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Quantum Real Projective Space, Disc and Spheres

Dedicated to the memory of Stanisław Zakrzewski

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Abstract. We define the C^*-algebra of quantum real projective space $\mathbb{R}P_q^2$, classify its irreducible representations, and compute its K-theory. We also show that the q-disc of Klimek and Lesniewski can be obtained as a non-Galois $\mathbb{Z}_2$-quotient of the equator Podleś quantum sphere. On the way, we provide the Cartesian coordinates for all Podleś quantum spheres and determine an explicit form of isomorphisms between the C^*-algebras of the equilateral spheres and the C^*-algebra of the equator one.

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1. Introduction

Classical spheres can be constructed by gluing two discs along their boundaries. Since an open disc is homeomorphic to $\mathbb{R}^2$, this fact is reflected in the following short exact sequence of C^*-algebras of continuous functions (vanishing at infinity where appropriate):

$$0 \longrightarrow C_0(\mathbb{R}^2) \oplus C_0(\mathbb{R}^2) \longrightarrow C(S^2) \longrightarrow C(S^1) \longrightarrow 0. \tag{1.1}$$

On the other hand, one can obtain a disc D^2 as the quotient of a sphere under the $\mathbb{Z}_2$-action given by the reflection with respect to the equator plane. Two copies of an open disc collapse to one copy, and we have the short exact sequence

$$0 \longrightarrow C_0(\mathbb{R}^2) \longrightarrow C(D^2) \longrightarrow C(S^1) \longrightarrow 0. \tag{1.2}$$

Similarly, real projective space $\mathbb{R}P^2$ can be constructed from the antipodal action of $\mathbb{Z}_2$ on the two-sphere. As for D^2, removing S^1 from $\mathbb{R}P^2$ also leaves an open disc, and again we have the short exact sequence

$$0 \longrightarrow C_0(\mathbb{R}^2) \longrightarrow C(\mathbb{R}P^2) \longrightarrow C(S^1) \longrightarrow 0. \tag{1.3}$$

The aim of this paper is to present the noncommutative geometry of a q-deformation of the aforementioned geometric setting. (This deformation is unique under some assumptions.) It turns out that the q-deformation changes $C_0(\mathbb{R}^2)$ in the above short exact sequences into the ideal $\mathcal{K}$ of compact operators (see (2.32), (3.27), (4.32)). Therefore, since $C_0(\mathbb{R}^2)$ and $\mathcal{K}$ behave in a similar way in K-theory, it is not surprising that the K-groups of these q-deformed surfaces coincide with the respective K-groups of their classical counterparts. Since D^2 has a boundary and $\mathbb{R}P^2$ is nonorientable, we hope that the study of their q-analogues will help one to understand the concept of a boundary and orientability in the general noncommutative setting.

Deformations of SL(2, $\mathbb{C}$) were studied in depth and classified [DL90, W-SL91, WZ94]. The choice of the compact $*$-structure and the requirement of the existence of the C^*-norm lead then to the celebrated deformation of SU(2), which we denote by $\mathrm{SU}_q(2)$. (The literature on this quantum group motivating and treating it from many different points of view is vast. E.g., see [KS97] for references.) Subsequently, the study of quantum homogeneous spaces of $\mathrm{SU}_q(2)$ leads to the classification of quantum spheres [P-P87]. (See [S-A91] for the Poisson aspects.) On the other hand, motivated by the Poisson geometry, noncommutative deformations of the unit disc were constructed in [KL92, KL93]. Gluings of quantum discs which produce quantum spheres were studied in [S-A91, MNW91, CM00]. Finally, quantum real projective space $\mathbb{R}P^2_q$ was defined in [H-PM96] within the framework of the Hopf–Galois theory to exemplify the concept of strong connections on algebraic quantum principal bundles (cf. [DGH01, Example 2.12]). It was obtained as the quantum quotient space from the antipodal $\mathbb{Z}_2$-action on the equator Podleś sphere. This action was already discovered in [P-P87], and is the only possible $\mathbb{Z}_2$-action on quantum spheres compatible with the actions of $\mathrm{SU}_q(2)$ (see above Section 6 therein).

In this paper, we continue along these lines. We begin in Section 2 by reviewing the relevant known results on quantum spheres (C^*-representations, K-theory). Then we provide the Cartesian coordinates and compute an explicit form of the C^*-isomorphisms between the C^*-algebra of the equator quantum sphere and the C^*-algebras of the equilateral Podleś spheres ($c \in (0, \infty)$). We also show that these isomorphisms commute with the U(1)-actions inherited from the actions of $\mathrm{SU}_q(2)$ on quantum spheres. In Section 3, we prove that the q-disc of Klimek and Lesniewski can be obtained as a noncommutative quotient of the equator quantum sphere by an appropriate $\mathbb{Z}_2$-action. More precisely, first we show that the polynomial algebra of the q-disc is a fixed-point subalgebra of the polynomial algebra of the equator quantum sphere under a non-Galois $\mathbb{Z}_2$-action. Then we extend this construction to the equilateral quantum spheres by employing the aforementioned

C^*-isomorphisms. Since these isomorphisms are non-polynomial, we handle the equilateral spheres only on the C^*-level. We complete this section by recalling the topological K-theory of the q-disc. The paper ends with Section 4 where we define the C^*-algebra of quantum $\mathbb{R}P^2$, study its representations, and compute the K-theory. Similarly to the quantum disc case, this C^*-algebra is obtained as a $\mathbb{Z}_2$-action fixed-point subalgebra of the C^*-algebra of the equator quantum sphere. For both the quantum disc and $\mathbb{R}P_q^2$ cases, we show that the $\mathbb{Z}_2$-actions are compatible with the above-mentioned actions of U(1).

Throughout the paper we use the jargon of Noncommutative Geometry referring to quantum spaces as objects dual to noncommutative algebras in the sense of the Gelfand–Naimark correspondence between spaces and function algebras. The unadorned tensor product means the completed (spatial) tensor product when placed between C^*-algebras, and the algebraic tensor product over $\mathbb{C}$ otherwise. The algebras are assumed to be associative and over $\mathbb{C}$. They are also unital unless the contrary is obvious from the context. By P(quantum space) we denote the polynomial algebra of a quantum space, and by C(quantum space) the corresponding C^*-algebra. In this paper, the C^*-completion (C^*-closure) of a $*$-algebra always means the completion with respect to the supremum norm over all $*$-representations in bounded operators.

2. Quantum Spheres

DEFINITION 2.1 ([P-P87]). The C^*-algebra $C(S^2_{q\infty})$ of the quantum sphere $S^2_{q\infty}$, $q \in \mathbb{R}$, $0 < |q| < 1$, is defined as the C^*-closure of the $*$-algebra $P(S^2_{q\infty}) := \mathbb{C}\langle A, B\rangle/I_{q\infty}$, where $I_{q\infty}$ is the (two-sided) $*$-ideal in the free $*$-algebra $\mathbb{C}\langle A, B\rangle$ generated by the relations

$$A^* = A, \qquad BA = q^2 AB, \tag{2.4}$$

$$B^*B = -A^2 + 1, \qquad BB^* = -q^4A^2 + 1. \tag{2.5}$$

The C^*-algebra $C(S^2_{qc})$ of the quantum sphere S^2_{qc}, $c \in [0, \infty)$ is defined analogously, with (2.5) replaced by

$$B_c^*B_c = A_c - A_c^2 + c, \qquad B_cB_c^* = q^2A_c - q^4A_c^2 + c. \tag{2.6}$$

The irreducible $*$-representations of the quantum spheres are determined in [P-P87]. Let us denote by $\pi^c_\pm$ and by π^c_θ the infinite-dimensional and one-dimensional representations of $C(S^2_{qc})$ ($c \in [0, \infty]$), respectively. (In the $c = \infty$ case, in agreement with the notation for generators in the above definition, we write $\pi_\pm$ and π_θ instead of $\pi^\infty_\pm$ and π^∞_θ, respectively.) The complete list of the irreducible $*$-representations of $C(S^2_{q\infty})$ is given by

$$\pi_\theta(A) = 0, \qquad \pi_\theta(B) = e^{i\theta}, \qquad \theta \in [0, 2\pi), \tag{2.7}$$

and

$$\pi_{\pm}(A)e_k = \pm q^{2k} e_k, \qquad \pi_{\pm}(B)e_k = (1-q^{4k})^{1/2} e_{k-1}, \qquad \pi_{\pm}(B)e_0 = 0. \quad (2.8)$$

Here $\{e_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of a Hilbert space. Similarly, the irreducible $*$-representations of $C(S^2_{qc})$, $c \in (0,\infty)$, are defined by

$$\pi^c_\theta(A_c) = 0, \qquad \pi^c_\theta(B_c) = c^{1/2}\mathrm{e}^{i\theta}, \qquad \theta \in [0, 2\pi), \quad (2.9)$$

and

$$\pi^c_{\pm}(A_c)e_k = \lambda_{\pm} q^{2k} e_k, \qquad \pi^c_{\pm}(B_c)e_k = c_{\pm}(k)^{1/2} e_{k-1}, \qquad \pi^c_{\pm}(B_c)e_0 = 0, \quad (2.10)$$

where

$$\lambda_{\pm} = \tfrac{1}{2} \pm \left(c + \tfrac{1}{4}\right)^{1/2}, \qquad c_{\pm}(k) = \lambda_{\pm} q^{2k} - (\lambda_{\pm} q^{2k})^2 + c. \quad (2.11)$$

The direct sums $\pi^c_+ \oplus \pi^c_-$, $0 < c \leqslant \infty$, are faithful representations. The representations $\pi^c_{\pm}$ can be considered as embeddings of quantum discs onto the northern and southern hemisphere, respectively, whereas the one-dimensional representations are the classical points (forming a circle). For $c = \infty$, the classical points are symmetric with respect to the hemispheres, i.e., they form the equator. With c decreasing, the circle of classical points shrinks to a pole. Thus, in the limit case $c = 0$, we can think of a quantum sphere as a quantum disc whose (classical) boundary is glued to a point. For $c = 0$, the formulas (2.9)–(2.11) still define $*$-representations. Now, however, π^0_θ coincide for all θ, and π^0_θ and π^0_+ are the only irreducible representations. The representation π^0_- becomes trivial, and π^0_+ becomes faithful. The cases $c = 0$, $0 < c < \infty$ and $c = \infty$ are referred to as the standard, equilateral and equator quantum sphere, respectively. Futhermore, it follows from [P-P87, Section 5] that the monomials $A^k_c B^l_c$, $A^m_c B^{*n}_c$, $k, l, m, n \in \mathbb{N}, n > 0, c \in [0, +\infty]$, viewed as elements of $C(S^2_{qc})$ are linearly independent and their span is dense in $C(S^2_{qc})$. Since they also span $P(S^2_{qc})$, they form a basis of $P(S^2_{qc})$. Thus $P(S^2_{qc}) \subseteq C(S^2_{qc})$. (The canonical map $P(S^2_{qc}) \to C(S^2_{qc})$ is injective.)

