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Quantum Serre Theorem as a Duality Between Quantum D-Modules

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We give an interpretation of quantum Serre theorem of Coates and Givental as a duality of twisted quantum D-modules. This interpretation admits a non-equivariant limit, and we obtain a precise relationship among (1) the quantum D-module of X twisted by a convex vector bundle E and the Euler class, (2) the quantum D-module of the total space of the dual bundle $E^{\vee} \to X$, and (3) the quantum D-module of a submanifold $Z \subset X$ cut out by a regular section of E. When E is the anticanonical line bundle K_X^{-1}, we identify these twisted quantum D-modules with second structure connections with different parameters, which arise as Fourier–Laplace transforms of the quantum D-module of X. In this case, we show that the duality pairing is identified with Dubrovin's second metric (intersection form).

1 Introduction

Genus-zero Gromov–Witten invariants of a smooth projective variety X can be encoded in different mathematical objects: a generating function that satisfies some system of

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partial differential equations (Witten–Dijkgraaf–Verlinde–Verlinde equations), an associative and commutative product called quantum product, the Lagrangian cone $\mathcal{L}_X$ of Givental [13] or in a meromorphic flat connection called quantum connection. These objects are all equivalent to each other; in this paper, we focus on the realization of Gromov–Witten invariants as a meromorphic flat connection.

Encoding Gromov–Witten invariants in a meromorphic flat connection defines the notion of quantum D-module [10], denoted by $\mathrm{QDM}(X)$, that is a tuple (F, ∇, S) consisting of a trivial holomorphic vector bundle F over $H^{\mathrm{ev}}(X) \times \mathbb{C}_z$ with fibre $H^{\mathrm{ev}}(X)$, a meromorphic flat connection ∇ on F given by the quantum connection:

$$\nabla = d + \sum_{\alpha=0}^{s} (T_\alpha \bullet_\tau) dt^\alpha + \left(-\frac{1}{z} (\mathfrak{E} \bullet) + \frac{\deg}{2} \right) \frac{dz}{z}$$

and a flat non-degenerate pairing S on F given by the Poicaré pairing (see Definition 2.1 and Remark 2.4). These data may be viewed as a generalization of a variation of Hodge structure (see [19]).

Quantum Serre theorem of Coates and Givental [4, Section 10] describes a certain relationship between *twisted* Gromov–Witten invariants. The data of a twist are given by a pair $(\mathbf{c}, E)$ of an invertible multiplicative characteristic class $\mathbf{c}$ and a vector bundle E over X. Since twisted Gromov–Witten invariants satisfy properties similar to usual Gromov–Witten invariants, we can define twisted quantum product, twisted quantum D-module $\mathrm{QDM}_{(\mathbf{c},E)}(X)$ and twisted Lagrangian cone $\mathcal{L}^{(\mathbf{c},E)}$ associated to the twist $(\mathbf{c}, E)$. Let $\mathbf{c}^*$ denote the characteristic class satisfying $\mathbf{c}(V)\mathbf{c}^*(V^\vee) = 1$ for any vector bundle V. Quantum Serre theorem (at genus zero) gives the equality of the twisted Lagrangian cones:

$$\mathcal{L}^{(\mathbf{c}^*, E^\vee)} = \mathbf{c}(E) \mathcal{L}^{(\mathbf{c},E)}. \tag{1.1}$$

Quantum Serre theorem of Coates and Givental was not stated as a duality. An observation in this paper is that this result can be restated as a duality between twisted quantum D-modules.

Theorem 1.1 (see Theorem 2.11 for more precise statements)**.** There exists a (typically non-linear) map $f\colon H^{\mathrm{ev}}(X) \to H^{\mathrm{ev}}(X)$ (see (2.7)) such that the following holds:

(1) The twisted quantum D-modules $\mathrm{QDM}_{(\mathbf{c},E)}(X)$ and $f^*\mathrm{QDM}_{(\mathbf{c}^*,E^\vee)}(X)$ are dual to each other; the duality pairing S^{QS} is given by the Poincaré pairing.

(2) The map $\mathrm{QDM}_{(\mathbf{c},E)}(X) \to f^*\mathrm{QDM}_{(\mathbf{c}^*,E^\vee)}(X)$ sending α to $\mathbf{c}(E) \cup \alpha$ is a morphism of quantum D-modules. □

Genus-zero twisted Gromov–Witten invariants were originally designed to compute Gromov–Witten invariants for Calabi–Yau hypersurfaces or non-compact local Calabi–Yau manifolds [2, 11, 25]. Suppose that E is a convex vector bundle and $\mathbf{c}$ is the equivariant Euler class $\mathbf{e}_\lambda$. In this case, non-equivariant limits of $(\mathbf{c}, E)$-twisted Gromov–Witten invariants yield Gromov–Witten invariants of a regular section $Z \subset X$ of E and non-equivariant limits of $(\mathbf{c}^*, E^\vee)$-twisted Gromov–Witten invariants yield Gromov–Witten invariants for the total space $E^\vee$. The original statement (1.1) of quantum Serre theorem does not admit a non-equivariant limit since the non-equivariant Euler class is not invertible. We see however that our restatement above passes to the non-equivariant limit as follows:

Corollary 1.2 (Theorem 3.14, Corollary 3.17)**.** Let E be a convex vector bundle and let e denote the (non-equivariant) Euler class. Let $h\colon H^{\mathrm{ev}}(X) \to H^{\mathrm{ev}}(X)$ be the map given by $h(\tau) = \tau + \pi\sqrt{-1}c_1(E)$ and let $\bar{f}\colon H^{\mathrm{ev}}(X) \to H^{\mathrm{ev}}(X)$ denote the non-equivariant limit of the map f of Theorem 1.1 in the case where $\mathbf{c} = \mathbf{e}_\lambda$. We have the following:

(1) The quantum D-modules $\mathrm{QDM}_{(\mathrm{e},E)}(X)$ and $(h \circ \bar{f})^*\mathrm{QDM}(E^\vee)$ are dual to each other.

(2) Let Z be the zero-locus of a regular section of E and suppose that Z satisfies one of the conditions in Lemma 3.15. Denote by $\iota : Z \hookrightarrow X$ the inclusion. Then the morphism $e(E)\colon \mathrm{QDM}_{(\mathrm{e},E)}(X) \to (h \circ \bar{f})^*\mathrm{QDM}(E^\vee)$ factors through the ambient part quantum D-module $\mathrm{QDM}_{\mathrm{amb}}(Z)$ of Z as:

$$\begin{tikzcd} \mathrm{QDM}_{(\mathrm{e},E)}(X) \arrow[r, "\mathrm{e}(E)\cup"] \arrow[d, "\iota^*"'] & (h \circ \bar{f})^*\mathrm{QDM}(E^\vee) \\ (\iota^*)^*\mathrm{QDM}_{\mathrm{amb}}(Z) \arrow[ur, hook, "\iota_*"] & \end{tikzcd} \tag{1.2}$$

□

What is non-trivial here is the existence of an embedding of $\mathrm{QDM}_{\mathrm{amb}}(Z)$ into $\mathrm{QDM}(E^\vee)$. This is reminiscent of the Knörrer periodicity [21, 32]: we expect that this would be a special case of a more general phenomenon which relates quantum cohomology of a non-compact space equipped with a holomorphic function W to quantum cohomology of the critical locus of W.

In Section 4, we introduce certain integral structures for the quantum D-modules $\mathrm{QDM}_{(\mathrm{e},E)}(X)$, $\mathrm{QDM}(E^\vee)$, and $\mathrm{QDM}_{\mathrm{amb}}(Z)$, generalizing the construction in [17, 19]. These integral structures are lattices in the space of flat sections which are isomorphic to the K-group $K(X)$ of vector bundles. We show in Propositions 4.4 and 4.5 that the duality pairing S^{QS} is identified with the Euler pairing on the K-groups, and that the maps appearing in the diagram (1.2) are induced by natural functorial maps between K-groups.

In Section 5, we consider the case where $E = K_X^{-1}$ and study quantum cohomology of a Calabi–Yau hypersurface in X and the total space of K_X. We show that the small quantum D-modules $\mathrm{SQDM}_{(\mathrm{e},K_X^{-1})}(X)$ and $\mathrm{SQDM}(K_X)$ are isomorphic to the second structure connections of Dubrovin [8] (Here small quantum D-modules are the restriction of quantum D-modules to the $H^2(X)$-parameter space). The second structure connections are meromorphic flat connections $\check{\nabla}^{(\sigma)}$ on the trivial vector bundle $\check{F}$ over $H^{\mathrm{ev}}(X) \times \mathbb{C}_x$ with fibre $H^{\mathrm{ev}}(X)$, which is obtained from the quantum connection of X via the *Fourier–Laplace transformation* with respect to z^{-1} (see (5.5)):

$$\partial_{z^{-1}} \rightsquigarrow x, \qquad z^{-1} \rightsquigarrow -\partial_x.$$

The second structure connection has a complex parameter σ; we will see that the two small quantum D-modules correspond to different values of σ.

Theorem 1.3 (see Theorems 5.16 and 5.19 for more precise statements)**.** Suppose that the anticanonical class $-K_X$ of X is nef. Let n be the dimension of X:

(1) There exist maps $\pi_{\mathrm{eu}}, \pi_{\mathrm{loc}} \colon H^2(X) \times \mathbb{C}_x \to H^2(X)$ and isomorphisms of vector bundles with connections:

$$\psi_{\mathrm{eu}} : (\check{F}, \check{\nabla}^{(\frac{n+1}{2})})|_{H^2(X)\times\mathbb{C}_x} \longrightarrow \pi_{\mathrm{eu}}^* \mathrm{SQDM}_{(\mathrm{e},K_X^{-1})}(X)|_{z=1}$$

$$\psi_{\mathrm{loc}} : (\check{F}, \check{\nabla}^{(-\frac{n+1}{2})})|_{H^2(X)\times\mathbb{C}_x} \longrightarrow \pi_{\mathrm{loc}}^* \mathrm{SQDM}(K_X)|_{z=1}$$

which are defined in a neighbourhood of the large radius limit point and for sufficiently large $|x|$.

(2) The duality pairing S^{QS} between $\pi_{\mathrm{eu}}^*\mathrm{SQDM}_{(\mathrm{e},K_X^{-1})}(X)$ and $\pi_{\mathrm{loc}}^*\mathrm{SQDM}(K_X)$ is identified with the second metric $\check{g}\colon (\mathcal{O}(\check{F}), \check{\nabla}^{(\frac{n+1}{2})}) \times (\mathcal{O}(\check{F}), \check{\nabla}^{(-\frac{n+1}{2})}) \to \mathcal{O}$ given by

$$\check{g}(\gamma_1, \gamma_2) = \int_X \gamma_1 \cup (c_1(X) \bullet_\tau - x)^{-1} \gamma_2$$

over $H^2(X) \times \mathbb{C}_x$. $\square$

Combined with the commutative diagram (1.2), this theorem gives an entirely algebraic description of the ambient part quantum D-module of a Calabi–Yau hypersurface Z in a Fano manifold X (Corollary 5.20). We will also describe the A-model Hodge filtration for these small quantum D-modules in terms of the second structure connection in Section 5.8. These results are illustrated for a quintic threefold in $\mathbb{P}^4$ in Section 6. Note that the second structure connection is closely related to the *almost dual* Frobenius manifold of Dubrovin (see [9, Proposition 3.3]) and our result may be viewed as a generalization of the example in [9, Section 5.4].

This paper arose out of our previous works [18, 28] on quantum D-modules of (toric) complete intersections. The embedding of $\mathrm{QDM}_{\mathrm{amb}}(Z)$ into $\mathrm{QDM}(E^\vee)$ appeared in [18, Remark 6.14] in the case where X is a weak Fano toric orbifold and $E^\vee = K_X$; in [28, Theorem 1.1], $\mathrm{QDM}_{\mathrm{amb}}(Z)$ was presented as the quotient $\mathrm{QDM}_{(\mathrm{e},E)}(X)$ by $\mathrm{Ker}(\mathrm{e}(E)\cup)$ when E is a direct sum of ample line bundles. We would also like to draw attention to a recent work of Borisov and Horja [1] on the duality of better behaved Gelfand Kapranov Zelevinsky systems. The conjectural duality in their work should correspond to a certain form of quantum Serre duality generalized to toric Deligne–Mumford stacks.

We assume that the reader is familiar with Givental's formalism and quantum cohomology Frobenius manifold. As preliminary reading for the reader, we list [4, 13; 27, Chapter I,II].

Notation 1.4. We use the following notation throughout the paper.

X	a smooth projective variety of complex dimension n.
E	a vector bundle over X of rank r with $E^\vee$ the dual vector bundle.
$(T_0, \dots, T_s)$	an homogeneous basis of $H^{\mathrm{ev}}(X) = \bigoplus_{p=0}^{n} H^{2p}(X, \mathbb{C})$ such that $T_0 = \mathbf{1}$ and $\{T_1, \dots, T_r\}$ form a nef integral basis of $H^2(X)$.
$(t^0, \dots, t^s)$	the linear coordinates dual to the basis $(T_0, \dots, T_s)$; we write $\tau := \sum_{\alpha=0}^{s} t^\alpha T_\alpha$ and $\partial_\alpha := (\partial/\partial t^\alpha)$.
$(T^0, \dots, T^s)$	the Poincaré dual basis such that $\int_X T_\alpha \cup T^\beta = \delta_\alpha^\beta$.
$\mathrm{Eff}(X)$	the set of classes in $H_2(X, \mathbb{Z})$ represented by effective curves.
$\gamma(d)$	the pairing $\int_d \gamma$ between $\gamma \in H^2(X)$ and $d \in H_2(X)$.
$\mathbf{e}_\lambda$	the equivariant Euler class.
e	the non-equivariant Euler class.

□

2 Quantum Serre Theorem as a Duality

In this section, we reformulate quantum Serre theorem of Coates and Givental [4] as a duality of quantum D-modules. After reviewing twisted Gromov–Witten invariants and twisted quantum D-modules, we give our reformulation in Theorem 2.11.

2.1 Notation

We introduce the notation we use throughout the paper. Let $\{T_0, \dots, T_s\}$ be a homogeneous basis of the cohomology group $H^{\mathrm{ev}}(X) = \bigoplus_{p=0}^{n} H^{2p}(X; \mathbb{C})$ of even degree. We assume $T_0 = \mathbf{1}$ and $T_1, \dots, T_r$ form a nef integral basis of $H^2(X)$ for $r = \dim H^2(X) \le s$. Let $\{t^0, \dots, t^s\}$ denote the linear co-ordinates on $H^{\mathrm{ev}}(X)$ dual to the basis $\{T_0, \dots, T_s\}$ and write $\tau = \sum_{\alpha=0}^{s} t^\alpha T_\alpha$ for a general point of $H^{\mathrm{ev}}(X)$. We write $\partial_\alpha = \partial/\partial t^\alpha$ for the partial derivative.

Let $\mathrm{Eff}(X)$ denote the set of classes of effective curves in $H_2(X; \mathbb{Z})$. Let K be a commutative ring. For $d \in \mathrm{Eff}(X)$, we write Q^d for the corresponding element in the group ring $K[Q] := K[\mathrm{Eff}(X)]$. The variable Q is called the Novikov variable. We write $K[\![Q]\!]$ for the natural completion of $K[Q]$. For an infinite set $\mathbf{s} = \{s_0, s_1, s_2, \dots\}$ of variables, we define the formal power series ring

$$K[\![\mathbf{s}]\!] = K[\![s_0, s_1, s_2, \dots]\!]$$

to be the (maximal) completion of $K[s_0, s_1, s_2, \dots]$ with respect to the additive valuation v defined by $v(s_k) = k + 1$. We write

$$\mathbb{C}[\![Q, \tau]\!], \quad \mathbb{C}[\![Q, \mathbf{s}, \tau]\!], \quad \text{and} \quad \mathbb{C}[z][\![Q, \mathbf{s}, \tau]\!]$$

for the completions of $\mathbb{C}[\![Q]\!][t^0, \dots, t^s]$, $\mathbb{C}[\![Q]\!][t^0, \dots, t^s, s_0, s_1, s_2, \dots]$, and $\mathbb{C}[z][\![Q]\!][t^0, \dots, t^s, s_0, s_1, s_2, \dots]$, respectively. We write $\tau_2 = \sum_{i=1}^{r} t^i T_i$ for the $H^2(X)$-component of τ and set $\tau = \tau_2 + \tau'$. Because of the divisor equation in Gromov–Witten theory, the Novikov variable Q and τ_2 often appear in the combination $(Q\,\mathrm{e}^{\tau_2})^d = Q^d\,\mathrm{e}^{t_1 T_1(d) + \cdots + t_r T_r(d)}$. Therefore we can also work with the subring

$$\mathbb{C}[\![Q\,\mathrm{e}^{\tau_2}, \tau']\!] = \mathbb{C}[\![Q\,\mathrm{e}^{\tau_2}]\!][\![t^0, t^{r+1}, \dots, t^s]\!] \subset \mathbb{C}[\![Q, \tau]\!].$$

The subrings $\mathbb{C}[\![Q\,\mathrm{e}^{\tau_2}, \mathbf{s}, \tau']\!] \subset \mathbb{C}[\![Q, \mathbf{s}, \tau]\!]$, $\mathbb{C}[z][\![Q\,\mathrm{e}^{\tau_2}, \mathbf{s}, \tau']\!] \subset \mathbb{C}[z][\![Q, \mathbf{s}, \tau]\!]$ are defined similarly.

2.2 Twisted quantum D-modules

Coates and Givental [4] introduced Gromov–Witten invariants twisted by a vector bundle and a multiplicative characteristic class. We consider the quantum D-module defined by genus-zero twisted Gromov–Witten invariants.

2.2.1 *Twisted Gromov–Witten invariants and twisted quantum product*

Let X be a smooth projective variety and let E be a vector bundle on X. Denote by $\mathrm{Eff}(X)$ the subset of $H_2(X,\mathbb{Z})$ of classes of effective curves. For $d\in\mathrm{Eff}(X)$ and $g,\ell\in\mathbb{N}$, we denote by $\bar{\mathcal{M}}_{g,\ell}(X,d)$ the moduli space of genus g stable maps to X of degree d and with ℓ marked points. Recall that $\bar{\mathcal{M}}_{g,\ell}(X,d)$ is a proper Deligne–Mumford stack and is equipped with a virtual fundamental class $[\bar{\mathcal{M}}_{g,\ell}(X,d)]^{\mathrm{vir}}$ in $H_{2D}(\bar{\mathcal{M}}_{g,\ell}(X,d),\mathbb{Q})$ with $D=(1-g)(\dim X-3)+(c_1(X)\cdot d)+\ell$. In this paper, we only consider the genus-zero moduli spaces. The universal curve of $\bar{\mathcal{M}}_{0,\ell}(X,d)$ is $\bar{\mathcal{M}}_{0,\ell+1}(X,d)$:

$$\begin{tikzcd}
\bar{\mathcal{M}}_{0,\ell+1}(X,d) \arrow[r, "\mathrm{ev}_{\ell+1}"] \arrow[d, "\pi"] & X \\
\bar{\mathcal{M}}_{0,\ell}(X,d) &
\end{tikzcd}$$

where π is the map that forgets the $(\ell+1)$th marked point and stabilizes, and $\mathrm{ev}_{\ell+1}$ is the evaluation map at the $(\ell+1)$th marked point.

The vector bundle E defines a K-class $E_{0,\ell,d}:=\pi_!e^*_{\ell+1}E\in K^0(\bar{\mathcal{M}}_{0,\ell}(X,d))$ on the moduli space, where $\pi_!$ denotes the K-theoretic push-forward. The restriction to a point $(f:C\to X)\in\bar{\mathcal{M}}_{0,\ell}(X,d)$ gives

$$E_{0,\ell,d}\,|_{(f:C\to X)}=[H^0(C,\,f^*E)]-[H^1(C,\,f^*E)].$$

For $i\in\{1,\dots,\ell\}$, let $\mathfrak{L}_i$ denote the universal cotangent line bundle on $\bar{\mathcal{M}}_{0,\ell}(X,d)$ at the ith marking. The fibre of $\mathfrak{L}_i$ at a point $(C,x_1,\dots,x_\ell,\ f\colon C\to X)$ is the cotangent space $T^*_{x_i}C$ at x_i. Put $\psi_i:=c_1(\mathfrak{L}_i)$ in $H^2(\bar{\mathcal{M}}_{0,\ell}(X,d),\mathbb{Q})$.

The universal invertible multiplicative characteristic class $\mathbf{c}(\cdot)$ is given by:

$$\mathbf{c}(\cdot)=\exp\left(\sum_{k=0}^{\infty}s_k\mathrm{ch}_k(\cdot)\right)$$

with infinitely many parameters $s_0,s_1,s_2,\dots$. In the discussion of Coates–Givental's quantum Serre theorem, we treat $s_0,s_1,s_2,\dots$ as formal infinitesimal parameters. On the other hand, the result we obtain later sometimes makes sense for non-zero values of the parameters. For example, we will use the equivariant Euler class for $\mathbf{c}(\cdot)$ in Section 3.

The *genus-zero* $(\mathbf{c}, E)$*-twisted Gromov–Witten invariants* are defined by the following formula. For any $\gamma_1, \ldots, \gamma_\ell \in H^{\mathrm{ev}}(X)$ and any $k_1, \ldots, k_\ell \in \mathbb{N}$, we put:

$$\langle \gamma_1 \psi^{k_1}, \ldots, \gamma_\ell \psi^{k_\ell} \rangle_{0,\ell,d}^{(\mathbf{c},E)} := \int_{[\bar{\mathcal{M}}_{0,\ell}(X,d)]^{\mathrm{vir}}} \left(\prod_{i=1}^{\ell} \psi_i^{k_i} \mathrm{ev}_i^* \gamma_i \right) \mathbf{c}(E_{0,\ell,d}).$$

We also use the following notation:

$$\langle\langle \gamma_1 \psi^{k_1}, \ldots, \gamma_\ell \psi^{k_\ell} \rangle\rangle_\tau^{(\mathbf{c},E)} := \sum_{d \in \mathrm{Eff}(X)} \sum_{k \geq 0} \frac{Q^d}{k!} \langle \gamma_1 \psi_1^{k_1}, \ldots, \gamma_\ell \psi_\ell^{k_\ell}, \tau, \ldots, \tau \rangle_{0,\ell+k,d}^{(\mathbf{c},E)}$$

The genus-zero twisted Gromov–Witten potential is

$$\mathcal{F}_{(\mathbf{c},E)}^0(\tau) = \langle\langle\rangle\rangle_\tau^{(\mathbf{c},E)} = \sum_{d \in \mathrm{Eff}(X)} \sum_{k \geq 0} \frac{Q^d}{k!} \langle \tau, \ldots, \tau \rangle_{0,k,d}^{(\mathbf{c},E)}.$$

The genus-zero-twisted potential $\mathcal{F}_{(\mathbf{c},E)}^0(\tau)$ lies in $\mathbb{C}[\![Q, \mathbf{s}, \tau]\!]$. By the divisor equation, we see that it lies in the subring $\mathbb{C}[\![Q\,\mathrm{e}^{\tau_2}, \mathbf{s}, \tau']\!]$. Introduce the symmetric bilinear pairing $(\cdot, \cdot)_{(\mathbf{c},E)}$ on $H^{\mathrm{ev}}(X) \otimes \mathbb{C}[\![\mathbf{s}]\!]$ by

$$(\gamma_1, \gamma_2)_{(\mathbf{c},E)} = \int_X \gamma_1 \cup \gamma_2 \cup \mathbf{c}(E).$$

The $(\mathbf{c}, E)$-twisted quantum product $\bullet_\tau^{(\mathbf{c},E)}$ is defined by the formula:

$$(T_\alpha \bullet_\tau^{(\mathbf{c},E)} T_\beta, T_\gamma)_{(\mathbf{c},E)} = \partial_\alpha \partial_\beta \partial_\gamma \mathcal{F}_{(\mathbf{c},E)}^0(\tau). \tag{2.1}$$

The structure constants lie in $\mathbb{C}[\![Q\,\mathrm{e}^{\tau_2}, \mathbf{s}, \tau']\!] \subset \mathbb{C}[\![Q, \mathbf{s}, \tau]\!]$. The product is extended bilinearly over $\mathbb{C}[\![Q, \mathbf{s}, \tau]\!]$ and defines the $(\mathbf{c}, E)$-twisted quantum cohomology $(H^{\mathrm{ev}}(X) \otimes \mathbb{C}[\![Q, \mathbf{s}, \tau]\!], \bullet_\tau^{(\mathbf{c},E)})$. It is associative and commutative, and has $T_0 = \mathbf{1}$ as the identity.

