ is an integer. Let $\Delta(z)$ denote the number of 1 s in binary string $z$ (the Hamming weight of $z$ ). It is an easy exercise to show that the resulting state after step 2 is:

$$
\begin{cases}\frac{1}{\sqrt{2^{n-1}}} \sum_{\Delta(a) \text { even }}|a\rangle & \text { if } \frac{\sum x_{i}}{2^{\ell}} \text { is even }  \tag{3.7}\\ \frac{1}{\sqrt{2^{n-1}}} \sum_{\Delta(a) \text { odd }}|a\rangle & \text { if } \frac{\sum x_{i}}{2^{\ell}} \text { is odd }\end{cases}
$$

Therefore, after the measurement of step 3 , the output of step 4 will satisfy:

$$
\begin{equation*}
\sum_{i=1}^{n} a_{i} \equiv \frac{\sum_{i}^{n} x_{i}}{2^{\ell}} \quad(\bmod 2) \tag{3.8}
\end{equation*}
$$

so the players always win.

### 3.3 Classical Players

Any winning strategy for the extended parity game yields a winning strategy for the parity game. Similarly, a winning strategy for the parity game yields a winning strategy for the Mermin-GHZ game. So, by showing that there is no classical winning strategy for the Mermin-GHZ game, we will have established the same result for all three games.

In the Mermin-GHZ game, consider a deterministic strategy in which $a_{x}$ is Alice's output on input $x$, and $b_{y}, c_{z}$ are defined similarly for Bob and Charlie. If Alice, Bob and Charlie receive $x=0, y=0, z=0$ as inputs, for instance, it follows from the winning condition that the sum of their outputs $a_{0}, b_{0}$ and $c_{0}$ must be even. This gives the first of the four constraints that correspond to the four legitimate inputs $000,011,101$ and 110. These constraints are summarized below, where the sums are taken modulo 2 .

$$
\begin{align*}
& a_{0}+b_{0}+c_{0} \equiv 0 \\
& a_{0}+b_{1}+c_{1} \equiv 1  \tag{3.9}\\
& a_{1}+b_{0}+c_{1} \equiv 1 \\
& a_{1}+b_{1}+c_{0} \equiv 1
\end{align*}
$$

If we add the four equations, we obtain a contradiction, since the sum on the lefthand side is even, and the sum on the right-hand side is odd. This proves that there is no classical deterministic winning strategy. It follows by Theorem 2 that there is no classical winning strategy. However, it is easy to invent a strategy that wins with probability $3 / 4$. It follows from Lemma 3 that $\widetilde{\omega}(G)=\omega(G)=3 / 4$, where $G$ stands for the Mermin-GHZ game.

Let $G_{n}$ be Mermin's parity game with $n$ players. It is shown in [36] that $\widetilde{\omega}\left(G_{n}\right)=\frac{1}{2}+2^{-\lceil n / 2\rceil}$. By applying Theorem 2, it follows that $\omega\left(G_{n}\right) \leq \frac{1}{2}+2^{-\lceil n / 2\rceil}$. But in fact, this upper-bound is tight [37, 39]: $\omega\left(G_{n}\right)=\frac{1}{2}+2^{-\lceil n / 2\rceil}$.

For the parity game, we also know exact values of $p_{*}$ and $\eta_{*}[37,39]$ :

$$
\begin{align*}
& p_{*}\left(G_{n}\right)= \begin{cases}\frac{1}{2}+2^{\frac{2-3 n}{2 n}} & \text { if } n \text { is even } \\
\frac{1}{2}+2^{\frac{1-3 n}{2 n}} & \text { if } n \text { is odd }\end{cases}  \tag{3.10}\\
& \eta_{*}\left(G_{n}\right)=\frac{1}{2} \sqrt[n]{4} \tag{3.11}
\end{align*}
$$

Since $\lim _{n \rightarrow \infty} p_{*}\left(G_{n}\right)=\frac{1}{2}+\frac{\sqrt{2}}{4}=\cos ^{2} \pi / 8 \approx 85 \%$, we conclude that if we use detectors that provide the correct answer with a probability greater than that, everything else being perfect, and if $n$ is sufficiently large, we can use game $G_{n}$ to observe a phenomenon that would be statistically impossible to explain in a classical world. Similarly, since $\lim _{n \rightarrow \infty} \eta_{*}\left(G_{n}\right)=1 / 2$, the same conclusion applies provided we have detectors that are better than $50 \%$ efficient.