To make the aforementioned geometric picture explicit, we need to find the *Cartesian coordinates for quantum spheres*. More precisely, we need to define self-adjoint generators x, y, z of $P(S^2_{qc})$, $c \in [0, \infty]$, which satisfy $x^2 + y^2 + z^2 = 1$. Note first that dividing (2.6) by c and rescaling the generators by $c^{-1/2}$ would lead to the formulas whose limit with $c \to \infty$ would be (2.5). To include also the $c = 0$ case, let us rescale the generators by $(1+\sqrt{c})^{-1}$, i.e.,

$$\widetilde{A}_c := \frac{A_c}{1+\sqrt{c}}, \qquad \widetilde{B}_c := \frac{B_c}{1+\sqrt{c}}. \quad (2.12)$$

Now, from (2.6), we have

$$\widetilde{B}^*_c \widetilde{B}_c = \frac{\widetilde{A}_c}{1+\sqrt{c}} - \widetilde{A}^2_c + \frac{c}{(1+\sqrt{c})^2},$$
$$\widetilde{B}_c \widetilde{B}^*_c = \frac{q^2 \widetilde{A}_c}{1+\sqrt{c}} - q^4 \widetilde{A}^2_c + \frac{c}{(1+\sqrt{c})^2}. \quad (2.13)$$

The relations (2.4) remain unchanged, that is, $\widetilde{A}_c^* = \widetilde{A}_c$, $\widetilde{B}_c\widetilde{A}_c = q^2\widetilde{A}_c\widetilde{B}_c$. Contrary to A_c and B_c, the generators $\widetilde{A}_c$ and $\widetilde{B}_c$ have limits with $c \to \infty$ when thought of as elements of $P(\mathrm{SU}_q(2))$. Indeed, remembering the definition of A_c, B_c [P-P87, pp. 196, 200] in terms of the spin 1 representation

$$D_1 := \begin{pmatrix} \delta^2 & -(1+q^2)\delta\gamma & -q\gamma^2 \\ -q^{-1}\beta\delta & 1+(q+q^{-1})\beta\gamma & \alpha\gamma \\ -q^{-1}\beta^2 & +(q+q^{-1})\beta\alpha & \alpha^2 \end{pmatrix} \tag{2.14}$$

of $\mathrm{SU}_q(2)$ (with $\alpha, \beta, \gamma, \delta$ being the generators of the algebra $P(\mathrm{SU}_q(2))$), we can write

$$\begin{aligned}\left(\widetilde{B}_c^*, \widetilde{A}_c, \widetilde{B}_c\right) &= \left(\frac{\sqrt{c}}{1+\sqrt{c}}, \frac{1}{1+\sqrt{c}}, \frac{\sqrt{c}}{1+\sqrt{c}}\right) D_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -(1+q^2)^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} + \\ &\quad + \left(0, \frac{1}{(1+\sqrt{c})(1+q^2)}, 0\right).\end{aligned} \tag{2.15}$$

It is clear now that the tilded generators are well-defined also for $c = \infty$. Since the relations among the tilded generators become for $c = \infty$ the relations among A and B, we can write $\widetilde{A}_\infty := A$, $\widetilde{B}_\infty := B$. Thus, we have a uniform description of quantum spheres for all $c \in [0, \infty]$. (See [BM00, Section 6] for a uniform parameterisation of Podleś spheres by the unit interval $[0, 1]$.) Remembering the geometrical meaning of $\widetilde{A}_c$, $\widetilde{B}_c$ (see [P-P87, pp. 196, 200, 201]), we put

$$\begin{aligned} &x = iQ_x(\widetilde{B}_c - \widetilde{B}_c^*), \qquad y = Q_y(\widetilde{B}_c + \widetilde{B}_c^*), \\ &z = Q_z(\widetilde{A}_c - \tilde{a}_0), \quad c \in [0, \infty]. \end{aligned} \tag{2.16}$$

Here Q_x, Q_y, Q_z and $\tilde{a}_0$ are real-valued functions of q and c, so that x, y, z are evidently self-adjoint. The condition $x^2 + y^2 + z^2 = 1$ and the linear independence of the monomials $A_c^k B_c^l$, $A_c^m B_c^{*n}$, $k, l, m, n \in \mathbb{N}, n > 0$ [P-P89, p. 116] imply that $Q_x^2 = Q_y^2$. Let us put $Q_h := |Q_x| = |Q_y|$. Then the sphere equation boils down to

$$2Q_h^2(\widetilde{B}_c\widetilde{B}_c^* + \widetilde{B}_c^*\widetilde{B}_c) + Q_z^2(\widetilde{A}_c - \tilde{a}_0)^2 = 1. \tag{2.17}$$

Plugging in (2.13) to the above formula yields

$$2Q_h^2\left(-(1+q^4)\widetilde{A}_c^2 + \frac{1+q^2}{1+\sqrt{c}}\widetilde{A}_c + \frac{2c}{(1+\sqrt{c})^2}\right) + Q_z^2(\widetilde{A}_c - \tilde{a}_0)^2 = 1. \tag{2.18}$$

Employing again the linear independence of the monomials A_c^k, one can compute:

$$\begin{aligned} &Q_h = -\frac{\sqrt{2(1+q^4)}}{1+q^2}(1+\sqrt{c})z_\infty, \qquad |Q_z| = -\frac{2(1+q^4)}{1+q^2}(1+\sqrt{c})z_\infty, \\ &\tilde{a}_0 = \frac{1+q^2}{2(1+q^4)}(1+\sqrt{c})^{-1}, \end{aligned} \tag{2.19}$$

where $z_\infty := -(8c\frac{1+q^4}{(1+q^2)^2} + 1)^{-1/2}$. (The meaning of this number will shortly become clear.) Let us choose $Q_x = Q_h = Q_y$, $Q_z = |Q_z|$. The formulas (2.16) read now:

$$\begin{aligned} x &= -i\frac{\sqrt{2(1+q^4)}}{1+q^2}(1+\sqrt{c})z_\infty(\widetilde{B}_c - \widetilde{B}_c^*),\\ y &= -\frac{\sqrt{2(1+q^4)}}{1+q^2}(1+\sqrt{c})z_\infty(\widetilde{B}_c + \widetilde{B}_c^*),\\ z &= -\frac{2(1+q^4)}{1+q^2}(1+\sqrt{c})z_\infty\widetilde{A}_c + z_\infty. \end{aligned} \tag{2.20}$$

(Observe that

$$z_\infty|_{c=\infty} = 0 \quad \text{and} \quad ((1+\sqrt{c})z_\infty)|_{c=\infty} = -\frac{1+q^2}{2\sqrt{2(1+q^4)}}.)$$

The eigenvalues $z_k^\pm$ of $\pi_\pm^c(z)$ are given by

$$\pi_\pm^c(z)e_k = \left(z_\infty - z_\infty\lambda_\pm\frac{2(1+q^4)}{1+q^2}q^{2k}\right)e_k, \quad c \in [0,\infty]. \tag{2.21}$$

(Note that

$$(z_\infty\lambda_\pm)|_{c=\infty} = \mp\frac{1+q^2}{2\sqrt{2(1+q^4)}}.)$$

It is evident that $\lim_{k\to\infty} z_k^\pm = z_\infty$. Since we also have $\pi_\theta^c(z) = z_\infty$, we can say that the eigenvalues of $\pi_\pm^c(z)$ converge (from both sides) to the circle of classical points (space of one-dimensional representations) given by π_θ^c. For $c = \infty$ we have $z_\infty = 0$, so that the circle is the equator, whereas for $c = 0$ the circle shrinks to the south pole ($z_\infty = -1$). Finally, let us remark that, as $(1+\sqrt{c})z_\infty \neq 0$ for any $c \in [0,\infty]$, the equations (2.20) can be solved for $\widetilde{A}_c$, $\widetilde{B}_c$, $\widetilde{B}_c^*$, and consequently x, y, z generate the algebra $P(S_{qc}^2)$. Since they are also self-adjoint and satisfy $x^2 + y^2 + z^2 = 1$, we call them the Cartesian coordinates of quantum spheres.

We recall from [S-A91, Proposition 1.2] that $\pi_+^c \oplus \pi_-^c$ is for all $c \in (0,\infty]$ a C^*-isomorphism of $C(S_{qc}^2)$ onto $C^*(\mathfrak{S}) \oplus_\sigma C^*(\mathfrak{S})$. Here $C^*(\mathfrak{S})$ is the C^*-algebra of the one-sided shift (Toeplitz algebra). It is the C^*-algebra generated by the shift operator $\mathfrak{S}e_i = e_{i+1}$, where $\{e_i\}_{i\in\mathbb{N}}$ is an orthonormal basis of a Hilbert space. The map $\sigma\colon C^*(\mathfrak{S}) \to C(S^1)$ is the so-called symbol map defined by $\mathfrak{S} \mapsto u$, where u is the unitary generator of $C(S^1)$. The algebra $C^*(\mathfrak{S}) \oplus_\sigma C^*(\mathfrak{S})$ is defined as the gluing of two copies of $C^*(\mathfrak{S})$ via σ, i.e.,

$$C^*(\mathfrak{S}) \oplus_\sigma C^*(\mathfrak{S}) := \{(a_1, a_2) \in C^*(\mathfrak{S}) \oplus C^*(\mathfrak{S}) \mid \sigma(a_1) = \sigma(a_2)\}. \tag{2.22}$$

Let

$$\chi_c := (\pi_+ \oplus \pi_-)^{-1} \circ (\pi_+^c \oplus \pi_-^c)\colon C(S_{qc}^2) \longrightarrow C(S_{q\infty}^2) \tag{2.23}$$

be the isomorphism composed from the isomorphisms

$$\pi_+ \oplus \pi_-\colon\ C(S^2_{q\infty}) \to C^*(\mathfrak{S}) \oplus_\sigma C^*(\mathfrak{S})$$

and

$$\pi^c_+ \oplus \pi^c_-\colon\ C(S^2_{qc}) \to C^*(\mathfrak{S}) \oplus_\sigma C^*(\mathfrak{S}).$$

An explicit form of the isomorphisms χ_c is given by:

PROPOSITION 2.2. *Let* $\eta_c(t) := \sqrt{t - t^2 + c}$ (cf. (2.11)), *and let* F_c *and* G_c *be functions given by*

$$F_c(x) := \begin{cases} \lambda_+ x, & \text{for } 0 \leqslant x \leqslant 1, \\ -\lambda_- x, & \text{for } -1 \leqslant x < 0, \end{cases} \tag{2.24}$$

$$G_c(x) := (1 - q^4x^2)^{-1/2} \begin{cases} \eta_c(q^2\lambda_+ x), & \text{for } 0 \leqslant x \leqslant 1, \\ \eta_c(-q^2\lambda_- x), & \text{for } -1 \leqslant x < 0. \end{cases} \tag{2.25}$$

Then $\chi_c(A_c) = F_c(A)$ *and* $\chi_c(B_c) = G_c(A)B$.

