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Abstract: Two pictures of BPS bound states in Calabi-Yau compactifications of type II string theory exist, one as a set of particles at equilibrium separations from each other, the other as a fusion of D-branes at a single point of space. We show how quiver quantum mechanics smoothly interpolates between the two, and use this, together with recent mathematical results on the cohomology of quiver varieties, to solve some nontrivial ground state counting problems in multi-particle quantum mechanics, including one arising in the setup of the spherical quantum Hall effect, and to count ground state degeneracies of certain dyons in supersymmetric Yang-Mills theories. A crucial ingredient is a non-renormalization theorem in $\mathcal{N} = 4$ quantum mechanics for the first order part of the Lagrangian in an expansion in powers of velocity.

Contents

1. Introduction

One of the big successes of string theory has been the counting of black hole microstates using D-brane constructions, starting with [1]. The basic idea is that black holes, i.e. solutions of the low energy supergravity theory, have a dual description as D-branes, which provide a reliable description of the physics in the limit of vanishing string coupling constant, $g_s \to 0$. Under the assumption that the number of states does not change when g_s is sent to zero, one can thus count black hole states by counting D-brane states, and one finds among other things a precise match with the Bekenstein-Hawking entropy formula. One way to obtain conservation of the number of states under change of g_s is to consider BPS black holes and D-brane states, since the number of supersymmetric ground states of a system is often invariant under continuous changes of the coupling constants, or at least well under control.

In this paper we will push further this idea to the counting of ground state degeneracies of objects more complicated than black holes. In four dimensional $\mathcal{N} = 2$ supergravities and gauge theories, there exists BPS states which are multicentered composites, bound together like atoms in a molecule [2, 3, 4, 5, 6]. These $\mathcal{N} = 2$ theories arise or can be geometrically engineered [7] as certain Calabi-Yau compactifications of type II string theory, in which BPS states appear in the limit of vanishing string coupling constant as wrapped D-branes, localized at a single position in space. Again there is a duality between these D-branes and the multi-centered solutions of the effective low energy theory, and one can follow the same reasoning as for ordinary black holes to count quantum ground state degeneracies.

The constituents of these multicentered composites do not necessarily have to be black holes to get interesting counting problems; they can be ordinary particles. The quantum dynamics of these systems is therefore often better under control, in both the multi-particle and the single D-brane pictures, making it possible to count states on both sides and compare the two. Using the quiver description of the wrapped D-branes [8, 9, 10, 11, 12, 13, 14] (in regimes where this description is valid), it becomes even possible to follow in detail the transition from one picture to the other.

One of these interesting counting problems which arises naturally in this framework is to find the lowest Landau level degeneracies of a "quantum Hall halo". A Hall halo can be thought of as a charge κ magnetic monopole surrounded by a number N of electrons bound to a sphere of fixed radius around the monopole. This system had been studied in detail in the condensed matter literature [15]. The generating function $G(t) = \sum_L n_L \, t^L$ for the number of ground states n_L with spin $J_3 = L/2 - N(\kappa - N)/2$ was found to be

$$G(t) = \frac{\prod_{j=1}^{\kappa}(1 - t^{2j})}{\prod_{j=1}^{N}(1 - t^{2j}) \prod_{j=1}^{\kappa-N}(1 - t^{2j})} \,. \tag{1.1}$$

Through the correspondence with microscopic D-brane states studied in this paper, this condensed matter problem gets mapped to counting supersymmetric ground states of quantum mechanics on the moduli space of a quiver with two nodes, κ arrows and dimension vector $(1, N)$ (see section 3.2 for definitions). This moduli space is the Grassmannian $Gr(N, \kappa)$, i.e. the space of N-dimensional planes in $\mathbb{C}^\kappa$. The supersymmetric ground states are in one-to-one correspondence with the cohomology of this space, which is classically known. The generating function for the betti numbers b_L, also known as the Poincaré polynomial, is [16]:

$$P(t) \equiv \sum_L b_L \, t^L = \frac{\prod_{j=1}^{\kappa}(1-t^{2j})}{\prod_{j=1}^{N}(1-t^{2j}) \prod_{j=1}^{\kappa-N}(1-t^{2j})} \,. \tag{1.2}$$

The cohomology is organized in Lefschetz SU(2) multiplets, which as we will see coincide here with the spatial spin multiplets, implying that an L-form has spin $J_3 = L/2 - N(\kappa - N)/2$. Comparing this to (1.1), we see that we have indeed exact agreement.

The main part of this paper is aimed at obtaining an understanding of why these two counting problems, and many generalizations thereof, are equivalent. We achieve this by modeling the systems under consideration as $\mathcal{N} = 4$ supersymmetric quiver quantum mechanical systems, obtained as the dimensional reduction of four dimensional $\mathcal{N} = 1$ gauge theories. These quantum mechanical models have both "Higgs" and "Coulomb" branches, where the Higgs branch is the one supporting the microscopic single D-brane picture of bound states, while the Coulomb branch supports the multi-centered picture. Classically, the Coulomb branch is trivially flat, but we will see that quantum effects induce an effective potential and magnetic interaction, which are precisely of the form needed to get multicentered "molecular" BPS bound states. The match in this respect between the supergravity and substringy regimes will be traced back to a non-renormalization theorem for the term linear in "velocities" in the $\mathcal{N} = 4$ multiparticle Lagrangian. We will furthermore identify a regime in which the state lives essentially on the Higgs branch, with quantum fluctuation effectively washing out the structure on the Coulomb branch, and a complementary regime in which the opposite happens. Lowering g_s down to zero corresponds then to "squeezing" the states from their life on the Coulomb branch to a new life on the Higgs branch.

The counting problem on the Higgs branch reduces to finding the Betti numbers of the cohomology of the associated quiver moduli space. For quivers without closed loops, this problem was recently solved in full generality [17], allowing us to do predictions of quantum ground state degeneracies of various highly nontrivial systems of generalized interacting "electron-monopole" systems, for which even the classical configuration moduli space can be extremely complicated. This includes ground state degeneracies of certain dyons in supersymmetric Yang-Mills theories.

Finally, this work also provides new insight in the correspondence between stability of multi-centered BPS configurations [2, 3] and stability of wrapped D-branes

[9]-[14],[18]-[35].

The organization of this paper is as follows. In section 2, we analyze in detail the structure of the molecular multi-centered bound states, both in supergravity and in abstract generality, and (re-)establish a supersymmetric non-renormalization theorem which fixes the part of the Lagrangian responsible for this kind of bound states. In section 3, we introduce the quiver model, recall some of its mathematical features, and give its physical interpretation. In section 4, we turn to the quantization of this model and explain how the two pictures of bound states are related in this framework. The case of the Hall halo, as summarized above, is studied in more detail as a non-trivial example of the correspondence. In section 5 we present more tests and applications. In particular, we formulate some predictions of ground state degeneracies of generalizations of the Hall halo, and reproduce and refine the ground state counting of the Stern-Yi dyon chain [36]. Our conclusions and open questions are discussed in section 6. The appendices give our conventions and various explicit expressions for quiver Lagrangians and their supersymmetries.

2. BPS bound states of particles in 3+1 dimensions

Consider a 3+1 dimensional $\mathcal{N} = 2$ supergravity theory containing a number of massless abelian vector multiplets coupled to a number of BPS particles with arbitrary corresponding electric and magnetic charges. Systems like this typically arise in the low energy limit of Calabi-Yau compactifications of type II string theory, where D-branes wrapped around nontrivial cycles manifest themselves as charged particles in the 3+1 dimensional low energy effective theory.

These particles have long distance interactions through their coupling to the metric, the vector fields, and the complex scalars of the vector multiplets. The relative strength of these forces depends on the choice of vacuum (expectation values of the complex scalars). The force between static BPS particles with proportional charge vectors always vanishes, but, for generic vacua, this is not true for particles with non-proportional charges. Moreover, the interactions are sufficiently complex to allow situations where nontrivial balancing between the different forces occurs at certain separations. Many of the resulting classical bound states are BPS, as studied in detail in [2, 3]. In quantum field theories without gravity, similar structures emerge, see for instance [4, 5, 6].

In this section, we will investigate such interacting multiparticle systems. We will take two approaches. The first one is based on solutions of the supergravity theory, and the second one on the constraints imposed by supersymmetry on the particle mechanics itself. The detailed supergravity picture is not really essential for the main purpose of this paper, but as it gives a concrete physical realization of these systems (and was in fact the original inspiration for this work), we include it here, though we will only briefly review the main features, referring to [2, 3] for details.

Let us first recall a few basic notions. Assume the supergravity theory under consideration has n $U(1)$ vectors apart from the graviphoton. A charge vector Q is then specified by $2n+2$ integers: $n+1$ electric charges $Q_{e,I}$ and $n+1$ magnetic charges Q_m^I. Two charges Q and $\tilde{Q}$ are called mutually nonlocal if their Dirac-Schwinger-Zwanziger product $\langle Q, \tilde{Q} \rangle \equiv Q_{m,I} \tilde{Q}_e^I - Q_e^I \tilde{Q}_{m,I}$ is nonzero. In the IIB wrapped D3-brane picture, the geometric interpretation of this product is the usual intersection product: it counts (with signs) the number of intersection points of the corresponding two 3-branes. A pair of mutually nonlocal charges can be thought of as a (generalized) monopole-electron system.[1]

A central role in the description of BPS states is played by the central charge. This is a function of the complex vector multiplet scalars z^a, $a = 1, \ldots, n$ (which are the complex structure or the complexified Kähler moduli of the CY in the IIB resp. IIA string theory context), and a linear function of the electromagnetic charge Q. We will denote it by $Z_Q(z)$. The dependence on z is holomorphic up to an overall normalization factor (the exponential of half the Kähler potential on the vector moduli space). The mass of a BPS particle of charge Q, in a vacuum specified by the vevs $\langle z^a \rangle = z^a|_{r=\infty} \equiv u^a$ is given by $M = |Z_Q(u)|/l_P$, where l_P is the four dimensional Planck length, defined here as the square root of the Newton constant. The interpretation of the phase of the central charge is the embedding angle of the residual $\mathcal{N}=1$ supersymmetry in the original $\mathcal{N}=2$ [37].

2.1 Single probe in supergravity background

We start with the simplest case: a (light) probe BPS particle of ($n+1$-component) charge q in the background produced by another (heavy) BPS particle of charge Q, fixed at the origin. We assume the particles are mutually nonlocal, i.e. $\langle q, Q \rangle \neq 0$. With $u^a = z^a|_{r=\infty}$, the mass of the probe is $m = |Z_q(u)|/l_P$.

The metric produced by the fixed source is of the form $ds^2 = -\rho^2 dt^2 + \rho^{-2} d\mathbf{x}^2$. The redshift factor ρ as well as the vector multiplet scalars z^a are function of the coordinate distance $r = |\mathbf{x}|$ only, and obtained as the solutions to the integrated BPS equations of motion (first derived in [38] and written in the following form in [2]):

$$2\,\rho^{-1}\,\mathrm{Im}[e^{-i\alpha} Z_{Q'}(z)]\big|_r = -\frac{l_P\,\langle Q', Q\rangle}{r} + 2\,\mathrm{Im}[e^{-i\alpha} Z_{Q'}(z)]\big|_{r=\infty} \qquad (2.1)$$

for arbitrary charges Q' (or equivalently for a basis of $2(n+1)$ charges Q'), with $\alpha \equiv \arg Z_Q$. For most Calabi-Yau manifolds, because of the complicated dependence of the central charges Z on the moduli z^a, it is in general not possible to find exact analytic solutions for $z^a(r)$ and $\rho(r)$, but several approximate analytical and numerical solutions have been obtained, and many properties can be infered directly from

[1] We will use the term "monopole-electron system" loosely in this paper. A more precise phrasing would be "a pair of BPS particles which have magnetic resp. electric charge with respect to the same $U(1)$, upon a suitable choice of charge basis".

the equations. As noted before, the details of this are not important for the purpose of this paper.

The action for a BPS probe is [39, 40]

$$S = -l_P^{-1} \int |Z_q| \, ds + \int \langle q, \mathcal{A} \rangle \,, \tag{2.2}$$

where $\mathcal{A}$ is the ($n+1$-component) electromagnetic connection of the background. Using the BPS equations of motion (2.1) for $z^a(r)$ and $\rho(r)$ and those for $\mathcal{A}$ (for which we refer to [2]), one gets a probe action of the form $S = \int (K - V + M) \, dt$ with kinetic, potential and magnetic parts given by [2]:

$$\begin{aligned}
K &= -l_P^{-1} \, \rho \, |Z_q| \, \left(\sqrt{1 - \rho^{-4} \, \dot{\mathbf{x}}^2} - 1 \right) && (2.3)\\
&\approx l_P^{-1} \, \rho^{-3} |Z_q| \, \dot{\mathbf{x}}^2/2 \quad \text{if } \rho^{-2} |\dot{\mathbf{x}}| \ll 1 \,, && (2.4)\\
V &= l_P^{-1} \, \rho \, |Z_q| \, (1 - \cos(\alpha_q - \alpha)) && (2.5)\\
&= 2 \, l_P^{-1} \, \rho \, |Z_q| \, \sin^2[(\alpha_q - \alpha)/2] && (2.6)\\
&\approx l_P^{-1} \, \rho \, |Z_q| (\alpha_q - \alpha)^2 \quad \text{if } |\alpha_q - \alpha| \ll 1 \,, && (2.7)\\
M &= \frac{1}{2} \, \langle q, Q \rangle \, \mathbf{A}^d \cdot \dot{\mathbf{x}} \,. && (2.8)
\end{aligned}$$

The dot denotes d/dt, α_q and α are the phases of Z_q resp. Z_Q, and $\mathbf{A}^d$ is a $U(1)$ vector potential for the Dirac magnetic monopole carrying one flux quantum:

$$\mathbf{A}^d \cdot \dot{\mathbf{x}} = \frac{1}{2} (\pm 1 - \cos \vartheta) \, \dot{\varphi} = \frac{1}{2} \, (\pm 1 - z/r) \, \frac{x \dot{y} - y \dot{x}}{x^2 + y^2} \,. \tag{2.9}$$

The approximations (2.4) and (2.7) correspond to the nonrelativistic limit (i.e. kinetic and interaction energies much smaller than the total mass of the system). Note that the various quantities appearing in (2.3) - (2.8) are r-dependent through the r-dependence of ρ and z. Fig. 1 shows some typical potentials $V(r)$; concrete examples for compactifications on the Quintic can be found in [2, 3].

If at a certain radius $r = R$ the phases of probe and source are equal (in other words, if the radial scalar flow passes through a (Q, q)-marginal stability wall, where $\alpha_q = \alpha$), the potential V reaches a zero energy minimum (cases (a) and (b) in the figure). Placed at this radius, the probe does not break more of the supersymmetry than the background already did, and the configuration is BPS. The value of R depends on the scalar vevs $u^a = z^a|_{r=\infty}$, and is immediately obtained from (2.1) by taking $Q' = q$:

$$R = \left. \frac{l_P \, \langle q, Q \rangle}{2 \, \mathrm{Im}[e^{-i\alpha} Z_q]} \right|_{r=\infty} = \left. \frac{\langle q, Q \rangle}{2 \, m \, \sin(\alpha_q - \alpha)} \right|_{r=\infty} \,, \tag{2.10}$$

Of course, R needs to be positive, so a necessary condition for the existence of such a classical BPS bound state with nonzero separation (in the probe approximation)

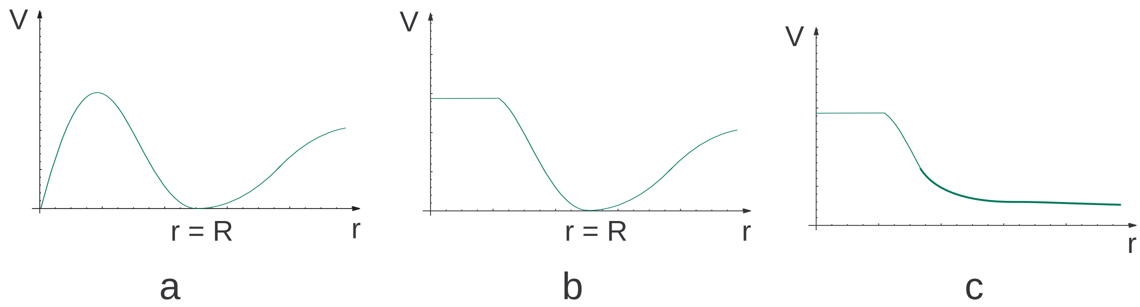

Figure 1: Typical examples of a potential for a test particle in the background field of another charge. The potential is taken to be zero when the total energy saturates the BPS bound. (a) corresponds to a black hole source (with horizon at $r = 0$), (b) and (c) to a source with an enhançon-like [41] core instead of a horizon (an "empty hole" in the terminology of [2]). The cases (a) and (b) have $\langle q, Q\rangle \sin(\alpha_q - \alpha)_{r=\infty} > 0$, while (c) has $\langle q, Q\rangle \sin(\alpha_q - \alpha)_{r=\infty} < 0$.

is that the DSZ intersection product $\langle q, Q\rangle$ and $\sin(\alpha_q - \alpha)_{r=\infty}$ have the same sign. Note also that the phenomenon of decay at marginal stability is a natural, smooth process in this picture: when we start with a BPS configuration in a certain vacuum characterized by $u^a = z^a|_{r=\infty}$, and the u^a are varied to approach a marginal stability wall, the radius diverges and the BPS bound state decays smoothly into a two particle state.