Let $G_{n}^{\prime}$ be the extended parity game with $n$ players. It is shown in [38] that $\eta_{*}\left(G_{n}^{\prime}\right) \leq 8 / n$ for all $n \geq 3$. Since $\lim _{n \rightarrow \infty} \eta_{*}\left(G_{n}^{\prime}\right)=0$, loophole-free experiments of nonlocality are possible with arbitrarily inefficient detectors! (Of course, as $n$ becomes larger, it becomes increasingly difficult to have "everything else being perfect".) The value of $p_{*}\left(G_{n}^{\prime}\right)$ is not known.

## 4 Deutsch-Jozsa Games

This family of two-player pseudo-telepathy games, based on the Deutsch-Jozsa problem [40], was first presented in [41]. These were the first games explicitly described in terms of pseudo-telepathy.

### 4.1 The Game

Alice and Bob face the following task: given bit strings $x$ and $y$ as input, they are requested to output bit strings $a$ and $b$, respectively, so that $a=b$ if and only if $x=y$. To prevent the trivial solution $a \leftarrow x$ and $b \leftarrow y$, the outputs must be exponentially shorter than the inputs. More precisely, $x$ and $y$ are strings of length $n=2^{m}$, for some parameter $m$, but $a$ and $b$ must be strings of length $m$. As such, this problem cannot be solved even with the help of entanglement, but it becomes a pseudo-telepathy game if we add the promise that either $x$ and $y$ are identical or that they differ in exactly half of the bit positions.

### 4.2 Quantum Winning Strategy

The players' strategy for any game in this family is to share entangled state $\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1}|j\rangle|j\rangle$. Note that this means that Alice and Bob each have an $m$-qubit register. After receiving their inputs $x=x_{0} x_{1} \cdots x_{n-1}$ and $y=y_{0} y_{1} \cdots y_{n-1}$ (it is more convenient to start the indices at 0 ), Alice and Bob do the following:

1. Alice applies to her quantum register the unitary transformation that maps

$$
\begin{equation*}
|j\rangle \mapsto(-1)^{x_{j}}|j\rangle \tag{4.1}
\end{equation*}
$$

for all $j$ between 0 and $n-1$; Bob does the same, but with $y_{j}$ instead of $x_{j}$
2. Alice and Bob apply the Walsh-Hadamard transform $H^{\otimes m}$ to their registers
3. Alice and Bob measure their registers in the computational basis. The resulting classical strings, $a$ and $b$, are their final output

It is straightforward to verify that this quantum strategy wins the game with certainty. Please consult [41] for details.

### 4.3 Classical Players

It follows immediately from [42] that classical players need to communicate at least $c 2^{m}$ bits, for some appropriate positive constant $c$ and all sufficiently large $m$, in order to win the Deutsch-Jozsa game with parameter $m$. Therefore, classical players cannot have a winning strategy if they are unable to communicate, again provided $m$ is large
enough. However, this asymptotic result does not help if we want to know which values of $m$ yield a pseudo-telepathy game.

It is easy to design a classical winning strategy for $m=1$ and $m=2$. It is more challenging to design one when $m=3$, but this is done in [43]. For the case $m=4$, it was shown in [44], by using an argument based on graph theory as well as a computerassisted case analysis, that there is no classical winning strategy. However, the proof of this result does not carry over to the case of larger values of $m$, so it still remains to determine explicitly for what other values of $m$ pseudo-telepathy occurs.

## 5 The Magic Square Game

The magic square game is a two-player pseudo-telepathy game that was presented by Padmanabhan Aravind [45, 46], who built on work by Mermin [47]. The most interesting feature of this game is that it is extremely easy to show that there cannot be a classical winning strategy (see Section 5.3). It follows that a successful implementation of the quantum winning strategy (see Section 5.2) would convince any observer that something classically impossible is happening, with no need for the observer to understand why the quantum strategy works.