Proof. First, note that, since $\pi_\pm(A)$ is diagonal and F_c and G_c are continuous functions defined on the spectrum of $\pi_\pm(A)$, the operators $F_c(\pi_\pm(A))$ and $G_c(\pi_\pm(A))$ make sense and are easily computable. Subsequently, notice that $\chi_c(A_c) = F_c(A)$ if and only if $\pi^c_\pm(A_c) = F_c(\pi_\pm(A))$. To verify the latter equality, we check that

$$\pi^c_\pm(A_c)e_k = \lambda_\pm q^{2k} e_k = F_c(\pi_\pm(A))e_k. \tag{2.26}$$

Similarly, to verify $\chi_c(B_c) = G_c(A)B$, we observe that

$$G_c(\pm q^{2k}) = \frac{c_\pm(k+1)^{1/2}}{(1 - q^{4(k+1)})^{1/2}}$$

and compute

$$\pi_\pm(G_c(A)B)e_k = G_c(\pi_\pm(A))\pi_\pm(B)e_k = \pi^c_\pm(B_c)e_k, \tag{2.27}$$

which proves the proposition. □

The above proposition shows that the isomorphisms χ_c are of non-polynomial nature. Therefore we suspect that:

CONJECTURE 2.3 *The polynomial* $*$*-algebras of the quantum spheres* S^2_{qc} *and* $S^2_{qc'}$ *are non-isomorphic for* $c \neq c'$.

Our next step is to consider the compatibility of the isomorphisms χ_c with the actions of U(1) inherited from the actions of $\mathrm{SU}_q(2)$ on quantum spheres. Let $c \in (0, \infty]$, and let

$$\delta := (\mathrm{id} \otimes \mathrm{p}) \circ \Delta_R\colon\ C(S^2_{qc}) \to C(S^2_{qc}) \otimes C(\mathrm{U}(1))$$

be the right coaction got from the coaction Δ_R: $C(S^2_{qc}) \to C(S^2_{qc}) \otimes C(\mathrm{SU}_q(2))$ [P-P87, p. 194] with the help of the C^*-epimorphism p: $C(\mathrm{SU}_q(2)) \to C(\mathrm{U}(1))$ sending α to u (the unitary generator of $C(S^1)$), δ to u^* and β, γ to 0. From (2.15), we have explicitly

$$\delta(A_c) = A_c \otimes 1, \qquad \delta(B_c) = B_c \otimes u^2. \tag{2.28}$$

Since δ: $C(S^2_{qc}) \to C(S^2_{qc}) \otimes C(\mathrm{U}(1))$ is a $*$-homomorphism, it is continuous. Therefore, identifying $C(S^2_{qc}) \otimes C(\mathrm{U}(1))$ with $C(\mathrm{U}(1), C(S^2_{qc}))$ (continuous functions on U(1) with values in $C(S^2_{qc})$; see [W-NE93, Proposition T.5.21]), we obtain, for any $g \in \mathrm{U}(1)$, a C^*-homomorphism

$$\delta_g\colon C(S^2_{qc}) \longrightarrow C(S^2_{qc}), \qquad \delta_g(a) := \delta(a)(g). \tag{2.29}$$

Furthermore, as δ is a coaction, each δ_g is a C^*-automorphism of $C(S^2_{qc})$. This defines an action of U(1) on S^2_{qc}. Contrary to the action of $\mathrm{SU}_q(2)$, the action of U(1) on S^2_{qc} is compatible with the quantum 'homeomorphisms' among the spheres S^2_{qc}, i.e., $(\chi^{-1}_{c'} \circ \chi_c) \circ \delta_g = \delta_g \circ (\chi^{-1}_{c'} \circ \chi_c)$. This follows from Proposition 2.4:

PROPOSITION 2.4. $\forall g \in \mathrm{U}(1)$: $\delta_g \circ \chi_c = \chi_c \circ \delta_g$.

Proof. Since both χ_c and δ_g are C^*-homomorphisms, it suffices to check this equality on the generators. It follows from (2.28) that $\delta_g(A_c) = A_c$ and $\delta_g(B_c) = g^2 B_c$. Taking advantage of Proposition 2.2, one can compute:

$$\begin{aligned}(\delta_g \circ \chi_c)(A_c) &= \delta_g(F_c(A)) = F_c(\delta_g(A)) = F_c(A) \\ &= \chi_c(A_c) = (\chi_c \circ \delta_g)(A_c), \end{aligned} \tag{2.30}$$
$$\begin{aligned}(\delta_g \circ \chi_c)(B_c) &= \delta_g(G_c(A)B) = G_c(\delta_g(A))\delta_g(B) \\ &= g^2 G_c(A)B = g^2 \chi_c(B_c) = (\chi_c \circ \delta_g)(B_c). \end{aligned} \tag{2.31}$$

This proves the proposition. □

For the sake of completeness (cf. (1.1), (3.27), (4.32)), let us end this section by recalling the topological K-theory of the quantum spheres. First, there is an exact sequence [S-A91, Proposition 1.2]:

$$0 \longrightarrow \mathcal{K} \oplus \mathcal{K} \longrightarrow C(S^2_{qc}) \longrightarrow C(S^1) \longrightarrow 0, \tag{2.32}$$

where $\mathcal{K}$ is the ideal of compact operators. It induces the 6-term exact sequence in K-theory, from which it follows that $K_0(C(S^2_{qc})) \cong \mathbb{Z} \oplus \mathbb{Z}$, $K_1(C(S^2_{qc})) \cong 0$ [MNW91, Proposition 4.1].

3. Quantum Disc

DEFINITION 3.1 ([KL93]). The C^*-algebra $C(D_q)$, $0 < q < 1$, of the quantum disc D_q is the C^*-closure (obtained from $*$-representations in bounded operators) of the algebra $P(D_q) := \mathbb{C}\langle x, x^*\rangle/J_q$. Here $\mathbb{C}\langle x, x^*\rangle$ is the unital free algebra generated by x and x^*, and J_q is the two-sided ideal in $\mathbb{C}\langle x, x^*\rangle$ generated by the relation

$$x^*x - qxx^* = 1 - q. \tag{3.1}$$

The goal of this section is to determine the relationship between the thus defined quantum discs and the equator and equilateral quantum spheres (cf. [NN94, p. 278] and references therein). The objects D_q form a one-parameter sub-family of the two-parameter family of quantum discs described in [KL93]. Explicitly, the latter family is given by

$$x^*x - qxx^* = 1 - q + \mu(xx^* - 1)(x^*x - 1).$$

It is known ([KL93, Proposition VI.1], [CM00, Proposition 15], [KL92, Theorem IV.7], [S-A91, p. 222]) that for

$$0 \leqslant \mu < 1 - q \quad \text{and} \quad q = 1, \quad 0 < \mu < 1,$$

the quantum-disc C^*-algebras are all isomorphic to the Toeplitz algebra (the C^*-algebra generated by the one-sided shift $\mathfrak{S}e_i = e_{i+1}$). Furthermore, we know from [KL93, Theorem IV.3] that every irreducible *bounded* $*$-representation of $P(D_q)$ is unitarily equivalent to a one-dimensional representation π^θ defined by

$$\pi^\theta(x) = \mathrm{e}^{i\theta}, \qquad \pi^\theta(x^*) = \mathrm{e}^{-i\theta}, \quad 0 \leqslant \theta < 2\pi, \tag{3.2}$$

or an infinite-dimensional representation π given on an orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ by the formulas

$$\pi(x)e_i = (1 - q^{i+1})^{1/2}\, e_{i+1}, \quad i \geqslant 0, \tag{3.3}$$

$$\pi(x^*)e_i = \begin{cases} 0, & i = 0, \\ (1 - q^i)^{1/2}\, e_{i-1}, & i \geqslant 1. \end{cases} \tag{3.4}$$

The infinite-dimensional representation π is faithful [KL93, p. 14]. Also, one can directly verify that π is faithful on the polynomial algebra $P(D_q)$, so that $P(D_q) \subseteq C(D_q)$ [CM00, p. 380]. Finally, let us mention that there are also unbounded representations of the relation (3.1). They are given, e.g., in [KS97, Section 5.2.6].

Now we are going to show that the above-defined q-disc can be obtained by collapsing the equator quantum sphere. In the classical case, the $\mathbb{Z}_2$-action on S^2 collapsing it to a disc is not free, as it leaves the equator invariant. This entails that the map

$$\psi\colon\ S^2 \times \mathbb{Z}_2 \ni (x, g) \longmapsto (x, xg) \in S^2 \times S^2 \tag{3.5}$$

is not injective, whence $S^2 \to D^2$ is not a principal fibration. (The considered $\mathbb{Z}_2$-action is not Galois.) The $\mathbb{Z}_2$-action on quantum spheres that we are looking for should identify 'points' of the same type and leave only the equator invariant. Recall that a classical point y is a homomorphism from the algebra to $\mathbb{C}$ and the action of -1 is an automorphism r_1 of the algebra. If y is not on the equator, then $y \circ r_1$ should be another classical point different from y. Therefore, since the standard sphere contains only one classical point (pole), we exclude it from our considerations. Our first step is to define the desired $\mathbb{Z}_2$-action on the polynomial algebra of the equator quantum sphere. As in the classical case, we define it as the reflection with respect to the equator plane, i.e., via the $*$-algebra automorphism r_1 of $P(S^2_{q\infty})$ sending B to itself, and A to $-A$. (It is immediate from the commutation relations of the equator quantum sphere that r_1 is well defined.) Dualizing the $\mathbb{Z}_2$-action r_1 on $S^2_{q\infty}$ we get the coaction $\Delta_1\colon\ P(S^2_{q\infty}) \to P(S^2_{q\infty}) \otimes P(\mathbb{Z}_2)$ making $P(S^2_{q\infty})$ a right $P(\mathbb{Z}_2)$-comodule algebra. (See [M-S93, Section 1.6] for generalities.) Explicitly, denoting by $\triangleright$ the action of $\mathbb{Z}_2$ on $P(S^2_{q\infty})$, we have

$$\begin{aligned} \Delta_1(p) &= (1 \triangleright p) \otimes 1^* + ((-1) \triangleright p) \otimes (-1)^* \\ &= \tfrac{1}{2}\big(p \otimes (1+\alpha) + r_1(p) \otimes (1-\alpha)\big). \end{aligned} \tag{3.6}$$

