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QUANTUM STOCHASTIC CALCULUS ON FULL FOCK SPACE

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Abstract. We present a new version of integration of time-adapted processes with respect to creation, annihilation and conservation processes on the full Fock space. Among the new features, in the first place, there is a new formulation of adaptedness which is both simpler and more general than the known ones. The new adaptedness allows for processes which are not restricted to be elements of some norm closure of the $*$-algebra which is generated by the basic creation processes.

1. Introduction. In Kümmerer, Speicher [5] and Speicher [8] a theory of integration with respect to the basic noise processes on the full Fock space was introduced. (See also Fagnola [3] where the calculus is treated by generalizing the methods of Accardi, Fagnola, Quaegebeur [1].) In these notes we present a new treatise of a quantum stochastic calculus on the full Fock space. Our motivation to return to this subject is to find a calculus on the full *Fock module* which is the analogue of the full Fock space in the category of Hilbert modules; see [2, 6]. Since we are restricted in space to few pages, we decided to restrict ourselves to the case of Hilbert spaces and publish the more general case separately in [7]. We emphasize, however, that any definition and statement in these notes generalizes literally to Hilbert modules. But also in the case of Hilbert spaces our method has sufficiently interesting new features.

In the case of the calculus on the *boson Fock space* $\Gamma(H)$ over a Hilbert space H adaptedness is defined by means of the functorial property $\Gamma(G \oplus H) = \Gamma(G) \otimes \Gamma(H)$; see [4]. An operator A on $\Gamma(G \oplus H)$ is *adapted* to the "past" G, if it is of the form $(A_G \otimes \mathrm{id})$ for a suitable operator A_G on $\Gamma(G)$. We find an analogue decomposition $\mathcal{F}(G \oplus H) = \mathcal{F}(G) \otimes \mathcal{F}_H$ for the full Fock space where, however, $\mathcal{F}_H$ is not just the full Fock space $\mathcal{F}(H)$ to the "future" H; see Equation (2). Like in the Bose case an operator on $\mathcal{F}(G \oplus H)$ is *adapted*, if

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it is of the form $(A_G \otimes \mathrm{id})$ for some $A_G \in \mathcal{F}(G)$. Like the new definition of adaptedness our whole theory does not rely on processes being contained in a closure of what is generated by basic creation and annihilation processes with respect to a suitable norm.

On the other hand, we restrict ourselves to processes which are continuous in the *$*$-strong topology.* (Recall that the $*$-strong topology on $\mathcal{B}(H)$ is given by the families $T \mapsto \|Tx\|, T \mapsto \|T^*x\|$ $(x \in H)$ of semi-norms. Observe that $\mathcal{B}(H)$ is complete in the $*$-strong topology.) This restriction has the advantage that we are concerned exclusively with vector-valued Riemann integrals of continuous functions. If one wishes to do so, our results may be extended in $*$-strong topology to step functions. This together with the estimate in Lemma 4.3 can be used as a basis for an extension of our theory to more general classes of processes which may approximated by step functions.

Our treatment contains the case of arbitrary many degrees of freedom. In other words, we consider the full Fock space over $L^2(\mathbf{R}, H)$ with H being an arbitrary Hilbert space. This is more general than [5, 8] where $H = \mathbf{C}$. Our results extend easily to the "projective tensor product" of the space of processes and the space of differentials; see [7] for details. Unlike in [3], our theory does not rely on an explicit orthonormal basis of H. On the other hand, we do not consider unbounded coefficients; see [3]. Due to the lack of space we also do not investigate quantum stochastic differential equations. This topic is contained in [7]. We do not consider an initial space, because this case is contained and generalized considerably in the case of Hilbert modules; see [7]. Also for reasons of space we leave the proof of the Ito formula (Equation (8)) a little bit undetailed.

In Section 2 we define the full Fock space and spaces of operators relevant to us. We consider the norm $\|\cdot\|_1$ introduced in [5] as one of the most fundamental ideas which plays a crucial role also for us. The norm $\|\cdot\|_1$ is based on the natural graduation of $\mathcal{F}(H)$. A more detailed analysis of this property can be found in [7]. In Section 3 we define adaptedness and study the algebraic consequences which lie at the heart of Ito's formula. Section 4 introduces the relevant spaces of processes. We define a generalized conservation integral and prove Ito's formula for this integral. In Section 5 we show that our generalized conservation integral splits into four cases which precisely correspond to the creation, the annihilation, the usual conservation and the time integral. Our single Ito formula is shown to contain the full 4×4-Ito table for these four types of integrals.

2. Operators on full Fock space

DEFINITION 2.1. Let H be a Hilbert space. Then the *full Fock space over H* is the Hilbert space

$$\mathcal{F}(H) = \bigoplus_{n=0}^{\infty} H^{\otimes n}$$

where $H^{\otimes 0} = \mathbf{C}\Omega$ and Ω is the *vacuum.* The *vacuum expectation* $\mathbf{E}_0 : \mathcal{B}(\mathcal{F}(H)) \to \mathbf{C}$ is the state defined by $\mathbf{E}_0(A) = \langle \Omega, A\Omega \rangle$.

$\mathcal{F}(H)$ has a natural *graded* vector subspace which is given by the vector space direct sum over the *homogeneous* subspaces $H^{(n)} = H^{\otimes n}$ $(n \in \mathbf{N}_0)$. Also the elements of $H^{(n)}$ are called *homogeneous.* We define the Banach space $\mathcal{F}_1(H)$ to be the Banach space L^1-

direct sum of all the $H^{(n)}$. This means $\mathcal{F}_1(H)$ consists of all families $\bigl(x^{(n)}\bigr)_{n\in\mathbf{Z}}$ ($x^{(n)} \in H^{(n)}$) for which $\|x\|_1 = \sum_{n\in\mathbf{Z}} \|x^{(n)}\| < \infty$. Since $\|x\| \le \|x\|_1$, we have $\mathcal{F}_1(H) \subset \mathcal{F}(H)$.

