Quantum query complexity of state conversion

Troy Lee* Rajat Mittal† Ben W. Reichardt† Robert Špalek‡ Mario Szegedy§

Quantum query complexity is a black-box model of quantum computation where the resource of interest is the number of queries made to the input. This model has proven very useful for the study of quantum computing—key algorithms like Grover's search and the period-finding routine of Shor's algorithm can be formulated in this model, while at the same time one can often show tight lower bounds, something still far off in the circuit model.

In the typical setting, one wishes to compute a function $f(x)$ while making as few queries to the input x as possible. Equivalently, one wishes to convert a state $|0\rangle$ which has no information about the input x into a state close to $|f(x)\rangle \otimes |0\rangle$, from which the answer $f(x)$ can be extracted with high probability. Viewed in this fashion, it is natural to consider a generalization to the *state-generation problem*, where the goal is to convert the starting state $|0\rangle$ into some input-dependent target state $|\sigma_x\rangle$. This problem was introduced by Shi [Shi02], and recently systematically studied by Ambainis et al. [AMRR11]. We study here the more general *state-conversion problem*, where the objective on input x is to convert a starting state $|\rho_x\rangle$ into a target state $|\sigma_x\rangle$, again by making queries to x. State-conversion problems arise naturally in algorithm design, generalizing classical subroutines (Figure 1).

We characterize the quantum query complexity of state conversion using an information-theoretic norm that can be expressed as a relatively simple semi-definite program. We call this the filtered factorization norm as it is a generalization of the factorization norm γ_2, also known as the Schur product operator norm. Given a set of initial states $\{|\rho_x\rangle\}_x$ and target states $\{|\sigma_x\rangle\}_x$, the query complexity of the state-conversion problem to depends only on the Gram matrices $\rho_{x,y} = \langle \rho_x | \rho_y \rangle$ and $\sigma_{x,y} = \langle \sigma_x | \sigma_y \rangle$. We show that the query complexity of this state-conversion problem is characterized by the *distance* between ρ and σ, as measured by the filtered factorization norm.

Recently, Reichardt showed that the general adversary bound [HLŠ07] characterizes the bounded-error quantum query complexity of any function with boolean output and binary input alphabet [Rei09, Rei11]. A corollary of our result is that the general adversary bound characterizes the bounded-error quantum query complexity of any function whatsoever:

Theorem 1. *Let $f : \mathcal{D} \to E$, where $\mathcal{D} \subseteq D^n$, and D and E are finite sets. Then the general adversary bound $\mathrm{Adv}^{\pm}(f)$ of f characterizes the bounded-error quantum query complexity $Q(f)$ of f:*

$$Q(f) = \Theta\big(\mathrm{Adv}^{\pm}(f)\big) \ .$$

We now give some further consequences of our characterization for function evaluation. Note that for a function f, the Gram matrix of the initial states is J, the all ones matrix, and the Gram matrix of the target states $|f(x)\rangle \otimes |0\rangle$ is $F_{x,y} = \delta_{f(x),f(y)}$.

*Centre for Quantum Technologies, Singapore.
†Institute for Quantum Computing, University of Waterloo.
‡Google, Inc.
§Rutgers University

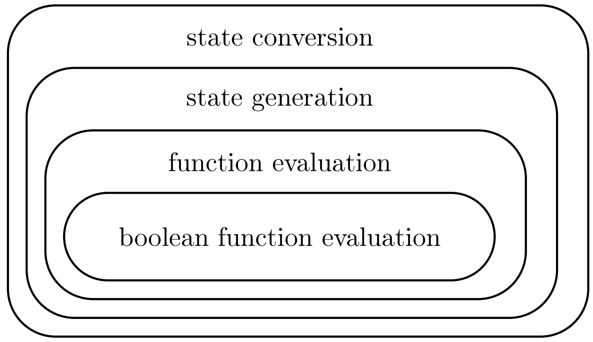

Figure 1: The state-conversion problem generalizes the state-generation problem, studied by Ambainis et al. [AMRR11], which in turn generalizes function evaluation. The quantum query complexity of evaluating boolean functions has been characterized by [Rei11].

- Straight-line property: If one can design an optimal algorithm for going from J to $\frac{99}{100}J + \frac{1}{100}F$, then one also obtains an optimal algorithm for evaluating f, that is going from J to F. This follows as quantum query complexity is given by the filtered factorization norm of the difference $J - F$.

- Relation to adversary bound: The semi-definite program giving the filtered factorization norm of $J - F$ is the semi-definite program for the general adversary bound of f, together with some additional constraints. These additional constraints change the objective value by at most a factor of two, leading to Theorem 1. This gives a new interpretation of the adversary bound as a distance measure. Here are the definitions:

$$\begin{aligned}
\gamma_2 \text{ norm:} \quad & \gamma_2(A) = \max_M \left\{ \|A \circ M\| : \|M\| \le 1 \right\} \\
\text{filtered } \gamma_2 \text{ norm:} \quad & \gamma_2(A|Z) = \max_M \left\{ \|A \circ M\| : \max_j \|Z_j \circ M\| \le 1 \right\} \\
\text{Adv}^\pm \text{ bound:} \quad & \text{Adv}^\pm(f) = \max_M \left\{ \|(J - F) \circ M\| : \max_j \|(J - F) \circ Z_j \circ M\| \le 1 \right\} .
\end{aligned}$$

  In our bound, A equals $J - F$ (or $\rho - \sigma$ for the general state-conversion problem), and Z_j are the query matrices, $(Z_j)_{x,y} = 1 - \delta_{x_j, y_j}$.

- Composition: We show that quantum query complexity possesses a remarkable composition property, inherited from the filtered factorization norm. Namely, $Q(f \circ g^n) = O(Q(f)Q(g))$ for a composed function $(f \circ g^n)(x^1, \ldots, x^n) = f(g(x^1), \ldots, g(x^n))$. Even though this is a result about functions, it is crucial to work with the filtered factorization norm rather than the adversary bound as heavy use is made of the additional constraints given by the former. When the input of f is boolean, we show a matching lower bound, extending and simplifying the proof for wholly boolean f, g [HLŠ07].

The full version of the paper may be found on the arXiv at `http://arxiv.org/abs/1011.3020`.

References

[AMRR11] Andris Ambainis, Loïck Magnin, Martin Roetteler, and Jérémie Roland. Symmetry-assisted adversaries for quantum state generation. In *26th IEEE Conference on Computational Complexity*, pages 167–177, 2011, arXiv:1012.2112 [quant-ph].

[HLŠ07] Peter Høyer, Troy Lee, and Robert Špalek. Negative weights make adversaries stronger. In *Proc. 39th ACM STOC*, pages 526–535, 2007, arXiv:quant-ph/0611054.

[Rei09] Ben W. Reichardt. Span programs and quantum query complexity: The general adversary bound is nearly tight for every boolean function. Extended abstract in *Proc. 50th IEEE FOCS*, pages 544–551, 2009, arXiv:0904.2759 [quant-ph].

[Rei11] Ben W. Reichardt. Reflections for quantum query algorithms. In *Proc. 22nd ACM-SIAM Symp. on Discrete Algorithms (SODA)*, pages 560–569, 2011, arXiv:1005.1601 [quant-ph].

[Shi02] Yaoyun Shi. Quantum lower bounds for the collision and the element distinctness problems. In *Proc. 43rd IEEE FOCS*, pages 513–519, 2002, arXiv:quant-ph/0112086.