### 5.1 The Game

A magic square is a $3 \times 3$ matrix whose entries are in $\{0,1\}$, with the property that the sum of each row is even and the sum of each column is odd. Such a square is magic because it cannot exist! Indeed, suppose we calculate the parity of the nine entries. According to the rows, the parity is even, yet according to the columns, the parity is odd. This is obviously impossible.

The task that the players face while playing the game is the following: Alice is asked to give the entries of a row $x \in\{1,2,3\}$ and Bob is asked to give the entries of a column $y \in\{1,2,3\}$. The winning condition is that the parity of the row must be even, the parity of the column must be odd, and the intersection of the given row and column must agree. It is an interesting feature of this game that it does not require a promise: all nine possible questions are legitimate.

### 5.2 Quantum Winning Strategy

The quantum winning strategy for the magic square game is not as simple as the classical impossibility proof. First, Alice and Bob share the entangled state

$$
\begin{equation*}
|\psi\rangle=\frac{1}{2}|0011\rangle-\frac{1}{2}|0110\rangle-\frac{1}{2}|1001\rangle+\frac{1}{2}|1100\rangle \tag{5.1}
\end{equation*}
$$

The first two qubits belong to Alice and the last two to Bob. Upon receiving their inputs $x$ and $y$, Alice and Bob apply unitary transformations $A_{x}$ and $B_{y}$, respectively, according to the following matrices.

$$
\begin{align*}
& A_{1}=\frac{1}{\sqrt{2}}\left[\begin{array}{rrrr}
\imath & 0 & 0 & 1 \\
0 & -\imath & 1 & 0 \\
0 & \imath & 1 & 0 \\
1 & 0 & 0 & \imath
\end{array}\right], \quad A_{2}=\frac{1}{2}\left[\begin{array}{rrrr}
2 & 1 & 1 & \imath \\
-\imath & 1 & -1 & \imath \\
\imath & 1 & -1 & -\imath \\
-\imath & 1 & 1 & -\imath
\end{array}\right], \quad A_{3}=\frac{1}{2}\left[\begin{array}{rrrr}
-1 & -1 & -1 & 1 \\
1 & 1 & -1 & 1 \\
1 & -1 & 1 & 1 \\
1 & -1 & -1 & -1
\end{array}\right]  \tag{5.2}\\
& B_{1}=\frac{1}{2}\left[\begin{array}{rrrr}
\imath \imath & -\imath & 1 & 1 \\
-\imath & -2 & 1 & -1 \\
-1 & 1 & -\imath & 2 \\
-2 & \imath & 1 & 1
\end{array}\right], \quad B_{2}=\frac{1}{2}\left[\begin{array}{rrrr}
-1 & \imath & 1 & \imath \\
1 & \imath & 1 & -\imath \\
1 & -2 & 1 & \imath \\
-1 & -2 & 1 & -\imath
\end{array}\right], \quad B_{3}=\frac{1}{\sqrt{2}}\left[\begin{array}{rrrr}
1 & 0 & 0 & 1 \\
-1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 \\
0 & 1 & -1 & 0
\end{array}\right] \tag{5.3}
\end{align*}
$$

Then, Alice and Bob measure their qubits in the computational basis. This provides 2 bits to each player, which are the first 2 bits of their respective output $a$ and $b$. Finally, Alice and Bob determine their third output bit from the first two so that their parity condition is satisfied.