Much of the interesting structure emerging here is in fact a direct consequence of supersymetry, as we will see in the following.

2.2 Supersymmetric mechanics of a single probe particle

An (effective) BPS particle in a 3+1 dimensional $\mathcal{N} = 2$ theory conserves four of the original eight supercharges. Its low energy dynamics can therefore be expected to be described by a $d = 1$, $\mathcal{N} = 4$ supersymmetric Lagrangian. The degrees of freedom appearing in this Lagrangian will always at least include the position coordinate $\mathbf{x}$ in the noncompact space, together with its fermionic superpartner, given by a 2-component spinor λ_α, $\alpha = 1, 2$ and its complex conjugate $\bar{\lambda}^\alpha \equiv (\lambda_\alpha)^*$. Together with an auxiliary bosonic variable D and a one dimensional connection A, these degrees of freedom form a vector multiplet (or linear multiplet) on the particle worldline, which can be thought of as the dimensional reduction of a $d = 4$, $\mathcal{N} = 1$ vector multiplet. This kind of supersymmetric quantum mechanics was first constructed in [42] for a flat target space. It was generalized to curved spaces in [43] as an effective description of the zero mode dynamics of supersymmetric QED, given a superfield description in [44], and obtained as an effective theory for chiral SQED in [45]. The superspace formulation of this model was rediscovered and used to prove a non-renormalization theorem for $\mathcal{N}=8$ quantum mechanics in [46].

The supersymmetry transformations are as follows (our conventions can be found in appendix A):

$$\delta A = i\,\bar{\lambda}\,\xi - i\,\bar{\xi}\,\lambda \tag{2.11}$$

$$\delta \mathbf{x} = i\,\bar{\lambda}\,\boldsymbol{\sigma}\,\xi - i\,\bar{\xi}\,\boldsymbol{\sigma}\,\lambda \tag{2.12}$$

$$\delta\lambda = \dot{\mathbf{x}}\cdot\boldsymbol{\sigma}\,\xi + i\,D\,\xi \tag{2.13}$$

$$\delta D = -\dot{\bar{\lambda}}\,\xi - \bar{\xi}\,\dot{\lambda} \tag{2.14}$$

The particle can have further internal degrees of freedom, but let us assume for now that these are absent (or that they do not couple to the position vector multiplet).

Regardless of its specific form, the Lagrangian can be expanded in powers of velocity:

$$L = L^{(1)} + L^{(2)} + \dots\,, \tag{2.15}$$

where we assign the following orders to the various quantities appearing in the Lagrangian:

$$\mathcal{O}(\mathbf{x}) = 0, \qquad \mathcal{O}(\frac{d}{dt}) = 1, \qquad \mathcal{O}(D) = 1, \qquad \mathcal{O}(\lambda) = 1/2\,. \tag{2.16}$$

With this assignment, and taking $\mathcal{O}(\xi) = -1/2$, the supersymmetry transformations preserve the order, so to have a supersymmetric total L, each individual $L^{(k)}$ has to be supersymmetric.

In [43, 46], it was shown that a wide class of quadratic Lagrangians $L^{(2)}$ can be obtained by choosing an arbitrary "Kähler potential" function $K(\mathbf{x})$, giving for the second order bosonic part of the Lagrangian:

$$L_B^{(2)} = \frac{1}{4}\nabla^2 K(\mathbf{x})\,(\dot{\mathbf{x}}^2 + D^2)\,. \tag{2.17}$$

The possible presence of a first order term $L^{(1)}$ was not considered in [46], but will be of crucial importance in the present work. In general it will be of the form

$$L^{(1)} = -U(\mathbf{x})\,D + \mathbf{A}(\mathbf{x})\cdot\dot{\mathbf{x}} + C(\mathbf{x})\,\bar{\lambda}\lambda + \mathbf{C}(\mathbf{x})\cdot\bar{\lambda}\boldsymbol{\sigma}\lambda. \tag{2.18}$$

The second term in this expression gives a Lorentz-type force, while the first term can be thought of as a position-dependent Fayet-Iliopoulos term. A similar term is well known to play a key role in two dimensional $\mathcal{N}=2$ linear sigma-models [47]. The one dimensional counterpart has been considered as well [42, 45, 48], but seems to be less widely known.

Requiring $L^{(1)}$ to be supersymmetric imposes rather strong constraints on U, $\mathbf{A}$, C and $\mathbf{C}$. A direct computation using (2.12)-(2.14) gives:

$$\mathbf{C} = \nabla U = \nabla \times \mathbf{A}\,, \qquad C = 0\,. \tag{2.19}$$

Allowing a singularity at the origin, the general spherically symmetric solution to these constraints is, with $r = |\mathbf{x}|$:

$$U = \frac{\kappa}{2r} + \theta\,, \qquad \mathbf{A} = -\kappa \mathbf{A}^d\,, \tag{2.20}$$

with θ and κ constants, and $\mathbf{A}^d$ a unit Dirac monopole vector potential as in (2.9). The Dirac quantization condition requires κ to be an integer. Non-spherically symmetric solutions are also possible of course, corresponding to dipoles, quadrupoles and so on, but since we assumed no further internal particle degrees of freedom, we do not consider those here. In the simplest case, flat space, the Lagrangian takes the following form:

$$L = \frac{m}{2}\left(\dot{\mathbf{x}}^2 + D^2 + 2i\bar{\lambda}\dot{\lambda}\right) - (\frac{\kappa}{2|\mathbf{x}|} + \theta)D - \kappa \mathbf{A}^d \cdot \dot{\mathbf{x}} - \frac{\kappa\,\mathbf{x}}{2|\mathbf{x}|^3} \cdot \bar{\lambda}\boldsymbol{\sigma}\lambda. \tag{2.21}$$

The physical potential energy for the position $\mathbf{x}$ is obtained by eliminating the auxiliary variable D from the Lagrangian. The precise form of the potential will therefore depend on the form of each of the terms $L^{(n)}$ in the expansion (2.15). For example if all $L^{(n)}$ with $n \geq 3$ are zero, the potential is $V(\mathbf{x}) = U^2/\nabla^2 K$. However, the presence and position of supersymmetric extrema of the Lagrangian only depends on the first order term. Indeed, from the supersymmetry variations (2.12)-(2.14), it follows that a classical supersymmetric configuration is given by a time-independent $\mathbf{x}$ with $\lambda = 0$ and $D = 0$. For this to be a solution to the equations of motion, one needs furthermore $\delta L/\delta D = 0$, which is the case if and only if $U = 0$, or with (2.20):

$$r = -\frac{\kappa}{2\theta}\,. \tag{2.22}$$

Identifying $\kappa = \langle Q, q\rangle$ and $\theta = m\sin(\alpha_q - \alpha)|_{r=\infty}$, this expression coincides with (2.10).

To make this match more precise, we would have to construct the full supersymmetric extention of the supergravity probe Lagrangian (2.3)-(2.8). This is quite complicated in the fully relativistic version, so we will restrict ourselves here to the non-relativistic approximation as given by (2.4) and (2.7), i.e. small velocities and small phase difference.[2] First observe that in this regime, the extended bosonic Lagrangian

$$L_B = \frac{1}{2l_P}\rho^{-3}|Z_q|(\dot{\mathbf{x}}^2 + D^2) - \rho^{-1}\,\mathrm{Im}[e^{-i\alpha}Z_q]\,D - \langle Q, q\rangle\,\mathbf{A}^d \cdot \dot{\mathbf{x}} \tag{2.23}$$

[2]The condition on the phases cannot be dropped in a consistent nonrelativistic approximation; for arbitarary phase differences, the supergravity probe Lagrangian has no supersymmetric extension quadratic in the velocities.

reduces to the nonrelativistic version of the probe Lagrangian (2.3)-(2.8) after eliminating the auxiliary field D. Note also that the background BPS equations of motion (2.1) imply that the coefficient of D in the linear term is of the form (2.20), with $\kappa = \langle Q, q \rangle$ and $\theta = l_P^{-1} \operatorname{Im}[e^{-i\alpha} Z_q]_\infty = m_q \sin(\alpha_q - \alpha)$. The first order terms thus satisfy the required constraints. Finally, the second order part is of the form (2.17), so we conclude that the (nonrelativistic) supergravity probe action indeed has a supersymmetric extention of the form analyzed in this section, with the above identifications of κ and θ.

2.3 General multicentered BPS bound states in supergravity

The supergravity probe considerations of section 2.1 can be enhanced to a full analysis of BPS solutions of the supergavity equations of motion, with an arbitrary number of centers $\mathbf{x}_p$, $p = 1, \ldots, N$, carrying arbitrary charges Q_p. This was investigated in detail in [2]. The analysis is technically quite involved (mainly due to the fact that the corresponding spacetimes are non-static — time-independent but with a non-diagonal metric), but fortunately we will only need some of the conclusions, which can be stated rather simply. The result which is most relevant for this paper is that the BPS requirement gives a set of constraints on the relative positions of the centers, generalizing (2.10), namely for every center p:

$$\sum_{q=1}^{N} \frac{\langle Q_p, Q_q \rangle}{|\mathbf{x}_q - \mathbf{x}_p|} = 2\, m_p \, \sin(\alpha_p - \alpha)\big|_{r=\infty} \,, \tag{2.24}$$

where $m_p = |Z_p|$ is the BPS mass of a particle with charge Q_p (in the given vacuum), α_p is the phase of Z_p, α the phase of the total central charge $Z = \sum_p Z_p$, and $\langle \cdot, \cdot \rangle$ denotes as before the DSZ intersection product. Note that only $N - 1$ constraints are independent, since summing (2.24) over all $p = 1, \ldots, N$ gives trivially $0 = 0$.

Another result of interest from [2] is the fact that these solutions, being non-static, carry an intrinsic angular momentum, given by the formula

$$\mathbf{J} = \frac{1}{2} \sum_{p<q} \langle Q_p, Q_q \rangle \, \frac{\mathbf{x}_p - \mathbf{x}_q}{|\mathbf{x}_p - \mathbf{x}_q|} \,. \tag{2.25}$$

2.4 Supersymmetric multi-particle mechanics

The general one-particle supersymmetric mechanics of section 2.2 can be generalized to an arbitrary number N of interacting BPS particles with arbitrary charges. As we saw in section 2.3 in the example of supergravity, such systems can have a moduli space of classical BPS configurations, conserving four of the original eight supercharges. The low energy dynamics of such a system can therefore be expected to be described by an $\mathcal{N} = 4$ supersymmetric multi-particle Lagrangian, with degrees of freedom including at least the position vector multiplets of the different particles involved. We will again assume that the particles do not have further internal low energy degrees of freedom (or at least that the latter don't couple to the position multiplets). In the context of string theory, where the particles arise as an effective description of wrapped D-branes, there can nevertheless still be additional relevant low energy degrees of freedom corresponding to strings stretching between the wrapped branes. For well-separated particles (compared to the string scale), such strings are very massive and can be integrated out safely. However, when the particles come close to each other, the stretched strings can become massless or even tachyonic, and the description in terms of commuting position multiplets only breaks down.

In this section, we will ignore this phenomenon and study the general form of $\mathcal{N}=4$ supersymmetric effective Lagrangians involving the abelian position multiplets $(\mathbf{x}_p, D_p, \lambda_p)_{p=1}^N$ only. A more refined analysis of the breakdown of this description and its stringy resolution will be presented in section 4.

The supersymmetry transformations are as in (2.11)-(2.14), with the addition of an index p. The general form of the Lagrangian to first order in the velocities becomes:

$$L^{(1)} = \sum_p \left(-U_p\, D_p + \mathbf{A}_p \cdot \dot{\mathbf{x}}_p\right) + \sum_{p,q} \left(C_{pq}\, \bar{\lambda}_p \lambda_q + \mathbf{C}_{pq} \cdot \bar{\lambda}_p \boldsymbol{\sigma} \lambda_q\right) . \tag{2.26}$$

Requiring the Lagrangian to be supersymmetric gives the following constraints, generalizing (2.18):

$$\mathbf{C}_{pq} = \nabla_p U_q = \nabla_q U_p = \frac{1}{2}\left(\nabla_p \times \mathbf{A}_q + \nabla_q \times \mathbf{A}_p\right); \qquad C_{pq} = 0 . \tag{2.27}$$

Allowing singularities when two centers coincide, these constraints are solved by

$$U_p = \sum_q \frac{\kappa_{pq}}{2r_{pq}} + \theta_p , \tag{2.28}$$

with $\kappa_{pq} = -\kappa_{qp}$, and $\mathbf{A}_p$ the vector potential produced at $\mathbf{x}_p$ by a set of magnetic monopoles with charges $\{\kappa_{pq}\}_q$, $q = 1, \ldots, N$ at respective positions $\{\mathbf{x}_q\}_q$. Plugged into the general form of the first order Lagrangian (2.26), this gives:

$$L^{(1)} = -\sum_p \theta_p D_p - \sum_{p<q} \kappa_{pq}\, L_{pq}^{int} \tag{2.29}$$

$$L_{pq}^{int} = \frac{1}{2r_{pq}} D_{pq} + \mathbf{A}^d(\mathbf{r}_{pq}) \cdot \dot{\mathbf{r}}_{pq} + \frac{1}{2r_{pq}^3}\, \mathbf{r}_{pq} \cdot \bar{\lambda}_{pq} \boldsymbol{\sigma} \lambda_{pq} , \tag{2.30}$$

where $\kappa_{pq} = -\kappa_{qp}$ are constants, $\mathbf{r}_{pq} = \mathbf{x}_p - \mathbf{x}_q$, $D_{pq} = D_p - D_q$, $\lambda_{pq} = \lambda_p - \lambda_q$, and $\mathbf{A}^d(\mathbf{r})$ as in (2.9). The Dirac quantization condition here is $\kappa_{pq} \in \mathbb{Z}$.