Here $\{1^*, (-1)^*\}$ denotes the basis dual to the basis $\{1, -1\}$ of the group ring $\mathbb{C}[\mathbb{Z}_2]$, and $\alpha(\pm 1) = \pm 1$. The main claim of this section is that the quantum disc is a non-Galois quotient of the equator quantum sphere:

PROPOSITION 3.2. *The polynomial algebra of the equator quantum sphere is a non-Galois $\mathbb{Z}_2$-extension of the polynomial algebra of the quantum disc via the above defined action r_1, i.e.,*

(1) $P(D^2_{q^4}) \cong P(S^2_{q\infty}/\mathbb{Z}_2) := \{a \in P(S^2_{q\infty}) \mid r_1(a) = a\}$ *($\mathbb{Z}_2$-extension).*

(2) *The canonical map* $P(S^2_{q\infty}) \otimes_{P(D^2_{q^4})} P(S^2_{q\infty}) \ni p \otimes_{P(D^2_{q^4})} p' \mapsto p\Delta_1(p') \in P(S^2_{q\infty}) \otimes P(\mathbb{Z}_2)$ *is* not *bijective. (The extension is not Galois.)*

Proof. (1) We know from [P-P89, p. 116] that the monomials

$$A^k B^l,\ A^m B^{*n}, \quad k, l, m, n \in \mathbb{N}, n > 0, \tag{3.7}$$

form a linear basis of $P(S^2_{q\infty}) \subseteq C(S^2_{q\infty})$. Since $r_1(A) = -A$ and $r_1(B) = B$, taking advantage of the above basis, one can see that $r_1(a) = a$ if and only if a is a linear combination of basis monomials that have A in the even power. It follows now from (2.5) that any r_1-invariant a is a polynomial in B and B^*. Thus, since every polynomial in B and B^* is r_1-invariant, $P(S^2_{q\infty}/\mathbb{Z}_2)$ is the $*$-subalgebra generated by B. On the other hand, one can conclude from (2.5) that

$$BB^* - q^4 B^* B = 1 - q^4. \tag{3.8}$$

This equation, together with (3.1), allows us to define a $*$-epimorphism

$$\varphi\colon\ P(D_{q^4}) \longrightarrow P(S^2_{q\infty}/\mathbb{Z}_2), \quad \varphi(x) = B^*. \tag{3.9}$$

To complete the proof, we need to show that φ is injective. It is immediate from formula (2.8) that $\pi_\pm \circ \varphi = \pi$, where π is defined by (3.3), (3.4). Hence the injectivity of π implies the injectivity of φ.

(2) It suffices to show that the map

$$\kappa\colon\ P(S^2_{q\infty}) \otimes P(S^2_{q\infty}) \ni p \otimes p' \mapsto p\Delta_1(p') \in P(S^2_{q\infty}) \otimes P(\mathbb{Z}_2) \tag{3.10}$$

is not surjective. (The considered $\mathbb{Z}_2$-action on $S^2_{q\infty}$ is not free.) Note first that in the classical case to detect the lack of surjectivity of the pullback map ψ^* (see (3.5)), we can use the function $1 \otimes \alpha$. Indeed, for any point x on the equator, we have

$$\psi^*(\text{anything})(x, 1) = (\text{anything})(x, x) = \psi^*(\text{anything})(x, -1), \tag{3.11}$$

whereas $(1 \otimes \alpha)(x, 1) = 1 \neq -1 = (1 \otimes \alpha)(x, -1)$. It turns out that $1 \otimes \alpha$ also does the job in the quantum case. Suppose that $1 \otimes \alpha$ is in the image of κ. Then there exists a tensor $\sum_i p_i \otimes p'_i$ such that

$$\frac{1}{2} \sum_i p_i \big(p'_i \otimes (1 + \alpha) + r_1(p'_i) \otimes (1 - \alpha)\big) = 1 \otimes \alpha. \tag{3.12}$$

Evaluating the right tensorands on both sides at 1 and -1 yields

$$\sum_i p_i p'_i = 1, \qquad \sum_i p_i r_1(p'_i) = -1. \tag{3.13}$$

Applying π_θ (see (2.7)) to these equations gives

$$\begin{aligned} &\sum_i \pi_\theta(p_i)\pi_\theta(p'_i) = 1, \\ &\sum_i \pi_\theta(p_i)\pi_\theta(r_1(p'_i)) = \sum_i \pi_\theta(p_i)\pi_\theta(p'_i) = -1, \end{aligned} \tag{3.14}$$

which is the desired contradiction. □

In order to define a $\mathbb{Z}_2$-action on the closure $C(S^2_{q\infty})$ of $P(S^2_{q\infty})$, note that the flip map

$$\tau\colon\ C(\mathfrak{S}) \oplus_\sigma C(\mathfrak{S}) \to C(\mathfrak{S}) \oplus_\sigma C(\mathfrak{S}), \quad \tau(a, b) = (b, a), \tag{3.15}$$

satisfies $\tau \circ (\pi_+ \oplus \pi_-) = (\pi_+ \oplus \pi_-) \circ r_1$. It is therefore natural to define the completion of r_1 to a C^*-algebra map by*

$$\bar{r}_1 := (\pi_+ \oplus \pi_-)^{-1} \circ \tau \circ (\pi_+ \oplus \pi_-). \tag{3.16}$$

* We owe this idea to S. L. Woronowicz.

PROPOSITION 3.3. *The C^*-subalgebra $C(S^2_{q\infty}/\mathbb{Z}_2) := \{a \in C(S^2_{q\infty}) \mid \bar{r}_1(a) = a\}$ of $\mathbb{Z}_2$-invariants in $C(S^2_{q\infty})$ coincides with the C^*-completion of $P(S^2_{q\infty}/\mathbb{Z}_2)$ inside $C(S^2_{q\infty})$, and is isomorphic to $C(D_{q^4})$.*

Proof. First let us argue that the map φ defined in the proof of Proposition 3.2 extends to a C^*-isomorphism of $C(D_{q^4})$ with the closure of $P(S^2_{q\infty}/\mathbb{Z}_2)$ inside $C(S^2_{q\infty})$. Since φ: $P(D^2_{q^4}) \to P(S^2_{q\infty}/\mathbb{Z}_2)$ is an isomorphism of $*$-algebras, the $*$-representations of $P(D^2_{q^4})$ can be turned to $*$-representations of $P(S^2_{q\infty}/\mathbb{Z}_2)$, and vice-versa. We have to show that φ determines a one-to-one correspondence between the $*$-representations used to define the norm on $C(D^2_{q^4})$ and $C(S^2_{q\infty})$ respectively. As every $*$-representation of $P(S^2_{q\infty})$ is bounded, it yields via φ a *bounded* $*$-representation of $P(D^2_{q^4})$. On the other hand, every bounded $*$-representation of $P(D^2_{q^4})$ gives a $*$-representation of $P(S^2_{q\infty}/\mathbb{Z}_2)$ which can be extended to $C(S^2_{q\infty})$. Indeed, let $P(D^2_{q^4}) \stackrel{\rho}{\to} B(H)$ be such a representation. Then $\widetilde{\rho} := \rho \circ \varphi^{-1}$ is a bounded $*$-representation of $P(S^2_{q\infty}/\mathbb{Z}_2)$, and since B and B^* satisfy the disc relation (3.8), it follows from [KL93, Proposition IV.1(I)] that $||\widetilde{\rho}(B^*B)|| = 1$. As a consequence, $1 - \widetilde{\rho}(B^*B) \geqslant 0$, and one can define $\widetilde{\rho}(A) = \sqrt{1 - \widetilde{\rho}(B^*B)}$. This gives the desired extension.

To complete the proof, note first that, since $\bar{r}_1$ is continuous, the C^*-closure of $P(S^2_{q\infty}/\mathbb{Z}_2)$ is contained in $C(S^2_{q\infty}/\mathbb{Z}_2)$. Thus it only remains to show that every $\mathbb{Z}_2$-invariant in $C(S^2_{q\infty})$ is in the closure of $P(S^2_{q\infty}/\mathbb{Z}_2)$. Let $a \in C(S^2_{q\infty}/\mathbb{Z}_2)$. Then, again by the continuity of $\bar{r}_1$ and density of $P(S^2_{q\infty})$ in $C(S^2_{q\infty})$, $a = \lim_{n\to\infty} a_n$ with $a_n \in P(S^2_{q\infty})$ and

$$a = \tfrac{1}{2}(\mathrm{id} + \bar{r}_1)(a) = \tfrac{1}{2}(\mathrm{id} + \bar{r}_1)\Big(\lim_{n\to\infty} a_n\Big) = \lim_{n\to\infty} \tfrac{1}{2}(\mathrm{id} + r_1)(a_n). \tag{3.17}$$

As $\frac{1}{2}(\mathrm{id} + r_1)(a_n) \in P(S^2_{q\infty}/\mathbb{Z}_2)$ for any n, a is in the closure of $P(S^2_{q\infty}/\mathbb{Z}_2)$, as claimed. □

Remark 3.4. Since all quantum disc algebras $C(D_q)$, $0 < q < 1$, are isomorphic as C^*-algebras to the C^*-algebra of the one-sided shift, in the above proposition q^4 can be replaced by any element of the interval $(0, 1)$.