Consider for example inputs $x=2$ and $y=3$. After Alice and Bob apply $A_{2}$ and $B_{3}$, respectively, the state evolves to

$$
\begin{align*}
&\left(A_{2} \otimes B_{3}\right)|\psi\rangle=\frac{1}{2 \sqrt{2}} {[|0000\rangle-|0010\rangle-|0101\rangle+|0111\rangle}  \tag{5.4}\\
&+|1001\rangle+|1011\rangle-|1100\rangle-|1110\rangle] \tag{5.5}
\end{align*}
$$

After measurement, Alice and Bob could obtain 10 and 01, for instance. In that case, Alice would complete with bit 1 so that her output $a=101$ has even parity and Bob would complete with bit 0 so that his output $b=010$ has odd parity. Xavier and Yolande will be satisfied with the answer since both Alice and Bob agree that the third entry of the second row is indeed the same as the second entry of the third column: $a_{3}=b_{2}=1$. It is easy to check that the seven other possible answers that Alice and Bob could have given on this example are all appropriate. The verification that this quantum strategy wins also on the other eight possible questions is tedious but straightforward.

### 5.3 Classical Players

The simple proof that a winning classical strategy cannot exist goes as follows. A deterministic classical strategy would have to assign definite binary values to each of the nine entries of the magic square, which is impossible. Therefore, there can be no deterministic classical winning strategy. It follows from Theorem 2 that there is no classical winning strategy, even probabilistic. Using Lemma 3, it is also straightforward to show that $\widetilde{\omega}(G)=\omega(G)=8 / 9$, where $G$ stands for the magic square game.

### 5.4 Related Work

There are other pseudo-telepathy games that are related to the magic square game. Adán Cabello's game [48, 49] does not resembles the magic square game on first approach. However, closer analysis reveals that the two games are totally equivalent!

Also, Aravind has generalized his own magic square idea [45] to a two-player pseudotelepathy game in which the players share $n$ Bell states, $n$ being an arbitrary odd number larger than 1.

## 6 Matching Games

This family of two-player games is a relatively recent development. It is based on an observation made by Harry Buhrman in response to a talk given by Iordanis Kerenidis at the 2004 Workshop on Quantum Information Processing [50], and on [51].

Definition $3 A$ perfect matching $M$ on $\{0,1, \ldots, m-1\}$, where $m$ is even, is a partition of $\{0,1, \ldots m-1\}$ into $\frac{m}{2}$ sets, each of size 2. We define $M_{m}$ as the set of all perfect matchings on $\{0,1, \ldots, m-1\}$.

### 6.1 The Game

In the matching game, Alice receives an $m$-bit string $x=x_{0} x_{1} \cdots x_{m-1}$ as input (again, it is more convenient to start the indices at 0 ), and Bob receives a perfect matching $M \in M_{m}$. The task that the players face is for Alice to output a string $a \in\{0,1\}^{\lceil\lg m\rceil}$, and Bob to output a pair $\{\alpha, \beta\} \in M$ as well as a string $b \in\{0,1\}^{\lceil\lg m\rceil}$ such that

$$
\begin{equation*}
x_{\alpha} \oplus x_{\beta}=(\alpha \oplus \beta) \cdot(a \oplus b) \tag{6.1}
\end{equation*}
$$

where $a \oplus b$ denotes the bitwise exclusive-or of $a$ and $b$, and $u \cdot v=\bigoplus_{i}\left(u_{i} \wedge v_{i}\right)$ denotes the parity of the bitwise "and" of $u$ and $v$.

This may seem at first to be a very bizarre game, but it is currently the only scalable game known - it depends on a parameter $m$-among those that do not require a promise.

### 6.2 Quantum Winning Strategy

The quantum players' strategy is to share entangled state $\frac{1}{\sqrt{m}} \sum_{j=0}^{m-1}|j\rangle|j\rangle$. After Alice receives her input $x=x_{0} x_{1} \cdots x_{m-1}$ and Bob his input $M \in M_{m}$, the players do the following:

1. Alice applies to her quantum register the unitary transformation that maps

$$
\begin{equation*}
|j\rangle \mapsto(-1)^{x_{j}}|j\rangle \tag{6.2}
\end{equation*}
$$

for all $j$ between 0 and $m-1$
2. Bob performs a projective partial measurement onto subspaces of dimension 2. Each subspace of the measurement is spanned by vectors $|k\rangle$ and $|\ell\rangle$, where $\{k, \ell\} \in M$. Bob outputs the classical outcome of this measurement, which is a pair $\{\alpha, \beta\} \in M$. In addition to producing a classical outcome, the measurement causes the quantum state shared with Alice to collapse to $\frac{1}{\sqrt{2}}(-1)^{x_{\alpha}}|\alpha \alpha\rangle+\frac{1}{\sqrt{2}}(-1)^{x_{\beta}}|\beta \beta\rangle$
3. Both Alice and Bob perform the Walsh-Hadamard transform $H^{\otimes\lceil\lg m\rceil}$
4. Alice measures in the computational basis and outputs $a$, the result of her measurement
5. Bob measures in the computational basis and outputs $b$, the result of his measurement

Details of why this works are left as an exercise.

### 6.3 Classical Players

By reducing the Hidden Matching Problem [38] to the matching game, it follows that there is no classical winning strategy for the latter, provided $m$ is chosen large enough.

Once again, this does not tell us exactly which values of $m$ yield a pseudo-telepathy game. For now, we are only able to say that there is a trivial classical winning strategy for $m=2$.

## 7 Other Games

We report briefly on two additional families of pseudo-telepathy games.
The first one is due to David DiVincenzo and Asher Peres [52]. By exploiting properties of quantum code words for $3,5,7$ and 9 qubits codes, they have given an elegant algebraic way of generating $3,5,7$ and 9 player pseudo-telepathy games, where the shared entangled state is a quantum code word. To the best of our knowledge, these games are no harder to win for classical players than the parity games of Section 3 for the same number of players.

The second game is due to Michel Boyer [53]. It is a multi-party pseudo-telepathy game for $n \geq 3$ players. Let $M$ be any even number. The task that the players face is the following: Each player $i$ receives as input an arbitrary integer $x_{i}$ and must output an integer $a_{i}$ between 0 and $M-1$. The promise is that $\sum x_{i}$ is even and the winning condition is that

$$
\begin{equation*}
\sum_{i=1}^{n} a_{i} \equiv \frac{\sum_{i=1}^{n} x_{i}}{2} \quad(\bmod M) \tag{7.1}
\end{equation*}
$$

This game is equivalent to a variant in which each input is a single bit. One interest of this game is that the quantum winning strategy uses a quantum Fourier transform modulo $M$. It may be more convenient to take $M$ a power of 2 .

## 8 The Inner Frontier

Throughout this survey, we have presented several pseudo-telepathy games, large and small according to a variety of metrics such as the cardinality of the input sets, of the output sets, and the dimensionality of the entangled state that must be shared at the outset of the game. It it interesting to investigate what are the smallest possible games that exhibit pseudo-telepathy.

Consider two-party pseudo-telepathy games. It is known that no such game can exist if Alice and Bob are requested to output a single bit each [27]. It follows that the impossible colouring game (Section 2) with $m=3$ is minimal in terms of the output
size since Alice outputs a trit and Bob outputs a bit. Another way of seeing this inner frontier is that the Mermin-GHZ game (Section 3) is also minimal since it exhibits pseudo-telepathy with a single bit of output per player... but it involves three players.

Similarly, two-party pseudo-telepathy games cannot exist if the entangled state that the quantum players share is of dimension $2 \times 3$ or smaller [29]. It follows that the impossible colouring game with $m=3$ is again minimal since it makes do with an entangled state of dimension $3 \times 3$. Or one could say that it's the Mermin-GHZ game that is again minimal since it exhibits pseudo-telepathy with a single qubit of entanglement in the hands of each of the three players. In terms of the total dimensionality, it is the Mermin-GHZ game that wins the minimalist Grand Prize since $8<9$, where $8=2 \times 2 \times 2$ and $9=3 \times 3$.

## 9 Conclusions and Open Problems

Einstein, Bohr, Bell, Kochen and Specker were all concerned with hidden variables ("elements of reality"). Pseudo-telepathy also deals with this issue: it provides alternate versions of Bell's argument against local realistic theories. But it's more than just that. Pseudo-telepathy games often provide a more concise and convincing argument than those along the lines of Bell. They may also prove useful in devising loophole-free experimental tests to rule out local realistic descriptions of the physical world.

