Progress of Theoretical Physics, Vol. 48, No. 5, November 1972

Quantum Space-Time

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(Received March 10, 1972)

Space-time is quantized so as to obtain a four-dimensional simple cubic lattice. The covariance under Poincaré transformation is guaranteed. The particles interact non-locally. The interaction region is spread so as to form a closed (Euclidean) area in the lattice of space-time.

§ 1. Introduction

One of the fundamental problems in elementary particle physics is to overcome the difficulties of the quantum theory of fields. Relativity and quantum mechanics do not seem to mesh together. It is probable that some concepts in relativity or quantum mechanics must be changed. Much work has been done to construct a theory of quantum space-time with a discrete structure for Minkowski-space.[1) All of them, however, have been defective in some respect. In this paper we develop a new idea for the construction of a quantum space-time. The remarkable results are that 1) fields necessarily interact non-locally and 2) the form factor of the corresponding interaction is of Euclidean type. The trouble in conventional non-local theory lies in the pseudo-Euclidean form factor. It is, therefore, worthwhile to re-examine negative results of non-local theory from the point of view of a quantum space-time.

§ 2. Covariance of quantum space-time

The space-time is regarded as a four-dimensional simple cubic lattice. Any event can occur only on lattice points, which are indicated by the set of four integers; $n=(n_0, n_1, n_2, n_3)$. A scalar field in continuous space-time is transformed according to

$$\varphi'(x')=\varphi(x), \tag{1}$$

where $x'=\Lambda x+a$ represents the Poincaré transformation of the coordinate system. Such a form is inapplicable in quantum space-time because the quantity $\Lambda n+a$ is in general not an integer and does not correspond to any lattice point. Here we introduce another type of transformation, under which a quantity $\varphi_n \equiv \varphi_{n_0, n_1, n_2, n_3}$ is called a "scalar" field in this quantum space-time:

$$\varphi'_{n'}=L^n_{n'}(\Lambda, a)\varphi_n, \tag{2}$$

where double index convention with respect to $\sum_n$ has been used. The term "scalar" does not refer to the index n but to the correspondence to the scalar in continuous space-time as will be shown soon later. The coefficient

$$L^n_{n'}(\Lambda, a) \equiv L^{n_0, n_1, n_2, n_3}_{n_0', n_1', n_2', n_3'}(\Lambda, a)$$

must become an infinite dimensional representation of the Poincaré transformation (Λ, a). To construct such a representation, we introduce a one variable real function $f(p)$ which has the following properties:

1) $-\pi < f(p) < \pi$ for $-\infty < p < +\infty$,

2) $f(p) < f(q)$ if $p < q$,

3) $f(-p) = -f(p)$,

4) $f(p) \sim a_0 p$ for $|p| \ll a_0^{-1}$ where a_0 means the lattice constant (fundamental length).

The coefficient $L^n_{n'}$ is given as follows:

$$L^n_{n'}(\Lambda, a) = (2\pi)^{-4} \int e^{-if(\Lambda p)n' + if(p)n + i(\Lambda p)a} \sqrt{df(p)\, df(\Lambda p)}, \tag{3}$$

where the following convention is used:

$$f(p)n \equiv f(p_0)n_0 - f(p_1)n_1 - f(p_2)n_2 - f(p_3)n_3,$$

$$f(\Lambda p)n' \equiv f(p')n', \qquad p' = \Lambda p,$$

$$df(p) = f'(p_0) f'(p_1) f'(p_2) f'(p_3)\, dp_0 dp_1 dp_2 dp_3.$$

It is easily proved that (3) really constructs a representation.

$$\begin{aligned} &L^{n'}_m(\Lambda', a') L^n_{n'}(\Lambda, a) \\ &\quad = (2\pi)^{-8} \int e^{-if(\Lambda' q)m + if(q)n' + i(\Lambda' q)a'} \\ &\qquad \times e^{-if(\Lambda p)n' + if(p)n + i(\Lambda p)a} \sqrt{df(q)\, df(p)\, df(\Lambda' q)\, df(\Lambda p)}. \end{aligned}$$

The sum over n' gives a δ function,

$$\begin{aligned} (2\pi)^{-4} \sum_{n'} e^{i[f(q) - f(\Lambda p)]n'} &= \sum_l \delta\{f(q) - f(\Lambda p) + 2\pi l\} \\ &= \delta\{f(q) - f(\Lambda p)\}. \end{aligned}$$