We extend the $\mathbb{Z}_2$-action to the equilateral quantum spheres by the formula

$$\bar{r}_1^c := \chi_c^{-1} \circ \bar{r}_1 \circ \chi_c. \tag{3.18}$$

It is now evident that we have the following corollary:

COROLLARY 3.5. *The subalgebra $C(S^2_{qc}/\mathbb{Z}_2) := \{a \in C(S^2_{qc}) \mid \bar{r}_1^c(a) = a\}$ of $\mathbb{Z}_2$-invariants of $C(S^2_{qc})$, $c \in (0, \infty]$, is isomorphic to the C^*-algebra $C(D_{q^4})$ of the quantum disc.*

Furthermore, it is clear from (2.23), (3.16) and (3.18) that

$$\bar{r}_1^c = (\pi_+^c \oplus \pi_-^c)^{-1} \circ \tau \circ (\pi_+^c \oplus \pi_-^c). \tag{3.19}$$

Explicitly, the above equality reads*

$$\pi_+^c(a) = \pi_-^c(\bar{r}_1^c(a)), \qquad \pi_-^c(a) = \pi_+^c(\bar{r}_1^c(a)), \quad a \in C(S_{qc}^2). \tag{3.20}$$

For $a = A_c$, these equations are solved by the formulas

$$\bar{r}_1^c(A_c) = f_c(A_c), \qquad f_c(x) = \begin{cases} \frac{\lambda_-}{\lambda_+}x, & x \geqslant 0, \\ \frac{\lambda_+}{\lambda_-}x, & x \leqslant 0. \end{cases} \tag{3.21}$$

Note that this piecewise linear function f_c can be replaced by any continuous function having the same values as f_c at the points $\lambda_\pm q^{2k}$. For $c \in (0, \infty)$, among these functions there is no polynomial, as they cannot be differentiable at 0. This shows that $\bar{r}_1^c$ does *not* leave $P(S_{qc}^2)$ invariant. Furthermore, considering the image of B_c^* under $\pi_+^c \oplus \pi_-^c$, one finds the polar decomposition $B_c^* = V_c|B_c^*|$ with $V_c = (\pi_+^c \oplus \pi_-^c)^{-1}(\sigma, \sigma)$:

$$\pi_\pm^c(V_c)e_k = e_{k+1}, \qquad \pi_\pm^c(|B_c^*|)e_k = c_\pm(k+1)^{1/2}e_k, \quad k \in \mathbb{N}. \tag{3.22}$$

It follows from (3.19) that $\bar{r}_1^c(V_c) = V_c$. Recall that the spectrum of $(\pi_+^c \oplus \pi_-^c)(A_c)$ and $(\pi_+^c \oplus \pi_-^c)(|B_c^*|)$ is

$$\{0\} \cup \{\lambda_\pm q^{2k} \mid k \in \mathbb{N}\} \quad \text{and} \quad \{\sqrt{c}\} \cup \{\sqrt{c_\pm(k+1)} \mid k \in \mathbb{N}\},$$

respectively. On the other hand,

$$\pi_\pm^c(\bar{r}_1^c(|B_c^*|))e_k = c_\mp(k+1)^{1/2}e_k, \quad k \in \mathbb{N}.$$

One can directly check that

$$\bar{r}_1^c(|B_c^*|) = g_c(A_c),$$

$$g_c(t) := \begin{cases} \eta_c\left(\frac{\lambda_-}{\lambda_+}q^2t\right), & 0 \leqslant t \leqslant \lambda_+, \\ \eta_c\left(\frac{\lambda_+}{\lambda_-}q^2t\right), & \lambda_- \leqslant t < 0, \end{cases} \qquad \eta_c(t) = \sqrt{t - t^2 + c}. \tag{3.23}$$

Clearly, g_c can be replaced by any continuous function having at the points $\lambda_\pm q^{2k}$ values $c_\mp(k+1)^{1/2}$, for any k. Note that we used A_c instead of $|B_c^*|$ to obtain $\bar{r}_1^c(|B_c^*|)$ as a continuous function of a generator because the assignment $\sqrt{c_+(k+1)} \mapsto \sqrt{c_-(k+1)}$ does not give a function, as $k_1 \neq k_2$ implies $c_-(k_1) \neq c_-(k_2)$, whereas it might happen that $c_+(k_1) = c_+(k_2)$ for $k_1 \neq k_2$. Indeed, let k_1, k_2 be any two different positive natural numbers. Then the equation $\sqrt{c_+(k_1)} = \sqrt{c_+(k_2)}$ is equivalent to the equation $q^{2k_1} + q^{2k_2} = \lambda_+^{-1}$. Since $\lambda_+^{-1} \in (0, 1)$, there exists $q \in (0, 1)$ solving this equality.

* We are grateful to S. L. Woronowicz for suggesting the reasoning below.

Remark 3.6. The formulas for f_c, F_c, g_c, G_c are consistent with one another by construction. Nevertheless, it is entertaining to verify this consistency in a direct manner. Taking into account Proposition 2.2 and formula (3.21), we obtain a sequence of equivalent equalities

$$\begin{aligned}
&\bar{r}_1^c(A_c) = (\chi_c^{-1} \circ \bar{r}_1 \circ \chi_c)(A_c),\\
&f_c(A_c) = (\chi_c^{-1} \circ \bar{r}_1)(F_c(A)),\\
&f_c(A_c) = \chi_c^{-1}(F_c(-A)),\\
&\pi_\pm^c(f_c(A_c)) = \pi_\pm(F_c(-A)),\\
&f_c(\pi_\pm^c(A_c)) = F_c(-\pi_\pm(A)),\\
&f_c(\pi_\pm^c(A_c))e_k = F_c(\pi_\mp(A))e_k.
\end{aligned}$$

Recalling (2.8) and (2.10) one can see that the last equality is true. Similarly, taking advantage of the polar decomposition $B_c^* = V_c|B_c^*|$, $\bar{r}_1^c(V_c) = V_c$ and (3.23), we get

$$\begin{aligned}
&\bar{r}_1^c(B_c^*) = (\chi_c^{-1} \circ \bar{r}_1 \circ \chi_c)(B_c^*),\\
&\bar{r}_1^c(V_c|B_c^*|) = (\chi_c^{-1} \circ \bar{r}_1)(\chi_c(B_c^*)),\\
&\bar{r}_1^c(V_c)\bar{r}_1^c(|B_c^*|) = (\chi_c^{-1} \circ \bar{r}_1)(B^* G_c(A)),\\
&V_c g_c(A_c) = \chi_c^{-1}(B^* G_c(-A)),\\
&\pi_\pm^c(V_c g_c(A_c)) = \pi_\pm(B^* G_c(-A)),\\
&\pi_\pm^c(V_c) g_c(\pi_\pm^c(A_c))e_k = \pi_\pm(B^*) G_c(\pi_\mp(A))e_k.
\end{aligned}$$

Remembering formulas (2.8), (2.10), (3.22), (3.23), (2.25), the last equality is evident.

Next, let us consider the rotational invariance with respect to the South–North Pole axis of the above-studied $\mathbb{Z}_2$-actions on quantum spheres. The U(1)-action on $S^2_{q\infty}$ (see (2.29)) is given on generators by $\delta_g(A) = A$, $\delta_g(B) = g^2 B$. Therefore, one can infer from Proposition 2.4 that the U(1)-action on S^2_{qc} ($c \in (0, \infty]$) and the reflection $\bar{r}_1^c$ are compatible:

$$\delta_g \circ \bar{r}_1^c = \bar{r}_1^c \circ \delta_g. \tag{3.24}$$

Remark 3.7. It follows already from (3.24) that $\delta \circ \bar{r}_1^c = (\bar{r}_1^c \otimes \mathrm{id}) \circ \delta$. Let us, however, provide also a direct proof. Since A_c and B_c^* generate $C(S^2_{qc})$ in the C^*-algebraic sense, and both $\bar{r}_1^c$ and δ are continuous, it suffices to check the desired equality on A_c and B_c^*. Taking advantage of (3.21) and using the fact that δ is a C^*-homomorphism, we obtain

$$\begin{aligned}
(\delta \circ \bar{r}_1^c)(A_c) &= \delta(f_c(A)) = f_c(\delta(A_c)) = f_c(A_c) \otimes 1\\
&= \bar{r}_1^c(A_c) \otimes 1 = ((\bar{r}_1^c \otimes \mathrm{id}) \circ \delta)(A_c).
\end{aligned} \tag{3.25}$$

To handle B_c^* it is useful to consider its polar decomposition $B_c^* = V_c|B_c^*|$ (see (3.22)). Now, $\delta(B_c^*) = B_c^* \otimes u^{*2}$ entails

$$\delta(B_c B_c^*) = B_c B_c^* \otimes u^2 u^{*2} = |B_c^*|^2 \otimes 1,$$

whence, by the continuity of the square root function, $\delta(|B_c^*|) = |B_c^*| \otimes 1$. Consequently,

$$\delta(V_c)(|B_c^*| \otimes 1) = \delta(B_c^*) = B_c^* \otimes u^{*2} = (V \otimes u^{*2})(|B_c^*| \otimes 1). \tag{3.26}$$

Thus, due to the invertibility of $|B_c^*| \otimes 1$, we have $\delta(V_c) = V_c \otimes u^{*2}$. (We chose B_c^* rather than B_c because, unlike $|B_c|$, $|B_c^*|$ is invertible.) On the other hand, $\bar{r}_1^c(V_c) = V_c$ (see above) and $\bar{r}_1^c(|B_c^*|) = g_c(A_c)$ (see (3.23)). To complete the proof, one can reason in the same way as for the generator A_c.

We end this section by recalling K-facts for the quantum disc. Since $C(D_q^2)$ is isomorphic to the Toeplitz algebra, the 'standard' exact sequence [B-B86, p. 78] (cf. (1.2), (2.32), (4.32)) is equivalent to:

$$0 \longrightarrow \mathcal{K} \longrightarrow C(D_q^2) \longrightarrow C(S^1) \longrightarrow 0, \tag{3.27}$$

from which it follows that

$$K_0(C(D_q^2)) \cong \mathbb{Z}, \qquad K_1(C(D_q^2)) \cong 0 \quad \text{([W-NE93, p. 123])}. \tag{3.28}$$

4. Quantum Real Projective Space

Our first aim is to define on the equator quantum sphere $S_{q\infty}^2$ a $\mathbb{Z}_2$-action mimicking the antipodal action of $\mathbb{Z}_2$ on S^2. The geometrical meaning of generators (see 2.20) hints at the formulas $r_2(A) = -A$, $r_2(B) = -B$. Owing to the even nature of algebraic relations in $P(S_{q\infty}^2)$, these equalities indeed define the desired action on $P(S_{q\infty}^2)$. (Note that this recipe would not work for $P(S_{qc}^2)$, $c \in [0, \infty)$.) The $*$-algebra of quantum real projective 2-space can now be defined by

DEFINITION 4.1 ([H-PM96]). $P(\mathbb{R}P_q^2) = \{a \in P(S_{q\infty}^2) \mid r_2(a) = a\}$.