The seeds of pseudo-telepathy were sowed more than twenty years ago by Heywood and Redhead (Section 2). Others followed. By setting these experiments in the framework of pseudo-telepathy, we are better able to grasp the essence of the arguments, compare them, and improve on previous results. For example, it is only recently, and in the context of pseudo-telepathy, that it was realized that the two earliest pseudotelepathy games were minimal in term of the number of possible outputs as well as the dimensionality of the required entanglement (Section 8).

Several interesting questions are still open. We have seen in Section 1.4 that noisy and inefficient detectors will unavoidably render the quantum players imperfect. For the parity games of Section 3, we have given exact values or bounds on the efficiencies that we can tolerate. In order to devise experiments that are less prone to these types of errors, it would be important to have an analysis for each game $G$ that provides exact values of $p_{*}(G)$ and $\eta_{*}(G)$. Also, imperfections can arise from the entangled state and the unitary evolution of the system. An analysis of these types of noise, as well as how the different types interact, would give us the tools necessary to devise better experiments.

Of course, it would be interesting to find new pseudo-telepathy games or families of games. It would be equally interesting to show how they relate to one another. For example, the three games of Section 3 can be parametrized so that they all fall into the same general description. The magic square game (Section 5) is equivalent to Cabello's game. Which other games, old or new, are similar? How do they differ?

The matching games of Section 6 come from a one-way communication complexity problem. Is it possible to find other links between one-way communication and pseudotelepathy? Still concerning the matching game $G_{m}$ with parameter $m$, it would be interesting to find values for $\widetilde{\omega}\left(G_{m}\right)$ and $\omega\left(G_{m}\right)$ for even $m \geq 4$. There are other games, such as the impossible colouring games of Section 2, the extended parity games of Section 3 and the Deutsch-Jozsa games of Section 4, for which, in general, precise values of $\widetilde{\omega}(G)$ and $\omega(G)$, as a function of the size of the game, are still unknown.

In Section 8, we gave bounds on the dimension of the shared entangled state and on the cardinality of the output sets in order for a two-player pseudo-telepathy game to exist. Can we find similar bounds on the cardinality of the input sets? Is the magic square game (Section 5) optimal in this respect?

Finally, it is not enough to find games $G$ that have a small classical success proportion $\widetilde{\omega}(G)$ or probability $\omega(G)$, or a good tolerance to detector noise $p_{*}(G)$ or inefficiencies $\eta_{*}(G)$. We must strive to find such games that are feasible experimentally. The most simplistic interpretation of this condition is that the classical success proportion or probability should be small, or the tolerance to detector noise or inefficiencies should be high, even for small sized entangled quantum systems. We would like to know bounds on what is achievable for pseudo-telepathy games, as well as games that reach these bounds. Failing that, it might be possible to find new pseudo-telepathy games that improve on any of the best known values for the above parameters, as a function of the size of the entangled quantum state.