As in the single particle case, the precise form of the interaction potential between the particles also depends on the higher order terms in the Lagrangian, but

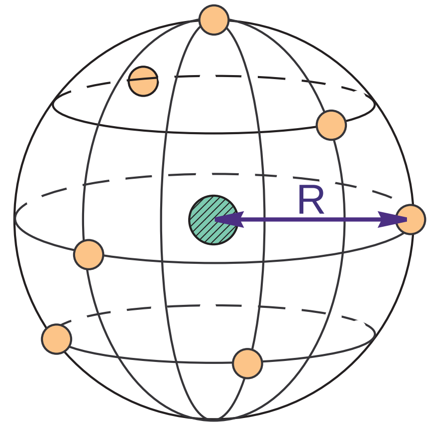

Figure 2: A Hall (or Hole) Halo consisting of a charge Q in the origin surrounded by $N = 7$ charges q on a sphere of radius $R = -\kappa/2c$, with κ units of magnetic flux through the sphere.

its supersymmetric minima, leading to classical BPS bound states, are entirely determined by $L^{(1)}$. They occur when U_p becomes zero, i.e. at positions satisfying for all p:

$$\sum_q \frac{\kappa_{pq}}{2r_{pq}} = -\theta_p \,. \tag{2.31}$$

Again, this expression is identical to the supergravity BPS constraint (2.24) with the identifications

$$\kappa_{pq} = \langle Q_q, Q_p \rangle \quad \text{and} \quad \theta_p = \mathrm{Im}(e^{-i\alpha} Z_p) = m_p \sin(\alpha_p - \alpha). \tag{2.32}$$

2.5 Special case: the hole or Hall halo.

A simple yet already quite interesting example is the system consisting of one particle (or black hole) of charge Q interacting with N particles (or black holes) of charge q, with $\kappa \equiv \langle Q, q \rangle \neq 0$. According to (2.31), the classical ground states of this system are configurations with all N particles of charge q on a sphere of radius $R = -\kappa/2\theta_q$ around the charge Q particle. In the supergravity context $\theta_q = l_P^{-1}\, \mathrm{Im}(\bar{Z}_Q Z_q)/|Z_Q + N Z_q|\,\big|_\infty = \mu \sin(\alpha_q - \alpha_Q)$, where μ is the "reduced mass", $\mu = m_Q m_q / m_{tot}$. One could call such a configuration a "Hole Halo", as illustrated in fig. 2. From equation (2.30), we furthermore see that the particles are moving in a uniform magnetic[3] field with κ units of flux through the sphere. This is the typical setup used to study the quantum Hall effect on a sphere (see for example [15]). One could therefore equally well call such a system a Hall Halo. Another system designed

[3]By "magnetic" we mean here magnetic relative to the charges on the sphere. With respect to a fixed chosen charge basis, the field is not necessarily purely magnetic; depending on the choice of basis, it could even be purely electric in that sense.

to reproduce the spherical quantum Hall effect was discussed in the context of string theory in [49]. The quantum Hall halo considered here seems to free of some of the undesired features of this other system [50], though we did not analyze this in detail.

3. BPS D-branes and classical quiver mechanics

In string theory, BPS states can often be analyzed perturbatively by describing them as D-branes, i.e. subspaces on which open strings can end (or their generalizations as CFT boundary states). This description of BPS states as infinitely thin superposed objects, with no backreaction on the ambient space, is exact in the limit of vanishing string coupling constant, $g_s = 0$. When the string coupling constant is turned on, the D-branes become dynamical objects of finite width, interacting with the ambient space, and in suitable regimes (typically at large $g_s \times$ (number of branes), outside the domain of validity of open string perturbation theory), they are believed to become well-described by the solitonic p-brane solutions of supergravity.

In the framework of type II string theory compactified on a Calabi-Yau manifold, we thus expect the bound states considered in section 2 to have a corresponding description as wrapped D-branes, which becomes accurate when $g_s \to 0$. At first sight, the fact that those multicentered, molecule-like bound states have a dual description as a wrapped D-brane localized at a single point in the noncompact space may seem a bit odd. However, note that all length scales appearing in section 2 are proportional to the four dimensional Planck length l_P, related to the string length $l_s = \sqrt{2\pi\alpha'}$ as

$$l_P = g_s l_s / \sqrt{v}\,, \tag{3.1}$$

where $v = 2V/\pi^2 l_s^6$, with V the volume of the Calabi-Yau. Thus, if we take $g_s \to 0$ while keeping all other parameters fixed, the equilibrium radii such as (2.10) become vanishingly small compared to the string length. The naive particle picture breaks down in this limit, as in particular open strings stretching between the different branes can become tachyonic, leading to the decay of the multicentered configuration into a single-centered wrapped brane. Conversely, this also indicates that quantum effects (i.e. open string loops) should produce rather drastic qualitative effects in order to match the two, allowing to go smoothly from single centered wrapped branes into multicentered configurations. Details of this D-brane bound state metamorphosis will be analyzed in section 4.

The classical ($g_s = 0$) D-brane description of BPS states in Calabi-Yau compactifications has been studied extensively (an incomplete list of references is [8]-[13], [18]-[35]). One of the results emerging from this work is that the picture of BPS D-branes as classical submanifolds (possibly carrying certain vector bundles), valid in the large radius limit, needs to be modified for generic moduli of the Calabi-Yau. The more general picture is that of a D-brane as an object in a certain category,

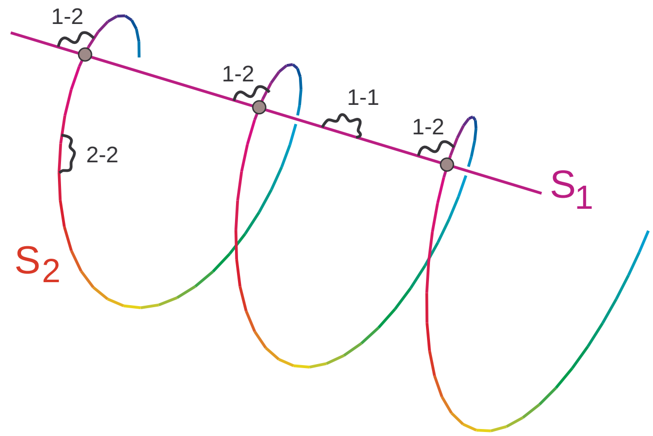

Figure 3: One dimensional caricature of the intersecting branes S_1 and S_2. The various light open string modes are indicated.

with massless fermionic strings playing the role of morphisms between the objects. The full story requires quite a bit of algebraic geometry, but fortunately, in many cases, the low energy D-brane dynamics can simply be described by a $d = 4$, $\mathcal{N} = 1$ quiver gauge theory dimensionally reduced to the effective particle worldline, and we can forget about the underlying geometrical structure. This is well known in the case of orbifold constructions [8]. Another (though overlapping) class of examples is given by collections of "parton" D-branes with nearly equal phases. Parton D-branes [14] are D-branes which can be considered elementary (at the given point in moduli space), in the sense that they come in the smallest massive BPS supermultiplets of the theory (hypermultiplets for an $\mathcal{N} = 2$ theory), and that other D-branes can be constructed as their bound states. This typically corresponds to D-branes that can become massless (in four dimensional Planck units) at some point in Calabi-Yau moduli space, such as a type IIB D3-brane wrapped around an S^3 cycle vanishing at a conifold point.[4]

3.1 An example

We start by considering a simple intuitive model in type IIB, in a regime where the classical geometric picture of D-branes as minimal volume manifolds is accurate, or at least where thinking of these branes geometrically gives the right results for our purposes.

[4]To make this statement more precise, we would have to use the category construction of [13]. Since we only want to give some intuition in how the quiver description arises, and are ultimately only interested in the resulting (dimensionally reduced) gauge theory, we will not get into this more deeply here.

Let S_1 and S_2 be two parton D3-branes with the topology of a 3-sphere, wrapped around two distinct cycles in a Calabi-Yau X. We assume S^3 topology for this example because $H_1(S^3, \mathbb{Z}) = 0$, implying [18, 51, 52] that the individual branes are *rigid*, i.e. they have no moduli of their own.[5] Assume furthermore that S_1 and S_2 intersect transversally in κ points, with all intersections positive, so the geometric intersection product $\langle S_1, S_2 \rangle = \kappa > 0$. The situation is sketched in fig. 3. The central charges of the branes are given by $Z_p = \int_{S_p} \Omega$, with Ω the holomorphic 3-form on X, normalized so $i \int \Omega \wedge \bar{\Omega} = 1$. The central charge phases are denoted as $\alpha_p = \arg Z_p$, $\alpha = \arg Z$. The combined brane system is supersymmetric (BPS) if $\alpha_1 = \alpha_2$.

There are three different kinds of open string modes: starting and ending on S_1, starting and ending on S_2, and stretching between S_1 and S_2. The massless $1-1$ and $2-2$ string modes correspond to changes in the positions $\mathbf{x}_p$ of the branes in the noncompact space, and possibly to further "internal" susy-preserving deformations of the individual branes. The latter are absent though in the case at hand, since we assumed the S_p rigid, leaving only the position modes. They come in two $d = 1$, $\mathcal{N} = 4$ vector multiplets (dimensionally reduced $d = 4$, $\mathcal{N} = 1$ $U(1)$ vector multiplets), which we will denote as in the previous section by $(A_p, \mathbf{x}_p, D_p, \lambda_p)$, $p = 1, 2$. The functions A_p are the one-dimensional $U(1)$ connections.

Assuming one can locally use the branes at angles setup of [53] to compute the light $1-2$ open string spectrum, one finds that in the case of coincident positions and phases ($\mathbf{x}_1 = \mathbf{x}_2$ and $\alpha_1 = \alpha_2$), there are κ massless modes, corresponding to stretched strings localized at the intersection points. These modes come in charged chiral multiplets, with charge $(-1, 1)$ under the $U(1) \times U(1)$ D-brane gauge group. The sign of the charge and the chirality are determined by the sign of the intersection. We denote these chiral multiplets by (ϕ^a, F^a, ψ^a), $a = 1, \ldots, \kappa$, where ϕ^a is a complex scalar, ψ^a a 2-component spinor, and F^a an auxiliary complex scalar.

When the branes are separated in the noncompact space, supersymmetry is preserved but the chiral multiplets become massive with mass $m_C = |\mathbf{x}_2 - \mathbf{x}_1|$ (in units such that $l_s \equiv 1$, which we will use from now on). On the other hand when $\alpha_1 \neq \alpha_2$, supersymmetry is broken and bose-fermi degeneracy in the chiral multiplets is lifted: while the mass of the fermionic chiral modes ψ^a remains unchanged, the mass of the bosonic modes ϕ^a is shifted to ${m_\phi}^2 = (\mathbf{x}_2 - \mathbf{x}_1)^2 + \alpha_2 - \alpha_1$. For sufficiently small separation and $\alpha_2 < \alpha_1$, these modes become tachyonic. Tachyon condensation corresponds in this case to the formation of a classical bound state, a single wrapped D-brane S produced by putting $\mathbf{x}_1 = \mathbf{x}_2$ and deforming the intersections to throats smoothly connecting S_1 and S_2. This breaks (classically) the $U(1) \times U(1)$ gauge

[5] Examples of non-rigid parton D-branes are given by flat D3 branes on T^6; they are partons because they come in the smallest possible $\mathcal{N} = 8$ massive BPS multiplet, they can be used to build more complicated branes [34] and their mass (in four dimensional Planck units) vanishes at appropriate large complex structure points, but they are not rigid since they have translation moduli.

symmetry to $U(1)$, and the corresponding $\kappa - 1$ massless Goldstone bosons provide the deformation moduli of S, consistent with the fact that $b^1(S = S_1 \# S_2) = \kappa - 1$. Note that the condition $\alpha_1 - \alpha_2 > 0$ for bound state formation here is the same as the one we found in the context of supergravity (remember we took $\kappa = \langle S_1, S_2 \rangle > 0$).

To keep these chiral multiplets much lighter than the infinite tower of excited open string modes, so they deliver the dominant contribution to the inter-brane interaction at low energies, we take $|\alpha_1 - \alpha_2| \ll 1$ and $|\mathbf{x}_1 - \mathbf{x}_2| \ll l_s$. Then the following classical $d = 1$, $\mathcal{N} = 4$ Lagrangian $L = L_V + L_C$, obtained by dimensional reduction from the $\mathcal{N} = 1$, $d = 4$ gauge theory with the field content outlined above, describes the low energy dynamics of these branes:

$$
\begin{aligned}
L_V &= \frac{m_p}{2}\left(\dot{\mathbf{x}}_p^2 + {D_p}^2 + 2i\bar{\lambda}_p\dot{\lambda}_p\right) - \theta_p D_p && (3.2)\\
L_C &= |\mathcal{D}_t\phi^a|^2 - \left((\mathbf{x}_2 - \mathbf{x}_1)^2 + D_2 - D_1\right)|\phi^a|^2 + |F^a|^2 + i\,\bar{\psi}^a\mathcal{D}_t\psi^a \\
&\quad -\bar{\psi}^a\,(\mathbf{x}_2 - \mathbf{x}_1)\cdot\boldsymbol{\sigma}\,\psi^a - i\sqrt{2}(\bar{\phi}^a\psi^a\epsilon(\lambda_2 - \lambda_1) - (\bar{\lambda}_2 - \bar{\lambda}_1)\epsilon\bar{\psi}^a)\phi^a\,, && (3.3)
\end{aligned}
$$

where summation over $p = 1, 2$ and $a = 1, \ldots, k$ is understood, the covariant derivative $\mathcal{D}_t\phi \equiv (\partial_t + i(A_2 - A_1))\,\phi$, $m_p = |Z_p|/l_P$, the θ_p are Fayet-Iliopoulos parameters, and we have put $l_s = 1$. We will see below that we have to take $\theta_p \equiv m_p(\alpha_p - \alpha_0)$, with $\alpha_0 \equiv (m_1\alpha_1 + m_2\alpha_2)/(m_1 + m_2)$, to match the string masses and to have zero energy for supersymmetric configurations.[6] The supersymmetry variations under which this action is invariant can be found in appendix B. This Lagrangian can also be considered to be the dimensional reduction of a two-dimensional linear sigma model [47]. These models have been analyzed very extensively in the literature, and since the *classical* features of the $d = 2$ and the $d = 1$ versions are essentially identical, the following discussion is in essence merely a review of well known facts. It is useful though to have things explicit for subsequent sections.

The Lagrangian can be split in a center of mass part and a relative part. Denoting the center of mass variables by $\mathbf{x}_0 \equiv (m_1\mathbf{x}_1 + m_2\mathbf{x}_2)/(m_1 + m_2)$ and so on, the center of mass Lagrangian is simply

$$
L_0 = \frac{m_1 + m_2}{2}\left({\dot{\mathbf{x}}_0}^2 + {D_0}^2 + 2i\bar{\lambda}_0\dot{\lambda}_0\right). \tag{3.4}
$$

There is no FI-term, hence no rest energy, because $\theta_1 + \theta_2 = 0$.

Denoting the relative variables by $\mathbf{x} = \mathbf{x}_2 - \mathbf{x}_1$ and so on, the relative part of the Lagrangian $L_{rel} = L_{rel,V} + L_{rel,C}$ becomes

$$
\begin{aligned}
L_{rel,V} &= \frac{\mu}{2}\left(\dot{\mathbf{x}}^2 + D^2 + 2i\bar{\lambda}\dot{\lambda}\right) - \theta D && (3.5)\\
L_{rel,C} &= |\mathcal{D}_t\phi^a|^2 - \left(\mathbf{x}^2 + D\right)|\phi^a|^2 + |F^a|^2 + i\,\bar{\psi}^a\mathcal{D}_t\psi^a \\
&\quad -\bar{\psi}^a\,\mathbf{x}\cdot\boldsymbol{\sigma}\,\psi^a - i\sqrt{2}(\bar{\phi}^a\psi^a\epsilon\lambda - \bar{\lambda}\epsilon\bar{\psi}^a\phi^a)\,, && (3.6)
\end{aligned}
$$

[6] This is a convention. The value of the rest energy does not influence the particle dynamics.

where the reduced mass $\mu = m_1 m_2/(m_1+m_2)$ and $\theta = -\theta_1 = \theta_2 = \mu(\alpha_2 - \alpha_1)$. The gauge group is the relative $U(1)$ between the branes. Eliminating the auxiliary fields D and F yields the following potential:

$$V(\phi, \mathbf{x}) = \frac{1}{2\mu}\left(|\phi|^2 + \theta\right)^2 + |\mathbf{x}|^2|\phi|^2\,, \tag{3.7}$$

with $|\phi|^2 \equiv \sum_a |\phi^a|^2$. In particular this gives for the mass[7] of the ϕ-modes $m_\phi{}^2 = |\mathbf{x}|^2 + \theta/\mu = |\mathbf{x}|^2 + \alpha_2 - \alpha_1$, correctly matching the stretched bosonic string masses as given earlier. As a further check, note that at $\phi = 0$, the potential energy equals $\mu(\alpha_2-\alpha_1)^2/2 \approx |Z_1| + |Z_2| - |Z_1+Z_2|$, in agreement with the Born-Infeld D-brane action if $\phi = 0$ is interpreted as having disconnected branes.