2.2.2 *Twisted quantum D-module and fundamental solution*

Definition 2.1. The $(\mathbf{c}, E)$*-twisted quantum D-module* is a triple

$$\mathrm{QDM}_{(\mathbf{c},E)}(X) = (H^{\mathrm{ev}}(X) \otimes \mathbb{C}[z][\![Q, \mathbf{s}, \tau]\!], \nabla^{(\mathbf{c},E)}, S_{(\mathbf{c},E)})$$

where $\nabla^{(\mathbf{c},E)}$ is the connection defined by

$$\nabla_\alpha^{(\mathbf{c},E)} \colon H^{\mathrm{ev}}(X) \otimes \mathbb{C}[z][\![Q, \mathbf{s}, \tau]\!] \to z^{-1} H^{\mathrm{ev}}(X) \otimes \mathbb{C}[z][\![Q, \mathbf{s}, \tau]\!]$$

$$\nabla_\alpha^{(\mathbf{c},E)} = \partial_\alpha + \frac{1}{z}(T_\alpha \bullet_\tau^{(\mathbf{c},E)}), \quad \alpha = 0, \ldots, s,$$

and $S_{(\mathbf{c},E)}$ is the "z-sesquilinear" pairing on $H^{\mathrm{ev}}(X)\otimes\mathbb{C}[z][\![Q,\mathbf{s},\tau]\!]$ defined by

$$S_{(\mathbf{c},E)}(u,v)=(u(-z),v(z))_{(\mathbf{c},E)}$$

for $u,v\in H^{\mathrm{ev}}(X)\otimes\mathbb{C}[z][\![Q,\mathbf{s},\tau]\!]$. The connection $\nabla^{(\mathbf{c},E)}$ is called the *quantum connection*. When $\mathbf{c}=1$ and $E=0$, the triple

$$\mathrm{QDM}(X)=(H^{\mathrm{ev}}(X)\otimes\mathbb{C}[z][\![Q,\tau]\!],\nabla=\nabla^{(\mathbf{c}=1,E=0)},S=S_{(\mathbf{c}=1,E=0)})$$

is called the *quantum D-module* of X. □

Remark 2.2. The module $H^{\mathrm{ev}}(X)\otimes\mathbb{C}[z][\![Q,\mathbf{s},\tau]\!]$ should be viewed as the module of sections of a vector bundle over the formal neighbourhood of the point $Q=\mathbf{s}=\tau=z=0$. Since the connection $\nabla^{(\mathbf{c},E)}$ does not preserve $H^{\mathrm{ev}}(X)\otimes\mathbb{C}[z][\![Q,\mathbf{s},\tau]\!]$, quantum D-module is *not* a D-module in the traditional sense. It can be regarded as a lattice in the D-module $H^{\mathrm{ev}}(X)\otimes\mathbb{C}[z^{\pm}][\![Q,\mathbf{s},\tau]\!]$ (see, e.g. [34, p. 18]). □

Remark 2.3. As discussed, structure constants of the quantum product belong to the subring $\mathbb{C}[\![Q\,\mathrm{e}^{\tau_2},\mathbf{s},\tau']\!]$. Therefore, the twisted quantum D-modules can be defined over $\mathbb{C}[z][\![Q\,\mathrm{e}^{\tau_2},\mathbf{s},\tau']\!]$. This will be important when we specialize Q to one in Section 3. □

Remark 2.4. For $\mathbf{c}=1$ and $E=0$, we can complete the quantum connection ∇ in the z-direction as a flat connection. We define

$$\nabla_{z\partial_z}=z\partial_z-\frac{1}{z}(\mathfrak{E}\bullet_\tau)+\frac{\deg}{2}$$

where $\mathfrak{E}=\sum_{\alpha=0}^{s}(1-\frac{1}{2}\deg T_\alpha)t^\alpha T_\alpha+c_1(TX)$ is the Euler vector field. □

The quantum connection $\nabla^{(\mathbf{c},E)}$ is known to be flat and admit a fundamental solution. The fundamental solution of the following form was introduced by Givental [11, Corollary 6.2]. We define $L_{(\mathbf{c},E)}(\tau,z)\in\mathrm{End}(H^{\mathrm{ev}}(X))\otimes\mathbb{C}[z^{-1}][\![Q,\mathbf{s},\tau]\!]$ by the formula:

$$L_{(\mathbf{c},E)}(\tau,z)\gamma=\gamma-\sum_{\alpha=0}^{s}\left\langle\!\!\left\langle\frac{\gamma}{z+\psi},T_\alpha\right\rangle\!\!\right\rangle_\tau^{(\mathbf{c},E)}\frac{T^\alpha}{\mathbf{c}(E)}$$

where $\gamma/(z+\psi)$ in the correlator should be expanded in the geometric series $\sum_{n=0}^{\infty}\gamma\psi^n(-z)^{-n-1}$.

Proposition 2.5 (see, e.g. [18, Proposition 2.1; 33, Section 2]).

The quantum connection $\nabla^{(\mathbf{c},E)}$ is flat and $L_{(\mathbf{c},E)}(\tau,z)$ gives the fundamental solution for $\nabla^{(\mathbf{c},E)}$. Namely we have

$$\nabla_\alpha^{(\mathbf{c},E)}(L_{(\mathbf{c},E)}(\tau,z)\gamma)=0, \quad \alpha=0,\dots,s$$

for all $\gamma\in H^{\mathrm{ev}}(X)$. Moreover, $S_{(\mathbf{c},E)}$ is flat for $\nabla^{(\mathbf{c},E)}$ and $L_{(\mathbf{c},E)}(\tau,z)$ is an isometry for $S_{(\mathbf{c},E)}$:

$$dS_{(\mathbf{c},E)}(u,v)=S_{(\mathbf{c},E)}(\nabla^{(\mathbf{c},E)}u,v)+S_{(\mathbf{c},E)}(u,\nabla^{(\mathbf{c},E)}v)$$

$$S_{(\mathbf{c},E)}(u,v)=S_{(\mathbf{c},E)}(L_{(\mathbf{c},E)}u,L_{(\mathbf{c},E)}v)$$

where $u,v\in H^{\mathrm{ev}}(X)\otimes\mathbb{C}[z][\![Q,\mathbf{s},\tau]\!]$. □

From the last point of Proposition 2.5, one can deduce that the inverse of $L_{(\mathbf{c},E)}$ is given by the adjoint of $L_{(\mathbf{c},E)}(\tau,-z)$. Explicitly,

$$L_{(\mathbf{c},E)}(\tau,z)^{-1}\gamma=\gamma+\sum_{\alpha=0}^{s}\left\langle\!\!\left\langle\frac{T_\alpha}{z-\psi},\gamma\right\rangle\!\!\right\rangle_\tau^{(\mathbf{c},E)}\frac{T_\alpha}{\mathbf{c}(E)}. \tag{2.2}$$

Definition 2.6. The $(\mathbf{c},E)$-*twisted J-function* is defined to be

$$\begin{aligned}J_{(\mathbf{c},E)}(\tau,z)&:=zL_{(\mathbf{c},E)}(\tau,z)^{-1}\mathbf{1}\\&=z+\tau+\sum_{\alpha=0}^{s}\left\langle\!\!\left\langle\frac{T_\alpha}{z-\psi}\right\rangle\!\!\right\rangle_\tau^{(\mathbf{c},E)}\frac{T^\alpha}{\mathbf{c}(E)}\end{aligned} \tag{2.3}$$

□

We deduce the following equality for $\alpha=0,\dots,s$:

$$L_{(\mathbf{c},E)}(\tau,z)^{-1}T_\alpha=L_{(\mathbf{c},E)}(\tau,z)^{-1}z\nabla_\alpha^{(\mathbf{c},E)}\mathbf{1}=\partial_\alpha J_{(\mathbf{c},E)}(\tau,z). \tag{2.4}$$

Remark 2.7. When $\mathbf{c}=1$ and $E=0$, we can complete the quantum connection ∇ in the z-direction as in Remark 2.4. The fundamental solution for flat sections, including in the z-direction, is given by $L(\tau,z)z^{-\deg/2}z^{c_1(TX)}$; see [18, Proposition 3.5]. □

Remark 2.8. The divisor equation for descendant invariants shows that

$$L_{(\mathbf{c},E)}(\tau,z)\gamma=e^{-\tau_2/z}\gamma+\sum_{\substack{(d,\ell)\neq(0,0)\\ d\in\mathrm{Eff}(X),\ell\geq 0}}\frac{Q^d e^{\tau_2(d)}}{\ell!}\left\langle\frac{e^{-\tau_2/z}\gamma}{-z-\psi},\tau',\dots,\tau',T_\alpha\right\rangle_{0,\ell+2,d}^{(\mathbf{c},E)}\frac{T^\alpha}{\mathbf{c}(E)}. \tag{2.5}$$

See, for example, [18, Section 2.5]. In particular, $L_{(\mathbf{c},E)}$ belongs to $\mathrm{End}(H^{\mathrm{ev}}(X))\otimes\mathbb{C}[z^{-1}][\![Q\,e^{\tau_2},\mathbf{s},\tau']\!][\tau_2]$. □

2.3 Quantum Serre theorem in terms of quantum D-modules

We formally associate to $\mathbf{c}(\cdot)$ another multiplicative class $\mathbf{c}^*(\cdot)$ by the formula:

$$\mathbf{c}^*(\cdot) = \exp\left(\sum_{k\geq 0}(-1)^{k+1}s_k \mathrm{ch}_k(\cdot)\right). \tag{2.6}$$

The class $\mathbf{c}^*$ corresponds to the choice of parameters $\mathbf{s}^* = (s_0^*, s_1^*, s_2^*, \ldots)$ with $s_k^* = (-1)^{k+1}s_k$. For any vector bundle G, we have

$$\mathbf{c}^*(G^\vee)\mathbf{c}(G) = \mathbf{1}.$$

Definition 2.9. Define the map $f\colon H^{\mathrm{ev}}(X) \to H^{\mathrm{ev}}(X)$ by the formula:

$$f(\tau) = \sum_{\alpha=0}^{s} \langle\langle T^\alpha, \mathbf{c}^*(E^\vee)\rangle\rangle_\tau^{(\mathbf{c},E)} T_\alpha. \tag{2.7}$$

More precisely, the formula defines a morphism $f\colon \mathrm{Spf}\mathbb{C}[\![Q, \mathbf{s}, \tau]\!] \to \mathrm{Spf}\mathbb{C}[\![Q, \mathbf{s}, \tau]\!]$ of formal schemes. □

Definition 2.10. The *quantum Serre pairing* S^{QS} is the z-sesquilinear pairing on $H^{\mathrm{ev}}(X) \otimes \mathbb{C}[z][\![Q, \mathbf{s}, \tau]\!]$ defined by:

$$S^{\mathrm{QS}}(u, v) = \int_X u(-z) \cup v(z)$$

for $u, v \in H^{\mathrm{ev}}(X) \otimes \mathbb{C}[z][\![Q, \mathbf{s}, \tau]\!]$. □

Theorem 2.11. (1) The twisted quantum D-modules $\mathrm{QDM}_{(\mathbf{c},E)}(X)$ and $f^*\mathrm{QDM}_{(\mathbf{c}^*,E^\vee)}(X)$ are dual to each other with respect to S^{QS}, that is,

$$\partial_\alpha S^{\mathrm{QS}}(u, v) = S^{\mathrm{QS}}(\nabla_\alpha^{(\mathbf{c},E)}u, v) + S^{\mathrm{QS}}(u, (f^*\nabla^{(\mathbf{c}^*,E^\vee)})_\alpha v) \tag{2.8}$$

for $u, v \in H^{\mathrm{ev}}(X) \otimes \mathbb{C}[z][\![Q, \mathbf{s}, \tau]\!]$.

(2) The isomorphism of vector bundles

$$\mathbf{c}(E) : \mathrm{QDM}_{(\mathbf{c},E)}(X) \to f^*\mathrm{QDM}_{(\mathbf{c}^*,E^\vee)}(X)$$

$$\alpha \mapsto \mathbf{c}(E) \cup \alpha$$

intertwines the connections $\nabla^{(\mathbf{c},E)}$, $f^*\nabla^{(\mathbf{c}^*,E^\vee)}$ and the pairings $S_{(\mathbf{c},E)}$, $f^*S_{(\mathbf{c}^*,E^\vee)}$.

(3) The fundamental solutions satisfy the following properties:

$$\mathbf{c}(E)L_{(\mathbf{c},E)}(\tau,z) = L_{(\mathbf{c}^*,E^\vee)}(f(\tau),z)\mathbf{c}(E),$$
$$S^{\mathrm{QS}}(u,v) = S^{\mathrm{QS}}(L_{(\mathbf{c},E)}u,\,(f^*L_{(\mathbf{c}^*,E^\vee)})v). \qquad \square$$

We will give a proof of this theorem in Section 2.5.

Remark 2.12. (1) The pull back $f^*\nabla^{(\mathbf{c}^*,E^\vee)}$ is defined to be

$$(f^*\nabla^{(\mathbf{c}^*,E^\vee)})_\alpha = \partial_\alpha + \frac{1}{z}\sum_{\beta=0}^{s}\frac{\partial f^\beta(\tau)}{\partial t^\alpha}(T_\beta \bullet^{(\mathbf{c}^*,E^\vee)}_{f(\tau)}).$$

where we set $f(\tau) = \sum_{\alpha=0}^{s} f^\alpha(\tau)T_\alpha$. The flatness (2.8) of S^{QS} implies a certain complicated relationship between the quantum products $\bullet^{(\mathbf{c},E)}_\tau$, $\bullet^{(\mathbf{c}^*,E^\vee)}_{f(\tau)}$.

(2) The map $\mathbf{c}(E)$ in the above theorem is obtained as the composition of the quantum Serre duality and the self-duality:

$$\begin{tikzcd}
& (-)^*(f^*\mathrm{QDM}_{(\mathbf{c}^*,E^\vee)}(X))^\vee \arrow[dr, "S_{(\mathbf{c}^*,E^\vee)}", "\cong"'] & \\
\mathrm{QDM}_{(\mathbf{c},E)}(X) \arrow[ur, "S^{\mathrm{QS}}", "\cong"'] \arrow[rr, "\mathbf{c}(E)"] & & f^*\mathrm{QDM}_{(\mathbf{c}^*,E^\vee)}(X)
\end{tikzcd}$$

where $(-)^*$ means the pull back by the change $z \mapsto -z$ of sign and $(\cdots)^\vee$ means the dual as $\mathbb{C}[z][\![Q,\mathbf{s},\tau]\!]$-modules. Therefore, part (1) of the theorem is equivalent to part (2). $\square$

2.4 Quantum Serre theorem of Coates–Givental

Coates and Givental [4] stated quantum Serre theorem as an equality of Lagrangian cones. We review the language of Lagrangian cones and explain quantum Serre theorem.

2.4.1 *Givental's symplectic vector space*

Givental's symplectic vector space for the $(\mathbf{c},E)$-twisted Gromov–Witten theory is an infinite dimensional $\mathbb{C}[\![Q,\mathbf{s}]\!]$-module:

$$\mathcal{H} = H^{\mathrm{ev}}(X)\otimes\mathbb{C}[z,z^{-1}][\![Q,\mathbf{s}]\!]$$

equipped with the anti-symmetric pairing

$$\Omega_{(\mathbf{c},E)}(f,g) = \mathrm{Res}_{z=0}(f(-z),g(z))_{(\mathbf{c},E)}dz.$$

The space $\mathcal{H}$ has a standard polarization

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$$

where $\mathcal{H}_\pm$ are $\Omega_{(\mathbf{c},E)}$-isotropic subspaces:

$$\mathcal{H}_+ = H^{\mathrm{ev}}(X) \otimes \mathbb{C}[z][\![Q, \mathbf{s}]\!]$$

$$\mathcal{H}_- = z^{-1} H^{\mathrm{ev}}(X) \otimes \mathbb{C}[z^{-1}][\![Q, \mathbf{s}]\!].$$

This polarization identifies $\mathcal{H}$ with the total space of the cotangent bundle $T^*\mathcal{H}_+$. A general element on $\mathcal{H}$ can be written in the form (note that the dual basis of $\{T_\alpha\}_{\alpha=0}^s$ with respect to the pairing $(\cdot,\cdot)_{(\mathbf{c},E)}$ is $\{\mathbf{c}(E)^{-1}T^\alpha\}_{\alpha=0}^s$):

$$\sum_{k=0}^{\infty}\sum_{\alpha=0}^{s} q_k^\alpha T_\alpha z^k + \sum_{k=0}^{\infty}\sum_{\alpha=0}^{s} p_{k,\alpha}\mathbf{c}(E)^{-1}T^\alpha \frac{1}{(-z)^{k+1}}$$

with $p_{k,\alpha}, q_k^\alpha \in \mathbb{C}[\![Q,\mathbf{s}]\!]$. The coefficients $p_{k,\alpha}, q_k^\alpha$ here give Darboux co-ordinates on $\mathcal{H}$ in the sense that $\Omega_{(\mathbf{c},E)} = \sum_{k,\alpha} dp_{k,\alpha} \wedge dq_k^\alpha$.

2.4.2 *Twisted Lagrangian cones*

The genus-zero gravitational descendant Gromov–Witten potential is a function on the formal neighbourhood of $-z\mathbf{1}$ in $\mathcal{H}_+$ defined by the formula:

$$\mathcal{F}^{0,\mathrm{grav}}_{(\mathbf{c},E)}(-z+\mathbf{t}(z)) = \langle\!\langle\rangle\!\rangle^{(\mathbf{c},E)}_{\mathbf{t}(\psi)} = \sum_{d\in \mathrm{Eff}(X)}\sum_{k=0}^{\infty}\frac{Q^d}{k!}\langle \mathbf{t}(\psi_1),\ldots,\mathbf{t}(\psi_k)\rangle_{0,k,d}$$

where $\mathbf{t}(z) = \sum_{k=0}^{\infty} t_k z^k$ with $t_k = \sum_{\alpha=0}^{s} t_k^\alpha T_\alpha$ is a formal variable in $\mathcal{H}_+$. The variables $\{t_k^\alpha\}$ are related to the variables $\{q_k^\alpha\}$ by $t_k^\alpha = q_k^\alpha + \delta_{k,1}\delta_{\alpha,0}$.

Definition 2.13. The $(\mathbf{c},E)$-twisted Lagrangian cone $\mathcal{L}_{(\mathbf{c},E)} \subset \mathcal{H}$ is the graph of the differential $\mathrm{d}\mathcal{F}^{0,\mathrm{grav}}_{(\mathbf{c},E)} : \mathcal{H}_+ \to T^*\mathcal{H}_+ \cong \mathcal{H}$. In terms of the Darboux co-ordinates above, $\mathcal{L}_{(\mathbf{c},E)}$ is cut out by the equations

$$p_{k,\alpha} = \frac{\partial \mathcal{F}^{0,\mathrm{grav}}_{(\mathbf{c},E)}}{\partial q_k^\alpha}.$$

In other words, it consists of points of the form:

$$-z + \mathbf{t}(z) + \sum_{\alpha=0}^{s}\left\langle\!\!\left\langle \frac{T^\alpha}{-z-\psi}\right\rangle\!\!\right\rangle^{(\mathbf{c},E)}_{\mathbf{t}(\psi)} \frac{T_\alpha}{\mathbf{c}(E)} \tag{2.9}$$

with $\mathbf{t}(z) \in \mathcal{H}_+$. □

Givental [13] showed that the submanifold $\mathcal{L}_{(\mathbf{c},E)}$ is in fact a cone (with vertex at the origin of $\mathcal{H}$). Moreover, he showed the following geometric property of $\mathcal{L}_{(\mathbf{c},E)}$. For every tangent space T of $\mathcal{L}_{(\mathbf{c},E)}$ (T is a linear subspace of $\mathcal{H}$),

- $zT = T \cap \mathcal{L}_{(\mathbf{c},E)}$;
- the tangent space of $\mathcal{L}_{(\mathbf{c},E)}$ at any point in $zT \subset \mathcal{L}_{(\mathbf{c},E)}$ is T.

Note that the twisted J-function (2.3) is a family of elements lying on $\mathcal{L}_{(\mathbf{c},E)}$:

$$J_{(\mathbf{c},E)}(\tau,-z) = -z + \tau + \sum_{\alpha=0}^{s} \left\langle\!\!\left\langle \frac{T_\alpha}{-z-\psi} \right\rangle\!\!\right\rangle_{\tau}^{(\mathbf{c},E)} \frac{T^\alpha}{\mathbf{c}(E)}$$

obtained from (2.9) by setting $\mathbf{t}(z) = \tau$.

Remark 2.14. In [3, Appendix B], $\mathcal{L}_{(\mathbf{c},E)}$ is defined as a formal scheme over $\mathbb{C}[\![Q,\mathbf{s}]\!]$. For a complete Hausdorff topological $\mathbb{C}[\![Q,\mathbf{s}]\!]$-algebra R, we have the notion of R-valued points on $\mathcal{L}_{(\mathbf{c},E)}$. An R-valued point on $\mathcal{L}_{(\mathbf{c},E)}$ is a point of the form (2.9) with $t_k^\alpha \in R$ such that t_k^α are topologically nilpotent, that is, $\lim_{n\to\infty}(t_k^\alpha)^n = 0$. A $\mathbb{C}[\![Q,\mathbf{s}]\!]$-valued point is given by $t_k^\alpha \in \mathbb{C}[\![Q,\mathbf{s}]\!]$ with $t_k^\alpha|_{Q=\mathbf{s}=0} = 0$. The J-function is a $\mathbb{C}[\![Q,\mathbf{s},\tau]\!]$-valued point on $\mathcal{L}_{(\mathbf{c},E)}$. In what follows we mean by a point (2.9) on $\mathcal{L}_{(\mathbf{c},E)}$ a $\mathbb{C}[\![Q,\mathbf{s}]\!]$-valued point, but the discussion applies to a general R-valued point. □

2.4.3 *Tangent space to the twisted Lagrangian cone*

Let $g = g(\mathbf{t})$ denote the point on $\mathcal{L}_{(\mathbf{c},E)}$ given in Equation (2.9). Differentiating $g(\mathbf{t})$ in t_k^α, we obtain the following tangent vector:

$$\frac{\partial g(\mathbf{t})}{\partial t_k^\alpha} = T_\alpha z^k + \sum_{\beta=0}^{s} \left\langle\!\!\left\langle T_\alpha \psi^k, \frac{T^\beta}{-z-\psi} \right\rangle\!\!\right\rangle_{\mathbf{t}(\psi)}^{(\mathbf{c},E)} \frac{T_\beta}{\mathbf{c}(E)} \quad \text{in } T_g\mathcal{L}_{(\mathbf{c},E)}. \tag{2.10}$$

The tangent space $T_g\mathcal{L}_{(\mathbf{c},E)}$ is spanned by these vectors. Since $T_g\mathcal{L}_{(\mathbf{c},E)}$ is complementary to $\mathcal{H}_-$, $T_g\mathcal{L}_{(\mathbf{c},E)}$ intersects with $\mathbf{1} + \mathcal{H}_-$ at a unique point. The intersection point is the one (2.10) with $k = \alpha = 0$:

$$(\mathbf{1} + \mathcal{H}_-) \cap T_g\mathcal{L}_{(\mathbf{c},E)} = \left\{ \frac{\partial g(\mathbf{t})}{\partial t_0^0} = \mathbf{1} - \frac{\tilde{\tau}(\mathbf{t})}{z} + O(z^{-2}) \right\}$$

where

$$\tilde{\tau}(\mathbf{t}) = \sum_{\alpha=0}^{s} \langle\!\langle \mathbf{1}, T_\alpha \rangle\!\rangle_{\mathbf{t}(z)}^{(\mathbf{c},E)} \frac{T^\alpha}{\mathbf{c}(E)}. \tag{2.11}$$

Givental [13] observed that each tangent space to the cone is uniquely parametrized by the value $\tilde{\tau}(\mathbf{t})$, that is, the tangent spaces at $g(\mathbf{t}_1)$ and $g(\mathbf{t}_2)$ are equal if and only if $\tilde{\tau}(\mathbf{t}_1) = \tilde{\tau}(\mathbf{t}_2)$. The string equation shows that $\tilde{\tau}(\mathbf{t}) = \tau$ when $\mathbf{t}(z) = \tau$. Hence $T_{g(\mathbf{t})}\mathcal{L}_{(\mathbf{c},E)}$ equals the tangent space at $g(\tilde{\tau}(\mathbf{t})) = J_{(\mathbf{c},E)}(\tilde{\tau}(\mathbf{t}), -z)$.

Proposition 2.15 ([13], see also [3, Proposition B.4]).

Let $g = g(\mathbf{t})$ denote the point of $\mathcal{L}_{(\mathbf{c},E)}$ given in Equation (2.9). The tangent space $T_g\mathcal{L}_{(\mathbf{c},E)}$ is a free $\mathbb{C}[z][\![Q,\mathbf{s}]\!]$-module generated by the derivatives of the twisted J-function:

$$\left.\frac{\partial J_{(\mathbf{c},E)}}{\partial t^\alpha}(\tau, -z)\right|_{\tau=\tilde{\tau}(\mathbf{t})}. \qquad \square$$

Proof. As discussed, $T_g\mathcal{L}_{(\mathbf{c},E)}$ equals the tangent space of $\mathcal{L}_{(\mathbf{c},E)}$ at $J(\tau,-z)$ with $\tau = \tilde{\tau}(\mathbf{t})$. On the other hand, the tangent space at $J_{(\mathbf{c},E)}(\tau,-z)$ is freely generated by the derivatives $\partial_\alpha J_{(\mathbf{c},E)}(\tau,-z)$ [3, Lemma B.5], and the result follows. ■

2.4.4 *Relations between Lagrangian cones and* $\mathrm{QDM}_{(\mathbf{c},E)}(X)$

Proposition 2.15 means that the quantum D-module can be identified with the family of tangent spaces to the Lagrangian cone $\mathcal{L}_{(\mathbf{c},E)}$ at the J-function $J_{(\mathbf{c},E)}(\tau,-z)$.

$$\begin{aligned}(-)^*\mathrm{QDM}_{(\mathbf{c},E)}(X)_\tau &\cong T_{J_{(\mathbf{c},E)}(\tau,-z)}\mathcal{L}_{(\mathbf{c},E)}\\ T_\alpha &\mapsto \partial_\alpha J_{(\mathbf{c},E)}(\tau,-z) = L_{(\mathbf{c},E)}(\tau,-z)^{-1}T_\alpha\end{aligned}$$

where $(-)^*$ denotes the pull back by the sign change $z \mapsto -z$ and we used (2.4). This identification preserves the pairing $S_{(\mathbf{c},E)}$ and intertwines the quantum connection ∇ on $\mathrm{QDM}_{(\mathbf{c},E)}(X)$ with the trivial differential d on $\mathcal{H}$: this follows from the properties of $L_{(\mathbf{c},E)}$ in Proposition 2.5.

2.4.5 *Quantum Serre theorem of Coates–Givental*

Theorem 2.16 (Coates–Givental [4, Corollary 9]). The multiplication by $\mathbf{c}(E)$ defines a symplectomorphism $\mathbf{c}(E)\colon (\mathcal{H}, \Omega_{\mathbf{c}(E)}) \to (\mathcal{H}, \Omega_{\mathbf{c}^*(E^\vee)})$ and identifies the twisted Lagrangian cones:

$$\mathbf{c}(E)\mathcal{L}_{(\mathbf{c},E)} = \mathcal{L}_{(\mathbf{c}^*,E^\vee)}. \qquad \square$$

For any $\gamma = \sum_{\alpha=0}^{s} \gamma^\alpha T_\alpha \in H^{\mathrm{ev}}(X)$, we write $\partial_\gamma := \sum_{\alpha=0}^{s}\gamma^\alpha\partial_\alpha$ for the directional derivative.