Remark 4.2. Recall that $\Delta_R \circ r_2 = (r_2 \otimes \mathrm{id}) \circ \Delta_R$ (see above Section 6 in [P-P87]), where Δ_R is the restriction to $P(S_{q\infty}^2)$ of the coproduct Δ in $P(\mathrm{SU}_q(2))$. (Since both r_2 and Δ_R are algebra homomorphisms, it suffices to check this formula on generators, where it is true due to (2.15) and $\Delta D_{1ij} = \sum_{k=1}^{3} D_{1ik} \otimes D_{1kj}$.) Thus the antipodal action and the $\mathrm{SU}_q(2)$-action on the equator quantum sphere are compatible. Consequently, just as quantum spheres themselves, $\mathbb{R}P_q^2$ is an (embeddable) quantum homogeneous space of $\mathrm{SU}_q(2)$, i.e., $\Delta(P(\mathbb{R}P_q^2)) \subseteq P(\mathbb{R}P_q^2) \otimes P(\mathrm{SU}_q(2))$.

Unlike the quantum disc, $\mathbb{R}P_q^2$ is a $\mathbb{Z}_2$-Galois quotient of the equator quantum sphere, i.e., $S_{q\infty}^2 \to \mathbb{R}P_q^2$ is an (algebraic) quantum principal bundle [H-PM96, Proposition 2.10]. As mentioned in the proof of Proposition 3.2, the elements A^kB^l, A^mB^{*n}, $n > 0$, form a basis of $P(S_{q\infty}^2)$. Taking this into account, it is straightforward that $P(\mathbb{R}P_q^2)$ is the $*$-subalgebra of $P(S_{q\infty}^2)$ generated by A^2, B^2 and AB. We put

$$P = A^2, \qquad R = B^2, \qquad T = AB, \tag{4.1}$$

and find immediately the following relations:

$$P = P^*,\ T^2 = q^2PR,\ RT^* = q^2T(-q^4P+1),\ R^*T = q^{-2}T^*(-P+1), \tag{4.2}$$

$$RR^* = q^{12}P^2 - q^4(1+q^4)P + 1,\ R^*R = q^{-4}P^2 - (1+q^{-4})P + 1, \tag{4.3}$$

$$TT^* = -q^4P^2 + P,\ T^*T = q^{-4}(P - P^2), \tag{4.4}$$

$$RP = q^8PR,\ RT = q^4TR,\ PT = q^{-4}TP. \tag{4.5}$$

PROPOSITION 4.3. *Let I_q be the $*$-ideal in the free $*$-algebra $\mathbb{C}\langle P, R, T\rangle$ generated by the relations* (4.2)–(4.5). *Then the $*$-algebra $\mathbb{C}\langle P, R, T\rangle/I_q$ is isomorphic to $P(\mathbb{R}P_q^2)$.*

Proof. There exists a $*$-algebra epimorphism $f\colon \mathbb{C}\langle P, R, T\rangle/I_q \to P(\mathbb{R}P_q^2)$ given on generators by $f(P) = A^2$, $f(R) = B^2$, $f(T) = AB$. On the other hand, we can define a linear map $g\colon P(\mathbb{R}P_q^2) \to \mathbb{C}\langle P, R, T\rangle/I_q$ by its values on the elements of a basis of $P(\mathbb{R}P_q^2)$: $g(A^{2k}B^{2l}) = P^kR^l$, $g(A^{2k+1}B^{2l+1}) = P^kTR^l$, $g(A^{2m}B^{*2(n+1)}) = P^mR^{*n+1}$, $g(A^{2m+1}B^{*2n+1}) = q^2P^mT^*R^{*n}$. Evidently, $f \circ g = \mathrm{id}$. Consequently g is injective and the above elements of $\mathbb{C}\langle P, R, T\rangle/I_q$ are linearly independent. To obtain the reverse equality $g \circ f = \mathrm{id}$, it suffices to show that these elements span $\mathbb{C}\langle P, R, T\rangle/I_q$. Assume inductively that every monomial in P, R, T, R^*, T^* of length at most n is in the span. This is clearly true for $n = 1$. Take now an arbitrary monomial M_{n+1} of length $n+1$. It can always be written as M_nW, where M_n is a monomial of length n and W is one of the elements P, R, T, R^*, T^*. By assumption, M_n is a linear combination of P^kR^l, P^kTR^l, P^mR^{*l+1}, $P^mT^*R^{*l}$. Using the commutation relations (4.2)–(4.5) among generators, it can be directly verified that each of the monomials M_nW is again in the span. □

In order to extend the antipodal $\mathbb{Z}_2$-action to $C(S_{q\infty}^2)$, note first that (2.29) entails

$$r_2(A) = (\bar{r}_1 \circ \delta_{\sqrt{-1}})(A), \qquad r_2(B) = (\bar{r}_1 \circ \delta_{\sqrt{-1}})(B). \tag{4.6}$$

Therefore, we can define the completion of r_2 by

$$\bar{r}_2 := \bar{r}_1 \circ \delta_{\sqrt{-1}}\colon C(S_{q\infty}^2) \longrightarrow C(S_{q\infty}^2). \tag{4.7}$$

Observe that we need to put $g = \sqrt{-1}$ rather than $g = -1$ because this U(1)-action comes from SU(2) which is the double-cover of SO(3). Therefore, to rotate the quantum sphere by the angle π (antipodal action is such a rotation composed with reflection), we take $g = \mathrm{e}^{i\pi/2}$ rather than $g = \mathrm{e}^{i\pi}$. Since both $\bar{r}_1$ and $\delta_{\sqrt{-1}}$ are C^*-homomorphisms, we can define the C^*-algebra of $\mathbb{R}P_q^2$ as

DEFINITION 4.4. $C(\mathbb{R}P_q^2) := \{a \in C(S_{q\infty}^2) \mid \bar{r}_2(a) = a\}$.

Arguing as in Proposition 3.3 (second part of the proof), we get that the completion of $P(\mathbb{R}P_q^2)$ with respect to the norm on $C(S_{q\infty}^2)$ coincides with the thus defined $C(\mathbb{R}P_q^2)$. To study the structure of this C^*-algebra, let us prove:

THEOREM 4.5. *There are no unbounded $*$-representations of the $*$-algebra $P(\mathbb{R}P_q^2)$. Up to the unitary equivalence, all irreducible (bounded) $*$-representations of this algebra are the following:*

(i) *A family of one-dimensional representations $\rho_\theta\colon P(\mathbb{R}P_q^2) \to \mathbb{C}$ parameterized by $\theta \in [0, 2\pi)$, which are given by*

$$\rho_\theta(P) = \rho_\theta(T) = 0, \qquad \rho_\theta(R) = \mathrm{e}^{i\theta}. \tag{4.8}$$

(ii) *An infinite-dimensional representation ρ (in a Hilbert space H with an orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$) given by*

$$\rho(P)e_k = q^{4k}e_k, \tag{4.9}$$

$$\rho(T)e_k = \begin{cases} 0, & k = 0, \\ q^{2(k-1)}(1 - q^{4k})^{1/2}e_{k-1}, & k > 1, \end{cases} \tag{4.10}$$

$$\rho(T^*)e_k = q^{2k}(1 - q^{4(k+1)})^{1/2}e_{k+1}, \quad k \geqslant 0, \tag{4.11}$$

$$\rho(R)e_k = \begin{cases} 0, & k = 0, 1, \\ (1 - q^{4k})^{1/2}(1 - q^{4(k-1)})^{1/2}e_{k-2}, & k > 1, \end{cases} \tag{4.12}$$

$$\rho(R^*)e_k = (1 - q^{4(k+1)})^{1/2}(1 - q^{4(k+2)})^{1/2}e_{k+2}, \quad k \geqslant 0. \tag{4.13}$$

Proof. Suppose that ρ is an unbounded $*$-representation ([S-K90, Definition 8.1.9]) of $P(\mathbb{R}P_q^2)$. The relations $T^*T = q^{-4}(P - P^2)$, $P = P^*$ and the formula $\mathrm{Sp}(f(X)) = f(\mathrm{Sp}(X))$ for a polynomial function f and the spectrum of a closed (not necessarily bounded) operator X whose spectrum is a proper subset of $\mathbb{C}$ [DS88, p. 604], e.g., for the self-adjoint operator $\rho(P)$, entail that $0 \leqslant \rho(P) \leqslant 1$, so that $\rho(P)$ is bounded. It follows then from the relations (4.3) and (4.4) that also $\rho(R)$ and $\rho(T)$ are bounded. Therefore, unlike $P(D_{q^4}^2)$, the $*$-algebra $P(\mathbb{R}P_q^2)$ has no unbounded representations.

Now, let ρ be an irreducible bounded $*$-representation in a Hilbert space H. As before, we have $0 \leqslant \rho(P) \leqslant 1$. Let $\rho(P) = 0$. Then $T^*T = q^{-4}(P - P^2)$ implies that also $\rho(T) = 0$. Hence, the only remaining relation is $\rho(R^*R) = 1 = \rho(RR^*)$, and we can see that the image of ρ is commutative. Since the only irreducible representations of a commutative algebra are one-dimensional, we arrive at (i).

Let us now assume $\rho(P) \neq 0$. It is immediate from $RP = q^8 PR$ and $PT = q^{-4}TP$ that $\operatorname{Ker}\rho(P)$ is ρ-invariant. Due to the irreducibility and the boundedness of ρ, either $\operatorname{Ker}\rho(P) = H$ or $\operatorname{Ker}\rho(P) = 0$. Since the first case is excluded by the assumption $\rho(P) \neq 0$, we have $\operatorname{Ker}\rho(P) = 0$. Using the characterization of elements of the spectrum by approximate eigenvectors and taking advantage of the relation $PT = q^{-4}TP$ we will show that the spectrum of $\rho(P)$ consists of the eigenvalues q^{4k}, $k \in \mathbb{N}$, and their limiting point 0. We already know that the spectrum of $\rho(P)$ lies in the interval [0, 1]. Next, note that 0 cannot be an isolated element of the spectrum (eigenvalue) because this would contradict $\operatorname{Ker}(\rho(P)) = 0$. If 1 would be the only element of the spectrum, we would have $\rho(P) = 1$, and consequently, due to $PT = q^{-4}TP$, $\rho(T)$ would vanish. This would contradict $\rho(TT^*) = 1 - q^4$ resulting from the relation $TT^* = -q^4P^2 + P$. Thus 1 cannot be the only element in the spectrum. Summing up, we have shown that there exists $\lambda \in \operatorname{Sp}(\rho(P)) \cap (0, 1)$.