Definition 2.2. For $n \in \mathbf{Z}$ we denote by $\mathcal{B}^{(n)} \subset \mathcal{B}(\mathcal{F}(H))$ the space consisting of all *operators with offset n in the number of particles*, i.e. $A^{(n)} \in \mathcal{B}^{(n)}$, if $A^{(n)}(H^{\otimes m}) \subset H^{\otimes(m+n)}$ (where we set $H^{\otimes m} = \{0\}$ for $m < 0$). Also $\mathcal{B}(\mathcal{F}(H))$ has a natural graded vector subspace $\mathcal{B}_0$ with $\mathcal{B}^{(n)}$ ($n \in \mathbf{Z}$) being the *homogeneous* subspaces. Any $A \in \mathcal{B}(\mathcal{F}(H))$ allows a (strong) decomposition into $A = \sum_{n\in\mathbf{Z}} A^{(n)}$ with $A^{(n)} \in \mathcal{B}^{(n)}$. We define the Banach space $\mathcal{B}_1$ as the space consisting of all $A \in \mathcal{B}(\mathcal{F}(H))$ for which $\|A\|_1 = \sum_{n\in\mathbf{Z}} \|A^{(n)}\| < \infty$. Again, we have $\|A\| \le \|A\|_1$, so that $\mathcal{B}_1 \subset \mathcal{B}(\mathcal{F}(H))$.

A bilinear mapping $j\colon \mathcal{B}_0 \times \mathcal{B}_0 \to \mathcal{B}_0$ is called *even*, if $j(\mathcal{B}^{(n)}, \mathcal{B}^{(m)}) \subset \mathcal{B}^{(n+m)}$.

Obviously, $\mathcal{B}^{(n)}\mathcal{B}^{(m)} \subset \mathcal{B}^{(n+m)}$ so that the multiplication on $\mathcal{B}_0$ is an even mapping. Notice also that $\mathcal{B}^{(n)^*} \subset \mathcal{B}^{(-n)}$. Clearly, also $\mathcal{B}^{(n)}$ is $*$-strongly complete.

Lemma 2.3 ([5]). *Let $j\colon \mathcal{B}_0 \times \mathcal{B}_0 \to \mathcal{B}_0$ be an even bilinear mapping and $M > 0$ a constant such that*

$$\|j(A,B)\| \le M\,\|A\|\,\|B\| \tag{1}$$

for all homogeneous elements $A, B \in \mathcal{B}_0$. Then j extends to a (*unique*) *bilinear mapping $\mathcal{B}_1 \times \mathcal{B}_1 \to \mathcal{B}_1$, also denoted by j, such that (1) is fulfilled for all $A, B \in \mathcal{B}_1$. An analogue statement is true for even multi-linear mappings.*

Proof. We show that (1) extends to arbitrary $A, B \in \mathcal{B}_0$. (Of course, such a mapping j extends by means of continuity to a unique bilinear mapping on $\mathcal{B}_1 \times \mathcal{B}_1$ also fulfilling (1).) Indeed,

$$\begin{aligned}\|j(A,B)\| &= \Bigl\|\sum_{n\in\mathbf{Z}}\sum_{m\in\mathbf{Z}} j(A^{(m)}, B^{(n-m)})\Bigr\| \le M \sum_{n\in\mathbf{Z}}\sum_{m\in\mathbf{Z}} \|A^{(m)}\|\,\|B^{(n-m)}\| \\ &= M\,\|A\|\,\|B\|\,. \ \blacksquare\end{aligned}$$

Corollary 2.4 ([5]). *$\mathcal{B}_1$ is a Banach $*$-algebra.*

Lemma 2.5. *Let $\bigl(j_\lambda\bigr)_{\lambda\in\Lambda}$ be an increasing net of even bilinear mappings $j_\lambda\colon \mathcal{B}_1\times\mathcal{B}_1 \to \mathcal{B}_1$ all fulfilling (1) with a constant $M > 0$ which is independent of λ. Furthermore, suppose that for homogeneous A and B the net $j_\lambda(A,B)$ converges $*$-strongly* (*of course, to a homogeneous element in $\mathcal{B}_0$*). *Then $j_\lambda(A,B)$ converges $*$-strongly to an element $j(A,B)$ in $\mathcal{B}_1$ for all $A, B \in \mathcal{B}_1$ where j is the extension to $\mathcal{B}_1 \times \mathcal{B}_1$ of the mapping $(A,B) \mapsto \lim_\lambda j_\lambda(A,B)$ on $\mathcal{B}_0 \otimes \mathcal{B}_0$ according to Lemma 2.3. As in Lemma 2.3 the statements remain true for even multi-linear mappings.*

Proof. Let $A, B \in \mathcal{B}_1$, $X \ne 0$ in $\mathcal{F}(H)$ and $\varepsilon > 0$. We may choose $A_0, B_0 \in \mathcal{B}_0$ such that

$$\|j_\lambda(A,B) - j_\lambda(A_0,B_0)\| < \frac{\varepsilon}{3\,\|X\|} \quad \text{and} \quad \|j(A,B) - j(A_0,B_0)\| < \frac{\varepsilon}{3\,\|X\|}$$

for all $\lambda \in \Lambda$. Furthermore, choose $\lambda_0 \in \Lambda$ such that

$$\|j(A_0,B_0)X - j_\lambda(A_0,B_0)X\| < \frac{\varepsilon}{3} \quad \text{and} \quad \|j(A_0,B_0)^*X - j_\lambda(A_0,B_0)^*X\| < \frac{\varepsilon}{3}$$

for all $\lambda \ge \lambda_0$. $\blacksquare$

Remark 2.6. We mention without proof that Lemmata 2.3 and 2.5 remain true for $\mathcal{B}_1$-valued multilinear mappings on the spaces $\mathbf{P}_1$ and $\mathbf{A}_1$ of processes to be introduced in Section 4.