Corollary 2.17. Let T_τ denote the tangent space of $\mathcal{L}_{(\mathbf{c},E)}$ at $J_{(\mathbf{c},E)}(\tau,-z)$ and let T^*_τ denote the tangent space of $\mathcal{L}_{(\mathbf{c}^*,E^\vee)}$ at $J_{(\mathbf{c}^*,E^\vee)}(\tau,-z)$. Then we have the following equations:

(1) $\mathbf{c}(E)T_\tau = T^*_{f(\tau)}$, where f was defined in (2.7);

(2) $\mathbf{c}(E)\partial_\alpha J_{(\mathbf{c},E)}(\tau,-z) = (\partial_{\mathbf{c}(E)\cup T_\alpha} J_{(\mathbf{c}^*,E^\vee)})(\tau^*,-z)|_{\tau^*=f(\tau)}$ for $\alpha = 0,1,\dots,s$. □

Remark 2.18. Exchanging the twist $(\mathbf{c},E)$ with $(\mathbf{c}^*,E^\vee)$ in the above Corollary, we obtain

$$\mathbf{c}^*(E^\vee)z\partial_{\mathbf{c}(E)}J_{(\mathbf{c}^*,E^\vee)}(\tau,z) = J_{(\mathbf{c},E)}(\tilde{f}(\tau),z)$$

where $\tilde{f}(\tau) = \sum_{\alpha=0}^{s}\langle\langle T_\alpha,\mathbf{c}(E)\rangle\rangle_\tau^{(\mathbf{c}^*,E^\vee)}T^\alpha$. This is exactly [4, Corollary 10]. □

Proof of Corollary 2.17. (1) Theorem 2.16 implies that $\mathbf{c}(E)T_\tau$ is a tangent space to the cone $\mathcal{L}_{(\mathbf{c}^*,E^\vee)}$. Therefore, by the discussion in the previous section Section 2.4.3, $\mathbf{c}(E)T_\tau$ equals T^*_σ with σ given by the intersection point:

$$(\mathbf{1}+\mathcal{H}_-)\cap\mathbf{c}(E)T_\tau = \left\{\mathbf{1}-\frac{\sigma}{z}+O(z^{-2})\right\}. \tag{2.12}$$

Note that

$$\partial_{\mathbf{c}^*(E^\vee)}J_{(\mathbf{c},E)}(\tau,-z) = \mathbf{c}^*(E^\vee) - \frac{1}{z}\sum_{\alpha=0}^{s}\langle\langle T^\alpha,\mathbf{c}^*(E^\vee)\rangle\rangle_\tau\frac{T_\alpha}{\mathbf{c}(E)} + O(z^{-2})$$

lies in T_τ. Multiplying this by $\mathbf{c}(E)$, we obtain the intersection point in (2.12) and we have $\sigma = f(\tau)$ as required (recall that $\mathbf{c}(E)\mathbf{c}^*(E^\vee)=1$).

(2) By part (1), the vector $\mathbf{c}(E)\partial_\alpha J_{(\mathbf{c},E)}(\tau,-z)$ belongs to the tangent space $T^*_{f(\tau)}$ of $\mathcal{L}_{(\mathbf{c}^*,E^\vee)}$. It has the following asymptotics:

$$\mathbf{c}(E)\partial_\alpha J_{(\mathbf{c},E)}(\tau,-z) = \mathbf{c}(E)\cup T_\alpha + O(z^{-1}).$$

By the description of tangent spaces in Section 2.4.3, a tangent vector in $T^*_{f(\tau)}$ with this asymptotics is unique and is given by $(\partial_{\mathbf{c}(E)\cup T_\alpha}J_{(\mathbf{c}^*,E^\vee)})(\tau^*,-z)$ with $\tau^*=f(\tau)$. ■

2.5 A proof of Theorem 2.11

We use the correspondence in Section 2.4.4 between quantum D-module and tangent spaces to the Givental cone. Then Corollary 2.17 implies that the map $\mathbf{c}(E)\colon \mathrm{QDM}_{(\mathbf{c},E)}(X)\to f^*\mathrm{QDM}_{(\mathbf{c}^*,E^\vee)}(X)$ respects the quantum connection. Also it is obvious that the map $\mathbf{c}(E)$ intertwines the pairings $S_{(\mathbf{c},E)}$ and $f^*S_{(\mathbf{c}^*,E^\vee)}$. This shows part

(2) of the theorem. Also part (2) of Corollary 2.17 implies, in view of (2.4)

$$\mathbf{c}(E)L_{(\mathbf{c},E)}(\tau,-z)^{-1}T_\alpha = L_{(\mathbf{c}^*,E^\vee)}(f(\tau),-z)^{-1}(\mathbf{c}(E)\cup T_\alpha)$$

for $\alpha=0,1,\dots,s$. This implies the first equation of part (3). To see the second equation of part (3), we calculate:

$$\begin{aligned} S^{\mathrm{QS}}(u,v) &= S_{(\mathbf{c},E)}(u,\mathbf{c}(E)^{-1}v) = S_{(\mathbf{c},E)}(L_{(\mathbf{c},E)}u, L_{(\mathbf{c},E)}\mathbf{e}(E)^{-1}v) \\ &= S_{(\mathbf{c},E)}(L_{(\mathbf{c},E)}u,\mathbf{c}(E)^{-1}(f^*L_{(\mathbf{c}^*,E^\vee)})v) = S^{\mathrm{QS}}(L_{(\mathbf{c},E)},(f^*L_{(\mathbf{c}^*,E^\vee)})v) \end{aligned}$$

where we used Proposition 2.5. Part (1) of the theorem is equivalent to part (2), as explained in Remark 2.12.

3 Quantum Serre Duality for Euler-Twisted Theory

In this section we apply Theorem 2.11 to the equivariant Euler class $\mathbf{e}_\lambda$ and a convex vector bundle E. By taking the non-equivariant limit, we obtain a relationship among the quantum D-module twisted by the Euler class and the bundle E, the quantum D-module of the total space of $E^\vee$, and the quantum D-module of a submanifold $Z\subset X$ cut out by a regular section of E.

To ensure the well-defined non-equivariant limit, we assume that our vector bundle $E\to X$ is *convex*, that is, for every genus-zero stable map $f\colon C\to X$ we have $H^1(C,f^*E)=0$. The convexity assumption is satisfied, for example, if $\mathcal{O}(E)$ is generated by global sections.

3.1 Equivariant Euler class

In this section, we take $\mathbf{c}$ to be the $\mathbb{C}^\times$-equivariant Euler class $\mathbf{e}_\lambda$. Given a vector bundle G, we let $\mathbb{C}^\times$ act on G by scaling the fibres and trivially on X. With respect to this $\mathbb{C}^\times$-action we have

$$\mathbf{e}_\lambda(G)=\sum_{i=0}^{\mathrm{rk}G}\lambda^{r-i}c_i(G)$$

where λ is the $\mathbb{C}^\times$-equivariant parameter: the $\mathbb{C}^\times$-equivariant cohomology of a point is $H^*_{\mathbb{C}^\times}(\mathrm{pt})=\mathbb{C}[\lambda]$. Choosing $\mathbf{e}_\lambda$ means the following specialization:

$$s_k:=\begin{cases}\log\lambda & \text{if } k=0\\ (-1)^{k-1}(k-1)!\lambda^{-k} & \text{if } k>0\end{cases}$$

Although the parameters s_k contain $\log\lambda$ and negative powers of λ, we will see that the $(\mathbf{e}_\lambda, E)$-twisted theory and $(\mathbf{e}_\lambda^{-1}, E^\vee)$-twisted theory are defined over the polynomial ring in λ, and hence admit the non-equivariant limit $\lambda \to 0$. Here the convexity of E plays a role.

3.2 Specialization of the Novikov variable

We henceforth specialize the Novikov variable Q to one. By Remark 2.3, the specialization $Q = 1$ is well defined: one has

$$T_\alpha \bullet_\tau^{(\mathbf{c},E)} T_\beta|_{Q=1} \in H^{\mathrm{ev}}(X) \otimes \mathbb{C}[\![\mathrm{e}^{\tau_2}, \mathbf{s}, \tau']\!]$$

$$L_{(\mathbf{c},E)}(\tau, z)|_{Q=1} \in \mathrm{End}(H^{\mathrm{ev}}(X)) \otimes \mathbb{C}[z^{-1}][\![\mathrm{e}^{\tau_2}, \mathbf{s}, \tau']\!][\tau_2]$$

$$f(\tau)|_{Q=1} \in H^{\mathrm{ev}}(X) \otimes \mathbb{C}[\![\mathrm{e}^{\tau_2}, \mathbf{s}, \tau']\!] \quad \text{see (2.7)}$$

where $\mathbb{C}[\![\mathrm{e}^{\tau_2}, \mathbf{s}, \tau']\!]$ is the completion of $\mathbb{C}[\mathrm{e}^{t^1}, \dots, \mathrm{e}^{t^s}, t^0, t^{r+1}, \dots, t^s, s_0, s_1, s_2, \dots]$. Since we chose $T_1, \dots, T_r$ to be a nef integral basis of $H_2(X, \mathbb{Z})$, we have only nonnegative integral powers of $\mathrm{e}^{t^1}, \dots, \mathrm{e}^{t^r}$ in the structure constants of quantum cohomology. The Euler-twisted quantum D-module will be defined over $\mathbb{C}[z][\![\mathrm{e}^{\tau_2}, \tau']\!]$. In what follows, we shall omit $(\cdots)|_{Q=1}$ from the notation.

3.3 Non-Equivariant limit of $\mathrm{QDM}_{(\mathbf{e}_\lambda, E)}(X)$

Let $\mathrm{e} = \lim_{\lambda\to 0} \mathbf{e}_\lambda$ denote the non-equivariant Euler class. We first discuss the non-equivariant limit of $\mathrm{QDM}_{(\mathbf{e}_\lambda, E)}(X)$. Recall the K-class $E_{0,\ell,d}$ on the moduli space $\bar{\mathcal{M}}_{0,\ell}(X, d)$ introduced in Section 2.2.1. The convexity assumption for E implies that $E_{0,\ell,d}$ is represented by a vector bundle. Moreover, the natural evaluation morphism $E_{0,\ell,d} \to \mathrm{ev}_j^* E$ at the jth marking is surjective for all $j \in \{1, \dots, \ell\}$. Define $E_{0,\ell,d}(j)$ by the exact sequence:

$$0 \longrightarrow E_{0,\ell,d}(j) \longrightarrow E_{0,\ell,d} \longrightarrow \mathrm{ev}_j^* E \longrightarrow 0 \quad . \tag{3.1}$$

We use the following variant of $(\mathbf{e}_\lambda, E)$-twisted invariants (see [33]). For any $\gamma_1, \dots, \gamma_\ell \in H^{\mathrm{ev}}(X)$ and any $k_1, \dots, k_\ell \in \mathbb{N}$, we put:

$$\langle \gamma_1 \psi_1^{k_1}, \dots, \widetilde{\gamma_j \psi_j^{k_j}}, \dots, \gamma_\ell \psi_\ell^{k_\ell} \rangle_{0,\ell,d}^{(\mathbf{e}_\lambda, E)} := \int_{[\bar{\mathcal{M}}_{0,\ell}(X,d)]^{\mathrm{vir}}} \left(\prod_{i=1}^{\ell} \psi_i^{k_i} \mathrm{ev}_i^* \gamma_i \right) \mathbf{e}_\lambda(E_{0,\ell,d}(j)).$$

This lies in the polynomial ring $\mathbb{C}[\lambda]$.

Lemma 3.1 ([33]). Suppose that E is convex. The $(\mathbf{e}_\lambda, E)$-twisted quantum product $T_\alpha \bullet_\tau^{(\mathbf{e}_\lambda, E)} T_\beta$ lies in $H^{\mathrm{ev}}(X) \otimes \mathbb{C}[\lambda][\![\mathrm{e}^{\tau_2}, \tau']\!]$ and admits the non-equivariant limit $T_\alpha \bullet_\tau^{(\mathrm{e}, E)} T_\beta := \lim_{\lambda\to 0}(T_\alpha \bullet_\tau^{(\mathbf{e}_\lambda, E)} T_\beta)$. □

Proof. Recall that the twisted quantum product (2.1) is given by:

$$\gamma_1 \bullet_\tau^{(\mathbf{e}_\lambda, E)} \gamma_2 = \sum_{\alpha=0}^{s} \left\langle\!\!\left\langle \gamma_1, \gamma_2, \frac{T_\alpha}{\mathbf{e}_\lambda(E)} \right\rangle\!\!\right\rangle_\tau^{(\mathbf{e}_\lambda, E)} T^\alpha.$$

From the exact sequence (3.1), we deduce that

$$\frac{\mathbf{e}_\lambda(E_{0,\ell,d})}{\mathrm{ev}_3^* \mathbf{e}_\lambda(E)} = \mathbf{e}_\lambda(E_{0,\ell,d}(3)).$$

Therefore, we have

$$\gamma_1 \bullet_\tau^{(\mathbf{e}_\lambda, E)} \gamma_2 = \sum_{\alpha=0}^{s} \langle\!\langle \gamma_1, \gamma_2, \tilde{T}_\alpha \rangle\!\rangle_\tau^{(\mathbf{e}_\lambda, E)} T^\alpha$$

and this lies in $H^{\mathrm{ev}}(X) \otimes \mathbb{C}[\lambda][\![\mathrm{e}^{\tau_2}, \tau']\!]$. ■

Lemma 3.2. Suppose that E is convex. For $(\mathbf{c}, E) = (\mathbf{e}_\lambda, E)$, the map $f(\tau)$ in (2.7) lies in $H^{\mathrm{ev}}(X) \otimes \mathbb{C}[\lambda][\![\mathrm{e}^{\tau_2}, \tau']\!]$ and admits the non-equivariant limit $\bar{f}(\tau) := \lim_{\lambda\to 0} f(\tau)$. □

Proof. Note that $\mathbf{c}^*(E^\vee) = \mathbf{e}_\lambda^*(E^\vee)$ in (2.7) equals $\mathbf{e}_\lambda(E)^{-1}$. Arguing as in Lemma 3.1, we have

$$f(\tau) = \sum_{\alpha=0}^{s} \langle\!\langle T_\alpha, \tilde{1} \rangle\!\rangle_\tau^{(\mathbf{e}_\lambda, E)} T^\alpha.$$

The conclusion follows. ■

By Lemma 3.1, we deduce that the non-equivariant limit $\nabla^{(\mathrm{e}, E)} = \lim_{\lambda\to 0} \nabla^{(\mathbf{e}_\lambda, E)}$ of the quantum connection exists. Moreover, it can be completed in the z-direction in a flat connection:

$$\nabla_{z\partial_z}^{(\mathrm{e}, E)} = z\partial_z - \frac{1}{z}(\mathfrak{E}^{(\mathrm{e}, E)} \bullet_\tau^{(\mathrm{e}, E)}) + \frac{\deg}{2}$$

where $\mathfrak{E}^{(\mathrm{e}, E)}$ is the Euler vector field:

$$\mathfrak{E}^{(\mathrm{e}, E)} = \sum_{\alpha=0}^{s} \left(1 - \frac{\deg T_\alpha}{2}\right) t^\alpha T_\alpha + c_1(TX) - c_1(E). \tag{3.2}$$

The non-equivariant limit $S_{(\mathrm{e}, E)}(u, v) := \int_X u(-z) \cup v(z) \cup \mathrm{e}(E)$ of the pairing $S_{(\mathbf{e}_\lambda, E)}$ becomes degenerate. The pairing $S_{(\mathrm{e}, E)}$ is not flat in the z-direction, but satisfies the following equation:

$$z\partial_z S_{(\mathrm{e},E)}(u,v) - S_{(\mathrm{e},E)}(\nabla^{(\mathrm{e},E)}_{z\partial_z} u, v) - S_{(\mathrm{e},E)}(u, \nabla^{(\mathrm{e},E)}_{z\partial_z} v) = -(\dim X - \mathrm{rk} E) S_{(\mathrm{e},E)}(u,v).$$

We refer to this by saying that $S_{(\mathrm{e},E)}$ *is of weight* $-(\dim X - \mathrm{rk}E)$. Note that $z^{\dim X - \mathrm{rk}E} S_{(\mathrm{e},E)}$ is flat in both τ and z.

Definition 3.3 (cf. Definition 2.1)**.** We call the triple

$$\mathrm{QDM}_{(\mathrm{e},E)}(X) := (H^{\mathrm{ev}}(X) \otimes \mathbb{C}[z][\![\mathrm{e}^{\tau_2}, \tau']\!], \nabla^{(\mathrm{e},E)}, S_{(\mathrm{e},E)})$$

the (e, E)-*twisted quantum D-module.* □

Remark 3.4. By a similar argument, the fundamental solution $L_{(\mathbf{e}_\lambda,E)}$ in Proposition 2.5 can be written as:

$$L_{(\mathbf{e}_\lambda,E)}(\tau,z)\gamma = \gamma - \sum_{\alpha=0}^{s} \left\langle\!\!\left\langle \frac{\gamma}{z+\psi}, \tilde{T}_\alpha \right\rangle\!\!\right\rangle_\tau^{(\mathbf{e}_\lambda,E)} T^\alpha$$

and therefore admits the non-equivariant limit $L_{(\mathrm{e},E)}$. The fundamental solution for $\nabla^{(\mathrm{e},E)}$, including in the z-direction, is given by $L_{(\mathrm{e},E)}(\tau,z) z^{-\frac{\deg}{2}} z^{c_1(TX)-c_1(E)}$. All the properties of Proposition 2.5 are true for the limit (see [28, Section 2] for a more precise statement). □

3.4 Quantum D-module of a section of E

In this section, we describe a relationship between the (e, E)-twisted quantum D-module and the quantum D-module of a submanifold $Z \subset X$ cut out by a regular section of E.

Let $\iota\colon Z \to X$ denote the natural inclusion. The functoriality of virtual classes [20]

$$[\bar{\mathcal{M}}_{0,\ell}(X,d)]^{\mathrm{vir}} \cap \mathrm{e}(E_{0,\ell,d}) = \sum_{\iota_*(d')=d} \iota_*[\bar{\mathcal{M}}_{0,\ell}(Z,d')]^{\mathrm{vir}}$$

together with the argument in [18, Corollary 2.5; 33] shows that

$$\iota^*(\gamma_1 \bullet^{(\mathrm{e},E)}_\tau \gamma_2) = (\iota^*\gamma_1) \bullet^{Z}_{\iota^*\tau} (\iota^*\gamma_2) \tag{3.3}$$

for $\gamma_1, \gamma_2 \in H^{\mathrm{ev}}(X)$. Define the ambient part of the cohomology of Z by $H^*_{\mathrm{amb}}(Z) = \mathrm{Im}(\iota\colon H^*(X) \to H^*(Z))$. Equation (3.3) shows that the ambient part $H^*_{\mathrm{amb}}(Z)$ is closed under the quantum product $\bullet^Z_\tau$ of Z as long as τ lies in the ambient part.

Definition 3.5. The *ambient part quantum D-module* of Z is a triple

$$\mathrm{QDM}_{\mathrm{amb}}(Z) := (H^{\mathrm{ev}}_{\mathrm{amb}}(Z) \otimes \mathbb{C}[z][\![\mathrm{e}^{\tau_2}, \tau']\!], \nabla^Z, S_Z)$$

where the parameter $\tau = \tau_2 + \tau'$ is restricted to lie in the ambient part $H^{\mathrm{ev}}_{\mathrm{amb}}(Z)$ and $S_Z(u, v) = \int_Z u(-z) \cup v(z)$. We complete the quantum connection ∇^Z in the z-direction as in Remark 2.4; then S_Z is of weight $(-\dim Z)$. □

Equation (3.3) proves the following proposition.

Proposition 3.6. The restriction map $\iota^* \colon H^{\mathrm{ev}}(X) \to H^{\mathrm{ev}}_{\mathrm{amb}}(Z)$ induces a morphism between the quantum D-modules

$$\iota^* \colon \mathrm{QDM}_{(\mathrm{e},E)}(X) \to (\iota^*)^* \mathrm{QDM}_{\mathrm{amb}}(Z)$$

which is compatible with the connection and the pairing. □

3.5 Quantum D-module of the total space of $E^\vee$

We explain that the non-equivariant limit of the $(\mathbf{e}_\lambda^{-1}, E^\vee)$-twisted quantum D-module is identified with the quantum D-module of the total space of $E^\vee$.

The $(\mathbf{e}_\lambda^{-1}, E^\vee)$-twisted Gromov–Witten invariants admit a non-equivariant limit under the *concavity* (a bundle $E^\vee$ is said to be *concave* if for every non-constant genus-zero stable map $f \colon C \to X$, one has $H^0(C, f^*E^\vee) = 0$) assumption for $E^\vee$ and they are called *local Gromov–Witten invariants* [2, 11, 12]. In this paper, we only impose the weaker assumption that E is convex (see Remark 3.9). In this case, a non-equivariant limit of $(\mathbf{e}_\lambda^{-1}, E^\vee)$-twisted invariants may not exist, but a non-equivariant limit of the twisted quantum product is still well-defined.

The virtual localization formula [14] gives the following proposition.

Proposition 3.7. For $\gamma_1, \dots, \gamma_\ell \in H^{\mathrm{ev}}(X)$ and non-negative integers $k_1, \dots, k_\ell$, we have

$$\langle \gamma_1 \psi^{k_1}, \dots, \gamma_\ell \psi^{k_\ell} \rangle_{0,\ell,d}^{(\mathbf{e}_\lambda^{-1}, E^\vee)} = \langle \gamma_1 \psi^{k_1}, \dots, \gamma_\ell \psi^{k_\ell} \rangle_{0,\ell,d}^{E^\vee, \mathbb{C}^\times}$$

where the right-hand side is the $\mathbb{C}^\times$-equivariant Gromov–Witten invariant of $E^\vee$ with respect to the $\mathbb{C}^\times$-action on $E^\vee$ scaling the fibres. □

The non-equivariant Gromov–Witten invariants for $E^\vee$ are ill-defined in general because the moduli space $\bar{\mathcal{M}}_{0,\ell}(E^\vee, d)$ can be non-compact. The following lemma, however, shows the existence of the non-equivariant quantum product of $E^\vee$.

Lemma 3.8. Let E be a vector bundle on X such that f^*E is generated by global sections for any stable maps $f\colon C \to X$ of genus g. Then the evaluation map $\mathrm{ev}_i\colon \bar{\mathcal{M}}_{g,\ell}(E^\vee, d) \to E^\vee$ is proper for all $i \in \{1, \dots, \ell\}$. In particular, when E is convex, $\mathrm{ev}_i\colon \bar{\mathcal{M}}_{0,\ell}(E^\vee, d) \to E^\vee$ is proper. □

Proof. The convexity of E implies that, for any map $u\colon \mathbb{P}^1 \to X$, u^*E is isomorphic to $\bigoplus_{i=1}^r \mathcal{O}(k_i)$ with $k_i \geq 0$. Thus the latter statement follows from the former.

Let us prove the former statement. We start with the remark that, for every stable map $f\colon C \to X$, the evaluation map $\mathrm{ev}_i\colon H^0(C, f^*E^\vee) \to E^\vee_{f(x_i)}$ at the ith marking $x_i \in C$ is injective. Suppose that a section $s \in H^0(C, f^*E^\vee)$ vanishes at x_i, that is, $\mathrm{ev}_i(s) = 0$. For every $u \in H^0(C, f^*E)$, the pairing $\langle s, u\rangle$ is a global section of $\mathcal{O}_C$ which vanishes at x_i. Then $\langle s, u\rangle$ must be identically zero on C. Since f^*E is generated by global sections, this implies that $s = 0$. Hence we have shown that the evaluation map $H^0(C, f^*E^\vee) \to E^\vee_{f(x_i)}$ is injective.

Giving a stable map to $E^\vee$ is equivalent to giving a stable map $f\colon C \to X$ and a section of $H^0(C, f^*E^\vee)$. Therefore, by the preceding remark, the moduli functor $\bar{\mathcal{M}}_{g,\ell}(E^\vee, d)$ is a subfunctor of $\bar{\mathcal{M}}_{g,\ell}(X, d) \times E^\vee$ via the natural projection $\bar{\mathcal{M}}_{g,\ell}(E^\vee, d) \to \bar{\mathcal{M}}_{g,\ell}(X, d)$ and the evaluation map $\mathrm{ev}_i\colon \bar{\mathcal{M}}_{g,\ell}(E^\vee, d) \to E^\vee$. Since $\bar{\mathcal{M}}_{g,\ell}(X, d)$ is proper, it suffices to show that the map $\bar{\mathcal{M}}_{g,\ell}(E^\vee, d) \to \bar{\mathcal{M}}_{g,\ell}(X, d) \times E^\vee$ is proper. We use the valuative criterion for properness (see [6, Theorem 4.19]). Let R be a DVR. Suppose that we are given a stable map $f\colon C_R \to X$ over $\mathrm{Spec}(R)$ and an R-valued point $v \in E^\vee(R)$. These data (f, v) give a map $\mathrm{Spec}(R) \to \bar{\mathcal{M}}_{g,\ell}(X, d) \times E^\vee$. Suppose moreover that there exists a section $s \in H^0(C_K, f^*E^\vee)$ over the field K of fractions of R such that $\mathrm{ev}_i(s) = v$ in $E^\vee(K)$, where $C_K = C_R \times_{\mathrm{Spec}(R)} \mathrm{Spec}(K)$. Then (f, s) defines a map $\mathrm{Spec}(K) \to \bar{\mathcal{M}}_{g,\ell}(E^\vee, d)$ such that the following diagram commutes:

$$\begin{tikzcd}
\bar{\mathcal{M}}_{g,\ell}(E^\vee, d) \arrow[r] & \bar{\mathcal{M}}_{g,\ell}(X, d) \times E^\vee \\
\mathrm{Spec}(K) \arrow[u, "(f,s)"] \arrow[r] & \mathrm{Spec}(R). \arrow[u, "(f,v)"'] \arrow[ul, dashed]
\end{tikzcd}$$

We will show that there exists a morphism $\mathrm{Spec}(R) \to \bar{\mathcal{M}}_{g,\ell}(E^\vee, d)$ which commutes with the maps in the above diagram. Since $\bar{\mathcal{M}}_{g,\ell}(E^\vee, d)$ is a subfunctor of $\bar{\mathcal{M}}_{g,\ell}(X, d) \times E^\vee$, it suffices to show the existence of a morphism $\mathrm{Spec}(R) \to \bar{\mathcal{M}}_{g,\ell}(E^\vee, d)$ which makes the upper-right triangle commutative, that is, v is the image of a section in $H^0(C_R, f^*E^\vee)$. Let $\pi : C_R \to \mathrm{Spec}(R)$ denote the structure map and $x_i : \mathrm{Spec}(R) \to C_R$ denote the ith marking.