The moduli space of classical ground states of this system is given by the local minima of V. If $\theta < 0$, there are two branches: one (the "Coulomb" branch) consisting of configurations with $\phi = 0$ and $\mathbf{x}^2 > \alpha_1 - \alpha_2$, and another one (the "Higgs" branch) consisting of configurations with $\mathbf{x} = 0$ and $|\phi|^2 = -\theta$ modulo the $U(1)$ gauge group. Configurations with $\phi = 0$ and $\mathbf{x}^2 < \alpha_1 - \alpha_2$ are unstable; the ϕ-modes become tachyonic and tend to "condense" into a Higgs branch ground state. Note that (for $\theta < 0$) the Higgs branch is $\mathbb{CP}^{\kappa-1}$, and the Coulomb branch $\mathbb{R}^3$ with a ball removed. If $\theta > 0$, there is no Higgs branch, and the Coulomb branch is $\mathbb{R}^3$.

A ground state is supersymmetric if $D = 0$ or equivalently $V = 0$, as can be seen directly from the supersymmetry transformation rules (B.3)-(B.9). Thus at the classical level, only the Higgs branch can provide supersymmetric ground states, unless $\theta = 0$; then it is the Coulomb branch. All this matches the string theory features discussed earlier.

We emphasize that these considerations are all classical. Quantum effects drastically alter this picture, as we will see in section 4.

3.2 General model and quiver mathematics

The model discussed in the previous subsection, describing the low energy dynamics of two rigid parton D-branes S_1 and S_2 with nearly coincident positions and phases, and with κ light chiral open strings between them, can be generalized in various ways. For example, one could increase the number of both types of branes, say to N_p branes of type S_p $(p = 1, 2)$. With $N_1 = 1$, this gives the D-brane counterpart of the Hole (Hall) Halo of section 2.5. The position coordinates of the stack of type S_p branes now become $N_p \times N_p$ hermitian matrices, with the off-diagonal elements representing strings stretching between different copies of the same S_p, while the strings stretching from an S_1 type brane to an S_2 type brane are now represented by κ $N_2 \times N_1$ complex matrices transforming in the $(\bar{\boldsymbol{N}}_{\boldsymbol{1}}, \boldsymbol{N}_{\boldsymbol{2}})$ of the $U(N_1) \times U(N_2)$ gauge group of the brane system. A further generalization is to include branes of

[7] In the context of particle mechanics, the "mass" of this mode is actually its oscillator frequency, but we will use the term mass as well, hoping that this will not lead to confusion.

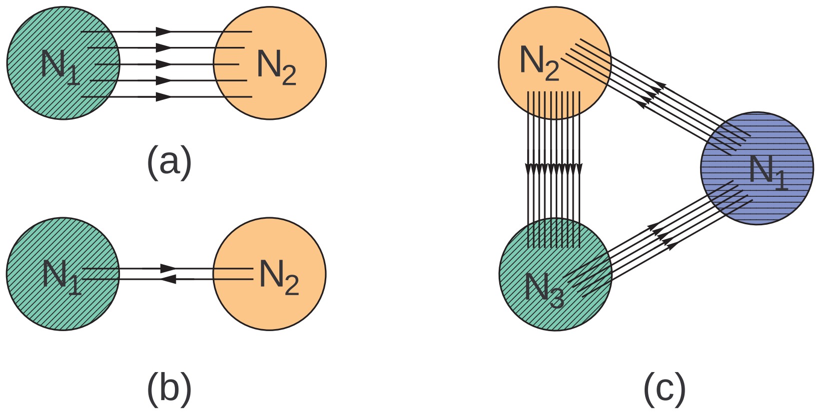

Figure 4: Some examples of quiver diagrams. The N_v indicate the dimensions of the vector spaces corresponding to the nodes. The quivers (a) and (c) appear in the description of certain D-branes on the quintic [12, 13, 35]. For example case (a) represents a bound state of N_1 pure $D6$-branes with N_2 $D6$ branes carrying the bundle $\mathcal{O}(-1)$. In the mirror picture the corresponding branes are D3-branes wrapped around two different cycles vanishing at the conifold point copies $\psi = 1$ resp. $\psi = e^{2\pi i/5}$ (in the notation of [3]). The two branes have intersection product equal to five, hence the five arrows.

other types. For simplicity we will always assume that the constituent branes S_p are rigid partons. In all these cases, for small separations and phase differences, the low energy dynamics is expected to be given by an $\mathcal{N} = 1$, $d = 4$ quiver gauge theory reduced to the effective particle worldline, with field content given by a quiver diagram.

Let us first recall some definitions from the mathematical theory of quivers [54, 55] (a more elaborate summary can be found e.g. in [10]).[8] A *quiver diagram* Q is an oriented graph, consisting of nodes (or vertices) $v \in V$ and arrows $a \in A$. Some examples are shown in fig. 4. A ($\mathbb{C}$-)*representation* $R = (X, \phi)$ of a quiver is given by a set of vector spaces $X_v = \mathbb{C}^{N_v}$ associated to the nodes $v \in V$, and a set of linear maps $\phi^a : V_v \to V_w$ associated to the arrows $a : v \to w \in A$. Such linear maps can be represented by $N_w \times N_v$ complex matrices. The vector $\mathbf{N} \equiv (N_v)_{v \in V}$ is called the *dimension vector* of the quiver representation. A *morphism* between two representations $(\tilde{X}, \tilde{\phi})$ and (X, ϕ) is a set of linear maps $(T_v)_{v \in V}$, with $T_v : \tilde{X}_v \to X_v$, such that $\phi^a T_v = T_w \tilde{\phi}^a$ for all arrows $a : v \to w$. A representation $(\tilde{X}, \tilde{\phi})$ is called a *subrepresentation* of (X, ϕ) if there exist an injective morphism from the former to the latter. Two representations (X, ϕ) and $(X, \tilde{\phi})$ are considered

[8] Some of the definitions we recall here, in particular those concerning θ-stability, are only given to make contact with the mathematical results of [17], which we will apply to solve ground state counting problems. They are however not strictly necessary to understand the main idea of this paper.

equivalent if they are isomorphic, in other words if there exist a "complexified gauge transformation" $(g_v)_{v\in V} \in \prod_{v\in V} GL(N_v) \equiv G_{\mathbb{C}}$ such that $\tilde{\phi}^a = g_w \phi^a g_v^{-1}$ for all arrows $a : v \to w \in A$. We will denote the subgroup of *unitary* isomorphisms ("ordinary gauge transformations") by G, that is, $G \equiv \prod_{v\in V} U(N_v)$.

An important concept in the study of quivers is the notion of θ-*(semi-)stability* [54]. Let $(\theta_v)_{v\in V}$ a set of real numbers associated to the nodes of Q, satisfying $\theta(\mathbf{N}) \equiv \sum_v N_v \theta_v = 0$ for a given dimension vector $\mathbf{N}$. Then a representation R is called θ-stable if every proper subrepresentation $\tilde{R}$ of R satisfies $\theta(\tilde{\mathbf{N}}) = \sum_v \tilde{N}_v \theta_v < 0$. Semi-stability is the same with the "$<$" replaced by "$\leq$". The moduli space of θ-semi-stable isomorphism classes of representations with dimension vector $\mathbf{N}$, denoted $\mathcal{M}^{ss}(Q, \mathbf{N}, \theta)$, is a projective variety, and the subspace obtained by considering only stable representations, denoted $\mathcal{M}^{s}(Q, \mathbf{N}, \theta)$ is a smooth open subvariety [54]. The connection with physics will be made below through a theorem by King [54], which states that if a representation is stable, it is equivalent to exactly one solution of

$$\sum_{a:v\to *} \phi^{a\dagger}\phi^a - \sum_{a:*\to v} \phi^a \phi^{a\dagger} = \theta_v \, \mathbf{1}_{N_v} \qquad \forall v \in V. \tag{3.8}$$

modulo the group of unitary gauge transformations G. A semi-stable representation is equivalent to *at most* one solution of these equations. A non-semi-stable representation on the other hand is never equivalent to a solution. If we denote the moduli space of solutions to this equation modulo G by $\mathcal{M}(Q, \mathbf{N}, \theta)$, we thus have $\mathcal{M}^s \subseteq \mathcal{M} \subseteq \mathcal{M}^{ss}$, and the three spaces do not necessarily coincide. This discrepancy is mathematically inconvenient, but one can remove it by introducing the notion of S-equivalence [54, 11], or, as we will do, by simply assuming the dimension vector and the θ_v to be sufficiently generic, such that all semi-stable representations are automatically stable. More precisely, as can be easily verified, this is achieved by taking g.c.d.$\{N_v\}_{v\in V} = 1$ and the θ_v linearly independent over $\mathbb{Q}$ except for the relation $\sum_v N_v \theta_v = 0$.

3.3 Quiver physics

We now turn to the physical interpretation of all this. A quiver diagram with given dimension vector $\mathbf{N}$ is associated to the field content of an $\mathcal{N} = 1$, $d = 4$ gauge theory (or, for our purposes, its dimensional reduction, to $d = 1$) in a natural way: each node v corresponds to a vector multiplet for gauge group $U(N_v)$, and each arrow $a : v \to w$ corresponds to a chiral multiplet transforming in the $(\bar{\boldsymbol{N}}_{\boldsymbol{v}}, \boldsymbol{N}_{\boldsymbol{w}})$ of $U(N_v) \times U(N_w)$. So a quiver representation as defined above is nothing but a particular configuration of the chiral multiplet scalars in a particular gauge theory, and gauge transformations correspond to the group G of unitary isomorphisms. The explicit Lagrangian for a given quiver, together with the relevant supersymmetry transformations, is given in appendix C.

The example considered in section 3.1 thus corresponds to a quiver with two nodes and κ arrows from the first one to the second, and dimension vector $\mathbf{N} = (1, 1)$. The generalization to N_1 branes of type S_1 and N_2 branes of type S_2 is represented by the same quiver with dimension vector (N_1, N_2). This is illustrated in fig. 4a for $\kappa = 5$. More generally one can think of a quiver with dimension vector $\mathbf{N}$ as arising from a collection of D3-branes[9] wrapped N_v times around supersymmetric, transversally intersecting parton 3-cycles S_v with nearly equal phase angles α_v,[10] where a positive intersection point between S_v and S_w corresponds to an arrow from v to w and a negative one to an arrow in the opposite direction. The arrows can be thought of as the light stretched strings localized near the intersection points.

Of course, the field content alone does not fix the gauge theory. One also needs to specify the Fayet-Iliopoulos coefficients $(\theta_v)_{v \in V}$, and the superpotential $W(\phi)$, which is a holomorphic function of the chiral fields ϕ^a, $a \in A$. When the quiver does not have closed loops (i.e. if one cannot return to the same place by following the arrows), the requirement of gauge invariance forbids a superpotential. When closed loops are present (like in fig. 4b and 4c) , this is no longer the case, and a superpotential will generically appear. Determining this superpotential from the D-brane data is a difficult problem in general, though significant progress has been made [35]. On the other hand, determining the FI-parameters is easy. As in our basic example, they are obtained by comparing the relevant parameters in the quiver Lagrangian (see appendix C) with the known masses of bosonic strings stretched between the different branes. This gives

$$\theta_v = m_v(\alpha_v - \alpha_0), \tag{3.9}$$

where $\alpha_v = \arg Z_v$, Z_v being the central charge of the brane labeled by v, and $\alpha_0 \equiv \sum_v N_v m_v \alpha_v / \sum_v N_v m_v$ (determined by requiring $\sum_v N_v \theta_v = 0$, equivalent to vanishing energy for supersymmetric configurations). Note that since all phases are nearly equal, we have $\alpha_0 \approx \alpha = \arg Z$ and $\theta_v \approx \mathrm{Im}(e^{-i\alpha} Z_v)$.

As can be seen from the Lagrangian and the supersymmetry transformations in appendix C, for generic FI-parameters, the classical ground states are given by commuting and coincident positions X, and values of ϕ which satisfy simultaneously the F-flatness conditions, i.e. $F^a = \partial_a W = 0$, and the D-flatness conditions, i.e. $D_v = 0$. The latter happen to be precisely the equations (3.8), as can be seen from the Lagrangian in appendix C. So if $W = 0$, the classical moduli space is $\mathcal{M}(Q, \mathbf{N}, \theta)$.

[9] Of course the quiver can also arise from a completely different geometric setup, e.g. from even dimensional branes with bundles in IIA, or from orbifold constructions or abstract conformal field theory considerations — the D3-brane construction just gives a particularly catchy geometrical picture.

[10] Note that generically, this can only be realized physically for some value of the closed string moduli if the number of different cycles is at most one more than the real dimension of the vector multiplet moduli space.

As discussed earlier, for sufficiently generic θ, this space coincides with $\mathcal{M}^{ss}$ and $\mathcal{M}^{s}$, guaranteeing that it is a smooth projective variety.

As an example, consider the quiver with two nodes and κ arrows, and dimension vector $\mathbf{N} = (1, N_2)$. The case $N_2 = 1$ is the example of section 3.1. For general N_2, we have the quiver corresponding in particle content to the Hall Halo of section 2.5. The gauge group is $U(1) \times U(N_2)$. The scalar fields are grouped in κ row vectors ϕ^a with N_2 entries each, transforming in the fundamental of $U(N_2)$ and with charge -1 under the first $U(1)$. The FI-parameters are $\theta_2 = -\theta_1/N_2 = \mu(\alpha_2 - \alpha_1)$, with reduced mass $\mu = m_1 m_2/(m_1 + N_2 m_2)$. There are no closed loops, so there is no superpotential. The D-flatness conditions (3.8) are

$$\sum_{a,n} \bar{\phi}^a_n \phi^a_n = \mu(\alpha_1 - \alpha_2) N_2 \tag{3.10}$$

$$\sum_{a} \phi^a_n \bar{\phi}^a_m = \mu(\alpha_1 - \alpha_2) \delta_{nm} \,. \tag{3.11}$$

The first equation follows from the second, corresponding to the fact that the sum over all v of (3.8) trivially leads to $0 = 0$. If $\alpha_2 < \alpha_1$ and $N_2 \leq \kappa$, the moduli space $\mathcal{M}$ of solutions to (3.11) modulo gauge transformations is nonempty. It is essentially (up to a normalization factor) the space of all possible orthonormal N_2-tuples $(\phi_n)_n$ of vectors in $\mathbb{C}^\kappa$, modulo $U(N_2)$-rotations. In other words, it is the space of all N_2-dimensional planes in $\mathbb{C}^\kappa$, also known as the *Grassmannian* $Gr(N_2, \kappa)$. Note that $\dim_{\mathbb{C}} Gr(N_2, \kappa) = N_2(\kappa - N_2)$, in agreement with straightforward counting of solutions to (3.11) minus the number of gauge symmetries broken by a generic solution. In the case $N_2 = 1$, we get $\mathcal{M} = Gr(1, \kappa) = \mathbb{CP}^{\kappa-1}$, reproducing the result of section 3.1. All this could alternatively be analyzed in terms of θ-stability, but we will not do this here.

The main advantage though of the description of quiver moduli spaces in the language of θ-stability is that it makes it possible to explore their properties in a systematic, algebraic way. For applications to D-brane physics, a particularly important property of these spaces is their cohomology, since cohomology classes can be identified with supersymmetric ground states, and betti numbers therefore with various ground state degeneracies. As the topology of these moduli spaces is typically extremely rich, computing their betti numbers is a difficult mathematical problem. Nevertheless, recently, Reineke succeeded in constructing an explicit formula for the generating function of these numbers (also known as the Poincaré polynomial), for arbitrary quivers without closed loops [17]. We will give some applications of this powerful result in section 5.

4. Quiver quantum mechanics

4.1 Relating the two pictures

We have developed two rather different classical pictures of supposedly the same BPS bound state of branes, on the one hand the picture of a set of particles at equilibrium separations from each other and on the other hand the picture of a fusion of D-branes (with nearly equal phases) at a single point of space. Upon (stringy) quantization, we should be able to connect the two by continuous variation of the coupling constants. The aim of this section is to find out how exactly this comes about.

Let us first go back to the example of section 3.1. At first sight, also the quantum mechanics traditionally associated to the two pictures looks very different: the first one gives a purely spatial Schrödinger equation of an electron-monopole pair bound to each other by a potential with finite distance minimum, whereas the second one gives rise to quantum mechanics on the quiver moduli space $\mathbb{CP}^{\kappa-1}$, with a trivially flat Coulomb branch.