Definition 2.7. Let $x \in H$. The *creation operator* (or *creator*) $\ell^*(x)$ on $\mathcal{F}(H)$ is defined by setting

$$\ell^*(x)x_n \otimes \cdots \otimes x_1 = x \otimes x_n \otimes \cdots \otimes x_1.$$

The *annihilation operator* (or *annihilator*) is the adjoint operator, i.e.

$$\ell(x)x_n \otimes \cdots \otimes x_1 = \langle x, x_n \rangle x_{n-1} \otimes \cdots \otimes x_1$$

for $n \geq 1$ and 0 otherwise.

Let $T \in \mathcal{B}(H)$. The *conservation operator* (or *conservator*) $p(T)$ on $\mathcal{F}(H)$ is defined by setting

$$p(T)x_n \otimes \cdots \otimes x_1 = (Tx_n) \otimes x_{n-1} \otimes \cdots \otimes x_1$$

for $n \geq 1$ and 0 otherwise.

Proposition 2.8. *We have $\ell^*(x) \in \mathcal{B}^{(1)}$, $p(T) \in \mathcal{B}^{(0)}$ and $\ell(x) \in \mathcal{B}^{(-1)}$. The mappings $x \mapsto \ell^*(x)$ and $T \mapsto p(T)$ depend linearly on their arguments. The mapping $x \mapsto \ell(x)$ depends anti-linearly on its argument.*

We have $\|\ell^*(x)\| = \|\ell(x)\| = \|x\|$ and $\|p(T)\| = \|T\|$.

We have

$$p(TT') = p(T)p(T') \quad \text{and} \quad p(T^*) = p(T)^*$$

so that $T \mapsto p(T)$ defines an injective homomorphism of C^*-algebras. Finally, we have the relations

$$p(T)\ell^*(x) = \ell^*(Tx) \quad \ell(x)p(T) = \ell(T^*x) \quad \ell(x)\ell^*(x') = \langle x, x' \rangle.$$

Proof. Obvious. ∎

Definition 2.9. For $X \in H^{\otimes n}$ we define the *generalized creator* $\mathbf{L}^*(X) \in \mathcal{B}^{(n)}$ by

$$\mathbf{L}^*(X)x_m \otimes \ldots \otimes x_1 = X \otimes (x_m \otimes \ldots \otimes x_1) \in H^{\otimes n} \otimes H^{\otimes m} \cong H^{\otimes(n+m)} \subset \mathcal{F}(H).$$

We define the *generalized annihilator* $\mathbf{L}(X) \in \mathcal{B}^{(-n)}$ to be the adjoint of $\mathbf{L}^*(X)$, i.e.

$$\mathbf{L}(X)x_{n+m} \otimes \ldots \otimes x_1 = \langle X, x_{n+m} \otimes \ldots \otimes x_{m+1} \rangle x_m \otimes \ldots \otimes x_1$$

and $\mathbf{L}(X)H^{\otimes m} = \{0\}$, if $m < n$.

Proposition 2.10. *Let $X \in H^{\otimes n}$. We have $\|\mathbf{L}^*(X)\| = \|\mathbf{L}(X)\| = \|X\|$. For $T \in \mathcal{B}(H)$ we have*

$$p(T)\mathbf{L}^*(X) = \mathbf{L}^*(p(T)X)$$

where we consider X also as an element of $\mathcal{F}(H)$. Moreover, for $Y \in H^{\otimes m}$ we have

$$\mathbf{L}(X)\mathbf{L}^*(Y) = \mathbf{L}^*(\mathbf{L}(X)Y) \quad \text{or} \quad \mathbf{L}(X)\mathbf{L}^*(Y) = \mathbf{L}(\mathbf{L}(Y)X)$$

depending on wether $n < m$ or $n > m$. For $n = m$ we have

$$\mathbf{L}(X)\mathbf{L}^*(Y) = \langle X, Y \rangle.$$

Proof. Clear. ∎

Remark 2.11. The definitions of $\mathbf{L}^*(X)$ and $\mathbf{L}(X)$ extend easily to elements $X \in \mathcal{F}_1(H)$, because $\|\mathbf{L}^*(X)\|_1 = \|\mathbf{L}(X)\|_1 = \|X\|_1$. In particular, we find for $A \in \mathcal{B}_1$ that $\|\mathbf{L}^*(A\Omega)\| \leq \|\mathbf{L}^*(A\Omega)\|_1 = \|A\Omega\|_1 \leq \|A\|_1$ so that $\mathbf{L}^*(A\Omega)$ is a well-defined element of $\mathcal{B}_1 \subset \mathcal{B}(\mathcal{F}(H))$. It is not difficult to see that $\mathbf{L}^*(X)$ is not necessarily a bounded operator when X is an arbitrary element of $\mathcal{F}(H)$.

Corollary 2.12. *Let $A_t \in \mathcal{B}_1$ such that $t \mapsto A_t$ is strongly continuous. Then both mappings $t \mapsto \mathbf{L}^*(A_t\Omega)$ and $t \mapsto \mathbf{L}(A_t\Omega)$ are $\|\cdot\|_1$-continuous.*

3. Adaptedness

Proposition 3.1. *Let G, H be Hilbert spaces. Then*

$$\mathcal{F}(G \oplus H) \;\cong\; \mathcal{F}(G) \otimes (\mathbf{C}\Omega \oplus (H \otimes \mathcal{F}(G \oplus H))) \tag{2}$$

in a canonical way.