Note that the composition $\mathrm{Spec}(R) \xrightarrow{v} E^\vee \to X$ coincides with $f \circ x_i$, since the two maps coincide when we compose them with $\mathrm{Spec}(K) \to \mathrm{Spec}(R)$ (by the existence of s) and by the separatedness of X. Thus v defines a section of $x_i^* f^* E^\vee$, which we denote again by v. We need to show that v is in the image of $R^0\pi_* f^* E^\vee \to x_i^* f^* E^\vee$. Let $p \in \mathrm{Spec}(R)$ denote the unique closed point and let $k(p)$ be the residue field at p. We claim that the maps $R^0\pi_* f^* E^\vee \bigotimes_R k(p) \to H^0(C_p, f^* E^\vee)$, $H^0(C_p, f^* E^\vee) \to (x_i^* f^* E^\vee) \bigotimes_R k(p)$ are injective. The injectivity of the latter map has been shown. To see the injectivity of the former, we take the so-called Grothendieck complex [31, Section 5, p. 46]: a complex $G^0 \to G^1$ of finitely generated free R-modules such that the sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & R^0\pi_* f^* E^\vee & \longrightarrow & G^0 & \xrightarrow{d^0} & G^1 \\ 0 & \longrightarrow & H^0(C_p, f^* E^\vee) & \longrightarrow & G^0 \bigotimes_R k(p) & \longrightarrow & G^1 \bigotimes_R k(p) \end{array}$$

are exact. Since R is a PID, the image of d^0 is a free R-module. Therefore $\mathrm{Tor}_1^R(\mathrm{Im} d^0, k(p)) = 0$ and we obtain the exact sequence:

$$0 \longrightarrow (R^0\pi_* f^* E^\vee) \bigotimes_R k(p) \longrightarrow G^0 \bigotimes_R k(p) \longrightarrow (\mathrm{Im} d^0) \bigotimes_R k(p) \longrightarrow 0.$$

Now the claim follows. The claim implies that $R^0\pi_* f^* E^\vee \to x_i^* f^* E^\vee$ is injective at the fibre of p. Then it follows that the cokernel M of $R^0\pi_* f^* E^\vee \to x_i^* f^* E^\vee$ is a free R-module. In fact, let N be the image of $R^0\pi_* f^* E^\vee \to x_i^* f^* E^\vee$; then the inclusion $N \subset x_i^* f^* E^\vee$ induces an injection $N \bigotimes_R k(p) \to (x_i^* f^* E^\vee) \bigotimes_R k(p)$. Because $x_i^* f^* E^\vee$ is a free R-module, we have the exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^R(M, k(p)) \longrightarrow N \bigotimes_R k(p) \longrightarrow x_i^* f^* E^\vee \bigotimes_R k(p) \longrightarrow M \bigotimes_R k(p) \longrightarrow 0.$$

Therefore, $\mathrm{Tor}_1^R(M, k(p)) = 0$; thus M is free. We know by assumption that the image of v in $M \bigotimes_R K$ vanishes. Thus v has to vanish in M. The conclusion follows. ■

Remark 3.9. The concavity of $E^\vee$ implies the convexity of E. This can be proved as follows. For a stable map $f\colon C \to X$ of genus zero, we have $H^1(C, f^* E) = H^0(C, f^* E^\vee \otimes \omega_C)^\vee$ by Serre duality, where ω_C is the dualizing sheaf on C. Suppose that $E^\vee$ is concave. Since the degree of ω_C on a tail component of C is negative, a section of $f^* E^\vee \otimes \omega_C$ has to vanish on tail components and defines a section of $f^* E^\vee \otimes \omega_{C'}$ where C' is obtained from C by removing all its tail components. By induction on the number of components, we can see that a section of $f^* E^\vee \otimes \omega_C$ vanishes and $H^0(C, f^* E^\vee \otimes \omega_C) = 0$. □

By the above lemma, we can define the quantum product of $E^\vee$ using the push-forward along the evaluation map ev_3:

$$T_\alpha \bullet_\tau^{E^\vee} T_\beta = \sum_{d \in \mathrm{Eff}(E^\vee)} \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \mathrm{ev}_{3*} \left(\mathrm{ev}_1^*(T_\alpha) \mathrm{ev}_2^*(T_\beta) \prod_{j=4}^{\ell+3} \mathrm{ev}_j^*(\tau) \cap [\bar{\mathcal{M}}_{0,3+\ell}(E^\vee, d)]^{\mathrm{vir}} \right). \tag{3.4}$$

The quantum product $\bullet_\tau^{E^\vee}$ defines a flat quantum connection $\nabla^{E^\vee}$ as in Definition 2.1.

Definition 3.10. The (non-equivariant) *quantum D-module* of $E^\vee$ is a pair

$$\mathrm{QDM}(E^\vee) = (H^{\mathrm{ev}}(X) \otimes \mathbb{C}[z][\![e^{\tau_2}, \tau']\!], \nabla^{E^\vee})$$

where the connection $\nabla^{E^\vee}$ is completed in the z-direction as in Remark 2.4:

$$\nabla^{E^\vee}_{z\partial_z} = z\partial_z - \frac{1}{z}(\mathfrak{E}^{E^\vee} \bullet_\tau) + \frac{\deg}{2}$$

where $\mathfrak{E}^{E^\vee} := \sum_{\alpha=0}^{s} (1 - \frac{1}{2} \deg T_\alpha) t^\alpha T_\alpha + c_1(TX) - c_1(E)$ is the Euler vector field (note that this is the same as $\mathfrak{E}^{(\mathrm{e},E)}$ in (3.2)). Here the standard identification $H^{\mathrm{ev}}(E^\vee) \cong H^{\mathrm{ev}}(X)$ is understood. □

We conclude the following proposition.

Proposition 3.11. Suppose that E is convex. Define the map $h \colon H^{\mathrm{ev}}(X) \to H^{\mathrm{ev}}(X)$ by

$$h(\tau) = \tau + \pi\sqrt{-1} c_1(E) \tag{3.5}$$

Then we have

(1) The non-equivariant limit of $\nabla^{(\mathbf{e}_\lambda^{-1}, E^\vee)}$ exists and coincides with $\nabla^{E^\vee}$.

(2) The non-equivariant limit of $\nabla^{(\mathbf{e}_\lambda^*, E^\vee)}$ exists and coincides with $h^*\nabla^{E^\vee}$. □

Proof. Part (1) follows from Proposition 3.7 and the existence of the non-equivariant quantum product for $E^\vee$. To see part (2), notice a small difference between $\mathbf{e}_\lambda^*$ and $\mathbf{e}_\lambda^{-1}$: for a vector bundle G we have

$$\mathbf{e}_\lambda^*(G) = \frac{1}{\mathbf{e}_\lambda(G^\vee)} = (-1)^{\mathrm{rk} G} \frac{1}{\mathbf{e}_{-\lambda}(G)}.$$

Since the virtual rank of $(E^\vee)_{0,\ell,d}$ equals $\mathrm{rk}E + c_1(E)(d)$, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{1}{n!}\langle T_\alpha, T_\beta, T_\gamma, \tau, \ldots, \tau\rangle_{0,n+3,d}^{(\mathbf{e}^*_\lambda, E^\vee)}\frac{T^\gamma}{\mathbf{e}^*_\lambda(E^\vee)} \\
&\quad = \sum_{n=0}^{\infty}\frac{1}{n!}(-1)^{\mathrm{rk}E + c_1(E)(d)}\langle T_\alpha, T_\beta, T_\gamma, \tau, \ldots, \tau\rangle_{0,n+3,d}^{(\mathbf{e}^{-1}_{-\lambda}, E^\vee)}\frac{T^\gamma}{\mathbf{e}^*_\lambda(E^\vee)} \\
&\quad = \sum_{n=0}^{\infty}\frac{1}{n!}\langle T_\alpha, T_\beta, T_\gamma, h(\tau), \ldots, h(\tau)\rangle_{0,n+3,d}^{(\mathbf{e}^{-1}_{-\lambda}, E^\vee)}\mathbf{e}_{-\lambda}(E^\vee)T^\gamma
\end{aligned}$$

where we used the divisor equation in the second line. This implies that the $(\mathbf{e}^*_\lambda, E^\vee)$-twisted quantum product is the pull-back of the $(\mathbf{e}^{-1}_{-\lambda}, E^\vee)$-twisted quantum product by h. The conclusion follows by taking the non-equivariant limit $\lambda \to 0$. ■

Remark 3.12. The pairing $S_{(\mathbf{e}^*_\lambda, E^\vee)}$ does not have a non-equivariant limit. □

Remark 3.13. We can define the fundamental solution for the quantum connection of $E^\vee$ using the push-forward along an evaluation map similarly to (3.4). Therefore, the fundamental solution $L_{(\mathbf{e}^{-1}_\lambda, E^\vee)}$ admits a non-equivariant limit $L^{E^\vee}$. Using the formula (2.5) and an argument similar to Proposition 3.11, we find that

$$L_{(\mathbf{e}^{-1}_{-\lambda}, E^\vee)}(h(\tau), z) = L_{(\mathbf{e}^*_\lambda, E^\vee)}(\tau, z) \circ \mathrm{e}^{-\pi\sqrt{-1}c_1(E)/z}$$

and thus

$$L^{E^\vee}(h(\tau), z) = \lim_{\lambda\to 0} L_{(\mathbf{e}^*_\lambda, E^\vee)}(\tau, z) \circ \mathrm{e}^{-\pi\sqrt{-1}c_1(E)/z}. \tag{3.6}$$

Then $L^{E^\vee}(\tau, z)z^{-\frac{\deg}{2}}z^{c_1(TX)-c_1(E)}$ is a fundamental solution of $\nabla^{E^\vee}$ including in the z-direction. □

3.6 Non-equivariant limit of quantum Serre duality

We will state a non-equivariant limit of Theorem 2.11 when $\mathbf{c}$ is $\mathbf{e}_\lambda$ and E is a convex vector bundle. From Sections 3.3 and 3.5, the quantum D-modules $\mathrm{QDM}_{(\mathbf{e}_\lambda, E)}(X)$ and $\mathrm{QDM}_{(\mathbf{e}^*_\lambda, E^\vee)}(X)$ have non-equivariant limits, and the limits are, respectively, $\mathrm{QDM}_{(\mathrm{e}, E)}(X)$ and $h^*\mathrm{QDM}(E^\vee)$. The map f in (2.7) also admits a non-equivariant limit by Lemma 3.2. The quantum Serre pairing in Definition 2.10 has an obvious non-equivariant limit:

$$S^{\mathrm{QS}} : \mathrm{QDM}_{(\mathrm{e}, E)}(X) \times (h \circ \bar{f})^*\mathrm{QDM}(E^\vee) \to \mathbb{C}[z][\![\mathrm{e}^{\tau_2}, \tau']\!]$$

defined by $S^{\mathrm{QS}}(u, v) = \int_X u(-z) \cup v(z)$.

Theorem 3.14. Let E be a convex vector bundle on X. Let h be the map in (3.5) and let $\bar{f}$ be the map in Lemma 3.2.

(1) The pairing S^{QS} is flat in the τ-direction and is of weight $(-\dim X)$.

(2) The map $\mathrm{e}(E)\cup: \mathrm{QDM}_{(\mathrm{e},E)}(X) \to (h\circ\bar{f})^*\mathrm{QDM}(E^\vee)$, $\alpha \mapsto \mathrm{e}(E)\cup\alpha$ respects the quantum connection in the τ-direction and is of weight $\mathrm{rk}E$, that is,

$$\nabla'_\alpha \mathrm{e}(E) = \mathrm{e}(E)\nabla_\alpha$$

$$\nabla'_{z\partial_z}\mathrm{e}(E) = \mathrm{e}(E)\nabla_{z\partial_z} + \mathrm{rk}(E)\mathrm{e}(E)$$

for $\nabla' = (h\circ\bar{f})^*\nabla^{E^\vee}$ and $\nabla = \nabla^{(\mathrm{e},E)}$.

(3) The fundamental solutions in Remarks 3.4, 3.13 satisfy the following relations:

$$\mathrm{e}(E)\circ L_{(\mathrm{e},E)}(\tau,z) = L^{E^\vee}(h\circ\bar{f}(\tau),z)\,\mathrm{e}^{\pi\sqrt{-1}c_1(E)/z}\circ\mathrm{e}(E)$$

$$(\gamma_1, \mathrm{e}^{-\pi\sqrt{-1}c_1(E)/z}\gamma_2) = (L_{(\mathrm{e},E)}(\tau,-z)\gamma_1, L^{E^\vee}(h(\bar{f}(\tau)),z)\gamma_2).$$

where $(u,v) = \int_X u\cup v$ is the Poincaré pairing. □

Proof of Theorem 3.14. Almost all the statements follow by taking the non-equivariant limit of Theorem 2.11. Note that part (3) follows from Theorem 2.11 (3), Remarks 3.4, 3.13, and Equation (3.6). What remains to show is the statement about weights of S^{QS} and $\mathrm{e}(E)$. Regarding $\mathfrak{E}^{(\mathrm{e},E)}$, $\mathfrak{E}^{E^\vee}$ (see (3.2)) as vector fields on $H^{\mathrm{ev}}(X)$, we can check that $(h\circ\bar{f})_*\mathfrak{E}^{(\mathrm{e},E)} = \mathfrak{E}^{E^\vee}$. Therefore,

$$\mathrm{gr} := \nabla^{(\mathrm{e},E)}_{z\partial_z} + \nabla^{(\mathrm{e},E)}_{\mathfrak{E}^{(\mathrm{e},E)}} = z\partial_z + \mathfrak{E}^{(\mathrm{e},E)} + \frac{\deg}{2} = ((h\circ\bar{f})^*\nabla^{E^\vee})_{z\partial_z} + ((h\circ\bar{f})^*\nabla^{E^\vee})_{\mathfrak{E}^{(\mathrm{e},E)}}.$$

On the other hand, we can check that

$$(z\partial_z + \mathfrak{E}^{(\mathrm{e},E)})S^{\mathrm{QS}}(u,v) - S^{\mathrm{QS}}(\mathrm{gr}u,v) - S^{\mathrm{QS}}(u,\mathrm{gr}v) = -(\dim X)S^{\mathrm{QS}}(u,v).$$

The flatness of S^{QS} in the $\mathfrak{E}^{(\mathrm{e},E)}$-direction shows that S^{QS} is of weight $-\dim X$. The discussion for $\mathrm{e}(E)$ is similar. ■

Let $Z\subset X$ be the zero-locus of a transverse section of E and let $\iota\colon Z\to X$ be the inclusion map. We consider the following conditions for Z.

Lemma 3.15. The following conditions are equivalent:

(1) the Poincaré pairing on $H^{\mathrm{ev}}_{\mathrm{amb}}(Z) = \mathrm{Im}(\iota^*\colon H^{\mathrm{ev}}(X) \to H^{\mathrm{ev}}(Z))$ is non-degenerate;

(2) we have the decomposition $H^{\mathrm{ev}}(Z) = \mathrm{Ker}\,\iota_* \oplus \mathrm{Im}\,\iota^*$;

(3) ι^* induces an isomorphism $H^{\mathrm{ev}}(X)/\mathrm{Ker}(\mathrm{e}(E)\cup) \cong H^{\mathrm{ev}}_{\mathrm{amb}}(Z)$. □

Proof. $(1) \Rightarrow (2)$: it suffices to see that $\mathrm{Ker}\,\iota_* \cap \mathrm{Im}\,\iota^* = \{0\}$. Suppose that $\alpha \in \mathrm{Ker}\,\iota_* \cap \mathrm{Im}\,\iota^*$. Then for every $\iota^*\beta \in H^{\mathrm{ev}}_{\mathrm{amb}}(Z)$ we have $(\iota^*\beta, \alpha) = (\beta, \iota_*\alpha) = 0$. By assumption we have $\alpha = 0$. $(2) \Rightarrow (3)$: we have $\iota^*\alpha = 0$ if and only if $\iota_*\iota^*\alpha = \mathrm{e}(E) \cup \alpha = 0$. Therefore, $\mathrm{Ker}(\iota^*) = \mathrm{Ker}(\mathrm{e}(E)\cup)$ and part (3) follows. $(3) \Rightarrow (1)$: since $(\iota^*\alpha, \iota^*\beta) = (\alpha, \iota_*\iota^*\beta) = (\alpha, \mathrm{e}(E) \cup \beta)$, the kernel of the Poincaré pairing on $H^{\mathrm{ev}}_{\mathrm{amb}}(Z)$ is $\iota^*(\mathrm{Ker}(\mathrm{e}(E)\cup))$, which is zero. ■

Remark 3.16. The conditions in Lemma 3.15 hold if E is the direct sum of ample line bundles by the Hard Lefschetz theorem. They also hold if X is a toric variety and Z is a regular hypersurface with respect to a semiample line bundle E on X by a result of Mavlyutov [29]. □

Corollary 3.17. Suppose that E is a convex vector bundle on X and $Z \subset X$ be the zero-set of a regular section of E satisfying one of the conditions in Lemma 3.15. Then the morphism $\mathrm{e}(E)\cup\colon \mathrm{QDM}_{(\mathrm{e},E)}(X) \to (h \circ \bar{f})^*\mathrm{QDM}(E^\vee)$ in Theorem 3.14 factors through $\mathrm{QDM}_{\mathrm{amb}}(Z)$ as:

$$\begin{tikzcd} \mathrm{QDM}_{(\mathrm{e},E)}(X) \arrow[r, "\mathrm{e}(E)\cup"] \arrow[d, two heads, "\iota^*"'] & (h \circ \bar{f})^*\mathrm{QDM}(E^\vee) \\ (\iota^*)^*\mathrm{QDM}_{\mathrm{amb}}(Z) \arrow[ur, hook, "\iota_*"'] & \end{tikzcd}$$

In particular, $\iota_*\colon (\iota^*)^*\mathrm{QDM}_{\mathrm{amb}}(Z) \to (h \circ \bar{f})^*\mathrm{QDM}(E^\vee)$ respects the quantum connection in the τ-direction and is of weight $\mathrm{rk}E$. □

Proof. We already showed that ι^* is a morphism of flat connections in Proposition 3.6. It suffices to invoke the factorization of the linear map $\mathrm{e}(E)\cup$:

$$\begin{tikzcd} H^{\mathrm{ev}}(X) \arrow[r, "\mathrm{e}(E)\cup"] \arrow[d, two heads, "\iota^*"'] & H^{\mathrm{ev}}(X) \\ H^{\mathrm{ev}}_{\mathrm{amb}}(Z) \cong H^{\mathrm{ev}}(X)/\mathrm{Ker}(\mathrm{e}(E)\cup) \arrow[ur, hook, "\iota_*"] & \end{tikzcd} \tag{3.7}$$

■

Remark 3.18. Recall that for a general non-compact space, the Poincaré duality pairs the cohomology with the cohomology with compact support. This analogy leads us to think of $\mathrm{QDM}_{(\mathrm{e},E)}(X)$ as the quantum D-module with compact support of the total space $E^\vee$. □

Remark 3.19. It would be interesting to study if ι_* always defines a morphism of quantum D-modules without assuming the conditions in Lemma 3.15. □

4 Quantum Serre Duality and Integral Structures

In this section, we study a relation between quantum Serre duality for the Euler-twisted theory and the $\hat{\Gamma}$-integral structure studied in [17–19, 28]. The $\hat{\Gamma}$-integral structure is a lattice in the space of flat sections for the quantum connection, which is isomorphic to the Grothendieck group $K(X)$ of vector bundles on X. After introducing a similar integral structure in the Euler-twisted theory, we see that the quantum Serre pairing is identified with the Euler pairing on K-groups, and that the morphisms of flat connections in Corollary 3.17 are induced by natural maps between K-groups. In this section, the Novikov variable Q is specialized to one (see Section 3.2).

Recall 4.1. Recall the classical self-intersection formula in K-theory. Let $j : X \hookrightarrow Y$ be a closed embedding with normal bundle N between quasi-compact and quasi-separated schemes. In Theorem 3.1 of [36], Thomason proves that we have for any $[V] \in K(X)$

$$j^* j_*[V] = [\lambda_{-1} N^\vee] \cdot [V] \tag{4.1}$$

where

$$[\lambda_{-1} N^\vee] := \sum_{k\geq 0} (-1)^k [\wedge^k N^\vee] \in K(X).$$

□

Definition 4.2. For a vector bundle G with Chern roots $\delta_1, \dots, \delta_r$, we define the $\hat{\Gamma}$-class to be

$$\hat{\Gamma}(G) = \prod_{i=1}^{r} \Gamma(1 + \delta_i).$$

We also define a $(2\pi\sqrt{-1})$-modified Chern character by

$$\mathrm{Ch}(G) = (2\pi\sqrt{-1})^{\frac{\deg}{2}} \mathrm{ch}(G) = \sum_{i=1}^{r} e^{2\pi\sqrt{-1}\delta_i}.$$

□

Suppose that E is a convex vector bundle on X. Let $Z \subset X$ be a submanifold cut out by a transverse section s of E. For the (twisted) quantum connection ∇, we write

$\mathrm{Ker}\nabla$ for the space of flat sections:

$$\mathrm{Ker}\nabla^{(\mathrm{e},E)} = \{s \in H^{\mathrm{ev}}(X) \otimes \mathbb{C}[z^{\pm}][\![\mathrm{e}^{\tau_2}, \tau']\!][\log z] : \nabla^{(\mathrm{e},E)} s = 0\},$$

$$\mathrm{Ker}\nabla^{Z} = \{s \in H^{\mathrm{ev}}_{\mathrm{amb}}(Z) \otimes \mathbb{C}[z^{\pm}][\![\mathrm{e}^{\tau_2}, \tau']\!][\log z] : \nabla^{Z} s = 0\},$$

$$\mathrm{Ker}\nabla^{E^\vee} = \{s \in H^{\mathrm{ev}}(X) \otimes \mathbb{C}[z^{\pm}][\![\mathrm{e}^{\tau_2}, \tau']\!][\log z] : \nabla^{E^\vee} s = 0\}.$$

Definition 4.3. The *K-group framing* is a map from a K-group to the space of flat sections defined as follows:

(1) for the twist (e, E), the K-group framing $Z^{(\mathrm{e},E)} \colon K(X) \to \mathrm{Ker}\nabla^{(\mathrm{e},E)}$ is:

$$Z^{(\mathrm{e},E)}(V) = \frac{1}{(2\pi\sqrt{-1})^{\dim X - \mathrm{rk} E}} L_{(\mathrm{e},E)}(\tau, z) z^{-\frac{\deg}{2}} z^{c_1(TX) - c_1(E)} \frac{\hat{\Gamma}(TX)}{\hat{\Gamma}(E)} \mathrm{Ch}(V);$$

(2) for a smooth section $Z \subset X$ of E, the K-group framing $Z^{\mathrm{amb}} \colon K_{\mathrm{amb}}(Z) \to \mathrm{Ker}\nabla^{Z}$ is:

$$Z^{\mathrm{amb}}(V) = \frac{1}{(2\pi\sqrt{-1})^{\dim Z}} L^{Z}(\tau, z) z^{-\frac{\deg}{2}} z^{c_1(TZ)} \hat{\Gamma}(TZ) \mathrm{Ch}(V);$$

(3) for the total space $E^\vee$, the K-group framing $Z^{E^\vee} \colon K(X) \to \mathrm{Ker}\nabla^{E^\vee}$ is:

$$Z^{E^\vee}(V) = \frac{1}{(2\pi\sqrt{-1})^{\dim E^\vee}} L^{E^\vee}(\tau, z) z^{-\frac{\deg}{2}} z^{c_1(TX) - c_1(E)} \hat{\Gamma}(TE^\vee) \mathrm{Ch}(V).$$

where $K_{\mathrm{amb}}(Z) = \mathrm{Im}(\iota^* \colon K(X) \to K(Z))$ and $L^Z, L^{E^\vee}$ are the fundamental solutions for Z and $E^\vee$, respectively. Recall from Remarks 2.7, 3.4, and 3.13 that these formula define a section which is flat in both τ and z. □

Proposition 4.4. For any vector bundles V, W on X, we have

$$\chi(V \otimes W^\vee) = (-2\pi\sqrt{-1}z)^{\dim X} S^{\mathrm{QS}}(Z^{(\mathrm{e},E)}(V)(\tau, \mathrm{e}^{\pi\sqrt{-1}}z), Z^{E^\vee}(W)(h \circ \bar{f}(\tau), z))$$

where $\chi(V \otimes W^\vee) = \sum_{i=0}^{\dim X} (-1)^i \dim \mathrm{Ext}^i(W, V)$ is the holomorphic Euler characteristic. □

Proof. This is analogous to [17, Proposition 2.10]. Since the pairing $z^{\dim X} S^{\mathrm{QS}}$ is flat, the right-hand side is constant with respect to τ and z. Evaluating the right-hand side at

$z=1$, we obtain

$$\frac{1}{(-2\pi\sqrt{-1})^{\dim X}}\left(L_{(\mathrm{e},E)}(\tau,-1)\,\mathrm{e}^{-\pi\sqrt{-1}\frac{\deg}{2}}\,\mathrm{e}^{\pi\sqrt{-1}(c_1(TX)-c_1(E))}\gamma_1,\,L^{E^\vee}(h\circ\bar{f}(\tau),1)\gamma_2\right)$$

with $\gamma_1=\frac{\hat{\Gamma}(TX)}{\hat{\Gamma}(E)}\mathrm{Ch}(V)$, $\gamma_2=\hat{\Gamma}(TX)\hat{\Gamma}(E^\vee)\mathrm{Ch}(W)$, where $(\cdot,\cdot)$ is the Poincaré pairing on X. By Theorem 3.14(3), we find that this equals

$$\frac{1}{(-2\pi\sqrt{-1})^{\dim X}}\left(\mathrm{e}^{-\pi\sqrt{-1}\frac{\deg}{2}}\,\mathrm{e}^{\pi\sqrt{-1}(c_1(TX)-c_1(E))}\gamma_1,\,\mathrm{e}^{-\pi\sqrt{-1}c_1(E)}\gamma_2\right).$$

Since the adjoint of $\frac{\deg}{2}$ is $\dim X-\frac{\deg}{2}$, this is

$$\frac{1}{(2\pi\sqrt{-1})^{\dim X}}\left(\mathrm{e}^{\pi\sqrt{-1}c_1(TX)}\gamma_1,\,\mathrm{e}^{\pi\sqrt{-1}\frac{\deg}{2}}\gamma_2\right).$$

Using the following identities:

$$\mathrm{e}^{\pi\sqrt{-1}\frac{\deg}{2}}\Gamma(1+\delta)=\Gamma(1-\delta)\quad\text{and}\quad\mathrm{e}^{\pi\sqrt{-1}\frac{\deg}{2}}\mathrm{ch}(W)=\mathrm{ch}(W^\vee)$$

$$(2\pi\sqrt{-1})^{-\dim X}\int_X\gamma=\int_X(2\pi\sqrt{-1})^{-\frac{\deg}{2}}\gamma$$

$$(2\pi\sqrt{-1})^{-\frac{\deg}{2}}\Gamma(1+\delta)=(2\pi\sqrt{-1})^{\frac{\deg}{2}}\Gamma\left(1+\frac{\delta}{2\sqrt{-1}\pi}\right)$$

with δ a degree-two cohomology class, we deduce that the right-hand side of the proposition is

$$\int_X\mathrm{ch}(V\otimes W^\vee)\mathrm{e}^{\rho/2}\prod_{i=1}^{n}\Gamma\left(1+\frac{\rho_i}{2\sqrt{-1}\pi}\right)\Gamma\left(1-\frac{\rho_i}{2\sqrt{-1}\pi}\right)$$

where $\rho_1,\ldots,\rho_n$ are the Chern roots of TX and $\rho=c_1(TX)=\rho_1+\cdots+\rho_n$. Finally, we use $\Gamma(x)\Gamma(1-x)=\pi/\sin(\pi x)$ to get

$$\mathrm{e}^{\rho/2}\prod_{i=1}^{n}\Gamma\left(1+\frac{\rho_i}{2\sqrt{-1}\pi}\right)\Gamma\left(1-\frac{\rho_i}{2\sqrt{-1}\pi}\right)=\mathrm{Td}(TX).\tag{4.2}$$

We conclude the proposition by the theorem of Hirzebruch–Riemann–Roch. ∎

The following proposition shows that the integral structures are compatible with the diagram in Corollary 3.17.