By [KR97, Lemma 3.2.13], there exists a sequence $(\xi_n)_{n\in\mathbb{N}}$ of unit vectors in the representation space H such that

$$\lim_{n\to\infty} \|\rho(P)\xi_n - \lambda\xi_n\| = 0. \tag{4.14}$$

We will now show that there exist $N \in \mathbb{N}$ and $C > 0$ such that $\|\rho(T)\xi_n\| \geqslant C$ for $n \geqslant N$. To estimate $\|\rho(T)\xi_n\|$, we use $T^*T = q^{-4}(P - P^2)$. Now, the right-hand side of this equality we want to put in a form allowing us to apply (4.14). Adding and subtracting $\lambda^2 - \lambda$ gives:

$$\begin{aligned}(P - P^2) &= (P - \lambda) + (\lambda^2 - P^2) - (\lambda^2 - \lambda)\\ &= (1 - \lambda - P)(P - \lambda) + \lambda(1 - \lambda).\end{aligned} \tag{4.15}$$

Therefore, using the triangle inequality and $\|a\|\|\eta\| \geqslant \|a\eta\|$, we obtain

$$\begin{aligned}&\|(\rho(P) - \rho(P^2))\xi_n\|\\ &\quad\geqslant |\lambda(1 - \lambda)| - \|1 - \lambda - \rho(P)\|\|(\rho(P) - \lambda))\xi_n\|.\end{aligned} \tag{4.16}$$

On the other hand,

$$\|\rho(T^*)\|\|\rho(T)\xi_n\| \geqslant \|\rho(T^*T)\xi_n\| = q^{-4}\|(\rho(P) - \rho(P^2))\xi_n\|. \tag{4.17}$$

Combining (4.16) with (4.17) and remembering that $\|\rho(T)^*\| = \|\rho(T)\| \neq 0$ ($\rho(T) = 0$ would contradict (4.4) and $\rho(P) \neq 0$), we get

$$\|\rho(T)\xi_n\| \geqslant \frac{|\lambda(1 - \lambda)|}{q^4\|\rho(T^*)\|} - \|(\rho(P) - \lambda)\xi_n\|\frac{\|1 - \lambda - \rho(P)\|}{q^4\|\rho(T^*)\|}. \tag{4.18}$$

Since $(|\lambda(1 - \lambda)|)/(q^4\|\rho(T^*)\|)$ is positive, the existence of the desired N and C follows from (4.14). Hence we conclude that

$$\eta_n := \frac{\rho(T)\xi_n}{\|\rho(T)\xi_n\|} \tag{4.19}$$

are well-defined unit vectors for $n \geqslant N$. Our goal now is to show

$$\lim_{n\to\infty} \|\rho(P)\eta_n - q^{-4}\lambda\eta_n\| = 0, \tag{4.20}$$

which is tantamount to $q^{-4}\lambda \in \mathrm{Sp}(\rho(P))$. Assume $n \geqslant N$. Then $\|\rho(T)\xi_n\| \geqslant C$. Using $PT = q^{-4}TP$, we have

$$\begin{aligned} \|\rho(P)\eta_n - q^{-4}\lambda\eta_n\| &= \frac{\|\rho(T)\rho(P)\xi_n - \lambda\rho(T)\xi_n\|}{q^4\|\rho(T)\xi_n\|} \\ &\leqslant \frac{\|\rho(T)\|}{q^4 C}\|\rho(P)\xi_n - \lambda\xi_n\|. \end{aligned} \tag{4.21}$$

Consequently, (4.20) follows from (4.14). Thus we have shown that

$$\lambda \in \mathrm{Sp}(\rho(P)) \cap (0,1) \Rightarrow q^{-4}\lambda \in \mathrm{Sp}(\rho(P)). \tag{4.22}$$

Therefore, there exists k such that $q^{-4k}\lambda = 1$, for otherwise we would get an unbounded sequence $\lambda, q^{-4}\lambda, q^{-8}\lambda, \ldots, q^{-4k}\lambda, \ldots \in \mathrm{Sp}(\rho(P))$ contradicting $\mathrm{Sp}(\rho(P)) \subseteq [0,1]$. Hence $\mathrm{Sp}(\rho(P)) \subseteq \{q^{4k} \mid k \in \mathbb{N}\} \cup \{0\}$. It also follows that $1 \in \mathrm{Sp}(\rho(P))$. Thus 1 is isolated in $\mathrm{Sp}(\rho(P))$, so that it is an eigenvalue, and there exists a vector ξ such that $\rho(P)\xi = \xi$, $\|\xi\| = 1$. (Notice that now we evidently have $\|\rho(P)\| = 1$.)

It follows from the relation $T^*P = q^{-4}PT^*$ that $\rho(T^{*k})\xi$ are eigenvectors of $\rho(P)$ corresponding to the eigenvalue q^{4k}. Let us prove inductively that all these eigenvectors are different from zero. For $n = 0$, the statement $\rho(T^{*n})\xi \neq 0$ is automatically true. Assume now that $\rho(T^{*n})\xi \neq 0$ for some $n \in \mathbb{N}$. Then, using $TT^* = -q^4P^2 + P$ and $T^*P = q^{-4}PT^*$, one obtains

$$TT^{*n+1} = T^{*n}(q^{4n}P - q^{8n+4}P^2). \tag{4.23}$$

Therefore, $\rho(T)\rho(T^{*n+1})\xi = (q^{4n} - q^{8n+4})\rho(T^{*n})\xi \neq 0$ and consequently $\rho(T^{*n+1})\xi \neq 0$. Hence, by induction, $\rho(T^{*n})\xi \neq 0$, $\forall n \in \mathbb{N}$. This proves that $\mathrm{Sp}(\rho(P)) = \{q^{4k} \mid k \in \mathbb{N}\} \cup \{0\}$. Since $\rho(P)$ is self-adjoint and $\rho(T^{*k})\xi$ are eigenvectors of different eigenvalues of $\rho(P)$, they are mutually orthogonal. Thus, the vectors

$$e_k := \frac{\rho(T^{*k})\xi}{\|\rho(T^{*k})\xi\|}, \quad k \in \mathbb{N}, \tag{4.24}$$

form an orthonormal system. Let us now show that the span of the e_k's is closed under the action of the entire algebra. We already know that

$$\rho(P)e_k = q^{4k}e_k. \tag{4.25}$$

On the other hand, the formula (4.23) entails

$$\begin{aligned} \|\rho(T^{*k+1})\xi\| &= \langle\rho(T)\rho(T^{*k+1})\xi, \rho(T^{*k})\xi\rangle^{1/2} \\ &= \langle\rho(T^{*k})(q^{4k} - q^{8k+4})\xi, \rho(T^{*k})\xi\rangle^{1/2} \\ &= q^{2k}(1 - q^{4(k+1)})^{1/2}\|\rho(T^{*k})\xi\|. \end{aligned} \tag{4.26}$$

Hence, from the definition (4.24) we have

$$\rho(T^*)e_k = q^{2k}(1-q^{4(k+1)})^{1/2}e_{k+1}. \tag{4.27}$$

The relation $T^*T = q^{-4}(P-P^2)$ implies that $\rho(T)e_0$ has zero length:

$$\|\rho(T)e_0\|^2 = \langle e_0, \rho(T^*)\rho(T)e_0\rangle = 0. \tag{4.28}$$

Thus $\rho(T)e_0 = 0$. Similarly, $R^*R = q^{-4}P^2 - (1+q^{-4})P + 1$ and $PT^* = q^4T^*P$ entail that $\rho(R)e_0 = \rho(R)e_1 = 0$. Using the equality $TT^* = -q^4P^2 + P$, one obtains

$$\rho(T)e_k = q^{2(k-1)}(1-q^{4k})^{1/2}e_{k-1}, \quad k > 0. \tag{4.29}$$

Furthermore, a straightforward computation taking advantage of relations among generators gives

$$\rho(R)e_k = (1-q^{4k})^{1/2}(1-q^{4(k-1)})^{1/2}e_{k-2}, \quad k \geqslant 2. \tag{4.30}$$

Similarly, remembering appropriate relations (in particular $T^2 = q^2PR$), we compute:

$$\begin{aligned}\rho(R^*)e_k &= \rho(R^*T^{*k})\xi\,\|T^{*k}\xi\|^{-1}\\ &= q^{-4k}\rho(T^{*k}R^*)\xi\,\|T^{*k}\xi\|^{-1}\\ &= q^{-4k-2}\rho(T^{*k}q^2R^*P)\xi\,\|T^{*k}\xi\|^{-1}\\ &= q^{-4k-2}\rho(T^{*(k+2)})\xi\,\|T^{*k}\xi\|^{-1}\\ &= (1-q^{4(k+1)})^{1/2}(1-q^{4(k+2)})^{1/2}r_{k+2}, \quad k \geqslant 0.\end{aligned} \tag{4.31}$$

Therefore the Hilbert space $H_e := \overline{\operatorname{span}\{e_k\}}$ is a closed invariant subspace of H, and we have $H_e = H$ by the irreducibility of the bounded representation ρ. Any other irreducible representation ρ' with $\rho'(P) \neq 0$ generates an orthonormal basis in the same way, so it has to be unitarily equivalent to ρ. □

Observe that the irreducible representations of the above theorem are restrictions of representations of $C(S^2_{q\infty})$. More precisely, we have that $\rho_{2\theta}$ is the restriction of π_θ, and ρ is the restriction of π_+. Thus all irreducible representations of $C(\mathbb{R}P^2_q)$ extend to representations of $C(S^2_{q\infty})$. Consequently, so does their direct sum which, by [KR97, Corollary 10.2.4], is faithful. Since the norm in any representation is always less or equal to the norm in a faithful representation and there exists a faithful representation $C(\mathbb{R}P^2_q)$ extending to $C(S^2_{q\infty})$, the norm of the universal C^*-algebra of $P(\mathbb{R}P^2_q)$ coincides with the norm inherited from $C(S^2_{q\infty})$. To see it more directly, note first⋆ that ρ is a restriction of both π_+ and $U\pi_-U^{-1}$, where U is the unitary defined by $Ue_k = (-1)^k e_k$. Hence the faithfulness of ρ follows from the faithfulness of $\pi_+ \oplus \pi_-$. Again, we can conclude that the universal and inherited norms coincide. Summarizing we have established:

⋆ We owe this observation to P. Podleś.

COROLLARY 4.6. *The* C^**-algebra* $C(\mathbb{R}P_q^2)$ *is the* universal C^**-algebra of* $P(\mathbb{R}P_q^2)$.

