

## Quantum stochastic processes as models for state vector reduction

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**Abstract.** An elementary introduction of quantum-state-valued Markovian stochastic processes (QSP) for  $N$ -state quantum systems is given. It is pointed out that a so-called master constraint must be fulfilled. For a given master equation a continuous and, as a new alternative possibility, a discontinuous QSP are derived. Both are discussed as possible models for state reduction during measurement.

### 1. Introduction

The theory of quantum measurements as stated in its classical form by von Neumann [1] represents a strange appendix to the causal equations of quantum mechanics. According to von Neumann's theory (cf in particular Lüders' interpretation [2]), the state  $\Psi$  of a given quantum system will undergo a stochastic change during the measurement. Given the state  $\Psi(t)$  at time  $t=0$  and then switching the measuring apparatus on, at the end of the measurement process, say at  $t=T$  ( $T>0$ ) that state  $\Psi(T)$  becomes stochastically reduced:

$$\Psi(0) \rightarrow \Psi(T) = \Phi_k \quad (1.1)$$

with probability  $x_k$  ( $k=1, 2, \dots$ ). Here the  $\Phi_k$  are the orthogonal 'pointer states' (i.e. the eigenstates of the operator which was measured) and  $x_k$  is connected with the initial quantum state by

$$x_k = |\langle \Phi_k, \Psi(0) \rangle|^2. \quad (1.2)$$

Seemingly von Neumann's prescription (1.1) and (1.2) cannot be incorporated into quantum theory nor into classical physics. It is rather an ingenious phenomenology bridging the microscopic and macroscopic theories.

The von Neumann machinery (1.1) and (1.2) is generally taken as a black box answering the purpose of the operative interpretation of quantum mechanics. Nevertheless, it seems reasonable to enquire what is actually happening during a quantum measurement. The reason for developing continuous reduction models is clear enough: if the wavefunction is supposed to be in one-to-one correspondence with the state of the system (this is not necessary but would be demanded, for example, on ontological grounds [3]) then the reduction process (1.1) and (1.2) must be taken as a realistic physical phenomenon.

A maximalist's goal would then be the detailed dynamical analysis of the interaction between the system and the apparatus. This task is, however, very complicated and it requires the inclusion of infinitely many degrees of freedom [4]. Therefore the phenomenology of the inside of the von Neumann black box deserves investigation even though it might not lead to an explanation of the elementary causes of the quantum state reduction.

Really, for the last two decades, several such reduction models (called continuous or, sometimes, dynamical) have been proposed. In the Bohm-Bub reduction theory [5] the state vector obeys a causal non-linear differential equation with constant 'secret parameters' which are, however, stochastically distributed. Recently, Pearle [6-11] and also Gisin [12, 13] have proposed Fokker-Planck diffusion equations (i.e., equivalently, Wiener processes) for the state vector  $\Psi(t)$  during a quantum measurement. There are other works, too, concerning phenomenology of quantum measurements distributed in time [14, 15] or even continuous in time [16, 17].

In the present paper we reconsider the common ingredient of the above-mentioned works: a formalism in which the evolution of the quantum state  $\Psi$  is controlled by stochastic rules in addition to the ordinary causal quantum dynamics. Such a quantum-stochastic process (QSP) was first introduced long ago [18]. A general theory of QSP has recently been constructed [19] in terms of quantum-stochastic differential equations (QSDE). The underlying idea is that the state  $\Psi$  of the system obtains its stochasticity by interaction with certain external fields (an idealised Markovian 'heat bath') represented by so-called quantum Wiener processes. In our paper, however, we do not discuss the origin of stochastic evolution of the system's  $\Psi$  so we do not restrict ourselves to systems immersed in a 'heat bath'. In principle, one should consider all possible  $\Psi$ -valued stochastic process but, for simplicity, we shall confine our investigations to Markovian ones. Our construction is elementary; we use ordinary differential equations instead of QSDE.

Section 2 will be devoted to our definition of QSP, where we shall adopt and rederive a constraint proposed earlier by Gisin but criticised recently by Pearle. In § 3 the generic Fokker-Planck equation will be derived for continuous (Wienerian) QSP. Section 4 will consider the equations for certain discontinuous QSP recently reported in the literature. Both of the above types of processes will be utilised in §§ 5-7 to construct simple quantum state reducing schemes.