Proposition 4.5. Let E be a convex vector bundle and $Z\subset X$ be a submanifold cut out by a regular section of E. Let $\iota\colon Z\hookrightarrow X$, $j\colon X\hookrightarrow E^\vee$ denote the natural inclusions. Assume that Z satisfies one of the conditions in Lemma 3.15. Then the diagram in Corollary 3.17 can be extended to the following commutative diagram

$$\begin{tikzcd}
K(X) \arrow[rrr, "j^*j_*"] \arrow[dr, "Z^{(\mathrm{e},E)}"] \arrow[dddr, bend right, "\iota^*"'] & & & K(X) \arrow[dl, "Z^{E^\vee}"'] \\
& \mathrm{Ker}\nabla^{(\mathrm{e},E)} \arrow[r, "c(z)\mathrm{e}(E)\cup"] \arrow[d, "\iota^*"] & \mathrm{Ker}(\bar{f}\circ h)^*\nabla^{E^\vee} & \\
& \mathrm{Ker}(\iota^*)^*\nabla^{\mathrm{amb}} \arrow[ur, hook, "c(z)\iota_*"'] & & \\
& K_{\mathrm{amb}}(Z) \arrow[u, "Z^{\mathrm{amb}}"] \arrow[uuurr, bend right, "(-1)^{\mathrm{rk}E}\,\det(E)\otimes\iota_*"'] & &
\end{tikzcd}$$

where $c(z)=1/(-2\pi\sqrt{-1}z)^{\mathrm{rk}E}$. □

Proof. We first prove that the top square is commutative. Recall the following equation from part (3) of Theorem 3.14:

$$\mathrm{e}(E)L_{(\mathrm{e},E)}(\tau,z)=L^{E^\vee}(h\circ\bar{f}(\tau),z)\,\mathrm{e}^{\pi\sqrt{-1}c_1(E)/z}\mathrm{e}(E).$$

So it remains to prove that for any $V\in K(X)$, we have

$$(-2\pi\sqrt{-1})^{\mathrm{rk}E}\mathrm{e}^{\pi\sqrt{-1}c_1(E)}\mathrm{e}(E)\hat{\Gamma}(TX)\hat{\Gamma}(E)^{-1}\mathrm{Ch}(V)=\hat{\Gamma}(TX)\hat{\Gamma}(E^\vee)\mathrm{Ch}(j^*j_*V)$$

This follows from (4.2) applied to the vector bundle E and from

$$\mathrm{ch}\,j^*j_*V=\mathrm{e}(E^\vee)\mathrm{Td}(E^\vee)^{-1}\mathrm{ch}(V),\quad \text{see (4.1).}$$

The commutativity of the left square follows from the properties of the $\hat{\Gamma}$-class and the following facts (see [18, Proposition 2.4] for the second property):

$$0\longrightarrow TZ\longrightarrow \iota^*TX\longrightarrow \iota^*E\longrightarrow 0\quad\text{is exact;}$$

$$L^Z(\iota^*\tau,z)\iota^*\gamma=\iota^*(L_{(\mathrm{e},E)}(\tau,z)\gamma)\quad\forall\gamma\in H^{\mathrm{ev}}(X).$$

The identity $j^*j_*=(-1)^{\mathrm{rk}E}\det(E)\otimes\iota_*\iota^*$ implies that the right square is commutative. ■

5 Quantum Serre Duality and Abstract Fourier–Laplace Transform

In this section, we study quantum Serre duality with respect to the anticanonical line bundle K_X^{-1}. We consider the (e, K_X^{-1})-twisted quantum D-module of X and the quantum D-module of the total space of K_X. On the small quantum cohomology locus $H^2(X)$, we identify these quantum D-modules with Dubrovin's second structure connections with different parameters σ. We show that the duality between them is given by the second metric $\check{g}$. Throughout the section, we assume that the anticanonical class $-K_X = c_1(X)$ of X is nef and the Novikov variable is specialized to one (Section 3.2). We also set $n := \dim_{\mathbb{C}} X$ and $\rho := c_1(X)$.

5.1 Convergence assumption

In this section, we assume certain analyticity of quantum cohomology of X. In Sections 5.2 and 5.3, we assume that the big quantum cohomology of X is convergent, that is, the quantum product (with Novikov variables specialized to one; see Section 3.2)

$$T_\alpha \bullet_\tau T_\gamma \in \mathbb{C}[\![\mathrm{e}^{\tau_2}, \tau']\!] = \mathbb{C}[\![\mathrm{e}^{t^1}, \dots, \mathrm{e}^{t^r}, t^0, t^{r+1}, \dots, t^s]\!]$$

converges on a region $U \subset H^{\mathrm{ev}}(X, \mathbb{C})$ of the form:

$$U = \{\tau \in H^{\mathrm{ev}}(X, \mathbb{C}) : |\mathrm{e}^{t^i}| < \epsilon \ (1 \le i \le r), |t^j| < \epsilon \ (r+1 \le j \le s)\}.$$

For the main results in this section, we only need the convergence of the *small* quantum product. This means that the quantum product $T_\alpha \bullet_\tau T_\beta$ restricted to $\tau = \tau_2$ to lie in $H^2(X, \mathbb{C})$ converges on a region $U_{\mathrm{sm}} \subset H^2(X, \mathbb{C})$ of the form

$$U_{\mathrm{sm}} = \{\tau_2 \in H^2(X, \mathbb{C}) : |\mathrm{e}^{t^i}| < \epsilon \ (1 \le i \le r)\}. \tag{5.1}$$

When X is Fano, that is, if $-K_X$ is ample, the convergence of small quantum cohomology is automatic because the structure constants are polynomials in $\mathrm{e}^{t^1}, \dots, \mathrm{e}^{t^r}$ for degree reason.

5.2 Quantum connection with parameter σ

We introduce a variant of the quantum connection parametrized by a complex number σ. Consider the trivial vector bundle

$$F = H^{\mathrm{ev}}(X) \times (U \times \mathbb{C}_z)$$

over $U \times \mathbb{C}_z$, where U is the convergence domain of the big quantum product in Section 5.1, and define a meromorphic flat connection $\nabla^{(\sigma)}$ of F by the formula (cf. Definition 2.1 and Remark 2.4)

$$\nabla^{(\sigma)}_\alpha = \partial_\alpha + \frac{1}{z}(T_\alpha \bullet_\tau)$$

$$\nabla^{(\sigma)}_{z\partial_z} = z\partial_z - \frac{1}{z}(\mathfrak{E}\bullet_\tau) + \left(\mu - \frac{1}{2} - \sigma\right)$$

where μ is an endomorphism of $H^{\mathrm{ev}}(X)$ defined by

$$\mu(T_\alpha) = \left(|\alpha| - \frac{n}{2}\right) T_\alpha \quad \text{with } |\alpha| = \frac{1}{2}\deg T_\alpha, \quad n = \dim_{\mathbb{C}} X.$$

Let $(-)\colon U \times \mathbb{C}_z \to U \times \mathbb{C}_z$ denote the map sending (τ, z) to $(\tau, -z)$. We note the following facts.

Proposition 5.1 ([15, Theorem 9.8(c)]). The $\mathcal{O}_{U\times\mathbb{C}_z}$-bilinear pairing

$$g\colon (-)^*(F, \nabla^{(\sigma)}) \times (F, \nabla^{(-1-\sigma)}) \to \mathcal{O}_{U\times\mathbb{C}_z}$$

defined by $g(T_\alpha, T_\beta) = \int_X T_\alpha \cup T_\beta$ is flat. □

Proposition 5.2 (see, e.g. [17, Proposition 2.4]). Let $L(\tau, z)$ be the fundamental solution for the quantum connection of X from Proposition 2.5 (with $\mathbf{c} = 1$, $E = 0$). We have that $L(\tau, z) z^{-(\mu - \frac{1}{2} - \sigma)} z^{c_1(TX)}$ is a fundamental solution of $\nabla^{(\sigma)}$ including in the z-direction. □

Remark 5.3. The variable z in this paper corresponds to z^{-1} in Hertling's book [15, Section 9.3]. For convenience of the reader, we made a precise link of notation with the book of Hertling:

$$\mathcal{U} = \mathfrak{E}\bullet_\tau, \quad D = 2 - n, \quad \mathcal{V} = -\mu - \frac{n}{2}. \tag{5.2}$$

□

Using the divisor equation, the inverse of the fundamental solution $L(\tau, z)$ for X (see (2.2)) can be written in the form:

$$L(\tau, z)^{-1} T_\alpha = e^{\tau_2/z} \left(T_\alpha + \sum_{\substack{(d,l)\neq(0,0)\\ \beta\in\{0,\dots,s\}}} \left\langle T_\alpha, \tau', \dots, \tau', \frac{T_\beta}{z - \psi} \right\rangle_{0,l+2,d} e^{\tau_2(d)} \frac{T^\beta}{l!} \right) \tag{5.3}$$

Denote by $K^{(\sigma)}_\alpha$ the αth column of the inverse fundamental solution matrix for $\nabla^{(\sigma)}$:

$$K^{(\sigma)}_\alpha(\tau, z) := z^{-c_1(TX)} z^{\mu - \frac{1}{2} - \sigma} L(\tau, z)^{-1} T_\alpha.$$

If we restrict τ to lie in $H^2(X)$, we have the following expression.

Lemma 5.4. For any $\alpha \in \{0, \dots, s\}$ and $\tau_2 \in H^2(X)$, we have

$$K_\alpha^{(\sigma)}(\tau_2, z) = \sum_{d \in \mathrm{Eff}(X)} N_{\alpha,d}(1) \frac{\mathrm{e}^{\tau_2 + \tau_2(d)}}{z^{\rho + \rho(d) - |\alpha| + \frac{n+1}{2} + \sigma}}$$

where $\rho = c_1(X)$ and

$$N_{\alpha,d}(z) := \begin{cases} \sum_{\beta=0}^{s} \left\langle T_\alpha, \dfrac{T_\beta}{z - \psi} \right\rangle_{0,2,d} T^\beta & \text{if } \ d \neq 0; \\ T_\alpha & \text{if } \ d = 0. \end{cases} \tag{5.4}$$

□

Proof. Restricting to $H^2(X)$ means setting $\tau' = 0$ in (5.3). For $d \neq 0 \in \mathrm{Eff}(X)$, we have

$$N_{\alpha,d}(z) = \sum_{\beta=0}^{s} \left\langle T_\alpha, \frac{T_\beta}{z - \psi} \right\rangle_{0,2,d} T^\beta = \sum_{k \geq 0} \sum_{\beta=0}^{s} \langle T_\alpha, \psi^k T_\beta \rangle_{0,2,d} \frac{T^\beta}{z^{k+1}}.$$

By the degree axiom for Gromov–Witten invariants, only the term with $k = n - |\beta| - |\alpha| + \rho(d) - 1$ contributes. Therefore, $z^\mu N_{\alpha,d}(z) = N_{\alpha,d}(1) z^{-(\rho(d) + \frac{n}{2} - |\alpha|)}$. Using the fact that $z^\mu \circ \mathrm{e}^{\tau_2/z} = \mathrm{e}^{\tau_2} \circ z^\mu$, we deduce the formula of the lemma. ■

5.3 The second structure connection

We introduce the *second structure connection* [8, lecture 3; 9, Section 2.3; 15, Section 9.2; 27, II, Section 1]. Let x be the variable Laplace-dual to z^{-1} and let $\mathbb{C}_x$ denote the complex plane with co-ordinate x. Consider the trivial vector bundle

$$\check{F} = H^{\mathrm{ev}}(X) \times (U \times \mathbb{C}_x)$$

over $U \times \mathbb{C}_x$. The *second structure connection* is a meromorphic flat connection on the bundle $\check{F}$ defined by

$$\begin{aligned} \check{\nabla}_\alpha^{(\sigma)} &= \partial_\alpha + \left(\mu - \frac{1}{2} - \sigma \right) ((\mathfrak{E} \bullet_\tau) - x)^{-1} (T_\alpha \bullet_\tau) \\ \check{\nabla}_{\partial_x}^{(\sigma)} &= \partial_x - \left(\mu - \frac{1}{2} - \sigma \right) ((\mathfrak{E} \bullet_\tau) - x)^{-1}. \end{aligned} \tag{5.5}$$

The connection has a singularity along the divisor $\Sigma \subset U \times \mathbb{C}$:

$$\Sigma := \{(\tau, x) \in U \times \mathbb{C}_x \mid \det((\mathfrak{E} \bullet_\tau) - x) = 0\}.$$

The second structure connection has an invariant pairing called the *second metric* (or the *intersection form*).

Proposition 5.5 ([15, Theorem 9.4.c]). The $\mathcal{O}_{U\times\mathbb{C}_x}$-bilinear pairing

$$\check{g}\colon(\check{F},\check{\nabla}^{(\sigma)})\times(\check{F},\check{\nabla}^{(-\sigma)})\to\mathcal{O}_{U\times\mathbb{C}_x}(\Sigma)$$

defined by $\check{g}(T_\alpha,T_\beta)=\int_X T_\alpha\cup((\mathfrak{E}\bullet_\tau)-x)^{-1}T_\beta$ is flat. This is called the *second metric*. □

We now explain how the second structure connection $\check{\nabla}^{(\sigma)}$ arises from the Fourier–Laplace transformation of the quantum connection $\nabla^{(\sigma-1)}$ (see [7, 1.b; 34, V]). Consider the module $M=H^{\mathrm{ev}}(X)\otimes\mathcal{O}_U[z]$ of sections of the trivial bundle F which are polynomials in z. The quantum connection $\nabla^{(\sigma-1)}$ equips $M[z^{-1}]$ with the structure of an $\mathcal{O}_U\langle\partial_\alpha,z^{\pm},\partial_z\rangle$-module by the assignment:

$$\partial_z\mapsto\nabla^{(\sigma-1)}_{\partial_z},\quad \partial_\alpha\mapsto\nabla^{(\sigma-1)}_{\alpha}.$$

Consider the isomorphism of the rings of differential operators:

$$\mathcal{O}_U\langle\partial_\alpha,z^{-1},\partial_{z^{-1}}\rangle\cong\mathcal{O}_U\langle\partial_\alpha,x,\partial_x\rangle$$

sending $\partial_{z^{-1}}=-z^2\partial_z$ to x and z^{-1} to $-\partial_x$. Via this isomorphism, we may regard $M[z^{-1}]$ as an $\mathcal{O}_U[x]\langle\partial_\alpha,\partial_x\rangle$-module. This is called the *abstract Fourier–Laplace transform*. The subset $M\subset M[z^{-1}]$ is closed under the action of $x=-z^2\partial_z$, and thus becomes an $\mathcal{O}_U[x]$-submodule of $M[z^{-1}]$. Note that $M[z^{-1}]$ is generated by M over $\mathcal{O}_U\langle x,\partial_x\rangle$ since $z^{-1}=-\partial_x$. Regard $T_\alpha\in H^{\mathrm{ev}}(X)$ as an element of M. Under the abstract Fourier–Laplace transformation, we have

$$\begin{aligned}(\partial_x x)\cdot T_\alpha&=\nabla^{(\sigma-1)}_{z\partial_z}T_\alpha=\partial_x\cdot(\mathfrak{E}\bullet_\tau T_\alpha)+(\mu+\tfrac{1}{2}-\sigma)T_\alpha\\ \partial_\beta\cdot T_\alpha&=\nabla^{(\sigma-1)}_{\beta}T_\alpha=-\partial_x\cdot(T_\beta\bullet_\tau T_\alpha)\end{aligned}$$

Regarding $(\mathfrak{E}\bullet_\tau)$, μ, $(T_\beta\bullet_\tau)$ as matrices written in the basis $\{T_\alpha\}$, we obtain

$$\begin{aligned}[\partial_x T_0,\dots,\partial_x T_s](x-\mathfrak{E}\bullet_\tau)&=[T_0,\dots,T_s](\mu-\tfrac{1}{2}-\sigma)\\ [\partial_\beta T_0,\dots,\partial_\beta T_s]&=-[\partial_x T_0,\dots,\partial_x T_s](T_\beta\bullet_\tau)\end{aligned}$$

Inverting $(x-\mathfrak{E}\bullet_\tau)$ in the first equation, we obtain the connection matrices for the second structure connection $\check{\nabla}^{(\sigma)}$. In other words, writing $\mathcal{O}(\check{F})$ for the sheaf of holomorphic sections of $\check{F}$ which are polynomials in x, the natural $\mathcal{O}_U[x]$-module map

$$\mathcal{O}(\check{F}) \to M, \qquad T_\alpha \mapsto T_\alpha \tag{5.6}$$

intertwines the meromorphic connection $\check{\nabla}^{(\sigma)}$ on $\check{F}$ with the action of ∂_x, ∂_α on M after inverting $\det(x - \mathfrak{E}\bullet_\tau) \in \mathcal{O}_U[x]$. On the other hand, $\{T_\alpha\}$ does not always give an $\mathcal{O}_U[x]$-basis of M and the map (5.6) is not always an isomorphism. A sufficient condition for the map (5.6) to be an isomorphism is given by a result of Sabbah [34].

Proposition 5.6 ([34, Proposition V.2.10]). Suppose that $\sigma \notin -\frac{n-1}{2} + \mathbb{Z}_{\geq 0}$ with $n = \dim_{\mathbb{C}} X$. Then the second structure connection $(\check{F}, \check{\nabla}^{(\sigma)})$ coincides with the abstract Fourier transform of the quantum connection $(F, \nabla^{(\sigma-1)})$, that is, the map (5.6) is an isomorphism. □

5.4 Fundamental solution for the second structure connection

We will henceforth restrict ourselves to the small quantum cohomology locus $H^2(X)$. We find an inverse fundamental solution for the second structure connection using a truncated Laplace transformation.

Definition 5.7. Consider a cohomology-valued power series of the form:

$$K(z) = z^{-\gamma} \sum_k a_k z^{-k}$$

with $a_k \in H^{\mathrm{ev}}(X)$ and $\gamma \in H^{\mathrm{ev}}(X)$, where $z^{-\gamma} = \mathrm{e}^{-\gamma \log z}$. We assume that the exponent k ranges over a subset of $\mathbb{C}$ of the form $\{k_0, k_0+1, k_0+2, \ldots\}$. Let ℓ be a complex number such that

- $\ell - k_0 \in \mathbb{Z}$ and,
- $0 \notin \{k_0+1, k_0+2, \ldots, \ell-1\}$ if $k_0 \leq \ell - 2$.

We define the *truncated Laplace transform* of $K(z)$ to be

$$\mathrm{Lap}^{(\ell)}(K)(x) := \sum_k a_k x^{-\gamma-k-1} \frac{\Gamma(\gamma+k+1)}{\Gamma(\gamma+\ell)}$$

where note that

$$\frac{\Gamma(\gamma+k+1)}{\Gamma(\gamma+\ell)} = \begin{cases} (\gamma+\ell)(\gamma+\ell+1)\cdots(\gamma+k) & \text{if } k \geq \ell; \\ 1 & \text{if } k = \ell - 1; \\ \dfrac{1}{(\gamma+k+1)(\gamma+k+2)\cdots(\gamma+\ell-1)} & \text{if } k \leq \ell - 2, \end{cases}$$

and the above condition for ℓ ensures that we do not have the division by γ when $k \le \ell - 2$ and that this expression is well defined. $\square$

The truncated Laplace transformation satisfies the following property:

$$\mathrm{Lap}^{(\ell)}(z^{-1}K) = (-\partial_x)\mathrm{Lap}^{(\ell)}(K)$$
$$\mathrm{Lap}^{(\ell)}(-z^2\partial_z K) = \mathrm{Lap}^{(\ell)}(\partial_{z^{-1}}K) = x\mathrm{Lap}^{(\ell)}(K) \tag{5.7}$$

Remark 5.8. Suppose that $K(z)$ is convergent for all $z \in \mathbb{C}^\times$, $\Re(k_0) > -1$ and that we have an estimate $|K(z)| \le C\, \mathrm{e}^{M/z}$ over the interval $z \in (0,1)$ for some $C, M > 0$. Then we can write the truncated Laplace transform as the actual Laplace transform:

$$\mathrm{Lap}^{(\ell)}(K)(x) = \frac{1}{\Gamma(\gamma+\ell)}\int_0^\infty K(z)\,\mathrm{e}^{-x/z}\,\mathrm{d}(z^{-1}). \qquad \square$$

Proposition 5.9. Let ℓ be a complex number such that $\ell \equiv \frac{n-1}{2} + \sigma \mod \mathbb{Z}$. Assume that we have either $\ell \notin \mathbb{Z}_{>0}$ or $\sigma \notin \frac{n-1}{2} + \mathbb{Z}_{\le 0}$. Then

(1) The truncated Laplace transform

$$\begin{aligned}\check{K}_\alpha^{(\sigma,\ell)}(\tau_2, x) &:= \mathrm{Lap}^{(\ell)}(K_\alpha^{(\sigma-1)}(\tau_2,\cdot)) \\ &= \sum_{d \in \mathrm{Eff}(X)} N_{\alpha,d}(1)\frac{\mathrm{e}^{\tau_2+\tau_2(d)}}{x^{\rho+\rho(d)-|\alpha|+\frac{n+1}{2}+\sigma}}\frac{\Gamma(\rho+\rho(d)-|\alpha|+\frac{n+1}{2}+\sigma)}{\Gamma(\rho+\ell)}.\end{aligned}$$

with $\tau_2 \in H^2(X)$ is well-defined. Here $\rho = c_1(X)$, $|\alpha| = \frac{1}{2}\deg T_\alpha$ and $N_{\alpha,d}(1)$ is given in (5.4).

(2) Under the convergence assumption for the small quantum cohomology of X (see Section 5.1), $\check{K}_\alpha^{(\sigma,\ell)}(\tau_2, x)$ converges on a region of the form $\{(\tau_2, x) : \tau_2 \in U_{\mathrm{sm}}, |x| > c\}$ where U_{sm} is a region of the form (5.1) and $c \in \mathbb{R}_{>0}$.

(3) These Laplace transforms define a cohomology-valued solution to the second structure connection $\check{\nabla}^{(\sigma)}$, that is, the multi-valued bundle map

$$\check{K}^{(\sigma,\ell)} : (\check{F}, \check{\nabla}^{(\sigma)}) \longrightarrow (\check{F}, d), \quad T_\alpha \longmapsto \check{K}_\alpha^{(\sigma,\ell)}$$

defined over $\{(\tau, x) \in U_{\mathrm{sm}} \times \mathbb{C} : |x| > c\}$ intertwines $\check{\nabla}^{(\sigma)}$ with the trivial connection d. $\square$

Proof. The well-definedness of the truncated Laplace transforms $\mathrm{Lap}^{(\ell)}(K_\alpha^{(\sigma-1)})$ follows easily from Lemma 5.4 by checking the conditions in Definition 5.7. The coefficients

$N_{\alpha,d}(1)$ satisfy the following estimate [16, Lemma 4.1]:

$$|N_{\alpha,d}(1)| \leq C_1 C_2^{|d|+\rho(d)} \frac{1}{\rho(d)!} \tag{5.8}$$

for some constants $C_1, C_2 > 0$ independent of α and d, where $|\cdot|$ is a fixed norm on $H^{\mathrm{ev}}(X)$ and $H_2(X)$. The convergence of the series $\check{K}_\alpha^{(\sigma,\ell)}$ follows from this.

Next we show that $\check{K}^{(\sigma,\ell)}$ gives a solution to the second structure connection. Since $K_\alpha^{(\sigma-1)}$, $\alpha = 0, \dots, s$ are the columns of an inverse fundamental solution for $\nabla^{(\sigma-1)}$, they satisfy the same differential relations as T_α:

$$[z\partial_z K_0^{(\sigma-1)}, \dots, z\partial_z K_s^{(\sigma-1)}] = [K_0^{(\sigma-1)}, \dots, K_s^{(\sigma-1)}]\left(-\frac{1}{z}(\mathfrak{E}\bullet_\tau) + \mu + \frac{1}{2} - \sigma\right)$$

$$[\partial_\beta K_0^{(\sigma-1)}, \dots, \partial_\beta K_s^{(\sigma-1)}] = [K_0^{(\sigma-1)}, \dots, K_s^{(\sigma-1)}]\frac{1}{z}(T_\beta \bullet_\tau)$$

where we regard $(\mathfrak{E}\bullet_\tau)$, μ, $(T_\beta\bullet_\tau)$ as matrices written in the basis $[T_0, T_1, \dots, T_s]$. Applying the truncated Laplace transformation $\mathrm{Lap}^{(\ell)}$ to the above formulae and using (5.7), we find the following equations:

$$[\partial_x x\check{K}_0^{(\sigma,\ell)}, \dots, \partial_x x\check{K}_s^{(\sigma,\ell)}] = [\partial_x \check{K}_0^{(\sigma,\ell)}, \dots, \partial_x \check{K}_s^{(\sigma,\ell)}](\mathfrak{E}\bullet_\tau) + [\check{K}_0^{(\sigma,\ell)}, \dots, \check{K}_s^{(\sigma,\ell)}](\mu + \tfrac{1}{2} - \sigma),$$

$$[\partial_\beta \check{K}_0^{(\sigma-1)}, \dots, \partial_\beta \check{K}_s^{(\sigma-1)}] = -[\partial_x \check{K}_0^{(\sigma-1)}, \dots, \partial_x \check{K}_s^{(\sigma-1)}](T_\beta\bullet_\tau).$$

The first equation can be rewritten as

$$[\partial_x \check{K}_0^{(\sigma,\ell)}, \dots, \partial_x \check{K}_s^{(\sigma,\ell)}] = [\check{K}_0^{(\sigma,\ell)}, \dots, \check{K}_s^{(\sigma,\ell)}](\mu - \tfrac{1}{2} - \sigma)(x - \mathfrak{E}\bullet_\tau)^{-1}.$$

Together with the second equation, this implies that $\check{K}_\alpha^{(\sigma,\ell)}$, $\alpha = 0, \dots, s$ define a solution to the second structure connection $\check{\nabla}^{(\sigma)}$. ■

Remark 5.10. Note that the convergence region U_{sm} in the above proposition depends on c. The real positive number c can be chosen arbitrarily, but U_{sm} becomes smaller if we choose a smaller c. □

5.5 Small twisted quantum D-modules

In this section, we study the (e, K_X^{-1})-twisted quantum D-module $\mathrm{QDM}_{(\mathrm{e},K_X^{-1})}(X)$ and the quantum D-module $\mathrm{QDM}(K_X)$ of the total space of K_X over the small quantum cohomology locus $H^2(X)$ using quantum Lefschetz theorem [4].