Proof. Let $n, m \geq 0$, $x_i \in G$ $(i = 1, \cdots, n)$, $y \in H$,$z_j \in G \oplus H$ $(j = 1, \cdots, m)$. It is easily checked that the mapping sending $(x_n \otimes \ldots \otimes x_1) \otimes (y \otimes z_m \otimes \ldots \otimes z_1)$ on the right-hand side to $x_n \otimes \ldots \otimes x_1 \otimes y \otimes z_m \otimes \ldots \otimes z_1$ on the left-hand side (and sending $(x_n \otimes \ldots \otimes x_1) \otimes \Omega$ to $x_n \otimes \ldots \otimes x_1$ and $\Omega \otimes (y \otimes z_m \otimes \ldots \otimes z_1)$ to $y \otimes z_m \otimes \ldots \otimes z_1$) and, of course, sending $\Omega \otimes \Omega$ to Ω) extends as an isometry onto $\mathcal{F}(G \oplus H)$. ∎

Definition 3.2. An operator A in $\mathcal{B}(\mathcal{F}(G \oplus H))$ is called *adapted to* G, if there is a (unique) $A_G \in \mathcal{B}(\mathcal{F}(G))$ such that $A = (A_G \otimes \mathrm{id})$ in the decomposition according to (2).

Remark 3.3. The set of all operators adapted to G is precisely $\mathcal{B}(\mathcal{F}(G)) \otimes \mathrm{id} \cong \mathcal{B}(\mathcal{F}(G))$. The identification is, indeed, the canonical one. In particular, the creator $\ell^*(x) \in \mathcal{B}(\mathcal{F}(G))$ $(x \in G)$ embedded via $(\ell^*(x) \otimes \mathrm{id})$ into $\mathcal{B}(\mathcal{F}(G \oplus H))$ coincides with the creator $\ell^*(x) \in \mathcal{B}(\mathcal{F}(G \oplus H))$ where now x is considered as an element of $G \oplus H$. The $*$-algebra generated by all creators to elements $x \in G$ is strongly dense in $\mathcal{B}(\mathcal{F}(G))$ so that we may identify the $*$-subalgebra of $\mathcal{B}(\mathcal{F}(G \oplus H))$ consisting of all operators adapted to G with the strong closure of the $*$-algebra generated by all creators on $\mathcal{F}(G \oplus H)$ to elements in $G \subset G \oplus H$.

Under the above isomorphism also the Banach $*$-algebra $\mathcal{B}_1 \subset \mathcal{B}(\mathcal{F}(G))$ coincides (isometrically in $\|\cdot\|_1$) with the Banach $*$-algebra of all elements in $\mathcal{B}_1 \subset \mathcal{B}(\mathcal{F}(G \oplus H))$ which are adapted to G.

Corollary 3.4. *Let $x \in G$, $T \in \mathcal{B}(G)$ and $X \in \mathcal{F}_1(G)$. Then $\ell^*(x)$, $\ell(x)$, $p(T)$, $\mathbf{L}^*(X)$ and $\mathbf{L}(X)$ are adapted to G. Also the identity is adapted. Moreover, whenever $\mathbf{L}^*(X) \in \mathcal{B}_1$ is adapted to G then $X \in \mathcal{F}_1(G)$.*

Lemma 3.5. *Let $A \in \mathcal{B}_1$ be adapted to G and T in $\mathcal{B}(H)$. Then*

$$Ap(T) = \mathbf{L}^*(A\Omega)p(T) \tag{3}$$

and

$$p(T)A = p(T)\mathbf{L}(A^*\Omega). \tag{4}$$

Proof. It is sufficient to prove (3), because (4) follows by considering the adjoint of (3). Now (3) follows from the observation that the range of $p(T)$ is contained in $\big(H \otimes \mathcal{F}(G \oplus H)\big)$ and from $A\Omega = A_G\Omega$ where we identify $A_G\Omega \in \mathcal{F}(G)$ with the corresponding element in $\mathcal{F}(G \oplus H)$. ■

COROLLARY 3.6. *Let $A, B \in \mathcal{B}_1$ both be adapted to G and let T, T' be in $\mathcal{B}(H)$. Then*

$$p(T)ABp(T') = p(T\mathbf{E}_0(AB)T').$$

Proof. By Remark 2.11 we may assume that $A \in \mathcal{B}^{(n)}$ and $B \in \mathcal{B}^{(m)}$. First, suppose $n \neq m$. Then $\mathbf{E}_0(AB) = 0$. Without loss of generality assume $n < m$. From Proposition 2.10 and Lemma 3.5 we find

$$p(T)ABp(T') = \mathbf{L}^*(p(T)\mathbf{L}(A^*\Omega)B\Omega)p(T') = 0,$$

because $\mathbf{L}(A^*\Omega)B\Omega$ is an element of $G^{\otimes(m-n)}$ and T vanishes on G. If $n = m$, we find $p(T)ABp(T') = p(T)\mathbf{L}(A^*\Omega)\mathbf{L}^*(B\Omega)p(T') = p(T)\mathbf{E}_0(AB)p(T')$. Therefore, in both cases we obtain our claimed result. ■

COROLLARY 3.7. *Suppose $A \in \mathcal{B}^{(0)}$ is adapted to G and $T \in \mathcal{B}(H)$. Then*

$$Ap(T) = \mathbf{E}_0(A)p(T).$$

4. Processes

DEFINITION 4.1. Let H be a Hilbert space. We identify $L^2(\mathbf{R}^+) \otimes H$ with $L^2(\mathbf{R}^+, H)$. For $0 \leq t \leq \infty$ we denote $H_t = L^2([0,t), H) \subset L^2(\mathbf{R}^+, H) = H_\infty$. We work on the full Fock space $\mathcal{F} = \mathcal{F}(L^2(\mathbf{R}^+, H))$.

The *C^*-algebra of processes* $\mathbf{P}$ consists of all families $F = \big(F_t\big)_{t \geq 0}$ of elements $F_t \in \mathcal{B}(\mathcal{F})$ which are $*$-strongly continuous as mappings $t \mapsto F_t$ and for which

$$\|F\| = \sup_{t \in \mathbf{R}^+} \|F_t\| < \infty.$$

The *C^*-algebra of adapted processes* $\mathbf{A}$ consists of all $F \in \mathbf{P}$ such that F_t is adapted to H_t.