The mystery dissipates in a way similar to (but not quite the same as) the resolution of some puzzles in $\mathcal{N} = 2$, $d = 2$ sigma models [47, 57]. Note that the full quiver quantum mechanics involves both vector and chiral modes. If we take the constituent branes sufficiently far apart in space, the charged chiral modes become massive and can be integrated out, giving a contribution to the low energy effective Lagrangian of the (still massless) position multiplets alone. Given the non-renormalization theorem of section 2, it seems very plausible that this reproduces exactly the right first order part $L^{(1)}$ of the Lagrangian to make the match between the two pictures. Here we will verify this explicitly.

The Lagrangian (3.6) for the chiral modes is quadratic, so integrating out can be done exactly, either by Wick rotating and doing Gaussian integrals, or by computing one loop diagrams. With $r \equiv |\mathbf{x}|$, the mass (or oscillator frequency) of the bosonic chiral mode ϕ^a is $\sqrt{r^2 + D}$, the mass of the fermions ψ^a is r, and in total there are κ such pairs (ϕ^a, ψ^a). The resulting bosonic effective Lagrangian for constant $\mathbf{x}$ and D is

$$L_{B,eff,const.} = \frac{\mu}{2} D^2 - \theta D - \kappa \ln\det(-\partial_t^2 + r^2 + D) + \kappa \ln\det(-\partial_t^2 + r^2) \tag{4.1}$$

$$= \frac{\mu}{2} D^2 - \theta D - \kappa \sqrt{r^2 + D} + \kappa\, r. \tag{4.2}$$

By definition, the function U of (2.18) equals minus the coefficient of the term linear in D, i.e. $U = -\partial_D L|_{D=0}$, so we indeed get $U(r) = \theta + \kappa/2r$, with $r = |\mathbf{x}|$. Because of the non-renormalization theorem, this also determines the remainder of the first order Lagrangian $L^{(1)}$. In particular, a magnetic interaction is generated equivalent to the field of a charge κ Dirac monopole. The second order term can also be read off from (4.2). The "metric" or "effective mass" factor is $\mu(r) = \partial_D^2 L|_{D=0} = \mu + \kappa/4r^3$. Comparing this to the general bosonic Lagrangian (2.17), we get for the "Kähler

Figure 5: (a): the diagram generating U. (b): diagram generating $\mathbf{A}(\mathbf{x}+\delta\mathbf{x}) \cdot d\,\delta\mathbf{x}$ to second order in $\delta\mathbf{x}$.

potential" $K(r) = \mu r^2/3 - \kappa \ln r/2r$, which fixes the second order term [46]. To quadratic order in velocities and D, this gives the following low energy effective bosonic Lagrangian:

$$L_{B,eff} = \frac{1}{2}(\mu + \kappa/4r^3)(\dot{\mathbf{x}}^2 + D^2) - (\theta + \kappa/2r)D - \kappa \mathbf{A}^d \cdot \dot{\mathbf{x}}\,, \tag{4.3}$$

where $\mathbf{A}^d$ is once again the unit Dirac potential (2.9). Here $\kappa = \langle S_1, S_2 \rangle$ and $\theta = \mu(\alpha_2 - \alpha_1)$. Within our small phase approximation, this is exactly what we had in supergravity (see e.g. (2.32)), since $\sin(\Delta\alpha) \approx \Delta\alpha$ when $\Delta\alpha \ll 1$. Comparing with the supergravity probe Lagrangian (2.23), we see that on the other hand the quadratic part of the Lagrangian, unlike the linear part, does not match. This was to be expected, since supersymmetry does not sufficiently constrain these terms in the Lagrangian, so there is no reason to expect agreement between the substringy and the supergravity regimes at this order.

All this can be verified by computing one loop diagrams. For example the correction to the FI term, or in other words the $1/r$ part of U, comes from the diagram shown in fig. 5 (a): indeed,

$$\int \frac{1}{\omega^2 + r^2}\, d\omega \sim 1/r\,.$$

The factor κ in U comes from summing over the ϕ^a.

To conclude, we see that integrating out the charged chiral quiver modes yields an effective potential on the Coulomb branch, restoring supersymmetry at its minimum, and reproducing the features of the BPS particle mechanics discussed in section 2. Moreover, the first order part $L^{(1)}$, which determines the equilibrium separation and the magnetic interaction, is exactly the same in large scale and substringy regimes.

We now address the question in which (substringy) regime the chiral modes can indeed be integrated out to arrive at a good description of the low energy physics. From the discussion in section 3.1 (see for instance (3.7)), we know that the on-shell classical mass of ψ and ϕ is r and $\sqrt{r^2 + \Delta\alpha}$ respectively, with $\Delta\alpha \equiv \alpha_2 - \alpha_1$.

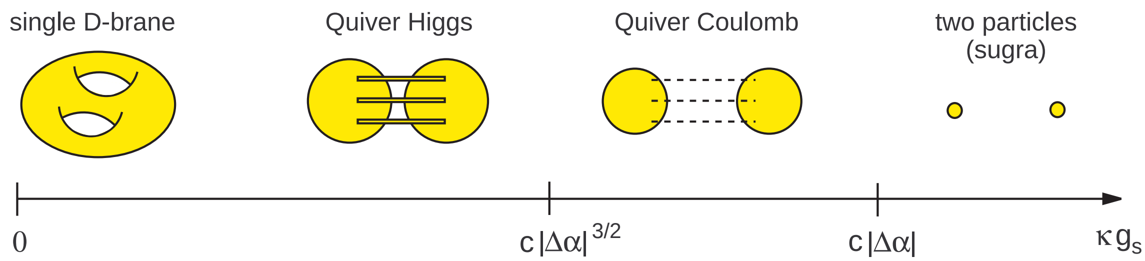

Figure 6: Metamorphosis of a single D-brane into a 2-particle configuration. In the quiver quantum mechanical model, the wave function lives on the Higgs branch when $\kappa g_s \ll c|\Delta\alpha|^{3/2}$, and on the Coulomb branch (with effective potential from integrating out the interconnecting strings) when $\kappa g_s \gg c|\Delta\alpha|^{3/2}$. The substringy quantum mechanical model is valid if $\kappa g_s \ll c|\Delta\alpha|$.

Therefore if $\Delta\alpha < 0$ and we only want to integrate out massive modes, we should restrict to positions with $r \gg \sqrt{-\Delta\alpha}$. In particular, this should be the case for the equilibrium radius $R = \frac{\kappa}{2\mu|\Delta\alpha|}$, giving the condition

$$\mu\,|\Delta\alpha|^{3/2} \ll \kappa. \tag{4.4}$$

On the other hand, if we want to stay in the substringy ("gauge theory") regime, where the quiver model gives a reliable description of the low energy physics, we need $R \ll 1$, i.e.

$$\mu\,|\Delta\alpha| \gg \kappa\,. \tag{4.5}$$

For sufficiently small $\Delta\alpha$, the two conditions are satisfied simultaneously for a certain range of μ, or equivalently for a certain range of the string coupling constant g_s, since $\mu = c/g_s$ with c a function of the Calabi-Yau moduli only; more precisely $c = \zeta\sqrt{v}/l_s$, with $\zeta = |Z_1||Z_2|/(|Z_1| + |Z_2|)$ and v defined as under (3.1).

Thus we see that by lowering the string coupling g_s down to zero, we first go smoothly and uneventfully from a macroscopic two centered configuration in the large scale regime to a microscopic two centered configurations in the substringy regime, but that if we keep on lowering g_s, the strings stretched between the branes in equilibrium become tachyonic and the two centered system becomes (classically) unstable with respect to decay into a configuration with nonzero ϕ — in other words, we fall from the Coulomb phase into the Higgs phase. Finally, at the extreme classical limit $g_s = 0$, the ϕ^a can be interpreted as the moduli of a suitable geometric object, for instance a supersymmetric D3-brane with κ connecting throat moduli ϕ^a [34]. The different regimes are illustrated in fig. 6.

Quantum mechanically, this phase transition is less sharp because of quantum fluctuations. However, we can still identify a regime in which the effective dynamics on the Coulomb branch captures the low energy physics consistently and a complementary regime in which the dynamics on the Higgs branch does so.

To have a consistent low energy description in terms of the vector multiplets only, one needs the oscillator frequencies of the integrated out chiral modes to be much higher than those of the vectors (in the resulting effective dynamics). In the regime (4.4), at the equilibrium distance, the frequency of the chiral modes is approximately $\omega_c \sim \frac{\kappa}{\mu|\Delta\alpha|}$, while from (4.3) (after integrating out D), it follows that the frequency of the radial $\mathbf{x}$-mode is $\omega_v \sim \frac{\mu|\Delta\alpha|^2}{\kappa}$. So we have in this regime

$$\frac{\omega_v}{\omega_c} \sim \frac{\mu^2|\Delta\alpha|^3}{\kappa^2}, \tag{4.6}$$

and the condition for this to be small is precisely again (4.4).

On the other hand, when $g_s \to 0$, one expects the more familiar semi-classical description of the low energy dynamics of D-branes to become valid, namely supersymmetric quantum mechanics on the classical moduli space. To check this in the case at hand, consider the Lagrangian after integrating out the auxiliary fields from (3.5)-(3.6):

$$\begin{aligned} L_{rel} = & \frac{\mu}{2}(\dot{\mathbf{x}}^2 + 2i\bar{\lambda}\dot{\lambda}) + |\mathcal{D}_t\phi^a|^2 + i\,\bar{\psi}^a\mathcal{D}_t\psi^a & (4.7) \\ & -|\mathbf{x}|^2|\phi^a|^2 - \frac{1}{2\mu}(|\phi|^2 + \mu\Delta\alpha)^2 & \\ & -\bar{\psi}^a\,\mathbf{x}\cdot\boldsymbol{\sigma}\,\psi^a - i\sqrt{2}(\bar{\phi}^a\psi^a\epsilon\lambda - \bar{\lambda}\epsilon\bar{\psi}^a\phi^a)\,, & (4.8) \end{aligned}$$

As discussed earlier, when $\Delta\alpha < 0$, the classical moduli space is non-empty. It is given by the zeros of the potential V, i.e. $\mathbf{x} = 0$ and $|\phi|^2 = -\mu\Delta\alpha$. If we want the semi-classical picture of a particle moving in this moduli space to be reliable, the spread of the particle wave function out of the zero locus of V has to be small compared to the scale of the Mexican hat potential. More precisely, if we write ϕ^a as $\phi^a = e^\sigma\phi_0^a$, with $|\phi_0|^2 = -\mu\Delta\alpha$, we should have $\langle\sigma^2\rangle \ll 1$. Furthermore, to justify putting $\mathbf{x} = 0$ and thus neglecting the $|\mathbf{x}|^2|\phi^a|^2$ contribution to the potential for ϕ, we need $\langle\mathbf{x}^2\rangle \ll -\Delta\alpha$.

The semi-classical approximation consists of treating the σ and $\mathbf{x}$ modes as harmonic oscillators. For the σ-mode, this is a harmonic oscillator with mass $\sim -\mu\Delta\alpha$ and spring constant $\partial_\sigma^2 V\,|_0 \sim \mu(\Delta\alpha)^2$. Then $\langle\sigma^2\rangle \sim \frac{1}{\mu|\Delta\alpha|^{3/2}}$. So to have $\langle\sigma^2\rangle \ll 1$, we need

$$\mu|\Delta\alpha|^{3/2} \gg 1\,. \tag{4.9}$$

For the $\mathbf{x}$-mode, we get a harmonic oscillator with mass μ and spring constant $-\mu\Delta\alpha$, so $\langle\mathbf{x}^2\rangle \sim \frac{1}{\mu|\Delta\alpha|^{1/2}}$. The condition for this to be much smaller than $-\Delta\alpha$ is again

(4.9). Note that this condition is complementary[11] to (4.4), and is indeed satisfied when we take $g_s \to 0$ (i.e. $\mu \to \infty$) while keeping $\Delta\alpha$ constant.

Finally, in this regime, the $\mathbf{x}$ and σ modes and their superpartners have frequencies $\sim \sqrt{-\Delta\alpha}$ and both should therefore be integrated out if we want an effective Lagrangian for the moduli space zero modes alone. This leaves us with a $d = 1$, $\mathcal{N} = 4$ nonlinear sigma model on $\mathbb{CP}^{\kappa-1}$.

4.2 Coulomb quiver quantum mechanics

In the quiver Coulomb regime of fig. 6, and near equilibrium, the second term in the metric factor in (4.3) is only a small correction. So as far as ground state counting is concerned (which is what we are ultimately interested in), we can safely drop this correction, and the full Lagrangian becomes of the simple form (2.21). After eliminating the auxiliary variable D, we get

$$L_{V,eff} = \frac{\mu}{2}\dot{\mathbf{x}}^2 + i\,\mu\,\bar{\lambda}\dot{\lambda} - \frac{1}{2\mu}(\theta + \kappa/2r)^2 - k\mathbf{A}^d \cdot \dot{\mathbf{x}} - \frac{k\,\mathbf{x}}{2r^3} \cdot \bar{\lambda}\boldsymbol{\sigma}\lambda\,. \qquad (4.10)$$

The (anti-) commutation relations can be read off from this Lagrangian, and in particular we have $\{\lambda_\alpha, \lambda_\beta\} = 0$, $\{\bar{\lambda}^\alpha, \bar{\lambda}^\beta\} = 0$, $\{\lambda_\alpha, \bar{\lambda}^\beta\} = \mu^{-1}\delta_\alpha^\beta$. The λ-operators can be represented as 4×4 matrices, acting on a vector space generated by $|0\rangle$, $\bar{\lambda}^\alpha|0\rangle$, $\bar{\lambda}^1\bar{\lambda}^2|0\rangle$, with $\lambda_\alpha|0\rangle = 0$. The supersymmetry generators are

$$Q_\alpha = \sigma_\alpha^{i,\beta}\lambda_\beta D_i - \lambda_\alpha U(r) \qquad (4.11)$$

$$\bar{Q}^\alpha = -\bar{\lambda}^\beta\sigma_\beta^{i,\alpha} D_i - \bar{\lambda}^\alpha U(r), \qquad (4.12)$$

where $D_i = \partial_i + i\kappa A_i^d$ and $U(r) = \kappa/2r + \theta$. The supercharges satisfy the usual algebra $\{Q_\alpha, Q_\beta\} = 0$, $\{\bar{Q}^\alpha, \bar{Q}^\beta\} = 0$, $\{Q_\alpha, \bar{Q}^\beta\} = 2\delta_\alpha^\beta H$, with Hamiltonian H given by

$$H = \frac{1}{2\mu}{D_i}^2 + \frac{1}{2\mu}(\theta + \kappa/2r)^2 + \frac{k\,\mathbf{x}}{2r^3} \cdot \bar{\lambda}\boldsymbol{\sigma}\lambda\,, \qquad (4.13)$$

A general wave function is of the form

$$F = \Phi(\mathbf{x})|0\rangle + \Psi_\alpha(\mathbf{x})\bar{\lambda}^\alpha|0\rangle + \tilde{\Phi}(\mathbf{x})\bar{\lambda}^1\bar{\lambda}^2|0\rangle\,. \qquad (4.14)$$

Supersymmetric ground states are given by wave functions annihilated by all four supercharges (4.11)-(4.12). One easily deduces that this requires $\Phi = 0, \tilde{\Phi} = 0$, while Ψ must satisfy

$$(\sigma^i D_i - U)\Psi = 0 \qquad (4.15)$$

This equation can be solved by standard separation of variables. As expected, when κ and θ have the same sign, there is no normalizable solution. If they have opposite

[11] At large κ, there can be overlap, in which case there are two consistent semi-classical descriptions. The full wave function will then presumably be a superposition of the two.

sign, say $\kappa > 0$ and $\theta < 0$, we get, in spherical coordinates (r, ϑ, φ),

$$\Psi_m \sim r^{\frac{\kappa}{2}-1} e^{\theta r} e^{im\varphi} (1-\cos\vartheta)^{\frac{m-1}{2}} (1+\cos\vartheta)^{-\frac{m+1}{2}+\frac{\kappa}{2}} \begin{pmatrix} 1-\cos\vartheta \\ -e^{i\varphi}\sin\vartheta \end{pmatrix}, \tag{4.16}$$

where $0 \leq m \leq \kappa - 1$. These κ solutions fill out a spin $(\kappa-1)/2$ multiplet. Note that this is 1/2 less than what one would get from quantizing a spinless point particle in the same setup. Physically, this is because the superparticle we are considering here minimizes its energy by going into a spin 1/2 state aligned with the radial magnetic field, thus giving a contribution of one spin quantum opposite to the intrinsic field angular momentum. This explains also why the spin zero components Φ and $\tilde{\Phi}$ vanish in a ground state.