Since $c_1(X)$ is assumed to be nef, the anticanonical line bundle K_X^{-1} is convex. Therefore, by the results of Sections 3.3 and 3.5, the quantum D-modules $\mathrm{QDM}_{(\mathrm{e},K_X^{-1})}(X)$

and $\mathrm{QDM}(K_X)$ are well-defined. We shall see that, under the convergence assumption for the small quantum cohomology in Section 5.1, the quantum connections for these quantum D-modules are convergent on a region $U_{\mathrm{sm}} \subset H^2(X)$ of the form (5.1). Therefore, we have the following *small quantum D-modules*:

$$\begin{aligned} \mathrm{SQDM}_{(\mathrm{e}, K_X^{-1})}(X) &:= (H^{\mathrm{ev}}(X) \otimes \mathcal{O}_{U_{\mathrm{sm}} \times \mathbb{C}_z}, \nabla^{\mathrm{eu}}, S_{\mathrm{eu}}) \\ \mathrm{SQDM}(K_X) &:= (H^{\mathrm{ev}}(X) \otimes \mathcal{O}_{U_{\mathrm{sm}} \times \mathbb{C}_z}, \nabla^{\mathrm{loc}}) \end{aligned} \tag{5.9}$$

where $\nabla^{\mathrm{eu}} = \nabla^{(\mathrm{e}, K_X^{-1})}$ is the (e, K_X^{-1})-twisted quantum connection, $S_{\mathrm{eu}} = S_{(\mathrm{e}, K_X^{-1})}$ is the (e, K_X^{-1})-twisted pairing $S_{\mathrm{eu}}(u, v) = \int_X u(-z) \cup v(z) \cup \rho$, and $\nabla^{\mathrm{loc}} = \nabla^{K_X}$ is the quantum connection of K_X. The superscript "eu" means "Euler" and "loc" means "local". We denote the fundamental solutions (in Proposition 2.5) for these quantum D-modules by

$$L_{\mathrm{eu}}(\tau, z) = L_{(\mathrm{e}, K_X^{-1})}(\tau, z) \quad \text{(see Remark 3.4)}$$

$$L_{\mathrm{loc}}(\tau, z) = L^{K_X}(\tau, z) \quad \text{(see Remark 3.13)}$$

where the Novikov variable is set to be one. For a smooth anticanonical hypersurface $Z \subset X$, we can similarly consider the small ambient part quantum D-module of Z (cf. Definition 3.5):

$$\mathrm{SQDM}_{\mathrm{amb}}(Z) := (H^{\mathrm{ev}}_{\mathrm{amb}}(Z) \otimes \mathcal{O}_{U_{\mathrm{sm}} \times \mathbb{C}_z}, \nabla^Z, S_Z).$$

Definition 5.11. For $\alpha \in \{0, \dots, s\}$, we put

$$I^{\mathrm{eu}}_\alpha(\tau_2, z) := \mathrm{e}^{\tau_2/z} \sum_{d \in \mathrm{Eff}(X)} N_{\alpha,d}(z)\, \mathrm{e}^{\tau_2(d)} \prod_{k=1}^{\rho(d)} (\rho + kz)$$

$$I^{\mathrm{loc}}_\alpha(\tau_2, z) := \mathrm{e}^{\tau_2/z} \sum_{d \in \mathrm{Eff}(X)} N_{\alpha,d}(z)\, \mathrm{e}^{\tau_2(d)} \prod_{k=0}^{\rho(d)-1} (-\rho - kz)$$

where recall that $\rho = c_1(X)$ and we set $\prod_{k=1}^{\rho(d)} (\rho + kz) = \prod_{k=0}^{\rho(d)-1} (-\rho - kz) = 1$ for $d = 0$. We call I^{eu}_α the (e, K_X^{-1})*-twisted I-function* of X and I^{loc}_α *the I-function* of K_X. □

Remark 5.12. (1) The *large radius limit* is the limit:

$$\Re(\tau_2(d)) \to 0 \quad \text{for } \forall d \in \mathrm{Eff}(X) \setminus \{0\}.$$

With our choice of co-ordinates, the large radius limit corresponds to $e^{t^i} \to 0$ for $1 \le i \le r$. The I-function satisfies the asymptotics

$$I_\alpha^{\mathrm{eu}}(\tau_2, z) \sim_{\mathrm{lrl}} e^{\tau_2/z} T_\alpha \quad \text{and} \quad I_\alpha^{\mathrm{loc}}(\tau_2, z) \sim_{\mathrm{lrl}} e^{\tau_2/z} T_\alpha$$

under the large radius limit (the subscript "lrl" stands for the large radius limit). Therefore, they are linearly independent in a neighbourhood of the large radius limit.

(2) Put $\partial_\rho := \sum_{\beta=0}^{s} \rho_\beta \partial_\beta$, where $\rho = c_1(TX) = \sum_{\beta=0}^{s} \rho_\beta T_\beta$. We have

$$e^{-\sqrt{-1}\pi\rho/z} z\partial_\rho I_\alpha^{\mathrm{loc}}(h(\tau_2), z) = \rho I_\alpha^{\mathrm{eu}}(\tau_2, z) \tag{5.10}$$

where h is the map in (3.5) with $c_1(E) = c_1(K_X^{-1}) = \rho$. □

Lemma 5.13. Suppose that the small quantum product of X is convergent as in Section 5.1. There exists a region $U_{\mathrm{sm}} \subset H^2(X)$ of the form (5.1) such that the I-functions $I_\alpha^{\mathrm{eu}}(\tau_2, z)$, $I_\alpha^{\mathrm{loc}}(\tau_2, z)$ are convergent and analytic on $U_{\mathrm{sm}} \times \mathbb{C}_z^\times$. □

Proof. This follows easily from the estimate (5.8) and $N_{\alpha,d}(z) = z^{-(\rho(d)-|\alpha|)} z^{-\frac{\deg}{2}} N_{\alpha,d}(1)$. ■

For $\alpha = 0$, the I-functions I_0^{eu} and I_0^{loc} have the following z^{-1}-expansions:

$$I_0^{\mathrm{eu}}(\tau_2, z) = F(\tau_2)\mathbf{1} + G(\tau_2)z^{-1} + O(z^{-2}),$$

$$I_0^{\mathrm{loc}}(\tau_2, z) = \mathbf{1} + H(\tau_2)z^{-1} + O(z^{-2}),$$

where $F(\tau_2)$ is a scalar-valued function and $G(\tau_2)$ and $H(\tau_2)$ are $H^2(X)$-valued functions. We define the mirror maps by

$$\mathfrak{m}_{\mathrm{eu}}(\tau_2) := \frac{G(\tau_2)}{F(\tau_2)}, \quad \mathfrak{m}_{\mathrm{loc}}(\tau_2) := H(\tau_2). \tag{5.11}$$

Note that $F(\tau_2)$ is invertible in a neighbourhood of the large radius limit point and the mirror maps take values in $H^2(X)$. The mirror maps have the asymptotic $\mathfrak{m}(\tau_2) \sim_{\mathrm{lrl}} \tau_2$ and thus induce isomorphisms between neighbourhoods of the large radius limit point. Quantum Lefschetz Theorem of Coates and Givental [4] gives the following proposition:

Proposition 5.14. For any $\alpha \in \{0, \ldots, s\}$, there exist $v_\alpha(\tau_2, z), w_\alpha(\tau_2, z) \in H^{\mathrm{ev}}(X) \otimes \mathbb{C}[z][\![e^{\tau_2}]\!]$ such that

$$I_\alpha^{\mathrm{eu}}(\tau_2, z) = L_{\mathrm{eu}}(\mathfrak{m}_{\mathrm{eu}}(\tau_2), z)^{-1} v_\alpha(\tau_2, z),$$

$$I_\alpha^{\mathrm{loc}}(\tau_2, z) = L_{\mathrm{loc}}(\mathfrak{m}_{\mathrm{loc}}(\tau_2), z)^{-1} w_\alpha(\tau_2, z).$$

Moreover, we have the asymptotics $v_\alpha \sim_{\mathrm{lrl}} T_\alpha$, $w_\alpha \sim_{\mathrm{lrl}} T_\alpha$ under the large radius limit, and v_α, w_α are homogeneous of degree $2|\alpha| = \deg T_\alpha$ with respect to the usual grading on $H^{\mathrm{ev}}(X)$, $\deg z = 2$ and $\deg \mathrm{e}^{\tau_2} = 0$. □

Proof. We will just prove the equality for the (e, K_X^{-1})-twisted theory. The same argument applies to the other case. Coates and Givental [4, Theorem 2, see also p. 27 and 34] introduced the following "big" I-function:

$$\mathbf{I}(\tau, z) := z\mathbf{1} + \tau + \sum_{\beta=0}^{s} \sum_{(d,\ell)\neq(0,0),(0,1)} \frac{Q^d}{\ell!} \left\langle \frac{T_\beta}{z-\psi}, \tau, \dots, \tau \right\rangle_{0,\ell+1,d} T^\beta \prod_{k=1}^{\rho(d)} (\rho + \lambda + kz)$$

and showed that $\mathbf{I}(\tau, -z)$ lies in the Lagrangian cone $\mathcal{L}_{(\mathbf{e}_\lambda, K_X^{-1})}$ of the $(\mathbf{e}_\lambda, K_X^{-1})$-twisted theory. This is related to our I-functions as

$$I_\alpha^{\mathrm{eu}}(\tau_2, z) = \partial_\alpha \mathbf{I}(\tau, z)|_{\tau=\tau_2, Q=1, \lambda=0}. \tag{5.12}$$

Note that $\partial_\alpha \mathbf{I}(\tau_2, -z)$ is a tangent vector to the cone $\mathcal{L}_{(\mathbf{e}_\lambda, K_X^{-1})}$ at $\mathbf{I}(\tau_2, -z)$. Moreover, $\partial_0 \mathbf{I}(\tau_2, -z)$ has the following expansion:

$$\partial_0 \mathbf{I}(\tau_2, -z) = \mathbf{F}(\tau_2) - z^{-1}\mathbf{G}(\tau_2) + O(z^{-2})$$

with $\mathbf{F}(\tau_2) \in \mathbb{C}[\![Q, \tau_2]\!]$ and $\mathbf{G}(\tau_2) \in H^{\mathrm{ev}}(X) \otimes \mathbb{C}[\lambda][\![Q, \tau_2]\!]$. Therefore, $\partial_0 \mathbf{I}(\tau_2, -z)/\mathbf{F}(\tau_2)$ gives the unique intersection point:

$$(\mathbf{1} + \mathcal{H}_-) \cap T_{\mathbf{I}(\tau_2, -z)} \mathcal{L}_{(\mathbf{e}_\lambda, K_X^{-1})}.$$

Set $\tilde{\tau} = \mathbf{G}(\tau_2)/\mathbf{F}(\tau_2)$. The discussion in Section 2.4.3 shows that the tangent space at $\mathbf{I}(\tau_2, -z)$ is generated by $\partial_\alpha J_{(\mathbf{e}_\lambda, K_X^{-1})}(\tilde{\tau}, -z) = L_{(\mathbf{e}_\lambda, K_X^{-1})}(\tilde{\tau}, -z)^{-1} T_\alpha$ over $\mathbb{C}[z, \lambda][\![Q, \tau_2]\!]$ (see (2.4)). Therefore, there exists $\mathbf{v}_\alpha(\tau_2, z) \in H^{\mathrm{ev}}(X) \otimes \mathbb{C}[z, \lambda][\![Q, \tau_2]\!]$ such that

$$\partial_\alpha \mathbf{I}(\tau_2, z) = L_{(\mathbf{e}_\lambda, K_X^{-1})}(\tilde{\tau}, z)^{-1} \mathbf{v}_\alpha(\tau_2, z).$$

It is easy to check that $\tilde{\tau} - \tau_2$ and $\mathbf{v}_\alpha$ in fact belong to $H^{\mathrm{ev}}(X) \otimes \mathbb{C}[z, \lambda][\![Q\,\mathrm{e}^{\tau_2}]\!]$. Under the non-equivariant limit $\lambda \to 0$ and the specialization $Q = 1$, $\tilde{\tau}$ becomes $\mathfrak{m}_{\mathrm{eu}}(\tau_2)$. Setting $v_\alpha(\tau_2, z) = \mathbf{v}_\alpha(\tau_2, z)|_{\lambda=0, Q=1}$ and using (5.12), we obtain the formula in the proposition. The asymptotics of $v_\alpha(\tau_2, z)$ follows from the asymptotics of $I_\alpha^{\mathrm{eu}}(\tau_2, z)$ in Remark 5.12. The homogeneity of $v_\alpha(\tau_2, z)$ follows from the homogeneity of $I_\alpha^{\mathrm{eu}}(\tau_2, z)$ and $L_{\mathrm{eu}}(\tau_2, z)$. ■

Lemma 5.15. Suppose that the small quantum cohomology of X is convergent as in Section 5.1. The flat connections ∇^{eu}, ∇^{loc} for the small quantum D-modules $\mathrm{SQDM}_{(\mathrm{e},K_X^{-1})}(X)$, $\mathrm{SQDM}(K_X)$ are convergent over a region $U_{\mathrm{sm}} \subset H^2(X)$ of the form (5.1). Also the functions v_α, w_α in Proposition 5.14 are convergent over the same region. □

Proof. We only discuss the convergence of the (e, K_X^{-1})-twisted theory. The other case is similar. From Lemma 5.13, it follows that the mirror map $\mathfrak{m}_{\mathrm{eu}}(\tau_2)$ is convergent on a region of the form (5.1). Recall that $L_{\mathrm{eu}}(\tau_2, z)$ is homogeneous of degree zero and that $L_{\mathrm{eu}}(\tau_2, z) = \mathrm{id} + O(z^{-1})$. Recall also that $v_\alpha(\tau_2, z)$ is homogeneous of degree $2|\alpha|$ from Proposition 5.14. Therefore, L_{eu} is lower-triangular and the matrix $[v_0, \dots, v_s]$ is upper-triangular with respect to the grading on $H^{\mathrm{ev}}(X)$. Therefore, the matrix equation

$$\begin{pmatrix} | & | & & | \\ I_0^{\mathrm{eu}} & I_1^{\mathrm{eu}} & \cdots & I_s^{\mathrm{eu}} \\ | & | & & | \end{pmatrix} = L_{\mathrm{eu}}(\mathfrak{m}_{\mathrm{eu}}(\tau_2), z)^{-1} \begin{pmatrix} | & | & & | \\ v_0^{\mathrm{eu}} & v_1^{\mathrm{eu}} & \cdots & v_s^{\mathrm{eu}} \\ | & | & & | \end{pmatrix}$$

in Proposition 5.14 can be viewed as the LU decomposition. Therefore, we can solve for L_{eu}^{-1} and v_α from $[I_0^{\mathrm{eu}}, \dots, I_s^{\mathrm{eu}}]$ by simple linear algebra, and Lemma 5.13 implies that both $L_{\mathrm{eu}}(\mathfrak{m}_{\mathrm{eu}}(\tau_2), z)$ and v_α are convergent. The conclusion follows. ■

The above lemma justifies the definition (5.9) at the beginning of Section 5.5.

5.6 Second structure connections are twisted quantum connections

We show that the small quantum D-modules $\mathrm{SQDM}_{(\mathrm{e},K_X^{-1})}(X)$ and $\mathrm{SQDM}(K_X)$ correspond to the second structure connections $\check{\nabla}^{(\frac{n+1}{2})}$ and $\check{\nabla}^{(-\frac{n+1}{2})}$, respectively.

By the divisor equation, the quantum connections ∇^{eu}, ∇^{loc} are invariant under the shift $\tau \mapsto \tau + 2\pi\sqrt{-1}v$ with $v \in H^2(X,\mathbb{Z})$. Therefore, the small quantum D-modules descend to the quotient space

$$U_{\mathrm{sm}}/2\pi\sqrt{-1}H^2(X,\mathbb{Z}) \cong \{(q^1,\dots,q^r) \in (\mathbb{C}^\times)^r : |q^i| < \epsilon\}.$$

Since the I-functions I_0^{eu}, I_0^{loc} satisfy the equation $I(\tau_2 + 2\pi\sqrt{-1}v, z) = e^{2\pi\sqrt{-1}v/z} I(\tau_2)$, the mirror maps $\mathfrak{m} = \mathfrak{m}_{\mathrm{eu}}$ or $\mathfrak{m}_{\mathrm{loc}}$ satisfy $\mathfrak{m}(\tau_2 + 2\pi\sqrt{-1}v) = \mathfrak{m}(\tau_2) + 2\pi\sqrt{-1}v$; therefore, the mirror maps descend to isomorphisms

$$\mathfrak{m}_{\mathrm{eu/loc}} : U'_{\mathrm{sm}}/2\pi\sqrt{-1}H^2(X,\mathbb{Z}) \xrightarrow{\cong} U''_{\mathrm{sm}}/2\pi\sqrt{-1}H^2(X,\mathbb{Z})$$

between neighbourhoods U'_{sm}, U''_{sm} of the form (5.1). Define the maps $\pi_{\mathrm{eu}}, \pi_{\mathrm{loc}}$ by

$$\begin{aligned} \pi_{\mathrm{eu}}(\tau_2, x) &= \mathfrak{m}_{\mathrm{eu}}(\tau_2 - \rho \log x), \\ \pi_{\mathrm{loc}}(\tau_2, x) &= \mathfrak{m}_{\mathrm{loc}}(\tau_2 - \rho \log x + \pi\sqrt{-1}\rho). \end{aligned} \tag{5.13}$$

Choosing smaller U'_{sm} if necessary, each of π_{eu} and π_{loc} defines a map

$$U'_{\mathrm{sm}} \times \{x \in \mathbb{C}^\times : |x| > c\} \longrightarrow U''_{\mathrm{sm}}/2\pi\sqrt{-1}H^2(X, \mathbb{Z}).$$

We need to choose sufficiently small large radius limit neighbourhoods of the form (5.1) which may vary in each case: we denote by U'_{sm}, U''_{sm} for such neighbourhoods.

Theorem 5.16. Suppose that the small quantum cohomology of X is convergent as in Section 5.1. We have the following isomorphisms of vector bundles with connections:

$$\begin{aligned} \psi_{\mathrm{eu}}\colon (\check{F}, \check{\nabla}^{(\frac{n+1}{2})})|_{U'_{\mathrm{sm}}\times\{|x|>c\}} &\cong \pi_{\mathrm{eu}}^* \mathrm{SQDM}_{(\mathrm{e}, K_X^{-1})}(X)|_{z=1} \\ \psi_{\mathrm{loc}}\colon (\check{F}, \check{\nabla}^{(-\frac{n+1}{2})})|_{U'_{\mathrm{sm}}\times\{|x|>c\}} &\cong \pi_{\mathrm{loc}}^* \mathrm{SQDM}(K_X)|_{z=1} \end{aligned}$$

where, as discussed above, we regard $\mathrm{SQDM}_{(\mathrm{e}, K_X^{-1})}(X)$, $\mathrm{SQDM}(K_X)$ as flat connections on the quotient space $U''_{\mathrm{sm}}/2\pi\sqrt{-1}H^2(X, \mathbb{Z})$. These maps are given by the following formulae:

$$\psi_{\mathrm{eu}}(T_\alpha) = ((-\pi_{\mathrm{eu}}^*\nabla^{\mathrm{eu}})_{\partial_x})^{n-|\alpha|}(x^{-1} v_\alpha(\tau_2 - \rho \log x, 1)) \tag{5.14}$$

$$\psi_{\mathrm{loc}}((-\check{\nabla}^{(-\frac{n+1}{2})}_{\partial_x})^{|\alpha|} T_\alpha) = w_\alpha(\tau_2 - \rho\log x + \pi\sqrt{-1}\rho, 1) \tag{5.15}$$

□

Proof. To show that $\psi_{\mathrm{eu/loc}}$ intertwines the connections, we compare the solution $\check{K}^{(\sigma,\ell)}$ to the second structure connection from Proposition 5.9 with the inverse fundamental solution $L^{-1}_{\mathrm{eu/loc}}$ of the small quantum D-modules. More precisely, we check the commutativity of a diagram of the form:

$$\begin{tikzcd} (\check{F}, \check{\nabla}^{(\sigma)})|_{U_{\mathrm{sm}}\times\{|x|>c\}} \arrow[rr, "\psi"] \arrow[dr, "\check{K}^{(\sigma,\ell)}"'] & & \pi^*\mathrm{SQDM}|_{z=1} \arrow[dl, "(\text{constant factor})\cdot\pi^*L^{-1}|_{z=1}"] \\ & (\check{F}, d) & \end{tikzcd}$$

for suitable ℓ; we shall take $\ell = 1$ for $\sigma = \frac{n+1}{2}$ and $\ell = 0$ for $\sigma = -\frac{n+1}{2}$.

We define the $\mathcal{O}$-module map ψ_{eu} by the formula (5.14) and show that it intertwines the connections. We have

$$
\begin{aligned}
&L_{\mathrm{eu}}(\pi_{\mathrm{eu}}(\tau_2, x), 1)^{-1}((-\pi_{\mathrm{eu}}^{*}\nabla^{\mathrm{eu}})_{\partial_x}^{n-|\alpha|}(x^{-1}v_\alpha(\tau_2-\rho\log x, 1)))\\
&\quad = (-\partial_x)^{n-|\alpha|}L_{\mathrm{eu}}(\mathfrak{m}_{\mathrm{eu}}(\tau_2-\rho\log x), 1)^{-1}(x^{-1}v_\alpha(\tau_2-\rho\log x, 1))\\
&\quad = (-\partial_x)^{n-|\alpha|}x^{-1}I_\alpha^{\mathrm{eu}}(\tau_2-\rho\log x, 1) \quad \text{(by Proposition 5.14)}\\
&\quad = \sum_d N_{\alpha,d}(1)\frac{\mathrm{e}^{\tau_2+\tau_2(d)}}{x^{\rho+\rho(d)+n-|\alpha|+1}}\prod_{k=1}^{\rho(d)+n-|\alpha|}(\rho+k) \qquad (5.16)\\
&\quad \sim_{\mathrm{lrl}} \frac{T_\alpha}{x^{\rho+n-|\alpha|+1}}(\rho+1)(\rho+2)\cdots(\rho+n-|\alpha|)\in H^{\geq 2|\alpha|}(X,\mathbb{C}) \qquad (5.17)
\end{aligned}
$$

From Proposition 5.9, we deduce that the expression in Formula (5.16) is exactly $\check{K}_\alpha^{(\frac{n+1}{2},1)}(\tau_2, x)$. This implies that the morphism ψ_{eu} is a morphism of vector bundles with connection. The asymptotics (5.17) at the large radius limit shows that it is an isomorphism.

Since $\{w_\alpha(\tau_2)\}$ form a basis in a neighbourhood of the large radius limit, we can define an $\mathcal{O}$-module map ψ_{loc}^{-1} such that the formula (5.15) holds. Note that $\check{\nabla}_{\partial_x}^{(\sigma)}$ has no singularities on $U'_{\mathrm{sm}}\times\{|x|>c\}$ if we take the neighbourhood U'_{sm} sufficiently small. We have

$$
\begin{aligned}
&\mathrm{e}^{-\pi\sqrt{-1}\rho}L_{\mathrm{loc}}(\pi_{\mathrm{loc}}(\tau_2, x), 1)^{-1}w_\alpha(\tau_2-\rho\log x+\sqrt{-1}\pi\rho, 1)\\
&\quad = \mathrm{e}^{-\pi\sqrt{-1}\rho}I_\alpha^{\mathrm{loc}}(\tau_2-\rho\log x+\sqrt{-1}\pi\rho, 1) \quad \text{(by Proposition 5.14)}\\
&\quad = \sum_d N_{\alpha,d}(1)\frac{\mathrm{e}^{\tau_2+\tau_2(d)}}{x^{\rho+\rho(d)}}\prod_{k=0}^{\rho(d)-1}(\rho+k) \qquad (5.18)\\
&\quad \sim_{\mathrm{lrl}} T_\alpha\,\mathrm{e}^{\tau_2}x^{-\rho}. \qquad (5.19)
\end{aligned}
$$

From Proposition 5.9, we deduce that the expression in Formula (5.18) is exactly

$$(-\partial_x)^{|\alpha|}\check{K}_\alpha^{(-\frac{n+1}{2},0)}(\tau_2, x).$$

This implies that the morphism ψ_{loc}^{-1} is a morphism of vector bundle with connection. The asymptotics (5.19) at the large radius limit shows that it is an isomorphism. ■

Remark 5.17. By construction in the proof, we have

$$\psi_{\mathrm{eu}} = L_{\mathrm{eu}}(\pi_{\mathrm{eu}}(\tau_2, x), 1)\circ\check{K}^{(\frac{n+1}{2},1)}$$

$$\psi_{\mathrm{loc}} = L_{\mathrm{loc}}(\pi_{\mathrm{loc}}(\tau_2, x), 1)\, e^{\pi\sqrt{-1}\rho} \circ \check{K}^{(-\frac{n+1}{2}, 0)}.$$

with $\check{K}^{(\sigma,\ell)}$ in Proposition 5.9. In particular, we have the following formula for $\psi_{\mathrm{loc}}(T_\alpha)$:

$$\psi_{\mathrm{loc}}(T_\alpha) = L_{\mathrm{loc}}(\pi_{\mathrm{loc}}(\tau_2, x), 1) \left[e^{\sqrt{-1}\pi\rho} \sum_d N_{\alpha,d}(1) \frac{e^{\tau_2 + \tau_2(d)}}{x^{\rho+\rho(d)-|\alpha|}} \frac{\prod_{k=-\infty}^{\rho(d)-|\alpha|-1}(\rho+k)}{\prod_{k=-\infty}^{-1}(\rho+k)} \right]. \tag{5.20}$$

□

5.7 Quantum Serre duality in terms of the second structure connections

In this section, we see that the quantum Serre duality between $\mathrm{QDM}_{(e,K_X^{-1})}(X)$ and $\mathrm{QDM}(K_X)$ from Theorem 3.14 can be rephrased in terms of the second structure connections. As a corollary, we obtain a description of the quantum D-module of an anti-canonical hypersurface $Z \subset X$ in terms of the second structure connection. When X is Fano, this gives an entirely algebraic description of the quantum connection of Z.