By $\mathbf{P}^{(n)}$ and $\mathbf{A}^{(n)}$ we denote the sets of all processes F in $\mathbf{P}$ and $\mathbf{A}$, respectively, for which $F_t \in \mathcal{B}^{(n)}$ for all $t \geq 0$. By $\mathbf{P}_1$ we denote the Banach $*$-algebra consistining all $F \in \mathbf{P}$ for which $\|F\|_1 = \sum_{n \in \mathbf{Z}} \|F^{(n)}\| < \infty$. We set $\mathbf{A}_1 = \mathbf{A} \cap \mathbf{P}_1$.

Let $\mathcal{F}^{01} = \mathbf{C}\Omega \oplus H_\infty$. We regard $\mathcal{F}^{01}$ as a Banach subspace of $\mathcal{F}_1 := \mathcal{F}_1(H_\infty)$. Our particular interest lies on processes in $\mathbf{L}^*(\mathcal{F}^{01})\mathbf{A}_1$ and $\mathbf{A}_1\mathbf{L}(\mathcal{F}^{01})$. Our interest to these spaces of processes will become clear in Section 5 where the integrals of the present Section are reduced to usual stochastic integrals. Notice that these processes are elements of $\mathbf{P}_1$ but, in general, not of $\mathbf{A}_1$.

DEFINITION 4.2. Let $\mathcal{T} > 0$. By $\mathcal{P}_\mathcal{T} = \big\{P = (t_0, \ldots, t_N)\colon 0 = t_0 < \ldots < t_N = \mathcal{T}\ (N \in \mathbf{N})\big\}$ we denote the *increasing net of partitions of the interval* $[0, \mathcal{T}]$ ordered by refinement. The norm of a partition P is $\|P\| = \max_{1 \leq k \leq N}(t_k - t_{k-1})$.

By $L^\infty(\mathbf{R}^+, \mathcal{B}(H))$ we understand the set of all functions on $\mathbf{R}^+$ with values in $\mathcal{B}(H)$ which are norm limits (in the supremum norm) of finitely-valued, measurable functions. We consider elements of $L^\infty(\mathbf{R}^+, \mathcal{B}(H))$ as operators on H_∞ which act by pointwise multiplication. For $T \in L^\infty(\mathbf{R}^+, \mathcal{B}(H))$ we set $dp_{t_k}(T) = p(\chi_{(t_{k-1}, t_k]}T)$.

LEMMA 4.3. *Let* $F = \bar{F}\mathbf{L}(X) \in \mathbf{A}_1\mathbf{L}(\mathcal{F}^{01})$ *and* $G = \mathbf{L}^*(Y)\bar{G} \in \mathbf{L}^*(\mathcal{F}^{01})\mathbf{A}_1$ *with* $\bar{F}, \bar{G} \in \mathbf{A}_1$ *and* $X, Y \in \mathcal{F}^{01}$. *Furthermore, let* $T \in L^\infty(\mathbf{R}^+, \mathcal{B}(H))$. *Then*

$$\Big\| \sum_{k=1}^n F_{t_{k-1}}\, dp_{t_k}(T)\, G_{t_{k-1}} \Big\|_1 \le \|T\|\, \|\bar{F}\|_1 \|\bar{G}\|_1\, \|X\|_1\, \|Y\|_1$$

for all $P \in \mathcal{P}_{\mathcal{T}}$ *and all* $\mathcal{T} > 0$.

Proof. The mapping $F \times G \mapsto \sum_{k=1}^n F_{t_{k-1}}\, dp_{t_k}(T)\, G_{t_{k-1}}$ gives rise to an even mapping $\mathbf{A}_1 \times \mathbf{L}(\mathcal{F}^{01}) \times \mathbf{L}^*(\mathcal{F}^{01}) \times \mathbf{A}_1 \to \mathcal{B}_1$. By Lemma 2.3 it is sufficient to show the inequality for processes $F = \bar{F}\mathbf{L}(X)$ and $G = \mathbf{L}^*(Y)\bar{G}$ where $\bar{F}, \bar{G}, X, Y$ are homogeneous elements.

We have to distinguish the four cases when X and Y are equal to Ω or elements of H_∞, respectively. First, suppose that $X = \Omega$. Observe that $F_t^* F_{t'}$ is an element of $\mathcal{B}^{(0)}$ which is adapted to $H_{\max(t,t')}$ and that $dp_{t_k}(T^*)\, dp_{t_\ell}(T) = \delta_{k\ell}\, dp_{t_k}(T^*T)$. Combining this with Corollary 3.7 we find

$$\begin{aligned}
\Big| \sum_{k=1}^n F_{t_{k-1}}\, dp_{t_k}(T)\, G_{t_{k-1}} \Big|^2 &= \sum_{k,\ell=1}^n G^*_{t_{k-1}}\, dp_{t_k}(T^*)\, F^*_{t_{k-1}} F_{t_{\ell-1}}\, dp_{t_\ell}(T)\, G_{t_{\ell-1}} \\
&= \sum_{k=1}^n G^*_{t_{k-1}}\, dp_{t_k}(T^* \mathbf{E}_0(F^*_{t_{k-1}} F_{t_{k-1}}) T)\, G_{t_{k-1}} \\
&\le \|T\|^2\, \|F\|^2 \sum_{k=1}^n G^*_{t_{k-1}}\, dp_{t_k}(\mathrm{id})\, G_{t_{k-1}} = \|T\|^2\, \|F\|^2 \Big| \sum_{k=1}^n dp_{t_k}(\mathrm{id})\, G_{t_{k-1}} \Big|^2. \qquad (5)
\end{aligned}$$

If also $y = \Omega$, we obtain by (5)

$$\Big\| \sum_{k=1}^n dp_{t_k}(\mathrm{id})\, G_{t_{k-1}} \Big\|^2 = \Big\| \sum_{k=1}^n G^*_{t_{k-1}}\, dp_{t_k}(\mathrm{id}) \Big\|^2 \le \|G\|^2 .$$