The last step is valid because

$$|f(q) - f(\Lambda p)| < 2\pi.$$

Therefore

$$\begin{aligned} &L^{n'}_m \cdot L^n_{n'} \\ &\quad = (2\pi)^{-4} \int e^{-if(\Lambda' q)m + if(p)n + i[(\Lambda' q)\cdot a' + (\Lambda p)a]} \\ &\qquad \times \delta\{f(q) - f(\Lambda p)\} \sqrt{df(q)\, df(p)\, df(\Lambda' q)\, df(\Lambda p)} \end{aligned}$$

$$= (2\pi)^{-4} \int e^{-if(\Lambda'\Lambda p)m + if(p)n + i(\Lambda'\Lambda p)(a'+\Lambda' a)} \sqrt{df(p)df(\Lambda'\Lambda p)}$$

$$= L_m{}^n(\Lambda'\Lambda, a' + \Lambda' a) = L_m{}^n\{(\Lambda', a')(\Lambda, a)\}.$$

We remark that (3) is real orthogonal and reducible. We discuss (3) in the limit $a_0 \to 0$ (continuous space-time). Approximation $f(p) \sim a_0 p$ is valid in this case.

$$L^n_{n'}(\Lambda, a) = (2\pi)^{-4} \int e^{-i(\Lambda p)a_0 n' + ipa_0 n + i(\Lambda p)\cdot a} d^4(a_0 p)$$

$$= (a_0)^4 (2\pi)^{-4} \int e^{ip\{x - \Lambda^{-1}(x'-a)\}} d^4 p$$

$$= (a_0)^4 \delta\{x - \Lambda^{-1}(x'-a)\}$$

in which continuous variables x, x' mean $a_0 n, a_0 n'$ respectively. The sum of n is expressed by the integration with respect to x,

$$\varphi'_{n'} = \sum_n L^n_{n'}\varphi_n = \int d^4 n L^n_{n'}\varphi_n = \int d^4 x \delta\{x - \Lambda^{-1}(x'-a)\}\varphi(x). \tag{4}$$

The field $\varphi'_{n'} \equiv \varphi'(x')$ now transforms as

$$\varphi'(x') = \varphi(\Lambda^{-1}(x'-a))$$

which really agrees with (1).

§ 3. Fundamental tensors

Fundamental tensors remain invariant under transformation. Examples are ε_{ijk}, ε^{ijk}, $\delta_i{}^j$ in $SU(3)$ or the metric tensor $g_{\mu\nu}$, $g^{\mu\nu}$ in relativity. They play an important role in each theory. We investigate them in the quantum space-time.

A fundamental tensor which is specified with an invariant function $a(p, p', \cdots)$ satisfying

$$a(\Lambda p, \Lambda p', \cdots) = a(p, p', \cdots)$$

can be obtained in the form

$$a_{ll'\cdots} = \int \delta(p + p' + \cdots) a(p, p', \cdots) e^{if(p)l + if(p')l' + \cdots} \sqrt{df(p)dp\,df(p')dp' \cdots}. \tag{5}$$

The fact that $a_{ll'\cdots}$ represents a fundamental tensor is seen as follows:

$$L^k_l(\Lambda, a) L^{k'}_{l'}(\Lambda, a) \cdots a_{kk'\cdots}$$

$$= (2\pi)^{-4} \cdots \int e^{-if(\Lambda p)l + if(p)k + i(\Lambda p)\cdot a} \sqrt{df(\Lambda p)df(p)}$$

$$\times e^{-if(\Lambda p')l' + if(p')k' + i(\Lambda p')\cdot a} \sqrt{df(\Lambda p')df(p')}$$

$$\times \cdots$$

$$\times\delta(s+s'+\cdots)a(s,s',\cdots)$$
$$\times e^{if(s)k+if(s')k'+\cdots}\sqrt{dsds'\cdots df(s)df(s')\cdots}$$
$$=\int e^{-if(\Lambda p)l-if(\Lambda p')l'-\cdots+i\Lambda(p+p'+\cdots)\cdot a}\sqrt{df(p)df(\Lambda p)df(p')df(\Lambda p')\cdots}$$
$$\times\delta\{f(p)+f(s)\}\delta\{f(p')+f(s')\}\cdots\delta(s+s'+\cdots)a(s,s',\cdots)$$
$$\times\sqrt{dsds'\cdots df(s)df(s')\cdots}$$
$$=\int e^{if(\Lambda s)l+if(\Lambda s')l'+\cdots-i\Lambda(s+s'+\cdots)\cdot a}$$
$$\times\delta(s+s'+\cdots)a(s,s',\cdots)\sqrt{dsds'\cdots df(\Lambda s)df(\Lambda s')\cdots}$$
$$=\int\delta(t+t'+\cdots)a(t,t',\cdots)e^{if(t)l+if(t')l'+\cdots}\sqrt{dtdt'\cdots df(t)df(t')\cdots}$$
$$=a_{ll'\cdots}.$$