To be complete, we also have to quantize the free center of mass degrees of freedom, governed by the Lagrangian (3.4). The fermionic zeromodes give two spin zero singlets and one spin 1/2 doublet. The total spin of the states and the supersymmetry representations they carry are then obtained as the direct product of the center of mass wave functions and the relative wave functions. Thus the case $\kappa = 1$ gives a (half) hypermultiplet, $\kappa = 2$ a vector multiplet, and so on.

Note that the radial probability density p_r derived from the wave functions (4.16),

$$p_r = \int |\Psi_m|^2 r^2 \sin\vartheta \, d\vartheta d\varphi \ \sim r^\kappa e^{2\theta r} \tag{4.17}$$

is peaked around the classical equilibrium point $r = -\kappa/2\theta$, and that the width of the wave function compared to the size of the configuration scales as $1/\sqrt{\kappa}$, as one would expect.

4.3 Higgs quiver quantum mechanics

We now turn to the more familiar counting of supersymmetric ground states in the Higgs regime, i.e. $\mathcal{N} = 4$ supersymmetric quantum mechanics on the classical moduli space $\mathcal{M} = \mathbb{CP}^{\kappa-1}$. We denote the $\kappa - 1$ bosonic degrees of freedom (complex coordinates on $\mathbb{CP}^{\kappa-1}$) as z^m, and their superpartners as χ^m. As is well known, the quantum supersymmetric ground states of a free particle moving in a Kähler manifold $\mathcal{M}$ are in one to one correspondence with the Dolbault cohomology classes of $\mathcal{M}$. This can be seen by identifying the quantized fermionic operators with the following wedge and contraction operations:

$$\chi_1^m \to g^{m\bar{n}} \frac{\partial}{\partial(d\bar{z}^{\bar{n}})}, \quad \chi_2^m \to dz^m, \quad \bar{\chi}^{\bar{m},1} \to d\bar{z}^{\bar{n}}, \quad \bar{\chi}^{\bar{m},2} \to g^{n\bar{m}} \frac{\partial}{\partial(dz^n)} \tag{4.18}$$

For example $\chi_2^3 \cdot dz^1 = dz^3 \wedge dz^1$ and $\bar{\chi}_2^3 \cdot dz^1 \wedge dz^3 = g^{1\bar{3}} dz^3 - g^{3\bar{3}} dz^1$. With these identifications, the fermionic canonical anticommutation relations are indeed satisfied: $\{\chi_\alpha^m, \bar{\chi}^{\bar{n},\beta}\} = \delta_\alpha^\beta \, g^{m\bar{n}}$, $\{\chi_\alpha^m, \chi_\beta^n\} = 0$, and $\{\bar{\chi}^{\bar{m},\alpha}, \bar{\chi}^{\bar{n},\beta}\} = 0$. The supersymmetry operators are correspondingly identified as follows:

$$Q_1 \to \bar{\partial}^\dagger, \quad Q_2 \to \partial, \quad \bar{Q}^1 \to \bar{\partial}, \quad \bar{Q}^2 \to \partial^\dagger, \tag{4.19}$$

so $\{Q_\alpha, z^m\} = \chi^m_\alpha$ and so on. In this way supersymmetric ground states are identified with the harmonic representatives in the Dolbeault cohomology classes of the target space.

The cohomology of $\mathbb{CP}^{\kappa-1}$ is well known; it has κ elements: 1, ω, $\omega \wedge \omega$, ..., $\omega^{\kappa-1}$, where ω is the Kähler form, $\omega = -i g_{m\bar{n}} dz^m \wedge d\bar{z}^{\bar{n}}$. The supersymmetric ground states we obtain in this way form a spin $(\kappa-1)/2$ multiplet. Indeed, since the z^m are invariant under spatial $SO(3)$ rotations and the χ^m are obtained from these by applying Q, which transforms in the **2**, the spinors χ^m transform likewise in the **2** under spatial $SO(3)$ rotations. The corresponding angular momentum operator is

$$\mathbf{S} = \frac{1}{2} g_{m\bar{n}} \, \bar{\chi}^{\bar{n}} \boldsymbol{\sigma} \chi^m. \tag{4.20}$$

With the identifications (4.18) and using the anticommutation relations, this gives:

$$S^3 = \frac{1}{2}(d\bar{z}^{\bar{n}} \frac{\partial}{\partial \, d\bar{z}^{\bar{n}}} + dz^m \frac{\partial}{\partial \, dz^n}) - \frac{\kappa - 1}{2} \tag{4.21}$$

$$= \frac{1}{2}(\text{form number} - \dim_{\mathbb{C}}) \tag{4.22}$$

$$S^+ = -g_{m\bar{n}} dz^m \wedge d\bar{z}^{\bar{n}} = -i\omega \tag{4.23}$$

$$S^- = g^{m\bar{n}} \frac{\partial}{\partial \, dz^m} \wedge \frac{\partial}{\partial \, d\bar{z}^{\bar{n}}} \,. \tag{4.24}$$

An $SU(2)$ action of this form always exists on the cohomology of Kähler manifolds; it is called the Lefschetz $SU(2)$. In this case we see that the Lefschetz $SU(2)$ coincides with the spatial $SU(2)$, and the cohomology of $\mathbb{CP}^{\kappa-1}$ forms a spin $(\kappa-1)/2$ multiplet under this group action, as announced.

So we get the same supersymmetric ground state degeneracy and spin as in the corresponding quantum mechanics on the Coulomb branch. By sending g_s to zero, the ground states simply underwent a continuous metamorphosis from living on the Coulomb branch to living on the Higgs branch, or in other words from a spatial molecular form to a geometric D-brane form.

4.4 General case

The generalization to arbitrary quivers is in principle straightforward. Let us take all constituent branes separated so stretched strings become massive. Integrating out strings between copies of the same brane will not produce an interaction at first velocity order because fermionic and bosonic string modes have the same mass. A string stretched between different species connected by arrow in the corresponding quiver diagram on the other hand generically does produce an effective first order interaction, with sign determined by the orientation of the corresponding arrow in the quiver. This can also be verified directly from the abelianized quiver Lagrangian at the end of appendix C. For each such string we integrate out, a two body interaction just like the one we had for the two particle case is induced. Thus the resulting contribution to the effective first order interaction Lagrangian for the position multiplets is simply a sum over all pairs of particles, with for each pair the two body interaction we found previously in the toy example. The terms in the sum are furthermore weighted by the intersection product $\langle v, w\rangle$ of the corresponding objects of type v and w, which is equal to the number of arrows $v \to w$ minus number of arrows $w \to v$. So we reproduce the Lagrangian (2.29)-(2.30) with $\kappa_{wv} = \langle v, w\rangle$.

One could worry about the possible presence of a superpotential (if the quiver has closed loops). If $W \neq 0$, the Lagrangian is no longer quadratic in the chiral multiplets, and integrating them out will involve higher loop diagrams. However, because of the nonrenormalization theorem for the first order part of the Lagrangian, we are allowed to compute this term at arbitrary large separations, where the $|\Delta x|^2|\phi|^2$ mass terms are certainly dominant and the superpotential terms can be ignored. Hence the superpotential will not affect $L^{(1)}_{eff}$.

As for the two center case, sending g_s all the way down to zero while keeping all other parameters fixed makes the multicentered structure collapse to mutual distances where bosonic strings become tachyonic and the description in terms of separate centers breaks down. Again, the appropriate picture then becomes the Higgs branch of the quiver quantum mechanics.[12]

However, in the opposite direction, starting at $g_s = 0$ with a classical wrapped D-brane bound state localized at a single point in space, and increasing the string coupling constant, things are not that simple. In general, the state will not necessarily "open up" and become a multicentered configuration of partons. Instead, it could for example stay centered in one point and turn into a black hole, or it could split into two black holes at a certain equilibrium distance, or into a black hole surrounded by a cloud of partons, and so on. In other words, there are many more D-brane states than there are multi-particle states (of the kind we have been describing), and this discrepancy is shows up in the form of emerging horizons.

The obvious question is then: what distinguishes between the different possibilities? We don't know the complete answer to that question, but we can argue for a certain class of quivers that they will certainly open up into a multicentered state when g_s increases, namely for quivers without closed loops, such as the one shown in fig. 7a. This excludes quiver diagrams with arrows going from a node to itself, diagrams with arrows going in two directions between a pair of nodes (fig. 4b) and diagrams such as fig. 7b or 4c. From (3.8) and (2.31), it is not hard to see that on both the D-brane and the particle side, having no closed loops implies that the moduli space is compact if the θ_v are finite, in the sense that the ϕ^a are bounded from above and the separations between mutually nonlocal particle species bounded from below[13], respectively. This is because having no closed loops implies that we

[12]Note that there can be intermediate stages which look like multicentered configurations with as centers bound states of branes in their Higgs phases.

[13]and from above for coprime dimension vector and sufficiently generic moduli such that none of

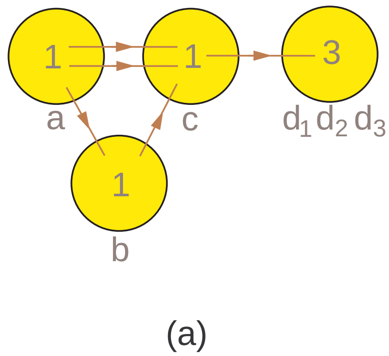

Figure 7: (a): A quiver with no loops and dimension vector (1,1,1,3). The arrows define the ordering $a < b < c < d$. (b): A quiver with a loop.

can consistently define a partial ordering of the nodes according to the direction of the arrows. A would-be divergence in the left hand side of (3.8) or (2.31) at a certain node would propagate through the chain of equations to a maximal or a minimal node, where a divergence is manifestly impossible since the left hand sides of (3.8) and (2.31) have only positive or only negative contributions there. As an example, consider the equilibrium equations (2.31) for the quiver of fig. 7, which has the ordering $a < b < c < d$:

$$\frac{1}{r_{ab}} + \frac{2}{r_{ac}} = 2\theta_a \tag{4.25}$$

$$\frac{1}{r_{bc}} - \frac{1}{r_{ab}} = 2\theta_b \tag{4.26}$$

$$\sum_{j=1}^{3} \frac{1}{r_{a,dj}} - \frac{2}{r_{ac}} - \frac{1}{r_{bc}} = 2\theta_c \tag{4.27}$$

$$-\sum_{j=1}^{3} \frac{1}{r_{a,dj}} = 2\theta_d. \tag{4.28}$$

It is clear that the first and the last equation prevent the full set of $r_{pq} = |\mathbf{x}_p - \mathbf{x}_q|$ to become arbitrary small. Thus, if two particles mutually interact, they stay away from each other.

Also, having no arrows from a node to itself means that the constituents have no internal moduli of their own. Since no moduli means no entropy, one therefore expects that the corresponding particles, when put together, do not form black holes. This is confirmed by examples [2, 3], where they form, through the enhançon mechanism, "empty" holes instead (giving interaction potentials like fig. 1b and 1c). Black holes can form if mutually interacting particles of different species come infinitely close in

the constituents has its phase equal to the overall phase of the system coordinate distance, but as we just saw this is not possible if the quiver has no closed loops. So we can apply the reasoning we made for the two-particle case, without black hole or non-compactness complications, and the transition of states living on the Higgs branch to states living on the Coulomb branch when g_s increases should be equally clean.

This is in contrast to cases with loops, such as fig. 7b. The equilibrium equations for this example are

$$\frac{1}{r_{ab}} - \frac{2}{r_{ac}} = 2\theta_a \tag{4.29}$$

$$\frac{1}{r_{bc}} - \frac{1}{r_{ab}} = 2\theta_b \tag{4.30}$$

$$\frac{2}{r_{ac}} - \frac{1}{r_{bc}} = 2\theta_c. \tag{4.31}$$

These equations always have solutions for arbitrarily small r_{pq}, so the interacting particles can come arbitrarily close to each other and form a black hole or at least a microscopic sort of bound state that falls outside the type of states we have been studying thus far. Note also that these configurations are more stable than the cases without closed loops: there will always be solutions to the above equations, no matter what values the θ_v have.

Finally, on the D-brane side, closed loops and the potential presence of a superporential go hand in hand. This suggests that black hole formation and the appearance of superpotentials are perhaps intrinsically linked, but we will not get into this in the present work. Instead, we will focus on quivers without closed loops, where the Coulomb-Higgs relation is most transparent.

Can we expect in general a detailed match between the supersymmetric ground state degeneracies in both pictures, as we found for the case of the quiver with two nodes, κ arrows and dimension vector $(1,1)$? As we will see below, this detailed correspondence also holds for a number of more involved examples, and it is tempting to conjecture that this is indeed the case in general, but we have no proof of this. The problem is as usual that during the continuous interpolation between the two regimes, boson-fermion pairs of states could in principle be moved in or out the ground state set. We will be able to show that in the Higgs regime, the supersymmetric ground states are either all fermionic or all bosonic. Showing the same for the Coulomb regime would therefore establish the detailed correspondence, but how to do this in general is an open question.

A quantity that should match in any case is of course the Witten index, defined here as the number of bosonic minus the number of fermionic ground states of the system before tensoring with the trivial center of mass half-hypermultiplet (after tensoring the index is trivially zero).

4.5 A nontrivial test: the Hall halo

For our basic example, corresponding to a quiver with two nodes, κ arrows, and dimension vector $(1,1)$, we were able to compute ground state degeneracies on both sides, with identical results. With a bit more effort, the same can be done for dimension vector $(1,N)$, providing a nontrivial test of the proposed correspondence. On the Coulomb side, this system can be thought of as a charge κ magnetic monopole surrounded by N mutually non-interacting electrons of charge 1, or in other words a Hall halo. To construct wave functions for this system, one has to make antisymmetric combinations of the one-particle wave function (4.16). A very similar problem was considered and explicitly solved in [15], with the purpose of studying the quantum Hall effect on a sphere. The only difference is that the one-particle wave function of [15] is that of a spinless particle, while ours is that of a superparticle frozen in a preferred energy-minimizing spin 1/2 state (as discussed in section 4.2), thus lowering the maximal spin with 1/2 and the ground state degeneracy with 1 compared to the spinless particle case. This can be taken into account effectively by simply subtracting one unit from the number of flux quanta wherever this quantity appears in [15].

In particular, this gives us without further effort a generating function for the number of ground states with given spin J_3 along the 3-axis.[14] If we denote by n_L the number of supersymmetric ground states with $J_3 = L/2 - N(\kappa - N)/2$, this is [15]:

$$G(t) \equiv \sum_L n_L \, t^L = \frac{\prod_{j=1}^{\kappa}(1-t^{2j})}{\prod_{j=1}^{N}(1-t^{2j}) \prod_{j=1}^{\kappa-N}(1-t^{2j})} \, . \tag{4.32}$$

The number of spin $j = L/2 - N(\kappa - N)/2$ multiplets equals $n_L - n_{L-1}$, and the total degeneracy is $n_{tot} = G(1) = \binom{\kappa}{N}$. The latter can be easily understood from the exclusion principle: since our N electrons are mutually noninteracting fermions, and the one-particle degeneracy is κ, states are labeled by filling up N slots out of a total of κ. Also because of the exclusion principle, the maximal spin $(N(\kappa-N)/2)$ is lower than the naive supergravity expectation given by (2.25) $(\kappa N/2)$. The highest spin in supergravity is obtained by putting all electrons on top of each other, which is not allowed quantum mechanically.