We begin with the following lemma.

Lemma 5.18. Let $\bar{f}$ be the map in Lemma 3.2 and h be the map in (3.5) in the case where $E = K_X^{-1}$. The map $h \circ \bar{f}$ relates the two mirror maps (5.11) as

$$(h \circ \bar{f})(\mathfrak{m}_{\mathrm{eu}}(\tau_2)) = \mathfrak{m}_{\mathrm{loc}}(h(\tau_2)).$$

In particular $h \circ \bar{f}|_{H^2(X)}$ is convergent and gives an isomorphism between neighbourhoods of the large radius limit point of the form (5.1). □

We will postpone the proof of the lemma until the end of this section. Consider the quantum Serre pairing S^{QS} from Theorem 3.14 in the case where $E = K_X^{-1}$. By the above lemma, $h \circ \bar{f}$ preserves $H^2(X)$ and therefore S^{QS} induces a flat pairing

$$S^{\mathrm{QS}} : \mathrm{SQDM}_{(e,K_X^{-1})}(X)|_{z=-1} \times (h \circ \bar{f})^* \mathrm{SQDM}(K_X)|_{z=1} \to \mathcal{O}_{U_{\mathrm{sm}}}.$$

Combined with the ∇^{eu}-flat shift $(-1)^{\frac{\deg}{2}} : \mathrm{SQDM}_{(e,K_X^{-1})}(X)|_{z=1} \cong \mathrm{SQDM}_{(e,K_X^{-1})}(X)|_{z=-1}$, we obtain a flat pairing

$$P : \mathrm{SQDM}_{(e,K_X^{-1})}(X)|_{z=1} \times (h \circ \bar{f})^* \mathrm{SQDM}(K_X)|_{z=1} \to \mathcal{O}_{U_{\mathrm{sm}}}$$

$$\text{defined by} \quad P(u, v) = \int_X \left((-1)^{\frac{\deg}{2}} u \right) \cup v. \tag{5.21}$$

The second structure connections satisfy a certain "difference equation" with respect to the parameter σ. It is easy to check that we have the following morphism of meromorphic flat connections [15, Theorem 9.4.b]:

$$\Delta_\sigma\colon (\check{F}, \check{\nabla}^{(\sigma+1)}) \longrightarrow (\check{F}, \check{\nabla}^{(\sigma)})$$
$$T_\alpha \longmapsto \check{\nabla}^{(\sigma)}_{\partial_x} T_\alpha = -(\mu - \tfrac{1}{2} - \sigma)(\mathfrak{E} \circ_\tau - x)^{-1} T_\alpha.$$

This is an isomorphism over $(U_{\mathrm{sm}} \times \mathbb{C}_x) \setminus \Sigma$ if $\mu - \frac{1}{2} - \sigma$ is invertible, that is, if $\sigma \notin \{-\frac{n+1}{2}, -\frac{n-1}{2}, \dots, \frac{n-1}{2}\}$.

Lemma 5.18 shows that $(h \circ \bar{f} \circ \pi_{\mathrm{eu}})(\tau_2, x) = \pi_{\mathrm{loc}}(\tau_2, x)$. Thus the pairing (5.21) induces the flat pairing

$$P\colon \pi_{\mathrm{eu}}^* \mathrm{SQDM}_{(\mathrm{e}, K_X^{-1})}(X)|_{z=1} \times \pi_{\mathrm{loc}}^* \mathrm{SQDM}(K_X)|_{z=1} \longrightarrow \mathcal{O}_{U_{\mathrm{sm}}}. \tag{5.22}$$

We also have the morphism of flat connections

$$\rho\colon \pi_{\mathrm{eu}}^* \mathrm{SQDM}_{(\mathrm{e}, K_X^{-1})}(X)|_{z=1} \longrightarrow \pi_{\mathrm{loc}}^* \mathrm{SQDM}(K_X)|_{z=1} \tag{5.23}$$

induced from the morphism in Theorem 3.14 (2).

Theorem 5.19. Via the isomorphisms ψ_{eu}, ψ_{loc} in Theorem 5.16, the pairing P (5.22) coincides with $(-1)^{n+1}\check{g}$, that is, $P(\psi_{\mathrm{eu}} T_\alpha, \psi_{\mathrm{loc}} T_\beta) = (-1)^{n+1}\check{g}(T_\alpha, T_\beta)$ and the morphism ρ (5.23) coincides with the composition:

$$\Delta := (-1)^{n+1} \Delta_{-\frac{n+1}{2}} \circ \Delta_{-\frac{n-1}{2}} \circ \cdots \circ \Delta_{\frac{n-1}{2}} \colon (\check{F}, \check{\nabla}^{(\frac{n+1}{2})}) \longrightarrow (\check{F}, \check{\nabla}^{(-\frac{n+1}{2})}).$$

Moreover, we have

$$\check{g}(\gamma_1, \gamma_2) = (-1)^{n+1} \int_X ((-1)^{\frac{\deg}{2}} \check{K}^{(\frac{n+1}{2}, 1)} \gamma_1) \cup \check{K}^{(-\frac{n+1}{2}, 0)} \gamma_2,$$
$$\check{K}^{(-\frac{n+1}{2}, 0)} \circ \Delta = \rho \circ \check{K}^{(\frac{n+1}{2}, 1)}$$

for $\gamma_1, \gamma_2 \in H^{\mathrm{ev}}(X)$. □

Proof. First, we prove that P corresponds to $(-1)^{n+1}\check{g}$. Since both pairings are flat, it is enough to compare the asymptotics of $P(\psi_{\mathrm{eu}} T_\alpha, \psi_{\mathrm{loc}} T_\beta)$ and $\check{g}(T_\alpha, T_\beta)$ at the large radius limit. Since the quantum product equals the cup product at the large radius limit, we have

$$\begin{aligned}\check{g}(T_\alpha, T_\beta)|_{x=1} &\sim_{\mathrm{lrl}} -\int_X (1-\rho)^{-1} \cup T_\alpha \cup T_\beta \\ &= \begin{cases} -\int_X \rho^{n-|\alpha|-|\beta|} \cup T_\alpha \cup T_\beta & \text{if } |\alpha|+|\beta| \le n; \\ 0 & \text{otherwise.} \end{cases}\end{aligned} \tag{5.24}$$

We then compute the asymptotics of $P(\psi_{\mathrm{eu}}(T_\alpha), \psi_{\mathrm{loc}}(T_\beta))$. By Remark 5.17, after some computation, we find

$$(-1)^{\frac{\deg}{2}} \psi_{\mathrm{eu}}(T_\alpha)|_{x=1} = L_{\mathrm{eu}}(\mathfrak{m}_{\mathrm{eu}}(\tau_2), -1) \left[\sum_d ((-1)^{\frac{\deg}{2}} N_{\alpha,d}(1))\, \mathrm{e}^{-\tau_2+\tau_2(d)} \prod_{k=1}^{\rho(d)+n-|\alpha|} (-\rho+k) \right]. \tag{5.25}$$

We have already found a similar formula (5.20) for $\psi_{\mathrm{loc}}(T_\alpha)$. In view of Lemma 5.18, Theorem 3.14 (3) gives the identity:

$$(L_{\mathrm{eu}}(\mathfrak{m}_{\mathrm{eu}}(\tau_2), -z)\gamma_1, L_{\mathrm{loc}}(\mathfrak{m}_{\mathrm{loc}}(h(\tau_2)), z)\gamma_2) = (\gamma_1, \mathrm{e}^{-\sqrt{-1}\pi\rho/z}\gamma_2)$$

for $\gamma_1, \gamma_2 \in H^{\mathrm{ev}}(X)$, where $(u, v) = \int_X u \cup v$ is the Poincaré pairing. Therefore, Equations (5.25) and (5.20) give

$$P(\psi_{\mathrm{eu}}(T_\alpha), \psi_{\mathrm{loc}}(T_\beta))|_{x=1} \sim_{\mathrm{lrl}} (-1)^n \int_X T_\alpha \cup T_\beta \cup \frac{\prod_{k=1}^{n-|\alpha|}(\rho-k)}{\prod_{k=1}^{|\beta|}(\rho-k)}.$$

To have non-zero asymptotics, we must have $|\alpha|+|\beta| \le n$; in this case, the right-hand side is

$$(-1)^n \int_X T_\alpha \cup T_\beta \cup \prod_{k=|\beta|+1}^{n-|\alpha|} (\rho-k) = (-1)^n \int_X T_\alpha \cup T_\beta \cup \rho^{n-|\alpha|-|\beta|}.$$

Comparing this with (5.24), we deduce that $P(\psi_{\mathrm{eu}} T_\alpha, \psi_{\mathrm{loc}} T_\beta) = (-1)^{n+1} \check{g}(T_\alpha, T_\beta)$. Note that the above computation shows

$$P(\psi_{\mathrm{eu}} T_\alpha, \psi_{\mathrm{loc}} T_\beta) = \int_X ((-1)^{\frac{\deg}{2}} \check{K}_\alpha^{(\frac{n+1}{2},1)}) \cup \check{K}_\beta^{(-\frac{n+1}{2},0)}.$$

We deduce the equality of the pairings.

Next, we prove that the morphism ρ corresponds to Δ, that is, $\rho \circ \psi_{\mathrm{eu}} = \psi_{\mathrm{loc}} \circ \Delta$. Recall from Remark 5.17 that $\psi_{\mathrm{eu}}, \psi_{\mathrm{loc}}$ are given by

$$\psi_{\mathrm{eu}} = L_{\mathrm{eu}}(\pi_{\mathrm{eu}}(\tau_2, x), 1) \circ \check{K}^{(\frac{n+1}{2},1)},$$

$$\psi_{\mathrm{loc}} = L_{\mathrm{loc}}(\pi_{\mathrm{loc}}(\tau_2, x), 1)\, \mathrm{e}^{\pi\sqrt{-1}\rho} \circ \check{K}^{(-\frac{n+1}{2},0)}.$$

Therefore, it suffices to prove the following formulae:

$$\rho \circ L_{\mathrm{eu}}(\pi_{\mathrm{eu}}(\tau_2, x), 1) = L_{\mathrm{loc}}(\pi_{\mathrm{loc}}(\tau_2, x), 1)\, \mathrm{e}^{\pi\sqrt{-1}\rho} \circ \rho,$$

$$\rho \circ \check{K}^{(\frac{n+1}{2},1)} = \check{K}^{(-\frac{n+1}{2},0)} \circ \Delta.$$

The first equation follows from Theorem 3.14(3) in view of Lemma 5.18. To see the second equation, it suffices to prove

$$\rho \circ \check{K}^{(\frac{n+1}{2},1)} = \check{K}^{(\frac{n+1}{2},0)} \quad \text{and} \quad \check{K}^{(\sigma+1,0)} = -\check{K}^{(\sigma,0)} \circ \Delta_\sigma.$$

The first formula is immediate from the definition. To see the second, we calculate:

$$\check{K}^{(\sigma,0)}(\Delta_\sigma T_\alpha) = \check{K}^{(\sigma,0)}(\check{\nabla}^{(\sigma)}_{\partial_x} T_\alpha) = \partial_x(\check{K}^{(\sigma,0)} T_\alpha) = \partial_x \check{K}^{(\sigma,0)}_\alpha = -\check{K}^{(\sigma+1,0)}_\alpha.$$

The conclusion follows. ■

Combined with Corollary 3.17, the above theorem implies the following corollary.

Corollary 5.20. Let Z be a smooth anticanonical hypersurface of X that satisfies one of the conditions in Lemma 3.15. Then the small quantum D-module $(\iota^* \circ \pi_{\mathrm{eu}})^* \mathrm{SQDM}_{\mathrm{amb}}(Z)|_{z=1}$ of Z is isomorphic to the image $\mathrm{Im}\Delta$ of the morphism:

$$\Delta\colon (\check{F}, \check{\nabla}^{(\frac{n+1}{2})})|_{U'_{\mathrm{sm}} \times \{|x|>c\}} \to (\check{F}, \check{\nabla}^{(-\frac{n+1}{2})})|_{U'_{\mathrm{sm}} \times \{|x|>c\}}.$$

where $\iota^* \circ \pi_{\mathrm{eu}}$ is regarded as a map

$$\iota^* \circ \pi_{\mathrm{eu}}\colon U'_{\mathrm{sm}} \times \{|x| > c\} \to \iota^*(U''_{\mathrm{sm}})/2\pi\sqrt{-1}H^2(Z, \mathbb{Z}).$$

The isomorphism sends $\Delta(T_\alpha) \in \mathrm{Im}(\Delta)$ to $\iota^* \psi_{\mathrm{eu}}(T_\alpha) \in (\iota^* \circ \pi_{\mathrm{eu}})^* \mathrm{SQDM}_{\mathrm{amb}}(Z)|_{z=1}$. □

Remark 5.21. Recall that the conditions in Lemma 3.15 are satisfied for an anticanonical hypersurface if X is Fano. □

Proof of Lemma 5.18. We consider the following equivariant I-functions (cf. Definition 5.11):

$$I_\alpha^{\mathrm{eu},\lambda}(\tau_2, z) = \mathrm{e}^{\tau_2/z} \sum_{d\in \mathrm{Eff}(X)} N_{\alpha,d}(z)\, \mathrm{e}^{\tau_2(d)} \prod_{k=1}^{\rho(d)} (\rho + \lambda + kz)$$

$$I_\alpha^{\mathrm{loc},\lambda}(\tau_2, z) = \mathrm{e}^{\tau_2/z} \sum_{d\in \mathrm{Eff}(X)} N_{\alpha,d}(z)\, \mathrm{e}^{\tau_2(d)} \prod_{k=0}^{\rho(d)-1} (-\rho - \lambda - kz)$$

and define the equivariant mirror maps $\mathfrak{m}_{\mathrm{eu}}^\lambda$, $\mathfrak{m}_{\mathrm{loc}}^\lambda$ as in (5.11). By exactly the same argument as Proposition 5.14, we have that

$$\begin{aligned} I_\alpha^{\mathrm{eu},\lambda}(\tau_2, z) &= L_{\mathrm{eu},\lambda}(\mathfrak{m}_{\mathrm{eu}}^\lambda(\tau_2), z)^{-1} v_\alpha^\lambda(\tau_2, z), \\ I_\alpha^{\mathrm{loc},\lambda}(\tau_2, z) &= L_{\mathrm{loc},\lambda}(\mathfrak{m}_{\mathrm{loc}}^\lambda(\tau_2), z)^{-1} w_\alpha^\lambda(\tau_2, z) \end{aligned} \tag{5.26}$$

for some $v_\alpha^\lambda(\tau_2, z)$, $w_\alpha^\lambda(\tau_2, z) \in H^{\mathrm{ev}}(X) \otimes \mathbb{C}[\lambda, z][\![\mathrm{e}^{\tau_2}]\!]$. Here we set $L_{\mathrm{eu},\lambda} = L_{(\mathrm{e}_\lambda, K_X^{-1})}$ and $L_{\mathrm{loc},\lambda} = L_{(\mathrm{e}_{-\lambda}^{-1}, K_X)}$. Similarly to (5.10), we have the following relationship:

$$\mathrm{e}^{-\pi\sqrt{-1}\rho/z}(z\partial_\rho + \lambda) I_\alpha^{\mathrm{loc},\lambda}(h(\tau_2), z) = (\rho + \lambda) I_\alpha^{\mathrm{eu},\lambda}(\tau_2, z). \tag{5.27}$$

We compute both sides of this equation. By (5.26), the left-hand side equals

$$\begin{aligned} &\mathrm{e}^{-\pi\sqrt{-1}\rho/z}(z\partial_\rho + \lambda) L_{\mathrm{loc},\lambda}(\mathfrak{m}_{\mathrm{loc}}^\lambda(h(\tau_2)), z)^{-1} w_\alpha^\lambda(h(\tau_2), z) \\ &= \mathrm{e}^{-\pi\sqrt{-1}\rho/z} L_{\mathrm{loc},\lambda}(\mathfrak{m}_{\mathrm{loc}}^\lambda(h(\tau_2)), z)^{-1} \tilde{w}_\alpha^\lambda(\tau_2, z) \end{aligned} \tag{5.28}$$

where $\tilde{w}_\alpha^\lambda(\tau_2, z)$ is obtained from $w_\alpha(h(\tau_2), z)$ by applying $z((\mathfrak{m}_{\mathrm{loc}}^\lambda \circ h)^* \nabla^{(\mathrm{e}_{-\lambda}^{-1}, K_X)})_\rho + \lambda$ and is an element of $H^{\mathrm{ev}}(X) \otimes \mathbb{C}[\lambda, z][\![\mathrm{e}^{\tau_2}]\!]$. On the other hand, by (5.26) again, the right-hand side of (5.27) is

$$\begin{aligned} &(\rho + \lambda) L_{\mathrm{eu},\lambda}(\mathfrak{m}_{\mathrm{eu}}^\lambda(\tau_2), z)^{-1} v_\alpha^\lambda(\tau_2, z) \\ &= L_{(\mathrm{e}_\lambda^*, K_X)}(f(\mathfrak{m}_{\mathrm{eu}}^\lambda(\tau_2)), z)^{-1} (\rho + \lambda) v_\alpha^\lambda(\tau_2, z) \quad \text{(by Theorem 2.11(3))} \\ &= \mathrm{e}^{-\pi\sqrt{-1}\rho/z} L_{\mathrm{loc},\lambda}(h(f(\mathfrak{m}_{\mathrm{eu}}^\lambda(\tau_2))), z)^{-1} (\rho + \lambda) v_\alpha^\lambda(\tau_2, z) \quad \text{(by Remark 3.13).} \end{aligned} \tag{5.29}$$

Comparing (5.28) and (5.29), we obtain

$$L_{\mathrm{loc},\lambda}(\tau_2', z)^{-1} \tilde{w}_\alpha^\lambda(\tau_2, z) = L_{\mathrm{loc},\lambda}(\tau_2'', z)^{-1} (\rho + \lambda) v_\alpha^\lambda(\tau_2, z)$$

with $\tau_2' = \mathfrak{m}^\lambda_{\mathrm{loc}}(h(\tau_2))$ and $\tau_2'' = h(f(\mathfrak{m}^\lambda_{\mathrm{eu}}(\tau_2)))$. Since $(\rho+\lambda)v^\lambda_\alpha(\tau_2, z)$, $\alpha = 0, \dots, s$ form a basis of $H^{\mathrm{ev}}(X)$ and $\tilde{w}^\lambda_\alpha(\tau_2, z)$ does not contain negative powers of z, we find that

$$L_{\mathrm{loc},\lambda}(\tau_2', z) L_{\mathrm{loc},\lambda}(\tau_2'', z)^{-1}$$

does not contain negative powers in z. By the asymptotics $L_{\mathrm{loc},\lambda}(\tau, z) = \mathrm{id} + O(z^{-1})$, we must have $L_{\mathrm{loc},\lambda}(\tau_2', z)^{-1} = L_{\mathrm{loc},\lambda}(\tau_2'', z)^{-1}$. The asymptotics $L_{\mathrm{loc},\lambda}(\tau, z)^{-1}\mathbf{1} = \mathbf{1} + \tau/z + O(z^{-2})$ shows that $\tau_2' = \tau_2''$. The conclusion follows by taking the non-equivariant limit. ■

Remark 5.22. Consider the family of connection $\check{\nabla}^{(\frac{n+1}{2}+k)}$ for $k \in \mathbb{Z}$. Via the morphisms Δ_σ, we have:

- for $k \in \mathbb{Z}_{\geq 0}$, $\check{\nabla}^{(\frac{n+1}{2}+k)}$ is isomorphic to $\check{\nabla}^{(\frac{n+1}{2})}$ as meromorphic connections;
- for $k \in \mathbb{Z}_{\geq 0}$, $\check{\nabla}^{(-\frac{n+1}{2}-k)}$ is isomorphic to $\check{\nabla}^{(-\frac{n+1}{2})}$ as meromorphic connections.

Theorem 5.16 above gives a geometric interpretation of these two connections. It would be interesting to understand the intermediate connections $\check{\nabla}^{(k)}$ for $k \in \{-\frac{n-1}{2}, \dots, \frac{n-1}{2}\}$. □

5.8 Hodge filtration for the second structure connection

The small quantum D-modules $\mathrm{SQDM}_{(\mathrm{e}, K_X^{-1})}(X)$ and $\mathrm{SQDM}(K_X)$ restricted to $z = 1$ have a natural filtration, called the *A-model Hodge filtration* [5, Section 8.5.4; 30, Lecture 7], and these small quantum D-modules are variations of Hodge structure. In this section, we identify the corresponding filtration on the second structure connection. See [22–24, 35] for related studies on the Hodge structure for local quantum cohomology.

We follow the notation in Theorem 5.16 and write U'_{sm} and U''_{sm} for large radius limit neighbourhoods in $H^2(X)$ on which (respectively) the second structure connection $(\check{F}, \check{\nabla}^{(\sigma)})$ and our small quantum D-modules are convergent.

Definition 5.23. We define the subbundle F^p of the trivial bundle $H^{\mathrm{ev}}(X) \times U''_{\mathrm{sm}} \to U''_{\mathrm{sm}}$ by

$$F^p := H^{\leq 2n-2p}(X) \times U''_{\mathrm{sm}}$$

and call it the *A-model Hodge filtration*. Because the small quantum product preserves the degree:

$$\deg(T_\alpha \bullet^{(\mathrm{e}, K_X^{-1})}_{\tau_2} T_\beta) = \deg(T_\alpha) + \deg(T_\beta)$$

$$\deg(T_\alpha \bullet^{K_X}_{\tau_2} T_\beta) = \deg(T_\alpha) + \deg(T_\alpha)$$

the filtration satisfies *Griffiths transversality* with respect to the small quantum connections:

$$\nabla^{\mathrm{eu}}_{\alpha}(F^p) \subset F^{p-1} \quad \text{and} \quad \nabla^{\mathrm{loc}}_{\alpha}(F^p) \subset F^{p-1}$$

for α with $|\alpha| = 1$. The Hodge filtration also satisfies the following orthogonality:

$$P(F^p, F^{n-p+1}) = 0$$

with respect to the pairing P in (5.21); in other words, the A-model Hodge filtrations on $\mathrm{SQDM}(K_X)|_{z=1}$ and $\mathrm{SQDM}_{(\mathrm{e},K_X^{-1})}(X)|_{z=1}$ are annihilators of each other. □

Next we introduce a filtration on the second structure connection.

Definition 5.24. Consider the second structure connection $(\check{F}, \check{\nabla}^{(-\frac{n+1}{2})})$ restricted to the small parameter space $U'_{\mathrm{sm}} \times \mathbb{C}_x$. Define $\check{F}^p_{\mathrm{loc}}$ to be the $\mathcal{O}_{U'_{\mathrm{sm}}\times\mathbb{C}_x}(*\Sigma)$-submodule of $\mathcal{O}(\check{F})(*\Sigma)$ generated by

$$\{(\check{\nabla}^{(-\frac{n+1}{2})}_{\partial_x})^k T_\alpha : |\alpha| \le k \le n-p\}.$$

Define $\check{F}^p_{\mathrm{eu}}$ to be the $\check{g}$-orthogonal of $\check{F}^{n-p+1}_{\mathrm{loc}}$, that is,

$$\check{F}^p_{\mathrm{eu}} := \{s \in \mathcal{O}(\check{F})(*\Sigma) : \check{g}(s,\gamma) = 0, \rangle\!\rangle \forall \gamma \in \check{F}^{n-p+1}_{\mathrm{loc}}\}.$$

These are decreasing filtrations. □

Lemma 5.25. The filtrations $\check{F}^p_{\mathrm{loc}}, \check{F}^p_{\mathrm{eu}}$ satisfy the Griffiths transversality: $\check{\nabla}^{(-\frac{n+1}{2})}\check{F}^p_{\mathrm{loc}} \subset \Omega^1_{U'_{\mathrm{sm}}\times\mathbb{C}_x} \otimes \check{F}^{p-1}_{\mathrm{loc}}$ and $\check{\nabla}^{(\frac{n+1}{2})}\check{F}^p_{\mathrm{eu}} \subset \Omega^1_{U'_{\mathrm{sm}}\times\mathbb{C}_x} \otimes \check{F}^{p-1}_{\mathrm{eu}}$. □

Proof. It suffices to prove the Griffiths transversality for $\check{F}^p_{\mathrm{loc}}$. We write $\check{\nabla}$ for $\check{\nabla}^{(-\frac{n+1}{2})}$ to save notation. The inclusion $\check{\nabla}_{\partial_x}\check{F}^p_{\mathrm{loc}} \subset \check{F}^{p-1}_{\mathrm{loc}}$ is obvious. We prove $\check{\nabla}_\beta \check{F}^p_{\mathrm{loc}} \subset \check{F}^{p-1}_{\mathrm{loc}}$ for β with $|\beta| = 1$. Take α and $k \in \mathbb{Z}_{\ge 0}$ satisfying $|\alpha| \le k \le n-p$. We have

$$\begin{aligned}\check{\nabla}_\beta(\check{\nabla}_{\partial_x})^k T_\alpha &= (\check{\nabla}_{\partial_x})^k \check{\nabla}_\beta T_\alpha = (\check{\nabla}_{\partial_x})^k \left(\mu + \frac{n}{2}\right)((\mathfrak{E}\bullet_\tau) - x)^{-1} T_\beta \bullet_\tau T_\alpha \\ &= -(\check{\nabla}_{\partial_x})^{k+1} T_\beta \bullet_\tau T_\alpha. \end{aligned} \tag{5.30}$$

Since $\rho = c_1(X)$ is nef, the small quantum product $T_\beta \bullet_\tau T_\alpha$ is a linear combination of classes of degree less than or equal to $2|\alpha| + 2$. Therefore, the expression (5.30) lies in $\check{F}^{p-1}$. ■

Theorem 5.26. There exists a small neighbourhood U'_{sm} of the form (5.1) such that we have

$$\psi_{\mathrm{loc}}(\check{F}^p_{\mathrm{loc}}) = F^p, \quad \psi_{\mathrm{eu}}(\check{F}^p_{\mathrm{eu}}) = F^p$$

over $U'_{\mathrm{sm}} \times \{|x| > c\}$, where $\psi_{\mathrm{loc}}, \psi_{\mathrm{eu}}$ are the isomorphisms in Theorem 5.16 and F^p is the A-model Hodge filtration of $\mathrm{SQDM}_{(\mathrm{e},K_X^{-1})}(X)|_{z=1}$ or of $\mathrm{SQDM}(K_X)|_{z=1}$. $\square$

Proof. Since the A-model Hodge filtration satisfies the orthogonality, it suffices to show that $\psi_{\mathrm{loc}}(\check{F}^p_{\mathrm{loc}}) = F^p$. When $|\alpha| \le k \le n-p$, we have

$$\psi_{\mathrm{loc}}((\check{\nabla}_{\partial_x})^k T_\alpha) = (-1)^{|\alpha|}((\pi^*_{\mathrm{loc}}\nabla)_{\partial_x})^{k-|\alpha|} w_\alpha(\tau_2 - \rho\log x + \pi\sqrt{-1}\rho, 1)$$

where $\check{\nabla} = \check{\nabla}^{(-\frac{n+1}{2})}$. This belongs to F^p by the Griffiths transversality for the A-model Hodge filtration. Considering the case $k = |\alpha|$, we can see that these sections span F^p. ■

Remark 5.27. It follows from the above theorem that the filtrations $\check{F}^p_{\mathrm{loc}}$, $\check{F}^p_{\mathrm{eu}}$ are subbundles over $U'_{\mathrm{sm}} \times \{|x| > c\}$ with U'_{sm} sufficiently small. It would be interesting to study where they are not subbundles, and how we can extend them along the singularity Σ. $\square$

Let $Z \subset X$ be a smooth anticanonical hypersurface. The small ambient part quantum D-module $\mathrm{SQDM}_{\mathrm{amb}}(Z)$ also admits the A-model Hodge filtration

$$F^p = H^{\le 2(n-1)-2p}_{\mathrm{amb}}(Z) \times U''_{\mathrm{sm}}.$$

Combined with Corollaries 3.17 and 5.20, we obtain the following corollary.