Using a special example of the function $f(p)$ given below, we infer that the fundamental tensor can have Euclidean features.

$$f_0(p)=\begin{cases}\pi & a_0p>\pi\,,\\ a_0p & \pi\geq a_0p\geq-\pi\,,\\ -\pi & -\pi>a_0p\,.\end{cases}\tag{6}$$

This function violates the conditions 1) and 2) given for $f(p)$ in § 2, and does not produce any representation of the Poincarè group. Nevertheless one can obtain a "true" function by adding some infinitesimal correction to $f_0(p)$, e.g.,

$$f(p)=f_0(p)-\varepsilon\cdot a_0p\cdot\exp(-a_0^2p^2/2\pi^2),\qquad 0<\varepsilon<1\,.$$

So it is expected that the asymptotic property is also inherited in the case of $f_0(p)$. Equation (6) is substituted into (5),

$$a_{mm'\cdots}=\int_{-(\pi/a_0)}^{\pi/a_0}\cdots\delta(q+q'+\cdots)a(q,q',\cdots)$$
$$\times e^{ipa_0m+iq'a_0m'+\cdots}d(a_0q)d(a_0q')\cdots$$
$$=a_0^4\times\cdots\int_\Omega a(q,q',\cdots)\Big|_{q+q'+\cdots=0}e^{iq'(m'-m)a_0+iq(m''-m)a_0+\cdots}dq'dq\cdots,$$

where Ω denotes some closed area in $q'\times q\times\cdots$. Integration in the closed area leads to an Euclidean property of the fundamental tensor, if the invariant function $a(q,q',\cdots)$ has no singularity.

§ 4. Interactions and commutation relations

The lagrangian density is written as

$$\mathcal{L}_n=\tfrac{1}{2}a_{njk}\varphi_j\varphi_k+\tfrac{1}{3}a_{njkl}\varphi_j\varphi_k\varphi_l+\cdots,\tag{7}$$

where a_{njk} and a_{njkl} are some appropriate fundamental tensors. The invariant lagrangian is given by

$$L=\sum_n \mathcal{L}_n=\tfrac{1}{2}a_{jk}\varphi_j\varphi_k+\tfrac{1}{3}a_{jkl}\varphi_j\varphi_k\varphi_l+\cdots, \tag{8}$$

where $a_{jk}=\sum_n a_{njk}$ and $a_{jkl}=\sum_n a_{njkl}$ are also fundamental tensors. The second term gives interaction. It is remarkable that the form factor a_{jkl} is of Euclidean type. The interacting point in continuous space-time can be expanded to form a closed area in this lattice space-time. This is in sharp contrast to the old theory:

$$L_{\rm int}=\int F(x-y,\ x-z)\varphi(x)\varphi(y)\varphi(z)\,dxdydz$$

in which F is of pseudo-Euclidean type.

The equation of motion is obtained by the variational principle

$$\delta L=0 \quad \text{for} \quad \varphi_j\to\varphi_j+\delta\varphi_j\,,$$

which gives

$$a_{jk}\varphi_k+a_{jkl}\varphi_k\varphi_l+\cdots=0\,. \tag{9}$$

The covariant commutation relation is given by a fundamental tensor Δ_{jk} in the form

$$[\varphi_j,\ \varphi_k]=\Delta_{jk}\,. \tag{10}$$

The explicit form of the commutation relation for the free field is given in the next section.