## 2. Quantum-stochastic processes: master constraint

In this section we consider a given  $N$ -state ( $N < \infty$ ) quantum system whose state vector satisfies a stochastic evolution equation.

A given element  $\Psi$  of the  $N$ -dimensional complex Hilbert space is parametrised by  $2N$  independent coordinates, usually the  $N$  complex orthogonal components  $\psi_1, \psi_2, \dots, \psi_N$  of the vector  $\Psi$ . In order to reflect the explicit gauge invariance we choose  $\psi \equiv (\psi_1, \psi_2, \dots, \psi_N)$  and  $\psi^* \equiv (\psi_1^*, \psi_2^*, \dots, \psi_N^*)$  as the independent coordinates. Quantum states are represented by the normalised vectors

$$\{\Psi \equiv (\psi, \psi^*); \|\Psi\|^2 = \psi_n^* \psi_n = 1\}. \quad (2.1)$$

Here and in the following we use Einstein's convention for summation  $\Sigma_1^N$  over doubly repeated free indices.

Let us introduce the probability distribution  $p$  of the quantum states (2.1) and denote its time dependence through the argument  $t$ . The following conditions must obviously be fulfilled:

$$p(\Psi; t) \geq 0 \tag{2.2a}$$

$$\int p(\Psi; t) d\Psi = 1 \tag{2.2b}$$

$$p(\Psi; t) = 0 \quad \text{if} \quad \|\Psi\| \neq 1 \tag{2.2c}$$

where  $d\Psi \equiv d^N\psi d^N\psi^* = \prod_{n=1}^N d(\text{Re } \psi_n) d(\text{Im } \psi_n)$ .

We assume that the quantum state  $\Psi$  of the system obeys a Markovian (and stationary) stochastic process. Accordingly, the general evolution equation for the distribution  $p$  has the following form:

$$\dot{p}(\Psi; t) = \int L(\Psi, \Psi') p(\Psi'; t) d\Psi' \equiv \hat{L}p(\Psi; t) \tag{2.3}$$

where  $\hat{L}$  is a certain linear evolution operator with kernel  $L(\Psi, \Psi')$ .

Equations (2.1)-(2.3) define the generic (Markovian)  $\Psi$ -valued stochastic process. In some cases it is more comfortable to replace (2.3) by the adjoint evolution equation. Let us define the expectation value  $\langle f \rangle$  of an arbitrary test function  $f(\Psi)$  of the quantum state:

$$\langle f \rangle \equiv \int f(\Psi) p(\Psi; t) d\Psi. \tag{2.4}$$

By applying the evolution equation (2.3), the time derivative of  $\langle f \rangle$  takes the following form:

$$d\langle f \rangle / dt = \langle \hat{L}^T f \rangle \tag{2.5}$$

where  $\hat{L}^T$  stands for the transpose of the evolution operator  $\hat{L}$ . The adjoint evolution equation (2.5) provides equivalent information with evolution equation (2.3).

Constraints (2.2a, b) are trivial and both are easy to keep by  $\hat{L}$ . The constraint (2.2c) needs, however, a more explicit form concerning the operator  $\hat{L}$ . It can be shown that the property (2.2c) can be kept if

$$\hat{L}^T f(\|\Psi\|^2) = 0 \tag{2.6}$$

at any choice of the function  $f$  depending now on  $\Psi$  through its norm.

One might think that, as for the rest, the evolution operator  $\hat{L}$  could arbitrarily be chosen. However, we shall show that, due to straightforward quantum physical arguments, one must take into account a further very strong non-trivial constraint.

Let us recall that in conventional quantum mechanics the set of observable quantities is very limited. The exclusively observable quantities are the components  $\rho_{mn}$  of the density matrix of Landau [20]:

$$\rho_{mn} \equiv \langle \psi_m \psi_n^* \rangle \tag{2.7}$$

where the expectation value on the RHS is defined by (2.4). If we adopt this set of observables for our quantum system too, i.e. if we assume that (i) the density matrix (2.7) is observable and (ii) any observable is the function of the density matrix components, then we have to include additional constraints for operator  $\hat{L}$ .