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## References

[1] A. Einstein, B. Podolsky, and N. Rosen, "Can quantum-mechanical description of physical reality be considered complete?", Physical Review 47:777-780, 1935.
[2] J. S. Bell, "On the Einstein Podolsky Rosen paradox", Physics 1(3):195-200, 1964.
[3] A. Aspect, P. Grangier, and G. Roger, "Experimental tests of realistic local theories via Bell's theorem", Physical Review Letters 47:460-463, 1981.
[4] A. Aspect, P. Grangier, and G. Roger, "Experimental realization of Einstein-Podolsky-Rosen-Bohm gedankenexperiment: A new violation of Bell's inequalities", Physical Review Letters 49:91-94, 1982.
[5] A. Aspect, J. Dalibard, and G. Roger, "Experimental test of Bell's inequalities using time-varying analyzers", Physical Review Letters 49:1804-1807, 1982.
[6] W. Tittel, J. Brendel, H. Zbinden, and N. Gisin, "Violation of Bell inequalities by photons more than 10 km apart", Physical Review Letters 81(17):3563-3566, 1998.
[7] H. Zbinden, J. Brendel, N. Gisin, and W. Tittel, "Experimental test of nonlocal quantum correlations in relativistic configurations", Physical Review A 63:022111.1-022111.10, 2001.
[8] N. D. Mermin, "Bringing home the atomic world: Quantum mysteries for anybody", American Journal of Physics 49:940-943, 1981.
[9] D. M. Greenberger, M. A. Horne, and A. Zeilinger, "Going beyond Bell's theorem", In Bell's Theorem, Quantum Theory and Conceptions of the Universe (M. Kafatos, editor), pages 69-72, 1988.
[10] L. Hardy, "Quantum mechanics, local realistic theories, and Lorentz-invariant realistic theories", Physical Review Letters 68(20):2981-2984, 1992.
[11] Z.-B. Chen, J.-W. Pan, Y.-D. Zhang, C. Brukner, and A. Zeilinger, "All-versusnothing violation of local realism for two entangled photons", Physical Review Letters 90:160408.1-160408.4, 2003.
[12] A. Peres, "Bayesian analysis of Bell inequalities", Fortschritte der Physik 48(5-7):531-535, 2000.
[13] A. C.-C. Yao, "Quantum circuit complexity", Proceedings of the 34th Annual IEEE Symposium on Foundations of Computer Science, pages 222-227, 1993.
[14] P. Heywood and M. L. G. Redhead, "Nonlocality and the Kochen-Specker paradox", Foundations of Physics 13(5):481-499, 1983.
[15] G. Brassard, "Quantum communication complexity", Foundations of Physics 33(11):1593-1616, 2003.
[16] N. Gisin and H. Zbinden, "Bell inequality and the locality loophole: Active versus passive switches", Physics Letters A 264(2-3):103-107, 1999.
[17] S. Massar, "Nonlocality, closing the detection loophole, and communication complexity", Physical Review A 65:032121.1-032121.5, 2002.
[18] J. Barrett, D. Collins, L. Hardy, A. Kent, and S. Popescu, "Quantum nonlocality, Bell inequalities and the memory loophole", Physical Review A 66(042111), 2002.
[19] S. Massar and S. Pironio, "Violation of local realism versus detection efficiency", Physical Review A 68:062109.1-062109.7, 2003.
[20] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information, (Cambridge University Press, Cambridge, 2000).
[21] S. Kochen and E.P. Specker, "The problem of hidden variables in quantum mechanics", Journal of Mathematics and Mechanics 17:59-87, 1967.
[22] J. S. Bell, "On the problem of hidden variables in quantum mechanics", Reviews of Modern Physics 38(3):447-452, 1966.
[23] E. Specker, "Die Logik nicht gleichzeitig entscheidbarer Aussagen", Dialectica 14:239-246, 1960.
[24] A. Gleason, "Measures on the closed subspaces of a Hilbert space", Journal of Mathematics and Mechanics 6:885-893, 1957.
[25] N. D. Mermin, "Quantum mysteries revisited", American Journal of Physics 58(8):731-734, 1990.
[26] A. Peres, Quantum Theory: Concepts and Methods, (Kluwer Academic Publishers, Dordrecht, 1993).
[27] R. Cleve, P. Høyer, B. Toner, and J. Watrous, "Consequences and limits of nonlocal strategies", Proceedings of the 19th IEEE Conference on Computational Complexity, pages 236-249, 2004.
[28] R. Renner and S. Wolf, "Quantum pseudo-telepathy and the Kochen-Specker theorem", Proceedings of IEEE International Symposium on Information Theory, page 322, 2004.
[29] G. Brassard, A. Méthot, and A. Tapp, "Minimal state dimension for pseudotelepathy", In preparation, 2004.