If $y \in H_\infty$, we find

$$\begin{aligned}
\Big\| \sum_{k=1}^n dp_{t_k}(\mathrm{id})\, G_{t_{k-1}} \Big\|^2 &= \Big\| \sum_{k=1}^n dp_{t_k}(\mathrm{id})\, \ell^*(y) \bar{G}_{t_{k-1}} \Big\|^2 \\
&\le \big\|\bar{G}^2\big\| \sum_{k=1}^n \|\ell(y)\, dp_{t_k}(\mathrm{id})\, \ell^*(y)\| = \|\bar{G}\|^2\, \|y\|^2 .
\end{aligned}$$

The converse case when $y = \Omega$ and $x \in H_\infty$ is treated by considering adjoints. The last yet missing case when $x, y \in H_\infty$ follows from the observation that

$$\sum_{k=1}^n \|\ell(x)\, dp_{t_k}(\mathrm{id})\, \ell^*(y)\| = \sum_{k=1}^n \big|\langle x, \chi_{[t_{k-1},t_k]} y\rangle\big| = \sum_{k=1}^n \big|\langle \chi_{[t_{k-1},t_k]} x, \chi_{[t_{k-1},t_k]} y\rangle\big| \le \|x\|\, \|y\|$$

by the Cauchy-Schwarz inequality for elements of $\mathbf{C}^n$. ∎

DEFINITION 4.4. We say an element $X \in \mathcal{F}^{01}$ is *simple*, if it is a finite linear combination of Ω and functions $x = \chi_I \xi \in H_\infty$ where $\xi \in H$ and $I \subset \mathbf{R}^+$ is a finite interval. We say an element $T \in L^\infty(\mathbf{R}^+, \mathcal{B}(H))$ is *simple*, if it is a finite linear combination of functions of the form $\chi_I \tau$ where $\tau \in \mathcal{B}(H)$ and I is an arbitrary measurable subset of $\mathbf{R}^+$.

The simple elements of $\mathcal{F}^{01}$ and $L^\infty(\mathbf{R}^+, \mathcal{B}(H))$ are dense in $\mathcal{F}^{01}$ and $L^\infty(\mathbf{R}^+, \mathcal{B}(H))$, repectively.

LEMMA 4.5. *Let F, G be as in Lemma 4.3 where, however, X, Y and $T = \chi_I \tau$ are simple. Furthermore, let $\mathcal{T}_0 > 0$. Then the net*

$$\Big(\sum_{k=1}^n F_{t_{k-1}}\, dp_{t_k}(T)\, G_{t_{k-1}}\Big)_{P\in\mathcal{P}_\mathcal{T}}$$

is a $*$-strong Cauchy net uniformly in $\mathcal{T} \in (0, \mathcal{T}_0]$. In other words, (denoting the sum to a certain partition P by Σ_P) for arbitrary $Z \in \mathcal{F}$ and $\varepsilon > 0$ we can find $\delta > 0$ such that

$$\|(\Sigma_P - \Sigma_{P'})Z\| < \varepsilon \tag{6}$$

for all $\mathcal{T} \in (0, \mathcal{T}_0]$ and $P, P' \in \mathcal{P}_\mathcal{T}$ with $\|P\| < \delta$ and $\|P'\| < \delta$ and an analogue statement for the adjoints of the sums.

Proof. Let us first reduce the problem. Clearly, by symmetry under adjoint it is sufficient to show only (6). By Lemma 4.3 the assumptions of Lemma 2.5 are fulfilled so that we may reduce to homogeneous elements as in the proof of Lemma 4.3. Finally, by Lemma 4.3 the net is bounded so that it is sufficient to check strong convergence on elementary tensors $Z = z_n \otimes \cdots \otimes z_1$ with z_i being simple elements of H_∞, because these Z span a dense subset of $\mathcal{F}$. The proof splits into the quite different cases when $X = \Omega$ and $X \in H_\infty$.

First, let $X = \Omega$. By (5) we find

$$\begin{aligned}\Big\|\sum_{k=1}^n F_{t_{k-1}}\, dp_{t_k}(T)\, G_{t_{k-1}} Z\Big\| &= \Big\|\sum_{k=1}^n \mathbf{L}^*(F_{t_{k-1}}\Omega)\, dp_{t_k}(T)\, G_{t_{k-1}} Z\Big\| \\ &\le \|T\|\,\|\mathbf{L}^*(F\Omega)\|\,\|GZ\|\,. \end{aligned}\tag{7}$$

Clearly, the same estimate also holds for step functions G^s_t and F^s_t. So by the usual technique of joint refinement of pairs of partitions used in Riemann integral our statements are clear, because $\mathbf{L}^*(F_t\Omega)$ (by Corollary 2.12) and $G_t Z$ (because G_t is strongly continuous) are norm continuous functions, which may be approximated in supremum norm by step functions (uniformly on $[0, \mathcal{T}_0]$ and stably under refinement of partitions). Notice that this argument does not depend on the special nature of Z.