As an example, the particle system corresponding to the $\kappa = 5$ quiver of fig. 4a, with dimension vector $(1,N)$, has a single spin $j=0$ ground state for $N=0$, a spin 2 multiplet for $N=1$, and a spin 3 multiplet plus a spin 1 multiplet for $N=2$. The cases $N = 3,4,5$ have the same structure as $N = 2,1,0$ respectively. There are no quantum supersymmetric ground states for $N > 5$, which is again a direct consequence of the exclusion principle. On the other hand, as shown in [3], there are *classical* supersymmetric ground states with $N > 5$ for this system (embedded in

[14] Here and in what follows, we count the states before tensoring with the trivial center of mass half-hypermultiplet.

supergravity). It was also pointed out there that these classical BPS configurations are not stable under monodromies, despite the fact that there is no line of marginal stability crossed. One of the proposed ways out of this paradox (also known as the “s-rule problem”) was that there was simply no BPS state with $N > 5$ at the *quantum* level to begin with. From our analysis here, we see that this was indeed the correct interpretation. Closely related problems were encountered for example in [6]. The relation between the s-rule, the exclusion principle and geometry was also pointed out (in a different context) in [56].

Finally note that the number of ground states grows exponentially fast with κ at constant filling fraction $\nu = N/\kappa$, since $n_{tot} = \binom{\kappa}{N} \approx \exp[\kappa(\nu \ln \nu^{-1} + (1-\nu)\ln(1-\nu)^{-1})]$.

We now turn to the Higgs branch description and see if the ground states there match those of the Coulomb regime. As argued under (3.10)-(3.11), the moduli space of the quiver with two nodes, κ arrows and dimension vector $(1, N)$ is the Grassmannian $Gr(N, \kappa)$, the space of all N-planes in $\mathbb{C}^\kappa$. The cohomology of this space is classically known. The generating function for the betti numbers b_L, also known as the Poincaré polynomial, is [16]:

$$P(t) \equiv \sum_L b_L\, t^L = \frac{\prod_{j=1}^{\kappa}(1-t^{2j})}{\prod_{j=1}^{N}(1-t^{2j})\prod_{j=1}^{\kappa-N}(1-t^{2j})}\,. \tag{4.33}$$

Furthermore, because of (4.22) and $\dim Gr(N, \kappa) = N(\kappa - N)$, we have that an L-form has spin $J_3 = L/2 - N(\kappa - N)/2$. Comparing this to (4.32) and the spin assignment there, we find perfect agreement between the two different pictures, even though the counting problems look very different at first sight. The framework presented in this paper provides a conceptual understanding of this remarkable match.

5. Further tests and applications

5.1 Counting cohomology classes of arbitrary quiver varieties

Thanks to a recent result of Reineke [17], it has become possible, at least in principle, to compute explicitly the Betti numbers of (semi-)stable quiver moduli spaces for arbitrary quivers $Q = (V, A)$ without closed loops, for arbitrary dimension vectors $\mathbf{N} = (N_v)_{v\in V}$, at arbitrary values of the FI parameters θ_v. The proof delves deep into the mathematics of finite fields, quantum groups and the Weil conjectures, but fortunately the final result can be stated as a down-to-earth explicit formula for the Poincaré polynomial $P(t) = \sum_L b_L\, t^L$, i.e. the generating function of the Betti numbers $b_L = \dim H^L(\mathcal{M}_d^{ss}(Q, \mathbf{N}, \theta), \mathbb{C})$. Denoting $[N, t] \equiv \frac{t^{2N}-1}{t^2-1}$ and $[N, t]! \equiv [1, t][2, t] \ldots [N, t]$, we have [17]

$$P(t) = (t^2-1)^{1-\sum_v N_v}\, t^{-\sum_v N_v(N_v-1)} \sum_{\mathbf{N}^*} (-1)^{s-1} t^{2\sum_{k\le l}\sum_{v\to w} N_v^l N_w^k} \prod_{k,v} ([N_v^k, t]!)^{-1} \tag{5.1}$$

where the sum runs over all ordered partitions of $\mathbf{N}$ by non-zero dimension vectors $\mathbf{N}^* = (\mathbf{N}^1 \dots \mathbf{N}^s)$ (i.e. $\mathbf{N} = \sum_{k=1}^s \mathbf{N}^k$, $\mathbf{N}^k \neq 0$), satisfying $\theta(\sum_{l=1}^k \mathbf{N}^l) > 0$ for $\kappa = 1 \dots s-1$. Note that only even powers of t occur. Physically this means that the supersymmetric ground states are either all bosonic or all fermionic (as usual before tensoring with the center-of-mass multiplet), and in particular that the Witten index (i.e. the Euler characteristic of the moduli space) equals the total number of ground states.

5.2 Bound states of N_1 "monopoles" with N_2 "electrons"

The quiver with two nodes and κ arrows from the first to the second node, and dimension vector $\mathbf{N} = (N_1, N_2)$ corresponds to a system of N_1 particles of one type and N_2 particles of another type, with intersection product between the two kinds of particles equal to κ — for example a system of N_1 "monopoles" of charge κ and N_2 "electrons" of charge 1. These particles are bound together (in the stable case $\theta_2 < 0$) by a potential of the form $V \sim (1/r + 2\theta_2/\kappa)^2$ between monopoles and electrons. We take N_1 and N_2 coprime, so stability and semi-stability are equivalent. (Physically, a common divisor d means that there is no potential preventing the state to split into d pieces of charge $\mathbf{N}/d$, giving rise to potential threshold bound state subtleties, which we wish to avoid in this paper.)

In this case the formula (5.1) can be simplified to [17]:

$$
\begin{aligned}
P^{\kappa}_{N_1,N_2}(t) &= (t^2-1)^{1-N_1}\; t^{-N_1(N_1-1)} \\
&\quad \times \sum_{N_1^*,N_2^*} (-1)^{s-1} t^{2\sum_{k<l}(\kappa N_1^l - N_2^l)N_2^k} \prod_{k=1}^{s} \frac{[\kappa N_1^k, t]!}{[N_1^k,t]![N_2^k,t]![\kappa N_1^k - N_2^k, t]!}
\end{aligned}
\tag{5.2}
$$

where the sum runs over all partitions $N_1^* = (N_1^1 \dots N_1^s)$, $N_2^* = (N_2^1 \dots N_2^s)$ of N_1 resp. N_2 such that $N_1^k \neq 0$ for all k, and $(N_1^1 + \dots + N_1^k)/N_1 > (N_2^1 + \dots + N_2^k)/N_2$ for all $k = 1 \dots s-1$.

As a check, consider the Hall halo case $N_1 = 1$. Then the formula collapses simply to

$$
P^{\kappa}_{1,N_2}(t) = \frac{[\kappa, t]!}{[N_2,t]![\kappa - N_2, t]!}, \tag{5.3}
$$

reproducing (4.33).

The formula (5.2) for $N_2 \geq 2$ gives new predictions for the structure of monopole-electron BPS bound states. This is a pretty nontrivial result, as even the *classical* ground state configurations are hard to obtain explicitly. To be concrete, let us consider the example $N_1 = 2, N_2 = 3$ with $\kappa = 5$. This example appears in the context of type IIA string theory compactified on the Quintic, see fig. 4. There are three contributing partition pairs $(N_1^*; N_2^*)$, namely $((2);(3))$, $((1,1);(1,2))$ and $((1,1);(0,3))$. The resulting Poincaré polynomial is, with $x \equiv t^2$:

$$
P^5_{2,3} = 1 + x + 3\,x^2 + 4\,x^3 + 7\,x^4 + 9\,x^5 + 14\,x^6 + 16\,x^7 + 20\,x^8 + 20\,x^9
$$

$$+20\,x^{10}+16\,x^{11}+14\,x^{12}+9\,x^{13}+7\,x^{14}+4\,x^{15}+3\,x^{16}+x^{17}+x^{18}.$$

The organization of ground states in spin multiplets can be read off directly from this polynomial, and is given in the following table, where j is the spin quantum number, n_j the number of spin j multiplets in the ground state Hilbert space, and $d_j = n_j(2j+1)$ the total dimension the spin j multiplets occupy. The total number of states is 170.

j	9	8	7	6	5	4	3	2	1	0
n_j	1	0	2	1	3	2	5	2	4	0
d_j	19	0	30	13	33	18	35	10	12	0

The full supermultiplet thus generated is obtained by taking the direct product with the center of mass wave function, which consists of two spin 0 singlets and one spin 1/2 doublet. The maximal total spin is thus $9 + 1/2$.

5.3 BPS states in $\mathcal{N} = 2$ $SU(2)$ Yang-Mills

A wide variety of constructions of BPS states in $\mathcal{N} = 2$ super Yang-Mills theories exist, including the two pictures considered in this paper. For the multi-particle picture in the (abelian) low energy effective field theory, see for instance [6]. For the quiver D-brane picture, see in particular [14], where the full set of (classical) BPS states was built from a certain set of parton branes, linked together by quiver diagrams derived from the orbifold construction associated to this theory through geometric engineering. In this construction, the number of arrows between two nodes again equals the DSZ intersection product of the objects corresponding to the nodes.

We will first consider the pure $SU(2)$ case. The low energy effective theory, solved exactly by Seiberg and Witten in [58], is a $U(1)$ gauge theory. The theory has a one complex dimensional moduli space, parametrized by a holomorphic coordiante u. This space has three singularities: one at infinity, one at $u = 1$, and one at $u = -1$. In the quiver picture [14], there are two partons, corresponding to a monopole and a dyon, which become massless at $u = 1$ and $u = -1$ respectively. The intersection product of the two partons is 2, giving a quiver with two nodes and two arrows between them, a special case of the class of quivers studied in section 5.2. The entire classical BPS spectrum of the theory can be built as stable representations of this quiver with various dimension vectors. As we saw, the stability depends on the moduli through the FI coefficient θ. In this theory $\theta = 0$ on an ellipse-like line containing the two singularities, $\theta < 0$ outside the line, and $\theta > 0$ inside. So to have a stable representations, we need to be outside this line of marginal stability, i.e. in the weak coupling region. Still, only a limited number of dimension vectors (n, m) support stable representations there. This follows already from simple counting of the degrees of freedom modulo complexified gauge transformations, giving for the

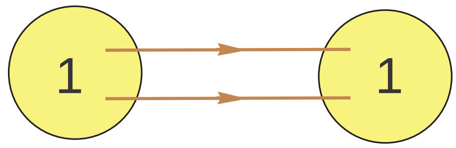

Figure 8: Quiver representation of a $W^{\pm}$ boson in $\mathcal{N} = 2$ $SU(2)$ Yang-Mills

expected dimension of the quiver moduli space $d(n, m) = 2nm - n^2 - m^2 + 1 = 1 - (n - m)^2$, which is zero for $n - m = \pm 1$ and one for $n = m$.

The case $n = m = 1$ is the only one with a nontrivial moduli space and g.c.d.$(n, m) = 1$. It corresponds to the $W^{\pm}$ gauge boson. The corresponding quiver representation is shown in fig. 8. The moduli space is $\mathbb{CP}^1$, so we get a spin 1/2 multiplet of BPS ground states, which multiplied with the center of mass states gives an $\mathcal{N} = 2$ vector multiplet, in agreement with the interpretation of this particle as a gauge boson. The quiver representations corresponding to the tower of dyons have a zero dimensional moduli space, so they all come in hypermultiplets, in agreement with their interpretation as cousins of the parton monopole and dyon.

The particle picture (in the abelian low energy effective field theory) should give the same result if the proposed duality is correct. Unfortunately, even finding the classical moduli space in this picture is very hard if more than one of each of the partons is present, and we will not attempt to solve this problem here. The $(1, n)$ or $(n, 1)$ cases on the other hand are again the Hall halo cases, and we found indeed that those agree with the quiver moduli picture. It would be interesting to extend this analysis to the general (n, m) case in the particle picture, and in particular to show how quantum mechanics corrects the fact that one finds way too many BPS bound states classically in this approach [6]. As was the case for the Hall halo, this is presumably due to the exclusion principle, but we do not know how exactly this comes about.

5.4 The Stern-Yi dyon chain

In [36], Stern and Yi studied the ground state counting problem for a collection of $k+1 \leq N$ distinct dyons in $\mathcal{N} = 2$ $SU(n)$ Yang-Mills with magnetic charges m_v given by an irreducible (sub)set of simple roots, $\beta_1, \ldots, \beta_{k+1}$. We take the roots normalized to satisfy the relations $\beta_v^2 = 2$, $\beta_v \cdot \beta_{v+1} = -1$, and $\beta_v \cdot \beta_w = 0$ for $|v - w| > 1$, with the dot denoting the inner product on weight space. The electric charges of the dyons are $e_v = n_v \beta_v$. The DSZ intersection product of two (magnetic, electric) charges (m, e) and (m', e') is $\langle (m, e), (m', e') \rangle \equiv m \cdot e' - e \cdot m'$. Thus we have for the above charges that $\langle (m_v, e_v), (m_{v+1}, e_{v+1}) \rangle = n_v - n_{v+1} \equiv \kappa_v$, and all other intersection products zero. Thus we obtain a "chain" of dyons, which can be represented by a quiver with

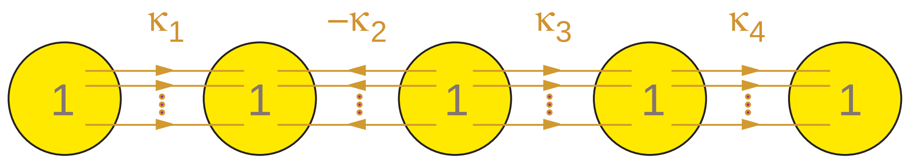

Figure 9: Quiver diagram corresponding to the $k = 4$ Stern-Yi dyon chain with $\kappa_1, \kappa_3, \kappa_4 > 0$ and $\kappa_2 < 0$.

$k+1$ nodes, κ_v arrows from node v to node $v+1$ if $\kappa_v > 0$, $-\kappa_v$ arrows from node $v+1$ to node v if $\kappa_v < 0$, and dimension vector $(1, 1, \dots, 1)$, as shown in fig. 9 for $k = 4$.

The Witten index $\mathcal{I}$ for this chain was computed in [36] in the framework of the moduli space approximation to the low energy dynamics of dyons as solitons in the classical nonabelian Yang-Mills theory. The main result was that, depending on the moduli, either $\mathcal{I} = 0$, or $\mathcal{I} = \prod_{v=1}^{k} |\kappa_v|$. It was also conjectured that four times this index is actually equal to the total number of BPS states (the four coming from the quantization of the center of mass fermions).

This result is reproduced in both of our pictures. We start with the quiver moduli space picture. An arrow a between v and $v+1$ corresponds to a complex variable $\phi^a_{v,v+1}$, where $a = 1, \dots, |\kappa_v|$. The moduli space $\mathcal{M}$ is, using the D-flatness description (3.8):

$$\mathcal{M} = \left\{ \phi : s_v \sum_{a=1}^{|\kappa_v|} |\phi^a_{v,v+1}|^2 - s_{v-1} \sum_{a=1}^{|\kappa_{v-1}|} |\phi^a_{v-1,v}|^2 = \theta_v \quad \text{for all nodes } v = 1, \dots, k+1 \right\} / \, G, \tag{5.4}$$

where $\kappa_0 \equiv 0$, $\kappa_{k+1} \equiv 0$, $s_v = \operatorname{sign} \kappa_v$, the gauge group $G = U(1)^{k+1}$, and $\theta_v = m_v(\alpha_v - \alpha_0)$ as in (3.9). This is equivalent to the equations

$$\begin{aligned} s_k \sum_{a=1}^{|\kappa_k|} |\phi^a_{k,k+1}|^2 &= -\theta_{k+1} \\ s_{k-1} \sum_{a=1}^{|\kappa_{k-1}|} |\phi^a_{k-1,k}|^2 &= -(\theta_k + \theta_{k+1}) \\ &\cdots \\ s_1 \sum_{a=1}^{|\kappa_1|} |\phi^a_{1,2}|^2 &= -(\theta_2 + \theta_3 + \cdots + \theta_{k+1}) = \theta_1 \end{aligned}$$

modulo G. A solution only exists if the partial θ-sums $s_{l-1} \sum_{v=l}^{k+1} \theta_v$, $l \geq 2$ are all negative. (As before, we discard nongeneric cases, i.e. having some of the partial sums equal to zero, to avoid complications of singularities and threshold bound states.)