§ 5. Correspondence to continuous space-time

(A) We begin our discussion with a plane-wave propagating in the one-dimensional lattice. While a differential equation is employed in the case of continuous space, a difference equation must be used in this case. It is written as

$$i\frac{1}{2a_0}(\phi_{n+1}-\phi_{n-1})=k\phi_n\,. \tag{11}$$

The integer n indicates a lattice point. Equation (14) has its solution in the form

$$\phi_n=\lambda^n, \qquad \lambda=i(a_0k)\pm\sqrt{1-(a_0k)^2}. \tag{12}$$

The solution ϕ_n has the property desired: $|\phi_n|=1$ only if $|k|\leqq a_0^{-1}$ (Debye cut). The case $|k|>a_0^{-1}$ is discarded because of the divergence of the solution: lim $|\phi_n|=\infty$. Using the function $f(p)$, which has been introduced in § 2, we rewrite (12) as

$$\lambda=e^{if(p)}, \qquad \phi_n=N(p)e^{if(p)n}. \tag{13}$$

We choose the normalization constant as

$$N(p)=\sqrt{\frac{1}{2\pi}f'(p)}$$

which leads to

$$\int \phi_n^*(p)\phi_m(p)\,dp=\delta_{nm}\,,$$

$$\sum_n \phi_n^*(p)\phi_n(q)=\delta(p-q).$$

A plane-wave in the four-dimensional quantum space-time is written as

$$\phi_{n_0,\,n_1,\,n_2,\,n_3}=(2\pi)^{-2}\sqrt{f'(p_0)f'(p_1)f'(p_2)f'(p_3)}\,e^{-i[f(p_0)n_0-f(p_1)n_1-f(p_2)n_2-f(p_3)n_3]}$$

which is abbreviated as

$$\phi_n=\phi_n(p)=(2\pi)^{-2}\sqrt{f'(p)}\,e^{-if(p)n}.$$

The plane-wave transforms as

$$\phi'_{n'}=L^n_{n'}(\Lambda,a)\phi_n$$

or

$$(2\pi)^{-2}\sqrt{f'(p)}\,e^{-if(p)n}\longrightarrow(2\pi)^{-2}\sqrt{f'(\Lambda p)}\,e^{-if(\Lambda p)n'+i(\Lambda p)a}. \tag{14}$$

Equation (14) has an analogy in continuous space-time,

$$e^{-ipx}\longrightarrow e^{-ip[\Lambda^{-1}(x'-a)]}=e^{-i(\Lambda p)x'+i(\Lambda p)a}. \tag{15}$$

(B) Secondly we introduce the operator of differentiation:

$$\begin{aligned}\partial^\mu&=\lim_{\epsilon_\mu\to0}-\{L_n{}^m(\Lambda=1,\,\epsilon_\mu\neq0)-L_n{}^m(\Lambda=1,\,\epsilon_\mu=0)\}\,\epsilon_\mu{}^{-1}\\&=-\frac{i}{(2\pi)^4}\int p^\mu e^{-if(p)(n-m)}df(p)\\&\equiv D^{\mu m}_{\ \ n}.\end{aligned} \tag{16}$$

The negative sign in the first line is necessary. The symbol ∂^μ represents the abbreviation

$$\partial^\mu\varphi_n\equiv D^{\mu m}_{\ \ n}\varphi_m\,.$$

The operator of difference is defined by

$$\Delta^\mu\varphi_n\equiv\tfrac{1}{2}[\varphi_{n_\mu+1}-\varphi_{n_\mu-1}]. \tag{17}$$

These two operators act on the plane-wave as

$$\partial^\mu\phi_n(p)=-ip^\mu\phi_n(p)$$

and

$$\Delta^\mu\phi_n(p)=-i\sin f(p^\mu)\phi_n(p).$$

The Δ^μ is therefore given by

$$i\Delta^\mu=\sin f(i\partial^\mu). \tag{18}$$

If we take a function G satisfying

$$G\{\sin f(p)\}=p\,,$$

∂^μ is given by

$$i\partial^\mu = G(i\Delta^\mu). \tag{19}$$

(B-1) The "vector" field

$$A_n{}^\mu = \partial^\mu \varphi_n$$

transforms as

$$A'^{\mu'}_{n'} = \Lambda_\mu{}^{\mu'} L^n_{n'}(\Lambda, a) A_n{}^\mu.$$

The "tensor" fields of higher rank can also be constructed.
(B-2) The ∂^μ acts on the product

$$\chi_n = a_{njk}\varphi_j\psi_k$$

to give

$$\partial^\mu \chi_n = a_{njk}\partial^\mu \varphi_j \psi_k + a_{njk}\varphi_j \partial^\mu \psi_k .$$

Therefore

$$\sum_n \partial^\mu \chi_n = \sum_n a_{njk}\partial^\mu \varphi_j \psi_k + \sum_n a_{njk}\varphi_j \partial^\mu \psi_k .$$