Now let $X = \chi_{I'}\xi$. We are finished, if we show that $F_{t_{k-1}}\, dp_{t_k}(T)\, G_{t_{k-1}} Z$ is dominated by a positive multiple of $\lambda([t_{k-1}, t_k])$ (where λ denotes the Lebesgue measure on $\mathbf{R}^+$). In the case when $Y = \chi_{I''}\zeta$ we find $\mathbf{L}(X)\, dp_{t_k}(T)\, \mathbf{L}^*(Y) = \langle \xi, \tau\zeta\rangle\, \lambda([t_{k-1}, t_k] \cap I \cap I' \cap I'')$. Notice that also this argument does not depend on the form of Z. Now suppose that $Y = \Omega$. We may assume that $G \in \mathbf{A}^{(-n)}$ where $n \ge 0$ (otherwise the sum is 0 by Lemma 3.5). Choose $Z = z_n \otimes \cdots \otimes z_1 \otimes z \otimes z'_m \otimes \cdots \otimes z'_1$ where $z = \chi_{I''}\eta$ and $\eta \in H$. We find

$$\begin{aligned}&\mathbf{L}(X)\, dp_{t_k}(T)\, G_{t_{k-1}} Z = \\ &\qquad \langle \xi, \tau \langle \Omega, G_{t_{k-1}} z_n \otimes \cdots \otimes z_1\rangle \eta\rangle z'_m \otimes \cdots \otimes z'_1\, \lambda([t_{k-1}, t_k] \cap I \cap I' \cap I'').\end{aligned}$$

We remark that (7), obviously, also holds here at least for our particular choice of Z. ■

THEOREM 4.6. *The integral*

$$\int_0^{\mathcal{T}} F_t\,dp_t(T)\,G_t \quad := \quad \lim_{P\in\mathcal{P}_{\mathcal{T}}} \sum_{k=1}^n F_{t_{k-1}}\,dp_{t_k}(T)\,G_{t_{k-1}}$$

extends in $\|\cdot\|_1$*-norm to arbitrary elements* $F \in \mathbf{L}^*(\mathcal{F}^{01})\mathbf{A}_1$, $G \in \mathbf{A}_1\mathbf{L}(\mathcal{F}^{01})$ *and* $T \in L^\infty(\mathbf{R}^+,\mathcal{B}(H))$ *by means of norm continuity. The process* $M = \big(M_t\big)_{t\geq 0}$ *defined by* $M_{\mathcal{T}} = \int_0^{\mathcal{T}} F_t\,dp_t(T)\,G_t$ *is an element of* $\mathbf{A}_1$.

Proof. Assume F and G are given like in Lemma 4.3. The claimed extension follows from Lemma 4.3. To see strong continuity (which by symmetry under building adjoints implies $*$-strong continuity), we may assume that X and Y are homogeneous. If $X \in H_\infty$ or $Y \in H_\infty$, then we see by Lemma 4.3 that M_t even is norm continuous. (In $\|M_t - M_{t'}\|$ we obtain a factor $\sqrt{|t-t'|}$.) If $X = Y = \Omega$, then we find strong continuity in the same manner, if we replace in (7) GZ with $\langle\Omega, Gz_n\otimes\cdots\otimes z_1\rangle(\chi_{(0,\mathcal{T}]}z)\otimes z'_m\otimes\cdots\otimes z'_1$ like in the last step of the preceding proof. Adaptedness follows from the observation that $dp_{t_k}(T)\ell^*(x) = \ell^*(\chi_{(t_{k-1},t_k]}Tx)$ is adapted to H_{t_k}. ∎

THEOREM 4.7. *Let* M, M' *be processes in* $\mathbf{A}_1$ *given by integrals*

$$M_{\mathcal{T}} = \int_0^{\mathcal{T}} F_t\,dp_t(T)\,G_t \quad \text{and} \quad M'_{\mathcal{T}} = \int_0^{\mathcal{T}} F'_t\,dp_t(T')\,G'_t$$

where $F,F' \in \mathbf{L}^*(\mathcal{F}^{01})\mathbf{A}_1$, $G,G' \in \mathbf{A}_1\mathbf{L}(\mathcal{F}^{01})$ *and* $T,T' \in L^\infty(\mathbf{R}^+,\mathcal{B}(H))$. *Then the product* $MM' = \big(M_tM'_t\big)_{t\geq 0} \in \mathbf{A}_1$ *is given by*

$$M_{\mathcal{T}}M'_{\mathcal{T}} = \int_0^{\mathcal{T}} F_t\,dp_t(T)\,G_tM'_t + \int_0^{\mathcal{T}} M_tF'_t\,dp_t(T')\,G'_t + \int_0^{\mathcal{T}} F_t\,dp_t(T\mathbf{E}_0(G_tF'_t)T')\,G'_t. \tag{8}$$

Proof. Again we may assume that F, G (and also F', G') are given like in Lemma 4.3 and, furthermore, that T, T' and the elements X, X', Y, Y' appearing in F, F', G, G' are simple. We investigate the product of the sums Σ_P and Σ'_P which approximate $M_{\mathcal{T}}$ and $M'_{\mathcal{T}}$, respectively, to the same partition. Since the approximation of $M_{\mathcal{T}}$ and $M'_{\mathcal{T}}$ by Σ_P and Σ'_P, respectively, is $*$-strong, the weak limit of $\Sigma_P\Sigma'_P$ is $M_{\mathcal{T}}M'_{\mathcal{T}}$. We are finished, if we show that $\Sigma_P\Sigma'_P$ converges weakly to right-hand side of (8).

We split the double sum over k and ℓ into the parts where $k > \ell$, $k < \ell$ and $k = \ell$

$$\Sigma_P\Sigma'_P = \Big[\sum_{1\leq\ell<k\leq n} + \sum_{1\leq k<\ell\leq n}\Big] F_{t_{k-1}}\,dp_{t_k}(T)\,G_{t_{k-1}}F'_{t_{\ell-1}}\,dp_{t_\ell}(T')\,G'_{t_{\ell-1}} + \sum_{k=1}^n F_{t_{k-1}}\,dp_{t_k}(T)\,G_{t_{k-1}}F'_{t_{k-1}}\,dp_{t_k}(T')\,G'_{t_{k-1}}. \tag{9}$$

Let us apply the first summand to Z

$$\sum_{1\leq\ell<k\leq n} F_{t_{k-1}}\,dp_{t_k}(T)\,G_{t_{k-1}}F'_{t_{\ell-1}}\,dp_{t_\ell}(T')\,G'_{t_{\ell-1}}Z. \tag{10}$$