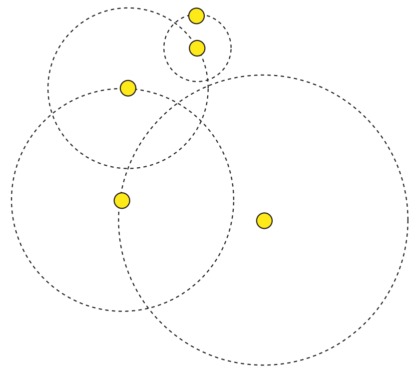

Figure 10: The Ptolemaean particle system corresponding to the $k = 4$ Stern-Yi dyon chain.

This is of course just the θ-stability condition discussed in section 3.2.[15] If this is satisfied, the moduli space is

$$\mathcal{M} = \mathbb{CP}^{|\kappa_1|} \times \mathbb{CP}^{|\kappa_2|} \times \cdots \times \mathbb{CP}^{|\kappa_k|}. \tag{5.5}$$

The Poincaré polynomial of $\mathcal{M}$ is therefore

$$P(t) = \prod_{v=1}^{k} \frac{t^{2|\kappa_v|} - 1}{t^2 - 1}\,, \tag{5.6}$$

from which we can directly read off the spin multiplet structure. In particular this gives for the total number of BPS ground states

$$\dim \mathcal{H}_{BPS} = 4 \times \prod_{v=1}^{k} |\kappa_v|\,, \tag{5.7}$$

where the factor of four comes again from the center of mass degrees of freedom. This is in exact agreement with [36], and confirms their conjecture.

In the corresponding particle mechanics picture we have one particle for each node of the quiver, interacting with its two nearest neighbors through (2.30)-(2.30). The equilibrium positions are given by equation (2.31):

$$\frac{\kappa_v}{2|\mathbf{x}_v - \mathbf{x}_{v+1}|} - \frac{\kappa_{v-1}}{2|\mathbf{x}_{v-1} - \mathbf{x}_v|} = \theta_v \quad \text{for all } v = 1, \ldots, k+1. \tag{5.8}$$

where again $\kappa_0 \equiv 0$ and $\kappa_{k+1} \equiv 0$. This is equivalent to

$$\begin{aligned}
|\mathbf{x}_k - \mathbf{x}_{k+1}| &= -\frac{\kappa_k}{2\theta_{k+1}} \\
|\mathbf{x}_{k-1} - \mathbf{x}_k| &= -\frac{\kappa_{k-1}}{2(\theta_k + \theta_{k+1})} \\
&\cdots \\
|\mathbf{x}_1 - \mathbf{x}_2| &= -\frac{\kappa_1}{2(\theta_2 + \theta_3 + \cdots + \theta_{k+1})} = \frac{\kappa_1}{2\theta_1}
\end{aligned}$$

[15]It should also match the stability condition of [36], but we didn't check this.

So the low energy motion of these particles is on some sort of Ptolemaean chain of epicycles (or more accurately epispheres), as shown in fig. 10. Without going into the formal details, it is intuitively clear that upon quantization of this chain, one will again find a direct product structure, resulting in the above generating function for the ground states. This picture also explains where the huge degeneracy of these states comes from.

6. Conclusions and discussion

We considered two seemingly very different pictures of BPS bound states, one as a set of particles at equilibrium separations from each other, the other as a fusion of D-branes at a single point of space, and we saw how the two are continuously related in the context of quantum quiver mechanics, by changing the string coupling constant g_s (with the single D-brane picture corresponding to $g_s \to 0$). We illustrated how this duality can be used to solve some quite nontrivial ground state counting problems in multi-particle "electron-monopole" quantum mechanics, and to count BPS degeneracies of certain dyons in supersymmetric Yang-Mills theories. Recent mathematical results on the cohomology of quiver varieties allowed us to give a general degeneracy formula for all such systems described by quivers without closed loops.

Strictly speaking, the quiver models discussed in this paper are only accurate for small phase differences (needed to cleanly separate the low energy string modes from the massive ones). However, even for bigger phase differences, the models can in many case still be expected to capture many of the qualitative features of the physics, especially since the most dramatic qualitative changes happen at marginal stability loci, where phase differences vanish rather than being bigger. In particular, one can in many cases expect the ground state counting to remain valid. This does not necessarily mean we can keep things under control for *arbitrary* phase differences, since by running around in moduli space, monodromies can occur which invalidate the quiver picture even qualitatively. For example in the case of dyons in $\mathcal{N} = 2$ $SU(2)$ Yang-Mills theory, running around the marginal stability line will change the dimension vector relevant for the description of the dyon, so at some point the original description must have broken down. On the other hand, the ground state counting should give identical results, so there must be a symmetry between these different representations. To get a satisfactory unified description of such situations, the framework of [13] is needed.

A problem left unanswered in this paper is whether the precise match between the ground state degeneracies in the Higgs and the Coulomb regimes extends from the examples considered here to the general case. We conjectured that this is indeed true. To prove this, it would be sufficient to show that the supersymmetric ground states in the Coulomb regime are either all fermionic or all bosonic, as is the case in the Higgs regime, but we did not do this.

Another open question is what can be said about quivers with closed loops (and hence possibly with superpotentials). Our counting results were all for quivers without closed loops, in part because the physics of those is more transparent, and in part because much of the mathematics of quivers with closed loops is still unknown. It would be very interesting to extend our results to those cases, especially since this includes the quivers describing black hole states.

Acknowledgments

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A. Notations and conventions

Our metric signature is $(-+++)$. Spinors with indices down transform in the $\mathbf{2}$ of the spatial $SO(3)$, spinors with indices up in the $\bar{\mathbf{2}}$. The unbarred spinors appearing in this paper all have indices down, the barred ones indices up. Barred and unbarred spinors are related through complex conjugation: $(\psi_\alpha)^* \equiv \bar{\psi}^\alpha$. We use the following notations:

$$\bar{\psi}\chi = \bar{\psi}^\alpha \chi_\alpha = -\chi_\alpha \bar{\psi}^\alpha = -\chi\bar{\psi} \tag{A.1}$$

$$\bar{\psi}\sigma^i\chi = \bar{\psi}^\alpha {\sigma^i}_\alpha{}^\beta \chi_\beta \tag{A.2}$$

$$\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}, \quad \epsilon_{\alpha\gamma}\epsilon^{\gamma\beta} = \delta_\alpha{}^\beta, \quad \epsilon^{12} = 1, \quad \epsilon_{12} = -1 \tag{A.3}$$

$$(\epsilon\psi)^\alpha = \epsilon^{\alpha\beta}\psi_\beta \tag{A.4}$$

$$(\bar{\psi}\epsilon)_\alpha = \bar{\psi}^\beta \epsilon_{\beta\alpha} \tag{A.5}$$

We do not define index lowering or raising; instead we always write the appropriate ϵ explicitly, as in (A.4)-(A.5). So for instance an invariant contraction of two lower index spinors will look like $\psi\epsilon\chi = \psi_\alpha \epsilon^{\alpha\beta}\chi_\beta$. The σ^i are the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.6}$$

B. Supersymmetry transformations for the $U(1)$ case

The relevant supersymmetry transformations for the different forms of the Lagrangian describing the example of section 3.1 are closely related. We give those for the relative

Lagrangian (3.5)-(3.6), which we copy here for convenience:

$$L_{rel} = \frac{\mu}{2}\left(\dot{\mathbf{x}}^2 + D^2 + 2i\bar{\lambda}\dot{\lambda}\right) - \theta D \tag{B.1}$$

$$+|\mathcal{D}_t\phi^a|^2 - \left(\mathbf{x}^2 + D\right)|\phi^a|^2 + |F^a|^2 + i\,\bar{\psi}^a\mathcal{D}_t\psi^a$$
$$-\bar{\psi}^a\,\mathbf{x}\cdot\boldsymbol{\sigma}\,\psi^a - i\sqrt{2}(\bar{\phi}^a\psi^a\epsilon\lambda - \bar{\lambda}\epsilon\bar{\psi}^a\phi^a)\,, \tag{B.2}$$

The corresponding supersymmetry transformations are:

$$\delta A = i\,\bar{\lambda}\,\xi - i\,\bar{\xi}\,\lambda \tag{B.3}$$
$$\delta\mathbf{x} = i\,\bar{\lambda}\,\boldsymbol{\sigma}\,\xi - i\,\bar{\xi}\,\boldsymbol{\sigma}\,\lambda \tag{B.4}$$
$$\delta\lambda = \dot{\mathbf{x}}\cdot\boldsymbol{\sigma}\,\xi + i\,D\,\xi \tag{B.5}$$
$$\delta D = -\dot{\bar{\lambda}}\,\xi - \bar{\xi}\,\dot{\lambda} \tag{B.6}$$
$$\delta\phi^a = \sqrt{2}\,\epsilon\xi\,\psi^a \tag{B.7}$$
$$\delta\psi^a = -i\sqrt{2}\,\bar{\xi}\epsilon\,\mathcal{D}_t\phi^a - \sqrt{2}\,\mathbf{x}\cdot\boldsymbol{\sigma}\,\bar{\xi}\epsilon\,\phi^a + \sqrt{2}\,\xi\,F^a \tag{B.8}$$
$$\delta F^a = -i\sqrt{2}\,\bar{\xi}\,\mathcal{D}_t\psi^a + \sqrt{2}\,\bar{\xi}\,\boldsymbol{\sigma}\,\psi^a\cdot\mathbf{x} - 2i\,\bar{\xi}\epsilon\,\bar{\lambda}\,\phi^a \tag{B.9}$$

C. General quiver mechanics Lagrangian

A quiver Q with nodes $v \in V$, arrows $a \in A$, and dimension vector $N = (N_v)_{v\in V}$ corresponds to an $\mathcal{N}=1$, $d=4$ gauge theory, or, in our setting, to an $\mathcal{N}=4$, $d=1$ matrix model, obtained by dimensional reduction from $d=4$. To each node v, we associate a linear (a.k.a. vector) multiplet $(A_v, X_v^i, \lambda_v, D_v)$, $i=1,2,3$, $v\in V$, with gauge group $U(N_v)$, and to each arrow $a: v\to w$, we associate a bifundamental chiral multiplet (ϕ^a, ψ^a, F^a), transforming in the $(\bar{\boldsymbol{N}}_{\boldsymbol{v}}, \boldsymbol{N}_{\boldsymbol{w}})$ of $U(N_v)\times U(N_w)$. The corresponding $\mathcal{N}=4$, $d=1$ Lagrangian is, in units with $l_s = 2\pi\alpha' = 1$:

$$L = L_V + L_{FI} + L_C + L_I + L_W \tag{C.1}$$

with

$$L_V = \sum_v \frac{m_v}{2}\mathrm{Tr}\Big((D_t X_v^i)^2 + {D_v}^2 - \frac{1}{2}[X^i,X^j]^2 + 2i{\lambda_v}^\dagger\mathcal{D}_t\lambda_v - 2{\lambda_v}^\dagger\sigma^i[X^i,\lambda_v]\Big),$$
$$L_{FI} = \sum_v -\theta_v \mathrm{Tr} D_v$$
$$L_C = \sum_a \mathrm{Tr}\Big(|\mathcal{D}_t\phi^a|^2 + |F^a|^2 + i\,\psi^{a\dagger}\mathcal{D}_t\psi^a\Big)$$
$$L_I = \sum_{a:v\to w} -\mathrm{Tr}\Big(|X_w^i\phi^a - \phi^a X_v^i|^2 + \phi^{a\dagger}(D_w\phi^a - \phi^a D_v) + \psi^{a\dagger}\sigma^i(X_w^i\psi^a - \psi^a X_v^i)$$
$$-i\sqrt{2}\left((\phi^{a\dagger}\lambda_w - \lambda_v\phi^{a\dagger})\epsilon\psi^a - \psi^{a\dagger}\epsilon({\lambda_w}^\dagger\phi^a - \phi^a{\lambda_v}^\dagger)\right)\Big)$$
$$L_W = \sum_a \mathrm{Tr}\Big(\frac{\partial W}{\partial\phi^a}\,F^a + \text{h.c.}\Big) + \frac{1}{2}\sum_{a,b}\mathrm{Tr}\Big(\frac{\partial^2 W}{\partial\phi^a\partial\phi^b}\,\psi^a\epsilon\psi^b + \text{h.c.}\Big)$$

where, for $a : v \to w$:

$$
\begin{aligned}
\mathcal{D}_t \phi^a &= \partial_t \phi^a + i\, A_w \phi^a - i \phi^a A_v \\
\mathcal{D}_t X^i_v &= \partial_t X^i_v + i [A_v, X^i_v]
\end{aligned}
$$

and similarly for the superpartners. The parameters θ_v in L_{FI} are the Fayet-Iliopoulos parameters, which for the D-brane model are given by (3.9). A (gauge invariant) holomorphic superpotential $W(\phi)$ can only appear if the quiver has no closed loops.

The supersymmetry transformations are

$$
\begin{aligned}
\delta A_v &= i\, \bar{\lambda}_v\, \xi - i\, \bar{\xi}\, \lambda_v \\
\delta X^i_v &= i\, \bar{\lambda}_v\, \sigma^i\, \xi - i\, \bar{\xi}\, \sigma^i\, \lambda_v \\
\delta \lambda_v &= \mathcal{D}_t X^i_v \sigma^i\, \xi + \frac{1}{2} \epsilon^{ijk} [X^i_v, X^j_v] \sigma^k \xi + i\, D_v\, \xi \\
\delta D &= -\mathcal{D}_t \bar{\lambda}_v\, \xi - i\, [X^i_v, \bar{\lambda}_v] \sigma^i \xi - \bar{\xi}\, \mathcal{D}_t \lambda_v - i\, \bar{\xi} \sigma^i [X^i_v, \lambda_v] \\
\delta \phi^a &= \sqrt{2} \epsilon \xi \psi^a \\
\delta \psi^a &= -i \sqrt{2}\, \bar{\xi} \epsilon\, \mathcal{D}_t \phi^a - \sqrt{2}\, \sigma^i\, \bar{\xi} \epsilon (X^i_w \phi^a - \phi^a X^i_v) + \sqrt{2}\, \xi\, F^a \\
\delta F^a &= -i \sqrt{2}\, \bar{\xi}\, \mathcal{D}_t \psi^a + \sqrt{2}\, \bar{\xi}\, \sigma^i (X^i_w \psi^a - \psi^a X^i_v) - 2i\, \bar{\xi} \epsilon (\bar{\lambda}_w \phi^a - \phi^a \bar{\lambda}_v)
\end{aligned}
$$

When the vector multiplets are restricted to diagonal matrices, the Lagrangian components simplify to

$$
\begin{aligned}
L_V &= \sum_v \sum_{n=1}^{N_v} \frac{m_v}{2} \Big((\dot{x}^i_{v,n})^2 + (D_{v,n})^2 + 2i \bar{\lambda}_{v,n} \dot{\lambda}_{v,n} \Big) \,, \\
L_{FI} &= - \sum_v \sum_{n=1}^{N_v} \theta_v D_{v,n} \\
L_C &= \sum_a \mathrm{Tr}(|\mathcal{D}_t \phi^a|^2 + |F^a|^2 + i\, \psi^{a\dagger} \mathcal{D}_t \psi^a) \\
L_I &= - \sum_{a:v \to w} \sum_{n,m=1}^{N_v, N_w} \Big(\Big((x^i_{w,m} - x^i_{v,n})^2 + D_{w,m} - D_{v,n} \Big) |\phi^a_{mn}|^2 \\
&\quad + (x^i_{w,m} - x^i_{v,n}) \overline{\psi^a_{mn}} \sigma^i \psi^a_{mn} \\
&\quad - i\sqrt{2} \Big(\overline{\phi^a_{mn}} (\lambda_{w,m} - \lambda_{v,n}) \epsilon \psi^a_{mn} - \overline{\psi^a_{mn}} \epsilon (\bar{\lambda}_{w,m} - \bar{\lambda}_{v,n}) \phi^a_{mn} \Big) \Big) \\
L_W &= \sum_a \mathrm{Tr} \Big(\frac{\partial W}{\partial \phi^a} F^a + \text{h.c.} \Big) + \frac{1}{2} \sum_{a,b} \mathrm{Tr} \Big(\frac{\partial^2 W}{\partial \phi^a \partial \phi^b}\, \psi^a \epsilon \psi^b + \text{h.c.} \Big)
\end{aligned}
$$

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