Corollary 5.28. Suppose that an anticanonical hypersurface Z of X satisfies one of the conditions in Lemma 3.15. Under the isomorphism

$$(\iota^* \circ \pi_{\mathrm{eu}})^* \mathrm{SQDM}_{\mathrm{amb}}(Z) \cong \mathrm{Im}(\Delta\colon (\check{F}, \check{\nabla}^{(\frac{n+1}{2})}) \to (\check{F}, \check{\nabla}^{(-\frac{n+1}{2})}))$$

in Corollary 5.20, the A-model Hodge filtration F^p on $\mathrm{SQDM}_{\mathrm{amb}}(Z)$ corresponds to $\Delta(\check{F}^{p+1}_{\mathrm{eu}})$, which is contained in $\check{F}^p_{\mathrm{loc}}$. $\square$

6 Quintic in $\mathbb{P}^4$

In this section, we make our result explicit in the case of $X = \mathbb{P}^4$ and $E = \mathcal{O}(5)$. This example was also studied by Dubrovin [9, Section 5.4]. Let $H = c_1(\mathcal{O}(1)) \in H^2(\mathbb{P}^4)$ be the hyperplane class and let t denote the co-ordinate on $H^2(\mathbb{P}^4)$ dual to H. We use the basis

$$\{T_0, T_1, T_2, T_3, T_4\} = \{1, H, H^2, H^3, H^4\}$$

of $H^{\mathrm{ev}}(\mathbb{P}^4)$. The small quantum connection of $\mathbb{P}^4$ is given by

$$\nabla^{(\sigma-1)}_{\partial_t} = \partial_t + z^{-1}(H\bullet_t), \quad \nabla^{(\sigma-1)}_{z\partial_z} = z\partial_z - z^{-1}5(H\bullet_t) + (\mu + \tfrac{1}{2} - \sigma)$$

where

$$H\bullet_t = \begin{pmatrix} 0 & 0 & 0 & 0 & \mathrm{e}^t \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mu = \begin{pmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

6.1 **Fourier–Laplace transformation**

We illustrate the Fourier–Laplace transformation in Section 5.3 for the small quantum connection of $\mathbb{P}^4$. We write ∂_t, $z\partial_z$ for the action of the small quantum connection $\nabla^{(\sigma-1)}_{\partial_t}$, $\nabla^{(\sigma-1)}_{z\partial_z}$, respectively. We have

$$\partial_t T_0 = z^{-1} T_1, \quad z\partial_z T_0 = -5z^{-1} T_1 - (\sigma + \tfrac{3}{2}) T_0$$

$$\partial_t T_1 = z^{-1} T_2, \quad z\partial_z T_1 = -5z^{-1} T_2 - (\sigma + \tfrac{1}{2}) T_1$$

$$\partial_t T_2 = z^{-1} T_3, \quad z\partial_z T_2 = -5z^{-1} T_3 - (\sigma - \tfrac{1}{2}) T_2$$

$$\partial_t T_3 = z^{-1} T_4, \quad z\partial_z T_3 = -5z^{-1} T_4 - (\sigma - \tfrac{3}{2}) T_3$$

$$\partial_t T_4 = \mathrm{e}^t z^{-1} T_0, \quad z\partial_z T_4 = -5\,\mathrm{e}^t z^{-1} T_0 - (\sigma - \tfrac{5}{2}) T_4$$

Under the Fourier–Laplace transformation $z\partial_z = x\partial_x + 1$ and $z^{-1} = -\partial_x$, we have

$$\partial_t T_0 = (-\partial_x) T_1, \quad x\partial_x T_0 = 5\partial_x T_1 - (\sigma + \tfrac{5}{2}) T_0$$

$$\partial_t T_1 = (-\partial_x) T_2, \quad x\partial_x T_1 = 5\partial_x T_2 - (\sigma + \tfrac{3}{2}) T_1$$

$$\partial_t T_2 = (-\partial_x) T_3, \quad x\partial_x T_2 = 5\partial_x T_3 - (\sigma + \tfrac{1}{2}) T_2$$

$$\partial_t T_3 = (-\partial_x) T_4, \quad x\partial_x T_3 = 5\partial_x T_4 - (\sigma - \tfrac{1}{2}) T_3$$

$$\partial_t T_4 = \mathrm{e}^t (-\partial_x) T_0, \quad x\partial_x T_4 = 5\,\mathrm{e}^t \partial_x T_0 - (\sigma - \tfrac{3}{2}) T_4 \tag{6.1}$$

These formulas define the second structure connection $\check{\nabla}^{(\sigma)}$:

$$\check{\nabla}^{(\sigma)}_{\partial_t} = \partial_t + \frac{1}{5^5\,\mathrm{e}^t - x^5}\left(\mu - \frac{1}{2} - \sigma\right)\begin{pmatrix} 5^4\,\mathrm{e}^t & 5^3\,\mathrm{e}^t x & 5^2\,\mathrm{e}^t x^2 & 5\,\mathrm{e}^t x^3 & \mathrm{e}^t x^4 \\ x^4 & 5^4\,\mathrm{e}^t & 5^3\,\mathrm{e}^t x & 5^2\,\mathrm{e}^t x^2 & 5\,\mathrm{e}^t x^2 \\ 5x^3 & x^4 & 5^4\,\mathrm{e}^t & 5^3\,\mathrm{e}^t x & 5^2\,\mathrm{e}^t x^2 \\ 5^2 x^2 & 5x^3 & x^4 & 5^4\,\mathrm{e}^t & 5^3\,\mathrm{e}^t x \\ 5^3 x & 5^2 x^2 & 5x^3 & x^4 & 5^4\,\mathrm{e}^t \end{pmatrix}$$

$$\check{\nabla}^{(\sigma)}_{\partial_x} = \partial_x - \frac{1}{5^5\,\mathrm{e}^t - x^5}\left(\mu - \frac{1}{2} - \sigma\right)\begin{pmatrix} x^4 & 5^4\,\mathrm{e}^t & 5^3\,\mathrm{e}^t x & 5^2\,\mathrm{e}^t x^2 & 5\,\mathrm{e}^t x^3 \\ 5x^3 & x^4 & 5^4\,\mathrm{e}^t & 5^3\,\mathrm{e}^t x & 5^2\,\mathrm{e}^t x^2 \\ 5^2 x^2 & 5x^3 & x^4 & 5^4\,\mathrm{e}^t & 5^3\,\mathrm{e}^t x \\ 5^3 x & 5^2 x^2 & 5x^3 & x^4 & 5^4\,\mathrm{e}^t \\ 5^4 & 5^3 x & 5^2 x^2 & 5x^3 & x^4 \end{pmatrix}$$

The second structure connection has poles along the divisor $\Sigma = \{5^5\,\mathrm{e}^t - x^5 = 0\}$. We replace all the ∂_x-actions in the second column of (6.1) with the ∂_t-actions using the first column and deduce:

$$e^{-t} x\partial_t T_4 = (5\partial_t + \sigma + \tfrac{5}{2})T_0$$

$$x\partial_t T_0 = (5\partial_t + \sigma + \tfrac{3}{2})T_1$$

$$x\partial_t T_1 = (5\partial_t + \sigma + \tfrac{1}{2})T_2$$

$$x\partial_t T_2 = (5\partial_t + \sigma - \tfrac{1}{2})T_3$$

$$x\partial_t T_3 = (5\partial_t + \sigma - \tfrac{3}{2})T_4$$

From this we find the following differential equation for T_0:

$$((x\partial_t)^5 - e^t(5\partial_t + \sigma + \tfrac{13}{2})(5\partial_t + \sigma + \tfrac{11}{2})(5\partial_t + \sigma + \tfrac{9}{2})(5\partial_t + \sigma + \tfrac{7}{2})(5\partial_t + \sigma + \tfrac{5}{2}))T_0 = 0. \quad (6.2)$$

A direct computation on computer (we used Maple) shows the following lemma.

Lemma 6.1. Let $\check{F}$ denote the trivial $H^*(\mathbb{P}^4)$-bundle over $\mathbb{C}^2 = H^2(\mathbb{P}^4) \times \mathbb{C}_x$. Suppose that $\sigma \notin \{-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\}$. Then the second structure connection $(\mathcal{O}(\check{F})(*\Sigma), \check{\nabla}^{(\sigma)})$ is generated by $T_0 = 1$ as an $\mathcal{O}(*\Sigma)\langle\partial_t\rangle$-module and is defined by the relation (6.2). □

6.2 Euler-twisted and local (small) quantum *D*-modules

Recall from Theorem 5.16 that the second structure connection corresponds to the $(\mathrm{e}, K_{\mathbb{P}^4}^{-1})$-twisted theory for $\sigma = \frac{5}{2}$ and to the local theory for $\sigma = -\frac{5}{2}$. For these cases, the differential Equation (6.2) specializes, respectively, to

$$D_{\mathrm{eu}} := (x\partial_t)^5 - e^t(5\partial_t+9)(5\partial_t+8)(5\partial_t+7)(5\partial_t+6)(5\partial_t+5) \quad (\text{for } \sigma = \tfrac{5}{2}),$$

$$D_{\mathrm{loc}} := (x\partial_t)^5 - e^t(5\partial_t+4)(5\partial_t+3)(5\partial_t+2)(5\partial_t+1)(5\partial_t) \quad (\text{for } \sigma = -\tfrac{5}{2}).$$

The I-functions in Definition 5.11 are given by

$$I_0^{\mathrm{eu}}(t,z) = \sum_{d=0}^{\infty} \mathrm{e}^{(d+H/z)t} \frac{\prod_{k=1}^{5d}(5H+kz)}{\prod_{k=1}^{d}(H+kz)^5},$$

$$I_0^{\mathrm{loc}}(t,z) = \sum_{d=0}^{\infty} \mathrm{e}^{(d+H/z)t} \frac{\prod_{k=0}^{5d-1}(-5H-kz)}{\prod_{k=1}^{d}(H+kz)^5}.$$

The mirror maps (5.11) are given by

$$\mathfrak{m}_{\mathrm{eu}}(t) = t + \frac{g_1(\mathrm{e}^t)}{g_0(\mathrm{e}^t)}, \quad \mathfrak{m}_{\mathrm{loc}}(t) = t + g_2(\mathrm{e}^t)$$

where we set

$$g_0(\mathrm{e}^t) = \sum_{d=0}^{\infty} \mathrm{e}^{dt} \frac{(5d)!}{(d!)^5}, \quad g_1(\mathrm{e}^t) = \sum_{d=1}^{\infty} \mathrm{e}^{dt} \frac{(5d)!}{(d!)^5} 5 \left(\sum_{m=d+1}^{5d} \frac{1}{m} \right), \quad g_2(\mathrm{e}^t) = 5 \sum_{d=1}^{\infty} \mathrm{e}^{dt} (-1)^d \frac{(5d-1)!}{(d!)^5}.$$

We define, as in (5.13),

$$\pi_{\mathrm{eu}}(t,x) = \mathfrak{m}_{\mathrm{eu}}(t - 5\log x) = t - 5\log x + \frac{g_1(\mathrm{e}^t x^{-5})}{g_0(\mathrm{e}^t x^{-5})},$$

$$\pi_{\mathrm{loc}}(t,x) = \mathfrak{m}_{\mathrm{loc}}(t - 5\log x + 5\pi\sqrt{-1}) = t - 5\log x + 5\pi\sqrt{-1} + g_2(-\mathrm{e}^t x^{-5}).$$

These maps converge when $|\mathrm{e}^t x^{-5}| < 5^{-5}$. Theorem 5.16 and Lemma 6.1 together give the following isomorphisms:

$$\pi_{\mathrm{eu}}^* \mathrm{SQDM}_{(\mathrm{e}, K_{\mathbb{P}^4}^{-1})}(\mathbb{P}^4)|_{z=1} \cong (\mathcal{O}(\check{F}), \check{\nabla}^{(\frac{5}{2})}) \cong \mathcal{O}\langle\partial_t\rangle / \mathcal{O}\langle\partial_t\rangle D_{\mathrm{eu}},$$

$$\pi_{\mathrm{loc}}^* \mathrm{SQDM}(K_{\mathbb{P}^4})|_{z=1} \cong (\mathcal{O}(\check{F}), \check{\nabla}^{(-\frac{5}{2})}) \cong \mathcal{O}\langle\partial_t\rangle / \mathcal{O}\langle\partial_t\rangle D_{\mathrm{loc}}$$

over the region $\{(t,x) \in \mathbb{C}^2 : |\mathrm{e}^t x^{-5}| < 5^{-5}\}$.

6.3 The small quantum D-module of a quintic

Recall from Theorem 3.14 and (5.23) that we have a natural morphism:

$$5H : \pi_{\mathrm{eu}}^* \mathrm{SQDM}_{(\mathrm{e}, K_{\mathbb{P}^4}^{-1})}(\mathbb{P}^4) \to \pi_{\mathrm{loc}}^* \mathrm{SQDM}(K_{\mathbb{P}^4})$$

By Theorem 5.19, this corresponds to the map $\varDelta$ between the second structure connections. Since $\varDelta$ maps T_0 in $(\check{F}, \check{\nabla}^{(\frac{5}{2})})$ to $(-\partial_x)^5 T_0 = \mathrm{e}^{-t}\partial_t^5 T_0$ in $(\check{F}, \check{\nabla}^{(-\frac{5}{2})})$, the above morphism corresponds to the map:

$$\delta\colon \mathcal{O}\langle\partial_t\rangle/\mathcal{O}\langle\partial_t\rangle D_{\mathrm{eu}} \to \mathcal{O}\langle\partial_t\rangle/\mathcal{O}\langle\partial_t\rangle D_{\mathrm{loc}}, \quad [f(t,x,\partial_t)] \mapsto [f(t,x,\partial_t)\,\mathrm{e}^{-t}\partial_t^5].$$

This is well-defined since $D_{\mathrm{eu}}\,\mathrm{e}^{-t}\partial_t^5 = \partial_t^5\,\mathrm{e}^{-t} D_{\mathrm{loc}}$. By Corollary 5.20, we have

$$\pi_{\mathrm{eu}}^* \mathrm{SQDM}_{\mathrm{amb}}(Z) \cong \mathrm{Im}(\delta).$$

for a quintic hypersurface $Z \subset \mathbb{P}^4$. We can therefore view $\mathrm{SQDM}_{\mathrm{amb}}(Z)$ either as a quotient of the Euler-twisted quantum D-module or as a sub-D-module of the local quantum D-module. The former viewpoint yields a presentation:

$$\pi_{\mathrm{eu}}^* \mathrm{SQDM}_{\mathrm{amb}}(Z) \cong \mathcal{O}\langle\partial_t\rangle/\mathcal{O}\langle\partial_t\rangle(x(x\partial_t)^4 - 5\,\mathrm{e}^t(5\partial_t+9)(5\partial_t+8)(5\partial_t+7)(5\partial_t+6))$$

and the latter yields a (more familiar) presentation:

$$\pi_{\mathrm{eu}}^* \mathrm{SQDM}_{\mathrm{amb}}(Z) \cong \mathcal{O}\langle\partial_t\rangle/\mathcal{O}\langle\partial_t\rangle(x(x\partial_t)^4 - 5\,\mathrm{e}^t(5\partial_t+4)(5\partial_t+3)(5\partial_t+2)(5\partial_t+1)).$$

6.4 Solutions

For the Euler-twisted theory ($\sigma = \frac{5}{2}$), the cohomology-valued function

$$\varphi(t,x) = (-\partial_x)^4 x^{-1} I_0^{\mathrm{eu}}(t - 5\log x, 1) = \sum_{d=0}^{\infty} \frac{\mathrm{e}^{t(H+d)}}{x^{5H+5d+5}} \frac{\prod_{k=1}^{5d+4}(5H+k)}{\prod_{k=1}^{d}(H+k)^5}$$

is a solution to the differential equation $D_{\mathrm{eu}}\varphi = 0$; for the local theory ($\sigma = -\frac{5}{2}$), the cohomology-valued function

$$\varphi(t,x) = I_0^{\mathrm{loc}}(t - 5\log x + 5\sqrt{-1}\pi, 1) = \mathrm{e}^{5\pi\sqrt{-1}H} \sum_{d=0}^{\infty} \frac{\mathrm{e}^{t(H+d)}}{x^{5H+5d}} \frac{\prod_{k=0}^{5d-1}(5H+k)}{\prod_{k=1}^{d}(H+k)^5}$$

is a solution to the differential equation $D_{\mathrm{loc}}\varphi = 0$. These functions are the images of T_0, respectively, under the maps $\check{K}^{(\frac{5}{2},1)}$ and $\mathrm{e}^{5\pi\sqrt{-1}H}\check{K}^{(-\frac{5}{2},0)}$ in Proposition 5.9. In terms of the quantum D-modules, these solutions correspond, respectively, to $L_{\mathrm{eu}}(\pi_{\mathrm{eu}}(t,x),1)^{-1}$ and $L_{\mathrm{loc}}(\pi_{\mathrm{loc}}(t,x),1)^{-1}$.

Table 1. Gromov–Witten Invariants $N_d = \langle H^3, \tilde{1} \rangle_{0,2,d}^{(e,K_X^{-1})}$

d	N_d
1	−650
2	−160,625
3	−337,216,250/3
4	−217,998,840,625/2
5	−125,251,505,498,880
6	−479,299,410,776,921,825/3
7	−1,531,227,197,616,745,455,000/7
8	−1,260,949,629,604,284,268,280,625/4

6.5 Mirror maps and f

Recall from Lemma 5.18 that the two mirror maps are related as follows:

$$\mathfrak{m}_{\mathrm{loc}}(t + 5\pi\sqrt{-1}) = \bar{f}(\mathfrak{m}_{\mathrm{eu}}(t)) + 5\pi\sqrt{-1}$$

$$\pi_{\mathrm{loc}}(t, x) = \bar{f}(\pi_{\mathrm{eu}}(t, x)) + 5\pi\sqrt{-1}$$

where $\bar{f}$ is the map appearing in Lemma 3.2:

$$\bar{f}(t) = t + \sum_{d=1}^{\infty} \mathrm{e}^{dt} \langle H^3, \tilde{1} \rangle_{0,2,d}^{(e,K_X^{-1})}.$$

Consider the exponentiated mirror maps and $\exp(\bar{f})$:

$$\mathfrak{M}_{\mathrm{eu}}(\mathrm{e}^t) := \exp(\mathfrak{m}_{\mathrm{eu}}(t)), \quad \mathfrak{M}_{\mathrm{loc}}(\mathrm{e}^t) := \exp(\mathfrak{m}_{\mathrm{loc}}(t)), \quad \bar{F}(\mathrm{e}^t) := \exp(\bar{f}(t)).$$

These maps are related by $\mathfrak{M}_{\mathrm{loc}}(-q) = -\bar{F}(\mathfrak{M}_{\mathrm{eu}}(q))$. Surprisingly, they have Taylor expansions in $q = \mathrm{e}^t$ with *integral* coefficients [26, 37]:

$$\mathfrak{M}_{\mathrm{eu}}(q) = q + 770q^2 + 1,014,275q^3 + 1,703,916,750q^4 + 3,286,569,025,625q^5 + \cdots$$

$$\mathfrak{M}_{\mathrm{loc}}(q) = q - 120q^2 + 63,900q^3 - 63,148,000q^4 + 85,136,103,750q^5 + \cdots$$

$$\bar{F}(q) = q - 650q^2 + 50,625q^3 - 5,377,000q^4 - 49,529,975,000q^5 + \cdots.$$

We can also deduce the Gromov–Witten invariants $N_d := \langle H^3, \tilde{1} \rangle_{0,2,d}^{(e,K_X^{-1})}$ as in Table 1.

6.6 Hodge filtration

Recall from Section 5.8 that we have Hodge filtrations on the second structure connections $(\check{F}, \nabla^{(\frac{5}{2})})$ and $(\check{F}, \nabla^{(-\frac{5}{2})})$ denoted, respectively, by $\check{F}^p_{\mathrm{eu}}$ and $\check{F}^p_{\mathrm{loc}}$. They are given by

$$\begin{aligned}
&\check{F}^0_{\mathrm{loc}} = \check{F}, \quad \check{F}^0_{\mathrm{eu}} = \check{F}\\
&\check{F}^1_{\mathrm{loc}} = \langle T_0, \partial_x T_0, \partial_x^2 T_0, \partial_x^3 T_0\rangle, \quad \check{F}^1_{\mathrm{eu}} = (\check{F}^4_{\mathrm{loc}})^{\perp}\\
&\check{F}^2_{\mathrm{loc}} = \langle T_0, \partial_x T_0, \partial_x^2 T_0\rangle, \quad \check{F}^2_{\mathrm{eu}} = (\check{F}^3_{\mathrm{loc}})^{\perp}\\
&\check{F}^3_{\mathrm{loc}} = \langle T_0, \partial_x T_0\rangle, \quad \check{F}^3_{\mathrm{eu}} = (\check{F}^2_{\mathrm{loc}})^{\perp}\\
&\check{F}^4_{\mathrm{loc}} = \langle T_0\rangle \quad \check{F}^4_{\mathrm{eu}} = (\check{F}^1_{\mathrm{loc}})^{\perp}
\end{aligned}$$

where $\perp$ means the orthogonal with respect to the second metric $\check{g}(\gamma_1, \gamma_2) = \int_{\mathbb{P}^4} \gamma_1 \cup (5H \bullet_t - x)^{-1}\gamma_2$ and ∂_x means $\nabla^{(-\frac{5}{2})}_{\partial_x}$ in the first column. Using Maple, we find that

$$\check{F}^4_{\mathrm{eu}} = \langle \tilde{T}_0\rangle, \quad \check{F}^3_{\mathrm{eu}} = \langle \tilde{T}_0, \partial_x\tilde{T}_0\rangle, \quad \check{F}^2_{\mathrm{eu}} = \langle \tilde{T}_0, \partial_x\tilde{T}_0, \partial_x^2\tilde{T}_0\rangle, \quad \check{F}^1_{\mathrm{eu}} = \langle \tilde{T}_0, \partial_x\tilde{T}_0, \partial_x^2\tilde{T}_0, \partial_x^3\tilde{T}_0\rangle$$

where $\partial_x = \nabla^{(\frac{5}{2})}_{\partial_x}$ and

$$\begin{aligned}
\tilde{T}_0 &:= T_0 - \tfrac{125}{3}x^{-1}T_1 + \tfrac{2125}{3}x^{-2}T_2 - 5,625x^{-3}T_3 + 15,000x^{-4}T_4,\\
\partial_x\tilde{T}_0 &= -5x^{-1}T_0 + \tfrac{565}{3}x^{-2}T_1 - \tfrac{8975}{3}x^{-3}T_2 + 22,875x^{-4}T_3 - 60,000x^{-5}T_4,\\
\partial_x^2\tilde{T}_0 &= 30x^{-2}T_0 - 1,030x^{-3}T_1 + 15,500x^{-4}T_2 - 115,500x^{-5}T_3 + 300,000x^{-6}T_4,\\
\partial_x^3\tilde{T}_0 &= -210x^{-3}T_0 + 6,610x^{-4}T_1 - 95,300x^{-5}T_2 + 697,500x^{-6}T_3 - 1,800,000x^{-7}T_4,\\
\partial_x^4\tilde{T}_0 &= 1,680x^{-4}T_0 - 48,680x^{-5}T_1 + 679,000x^{-6}T_2 - 4,905,000x^{-7}T_3 + 12,600,000x^{-8}T_4.
\end{aligned}$$

One can check that $\tilde{T}_0$ corresponds to a multiple of the twisted I-function I_0^{eu} under the solution in Section 6.4: we have $\check{K}^{(\frac{5}{2},1)}(\tilde{T}_0) = 24x^{-5}I_0^{\mathrm{eu}}(t - 5\log x, 1)$.

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