Remark 4.7. Similarly to the case of the reflection action $\bar{r}_1^c$, we want the diagram

$$\begin{array}{ccccc} C(S_{qc}^2) & \xrightarrow{\pi_+^c \oplus \pi_-^c} & C^*(\mathfrak{S}) \oplus_\sigma C^*(\mathfrak{S}) & \xleftarrow{\pi_+ \oplus \pi_-} & C(S_{q\infty}^2) \\ \big\downarrow \bar{r}_2^c & & \big\downarrow & & \big\downarrow \bar{r}_2 \\ C(S_{qc}^2) & \xrightarrow[\pi_+^c \oplus \pi_-^c]{} & C^*(\mathfrak{S}) \oplus_\sigma C^*(\mathfrak{S}) & \xleftarrow[\pi_+ \oplus \pi_-]{} & C(S_{q\infty}^2) \end{array}$$

to be commutative. To this end, we define the antipodal $\mathbb{Z}_2$-actions on the equilateral spheres by $\bar{r}_2^c := \chi_c^{-1} \circ \bar{r}_2 \circ \chi_c$. It is clear that $\{a \in C(S_{qc}^2) \mid \bar{r}_2^c(a) = a\} \cong C(\mathbb{R}P_q^2)$. Furthermore, it follows directly from definitions, Proposition 2.4 and (3.24) that $\bar{r}_2^c = \bar{r}_1^c \circ \delta_{\sqrt{-1}} = \delta_{\sqrt{-1}} \circ \bar{r}_1^c$. Remembering also that U(1) is Abelian, we have that the antipodal actions are compatible with the U(1)-actions on quantum spheres: $\delta_g \circ \bar{r}_2^c = \bar{r}_2^c \circ \delta_g$.

Let us turn now to the computation of K-groups of $C(\mathbb{R}P_q^2)$. Just as in the classical case (e.g., see [K-M78, Corollary IV.6.47]), we have the following theorem:

THEOREM 4.8. *The topological* K*-groups of the quantum real projective space* $\mathbb{R}P_q^2$ *are as follows:* $K_0(C(\mathbb{R}P_q^2)) \cong \mathbb{Z} \oplus \mathbb{Z}_2$, $K_1(C(\mathbb{R}P_q^2)) \cong 0$.

Proof. First we need to find an exact sequence analogous to (1.3). Let J be the closed two-sided $*$-ideal of $C(\mathbb{R}P_q^2)$ generated by P, and let p: $C(\mathbb{R}P_q^2) \to C(\mathbb{R}P_q^2)/J$ be the natural surjection. Arguing as in the proof of Theorem 4.5, we see that in the quotient all relations (see (4.2)–(4.5)) reduce to $p(RR^*) = 1 = p(R^*R)$. Consequently $p(R)$ is unitary, and we have $C(\mathbb{R}P_q^2)/J \cong C^*(p(R)) \cong C(S^1)$.

LEMMA 4.9. *The ideal* J *is isomorphic* (*via the faithful representation* ρ) *to the* C^**-algebra* $\mathcal{K}$ *of compact operators on a separable Hilbert space.*

Proof. The operator $\rho(P)$ is evidently compact (see Theorem 4.5 for an explicit formula), whence $\rho(J) \subseteq \mathcal{K}$. On the other hand, as $\rho(P)$ is a diagonal operator with eigenvalues of multiplicity one, all the one-dimensional projections P_k onto the vectors e_k are elements of $\rho(J)$. (One can apply the continuous functional calculus to $\rho(P)$ to generate any P_k.) Furthermore, since also $\rho(T)$ belongs to $\rho(J)$ (see (4.4)) and it is a weighted shift with nonvanishing coefficients, all matrix units E_{ij} belong to $\rho(J)$. (They can be obtained from the P_k and $\rho(T)$.) Therefore, $\mathcal{K} \subseteq \rho(J)$, and consequently $\mathcal{K} = \rho(J)$. The claim of the lemma follows from the faithfulness of ρ. □

Denote by i the inverse of the appropriate restriction of ρ, and again by p the canonical surjection p composed with the isomorphism $C(\mathbb{R}P_q^2)/J \cong C(S^1)$. With the help of Lemma 4.9, we obtain the desired exact sequence:

$$0 \longrightarrow \mathcal{K} \xrightarrow{i} C(\mathbb{R}P_q^2) \xrightarrow{p} C(S^1) \longrightarrow 0. \tag{4.32}$$

Since $K_1(\mathcal{K}) \cong 0$, $K_1(C(S^1)) \cong \mathbb{Z} \cong K_0(\mathcal{K})$, the 6-term exact sequence of K-theory yields

$$0 \longrightarrow K_1(C(\mathbb{R}P_q^2)) \longrightarrow \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \xrightarrow{i_*} K_0(C(\mathbb{R}P_q^2)) \xrightarrow{p_*} \mathbb{Z} \longrightarrow 0, \tag{4.33}$$

where $\partial\colon \mathbb{Z} \cong K_1(C(S^1)) \to K_0(\mathcal{K}) \cong \mathbb{Z}$ is the index map. Due to the exactness of the above sequence, to compute the K-groups it suffices to determine the index map ∂. We know from the preceding discussion that $p(R)$ is the unitary generator of $C(S^1)$. Hence $[p(R)]$ generates $K_1(C(S^1))$. We have $[p(R)] \cong 1$ via the identification of $K_1(C(S^1))$ with $\mathbb{Z}$. Thus all we need to complete the calculation is the value of ∂ on $[p(R)]$. In general, if A is a unital C^*-algebra, $0 \to I \to A \to A/I \to 0$ the short exact sequence inducing the 6-term exact sequence, u a unitary element of A/I, and $v \in A$ such that $vv^* = 1$ and $v/I = u$, then $\partial([u]) = [1 - v^*v]$ (see [B-B86, Section 8.3.2] or [W-NE93, Remark 8.1.4]). We need to lift the unitary $p(R)$ to an appropriate coisometry in $C(\mathbb{R}P_q^2)$. As $\rho(R)$ is a weighted double-shift, the desired coisometry could be given by $U(e_n) = e_{n-2}$, $Ue_1 = 0 = Ue_2$. The operator U satisfies the polar decomposition $\rho(R) = U|\rho(R)|$. Furthermore, since $p(|R|) = 1$ and ρ is faithful, if there exists $v_2 \in C(\mathbb{R}P_q^2)$ such that $\rho(v_2) = U$, then $p(R) = p(v_2)$ and v_2 is the desired coisometry. Thus we need to show that $U \in \rho(C(\mathbb{R}P_q^2))$. Let $\mathcal{U}$ be an open neighbourhood of $\{1, q^4\}$ such that $\mathcal{U} \cap \{q^{4k}\}_{k=2,\ldots,\infty} = \emptyset$, and let $\beta\colon \mathbb{R} \to \mathbb{R}$ be a continuous function given on $\mathbb{R} \setminus \mathcal{U}$ by the formula

$$\beta(x) = (1 - x)^{-1/2}(1 - q^{-4}x)^{-1/2}. \tag{4.34}$$

Then $U = \rho(R)\beta(\rho(P))$, whence $U \in \rho(C(\mathbb{R}P_q^2))$, as needed. Consequently,

$$\partial([p(R)]) = [1 - v_2^* v_2] \cong [\mathrm{diag}(1, 1, 0, \ldots)] \cong 2. \tag{4.35}$$

Here in the penultimate equality we identified $K_0(C(\mathbb{R}P_q^2))$ with $K_0(\rho(C(\mathbb{R}P_q^2)))$, and in the last step $K_0(\mathcal{K})$ with $\mathbb{Z}$. Finally, as ∂ is injective, (4.33) breaks into two exact sequences:

$$0 \longrightarrow K_1(C(\mathbb{R}P_q^2)) \longrightarrow 0 \tag{4.36}$$

and

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \xrightarrow{i_*} K_0(C(\mathbb{R}P_q^2)) \xrightarrow{p_*} \mathbb{Z} \longrightarrow 0. \tag{4.37}$$

The former gives immediately $K_1(C(\mathbb{R}P^2_q)) \cong 0$, and the latter splits, as $\mathbb{Z}$ is a free module over itself. Therefore we conclude that $K_0(C(\mathbb{R}P^2_q)) \cong \mathrm{Im}\, i_* \oplus \mathbb{Z}$. On the other hand, the exactness of (4.37) implies the exactness of

$$0 \longrightarrow 2\mathbb{Z} \longrightarrow \mathbb{Z} \xrightarrow{i_*} \mathrm{Im}\, i_* \longrightarrow 0. \tag{4.38}$$

Hence $\mathrm{Im}\, i_* \cong \mathbb{Z}_2$, and consequently $K_0(C(\mathbb{R}P^2_q)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}$. □

Remark 4.10. It follows from the exact sequence (4.37) that our isomorphism identifying $K_0(C(\mathbb{R}P^2_q))$ with $\mathbb{Z}_2 \oplus \mathbb{Z}$ maps [1] to $(x, 1), x \in \mathbb{Z}_2$. This is because $p_*([1]) = [1] \in K_0(C(S^1))$, and we identify the latter with $\mathbb{Z}$ via $[1] \mapsto 1$. On the other hand, the K_0-class of a minimal projection in $\mathcal{K}$ (viewed as $1 \in \mathbb{Z}$) provides via i_* the order 2 generator of $K_0(C(\mathbb{R}P^2_q))$.

It is an immediate consequence of Theorem 4.8 and (3.28) that $C(\mathbb{R}P^2_q)$ is *not* the standard extension of $C(S^1)$ by $\mathcal{K}$:

COROLLARY 4.11. *The C^*-algebras $C(\mathbb{R}P^2_q)$ and $C(D^2_q)$ are* not *isomorphic, i.e., $C(\mathbb{R}P^2_q)$ is not the Toeplitz algebra.*

Remark 4.12. The C^*-algebra $C(S^2_{q\infty})$ is generated by A and B. The antipodal $\mathbb{Z}_2$-action sends A to $-A$, B to $-B$, and $C(\mathbb{R}P^2_q)$ is the fixed-point subalgebra. There is a conditional expectation (e.g., see [KR97, pp. 570, 571]) $E\colon C(S^2_{q\infty}) \to C(\mathbb{R}P^2_q)$ sending even monomials to themselves and annihilating the odd ones. We have a 'quasi-basis' $\{u_1 := 1, u_2 := A, u_3 := B^*\}$ for E (in the sense of [W-Y90, p. 2]). This means that

$$a = \sum_{i=1}^{3} E(au_i)u_i^* = \sum_{i=1}^{3} u_i E(u_i^* a), \quad \forall a \in C(S^2_{q\infty}). \tag{4.39}$$

Hence

$$\mathrm{Index}(E) := \sum_{i=1}^{3} u_i u_i^* = 1 + A^2 + B^*B = 2. \tag{4.40}$$

We can think of $S^2_{q\infty}$ as a two-fold covering of $\mathbb{R}P^2_q$.

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