[30] J. Zimba and R. Penrose, "On Bell non-locality without probabilities: More curious geometry", Studies in History and Philosophy of Science, Part A 24(5):697-720, 1993.
[31] P. K. Aravind, "Impossible colorings and Bell's theorem", Physics Letters A 262(4-5):282-286, 1999.
[32] J. E. Massad and P. K. Aravind, "The Penrose dodecahedron revisited", American Journal of Physics 67(7):631-638, 1999.
[33] D. M. Greenberger, M. A. Horne, A. Shimony, and A. Zeilinger, "Bell's theorem without inequalities", American Journal of Physics 58(12):1131-1143, 1990.
[34] N. D. Mermin, "What's wrong with these elements of reality?", Physics Today 43:9-11, 1990.
[35] N. D. Mermin, "Extreme quantum entanglement in a superposition of macroscopically distinct states", Physical Review Letters 65(15):1838-1849, 1990.
[36] G. Brassard, A. Broadbent, and A. Tapp, "Multi-party pseudo-telepathy", Proceedings of the 8th International Workshop on Algorithms and Data Structures, Volume 2748 of Lecture Notes in Computer Science, pages 1-11, 2003.
[37] G. Brassard, A. Broadbent, and A. Tapp, "Recasting Mermin's multi-player game into the framework of pseudo-telepathy", Available as arXiv:quant-ph/0408052, August 2004.
[38] H. Buhrman, P. Høyer, S. Massar, and H. Röhrig, "Combinatorics and quantum nonlocality", Physical Review Letters 91(4):0479031-0479034, 2003.
[39] A. Broadbent, "Quantum pseudo-telepathy games", Master's thesis, Université de Montréal, 2004.
[40] D. Deutsch and R. Jozsa, "Rapid solution of problems by quantum computation", Proceedings of the Royal Society of London, Series A 439:553-558, 1992.
[41] G. Brassard, R. Cleve, and A. Tapp, "Cost of exactly simulating quantum entanglement with classical communication", Physical Review Letters 83(9):1874-1878, 1999.
[42] H. Buhrman, R. Cleve, and A. Wigderson, "Quantum vs. classical communication and computation", Proceedings of the 30th Annual ACM Symposium on Theory of Computing, pages 63-68, 1998.
[43] V. Galliard and S. Wolf, "Pseudo-telepathy, entanglement and graph colorings", Proceedings of IEEE International Symposium on Information Theory, page 101, 2002.
[44] V. Galliard, S. Wolf, and A. Tapp, "The impossibility of pseudo-telepathy without quantum entanglement", Proceedings of IEEE International Symposium on Information Theory, page 457, 2003. Full paper available as arXiv: quant-ph/0211011.
[45] P. K. Aravind, "Bell's theorem without inequalities and only two distant observers", Foundations of Physics Letters 15(4):397-405, 2002.
[46] P. K. Aravind, "A simple demonstration of Bell's theorem involving two observers and no probabilities or inequalities", Available as arXiv:quant-ph/0206070, revised January 2003.
[47] N. D. Mermin, "Simple unified form for the major no-hidden-variables theorems", Physical Review Letters 65(27):3373-3376, 1990.
[48] A. Cabello, "Bell's theorem without inequalities and without probabilities for two observers", Physical Review Letters 86(10):1911-1914, 2001.
[49] A. Cabello, "'All versus nothing' inseparability for two observers", Physical Review Letters 87(1):010403.1-010403.4, 2001.
[50] H. Buhrman and I. Kerenidis, Personal communication, January 2004.
[51] Z. Bar-Yossef, T. S. Jayram, and I. Kerenidis, "Exponential separation of quantum and classical one-way communication complexity", Proceedings of the 36th Annual ACM Symposium on Theory of Computing, pages 128-137, 2004.
[52] D. P. DiVincenzo and A. Peres. "Quantum code words contradict local realism", Physical Review A 55(6):4089-4092, 1997.
[53] M. Boyer, "Extended GHZ n-player games with classical probability of winning tending to 0", Available as arXiv:quant-ph/0408090, August 2004.


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[^1]:    ${ }^{1}$ For a quantum physicist, even relativity belongs to classical physics.

[^2]:    ${ }^{2}$ One could argue that pseudo-telepathy games do not solve Yao's problem as stated and therefore the first solution to Yao's problem came as late as 1997.

[^3]:    ${ }^{3}$ In some pseudo-telepathy games, such as the impossible colouring game of Section 2 , the expected output is not a bit string. The notion of noisy detector must be modified accordingly.