This is an analogue of a partial integral. The left-hand side reduces to a "surface" term which we can evaluate if an explicit form for G or f is given.
(C) The Lagrangian formalism (7), (8) and (9) can be written as

$$\mathcal{L}_n = \tfrac{1}{2} a_{njk}(\partial_\mu \varphi_j \partial^\mu \varphi_k - m^2 \varphi_j \varphi_k) + \cdots, \tag{20}$$

which leads to

$$a_{jk}(\partial_\mu \partial^\mu \varphi_k + m^2 \varphi_k) + \cdots = 0 \tag{21}$$

where

$$a_{jk} = \sum_n a_{njk} .$$

If a_{jk} has an inverse,

$$(\partial_\mu \partial^\mu + m^2)\varphi_k + \cdots = 0 . \tag{22}$$

(D) The fundamental tensor representing the commutation relation of a free field is given by

$$[\varphi_j, \varphi_k] = \Delta_{j-k} = \int \varepsilon(p_0)\delta(p^2 - m^2) e^{if(p)(j-k)} \sqrt{dp\,df(p)}$$

$$= i\int \sin\{f(p)\cdot(j-k)\} \sqrt{f'(p)} \frac{d^3p}{p_0}, \qquad p_0 = \sqrt{\mathbf{p}^2 + m^2}.$$

(E) Transformation property of the coordinate (lattice point) variable

$$X_n{}^\mu \equiv n^\mu = i a_0{}^{-1} \int \delta(q) \frac{\partial}{\partial q_\mu} e^{-if(q)n} dq$$

is given by

$$X_m^{\nu}=\Lambda_\mu^{\nu}L_m^{n}(\Lambda, a)X_n^{\mu}+a^{\nu}\cdot a_0^{-1}$$

which corresponds to

$$x'^{\nu}=\Lambda_\mu^{\nu}x^{\mu}+a^{\nu}$$

in continuous space-time. The other quantity $X_{n\mu}$ transforms appropriately.

§ 6. Conserved quantities

(A) The lagrangian density for a non-interacting field is written as

$$\mathcal{L}_n=\tfrac{1}{2}a_{njk}(\partial_\mu\varphi_j\partial^\mu\varphi_k-m^2\varphi_j\varphi_k).$$

Then the energy-momentum tensor

$$J_n^{\mu\nu}=a_{njk}\partial^\mu\varphi_j\partial^\nu\varphi_k-g^{\mu\nu}\mathcal{L}_n$$

satisfies the local conservation law

$$\partial_\mu J_n^{\mu\nu}=0\,.$$

If, however, an interaction term, for example $\frac{1}{3}a_{jkl}\varphi_j\varphi_k\varphi_l$, is added to $\mathcal{L}_n$ the energy momentum tensor

$$J_n^{\mu\nu}=a_n^{jk}\partial^\mu\varphi_j\partial^\nu\varphi_k-g^{\mu\nu}\mathcal{L}_n^{\text{total}}$$

never satisfies the local conservation law. However it guarantees the so-called macroscopic conservation law

$$\sum_n \partial_\mu J_n^{\mu\nu}=0\,.$$

The reason why the local conservation law breaks is obvious. Particles interact non-locally and they are annihilated into or created from vacuum at random so that any conservation law breaks down in the microscopic point of view.
(B) If the complex field obeys the Klein-Gordon equation, there exists a conserved current density.

$$j_n^{\mu}=ia_n^{lm}(\partial^\mu\varphi_l^{+}\varphi_m-\varphi_l^{+}\partial^\mu\varphi_m),$$

$$\partial_\mu j_n^{\mu}=0\,.$$

The macroscopic conservation law holds even if the interaction term exists:

$$\sum_n \partial_\mu j_n^{\mu}=0\,.$$

(C) The angular momentum tensor can be constructed for a free field.

$$M_n^{\mu\nu\sigma}=a_n^{lm}(J_l^{\mu\nu}X_m^{\sigma}-J_l^{\mu\sigma}X_m^{\nu})$$

whcih satisfies

$$\partial_\mu M_n^{\mu\nu\sigma}=0\,.$$

We have not yet succeeded, however, in constructing the macroscopically conserved angular momentum tensor if there exists any interaction term.

Acknowledgements

The author wishes to express his gratitude to Dr. T. Maskawa for his discussion of the general method for finding the infinite dimensional representation of the Poincaré group as is used in this paper.

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