Choosing a partition P of sufficiently small norm the part $\sum_{\ell=1}^{k-1} F'_{t_{\ell-1}}\,dp_{t_\ell}(T')\,G'_{t_{\ell-1}}Z$ is close to $M'_{t_{k-1}}Z$ (by Lemma 4.5) for all partitions finer than P. Of course, we would like to insert this into (10) and then perform the limit $\|P\| \to 0$, because in this way we would obtain the first summand (8) (applied to Z). Let us check that this procedure, indeed, is allowed. In other words, let us check, if the part $\sum_{\ell=1}^{k-1} F'_{t_{\ell-1}}\, dp_{t_\ell}(T')\, G'_{t_{\ell-1}} Z$ enters (10) in norm. This assertion, however, follows precisely as in the proof of Lemma 4.5 with $G_{t_{k-1}} Z$ replaced by $G_{t_{k-1}} \sum_{\ell=1}^{k-1} F'_{t_{\ell-1}}\, dp_{t_\ell}(T')\, G'_{t_{\ell-1}} Z$. In this way we even show that the convergence of the first summand is strong. An analogous procedure in the case $k < \ell$ yields that the second summand converges to the second summand of (8) at least weakly.

For the last summand of (8) observe that $\mathbf{E}_0(G_t F'_t) = 0$, whenever $Y \in H_\infty$ or $X' \in H_\infty$. Let us check if this is also true for the limit of the last summand of (9). For instance, assume that $Y = \chi_I \zeta \in H_\infty$. If $X = \Omega$, we find by computations like (5) that the last summand converges to 0 even in norm. If $X \in H_\infty$, the part $F_{t_{k-1}}\, dp_{t_k}(T)\, G_{t_{k-1}}$ is bounded by a multiple of $t_k - t_{k-1}$ and the part $F'_{t_{k-1}}\, dp_{t_k}(T')\, G'_{t_{k-1}} Z$ is bounded by a multiple of $\sqrt{t_k - t_{k-1}}$ (see proof of Lemma 4.5), so that in this case the sum converges to 0 strongly. By considering the adjoint we see that the summand converges to 0 at least weakly in the case when $X' \in H_\infty$. In the remaining case $Y = X' = \Omega$ we have $GF' \in \mathbf{A}^{(n)}$. Then our claim follows immediately from Corollary 3.6 and Theorem 4.5. ∎

Corollary 4.8. *In the usual differential notation where* $M_{\mathcal{T}} = \int_0^{\mathcal{T}} dM_t$ *with* $dM = F\, dp(T)\, G$ *and* $d(MM') = dM\, M' + M\, dM' + dM\, dM'$ *we find the Ito formula*

$$dM\, dM' = F\, dp(T\mathbf{E}_0(GF')T')\, G'.$$

5. The usual quantum stochastic integrals and Ito table. Our integrals apparently are concerned only with (slightly generalized) conservation integrals. It remains to extract the usual quantum stochastic integrals with respect to creators, annihilators and also dt from our notation. They correspond (together with the usual conservation integral) precisely to the four cases when X and Y are equal to Ω or elements of H_∞, respectively.

Definition 5.1. Let $\mathcal{T} > 0$ and $F, G \in \mathbf{A}_1$. Let $X = x$ and $Y = y$ be in H_∞. Define the bounded mesaure $\mu^{x,y}$ on $\mathbf{R}^+$ by $\mu^{x,y}(I) = \langle x, \chi_I y\rangle$. We define the $\mu^{x,y}$*-integral* as

$$\int_0^{\mathcal{T}} F\, d\mu^{x,y}\, G = \int_0^{\mathcal{T}} F_t \mathbf{L}(X)\, dp_t(\mathrm{id})\, \mathbf{L}^*(Y) G_t.$$

Let $X = x \in H_\infty$. We define the *creator integral* and the *annihilator integral* as

$$\int_0^{\mathcal{T}} F\, d\ell^*(x)\, G = \int_0^{\mathcal{T}} F_t\, dp_t(\mathrm{id})\, \mathbf{L}^*(X) G_t \quad\text{and}\quad \int_0^{\mathcal{T}} F\, d\ell(x)\, G = \int_0^{\mathcal{T}} F_t \mathbf{L}(X)\, dp_t(\mathrm{id})\, G_t,$$

respectively. Let $T \in L^\infty(\mathbf{R}^+, \mathcal{B}(H))$ we define the *conservator integral* as

$$\int_0^{\mathcal{T}} F\, dp(T)\, G = \int_0^{\mathcal{T}} F_t\, dp_t(T)\, G_t.$$

Theorem 5.2. *Let* $M_t = \int_0^t F\, dl\, G$ *be one of the integrals in Definition 5.1. Then* $M = \big(M_t\big)_{t \ge 0}$ *is an element of* $\mathbf{A}_1$. *Moreover, given another* $M'_t = \int_0^t F'\, dl'\, G'$. *Then* (*using the formal notation*) $dM\, dM' = F\, dl''\, G'$ *where* dl'' *has to be chosen according to* *the Ito table*

$dl \backslash dl'$	$d\mu^{x',y'}$	$d\ell^*(x')$	$d\ell(x')$	$dp(T')$
$d\mu^{x,y}$	0	0	0	0
$d\ell^*(x)$	0	0	0	0
$d\ell(x)$	0	$d\mu^{x,\mathbf{E}_0(GF')x'}$	0	$d\ell(T'^*\overline{\mathbf{E}_0(GF')}x)$
$dp(T)$	0	$d\ell^*(T\mathbf{E}_0(GF')x')$	0	$dp(T\mathbf{E}_0(GF')T')$.

Remark 5.3. In [8] the $\mathbf{C}$-valued function $\mathbf{E}_0(GF')$ is always written outside the integrators. This can be done without any effect. However, in view of [7] we have in mind a generalization to Hilbert modules where $\mathbf{E}_0$ is not an expectation with values in $\mathbf{C}$, but, a *conditional expectation* with values in an algebra. Only if we write our formulae precisely as we did, our definitions, results and proofs generalize to Hilbert modules.

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