From the structure of the adjoint evolution equation (2.5) we see that the moments of the distribution  $p$  satisfy a certain linear first-order differential equation system. For example, the second Hermitian moments, i.e. the density matrix components (2.7), satisfy an equation of the form:

$$\dot{\rho}_{mn} = L_{mnrs}\rho_{rs} + \text{linear combination of higher-order moments} \tag{2.8}$$

where  $L_{mnrs}$  are constant coefficients. Since  $\dot{\rho}_{mn}$  is observable itself it may depend on observable moments, i.e. on  $\{\rho_{rs}\}$ , but it must not depend on higher-order moments because they were assumed to be unobservable. Hence we have to require that the evolution equation (2.3) should provide the following closed equation for the density matrix:

$$\dot{\rho}_{mn} = L_{mnrs}\rho_{rs}. \tag{2.9}$$

This equation is the master equation [21] of the so-called open quantum systems. Its mathematical structure is well known (see the appendix).

Let us make the constraint (2.9) explicit for the evolution operator  $\hat{L}$ . Regarding (2.7) and by choosing  $f = \psi_m\psi_n^*$  in (2.5) and  $f = \psi_r\psi_s^*$  in (2.4) we can substitute  $\rho_{mn}$  and  $\rho_{rs}$ , respectively. Thus we obtain  $\langle \hat{L}^T \psi_m\psi_n^* \rangle = \langle L_{mnrs}\psi_r\psi_s^* \rangle$ . This equation must hold for arbitrary distribution  $p$  of  $\Psi$ ; hence we get the ‘master’ constraint

$$\hat{L}^T \psi_m\psi_n^* = L_{mnrs}\psi_r\psi_s^*. \tag{2.10}$$

Let us now summarise what we mean by the notion of a quantum-stochastic process (QSP). The state vector of a given quantum system is a stochastic variable governed by a  $\Psi$ -valued Markovian stochastic process according to the evolution equation (2.3) while the evolution operator satisfies the master constraint (2.10) in order for the closed master equation (2.9) to be kept for the density matrix (2.7).

We note that the necessity of a closed evolution equation for the density matrix was formerly derived [13] from the ‘peaceful coexistence’ [22] between quantum theory and special relativity. We have tried to offer a more elementary proof without referring to any disciplines outside quantum mechanics.

### 3. Continuous quantum-stochastic processes

It is known from mathematics [23] that an exhaustively large class of continuous Markovian stochastic processes (the Wienerian ones, in particular) are described by the Fokker-Planck equation (FPE). In our case, the most general gauge-invariant FPE looks like

$$\dot{p}(\Psi; t) = [\partial_m \partial_n^* G_{mn}(\Psi) - \partial_n v_n(\Psi) - \partial_n^* v_n^*(\Psi)]p(\Psi; t) \tag{3.1}$$

where  $\partial_m \equiv \partial/\partial\psi_m$  and, similarly,  $\partial_n^* \equiv \partial/\partial\psi_n^*$ .  $G_{mn}$  is the Hermitian non-negative diffusion matrix and the  $v_n$  are the drift coefficients. We are going to show that  $G_{mn}$  and  $v_n$  are completely determined by the coefficients  $L_{mnrs}$ .

For later convenience let us introduce the notation

$$g = G_{nn} \tag{3.2}$$

for the real trace of the diffusion matrix. By comparing (2.3) and (3.1) we obtain the evolution operator

$$\hat{L} = \partial_m \partial_n^* G_{mn} + \frac{1}{2}(\partial_n g \psi_n + \text{cc}) - (\partial_n u_n + \text{cc}) \tag{3.3}$$

where, with the notation given in (3.2), we have introduced modified drift coefficients by  $u_n = v_n - \frac{1}{2}g\psi_n$ . The transpose of the evolution operator (3.3) takes the form

$$\hat{L}^T = G_{mn}\partial_m^* \partial_n^* - \frac{1}{2}g(\psi_n \partial_n + c c) + (u_n \partial_n + c c). \tag{3.4}$$

The properties (2.2a, b) are kept by any FPE automatically. In order to assure the condition (2.2c), too, let us substitute expression (3.4) into constraint (2.6); we obtain

$$f' G_{mn} \psi_m^* \psi_n + f' [u_n \psi_n + c c] = 0.$$

For arbitrary  $f$  and because  $G_{mn}$  is non-negative this equation is equivalent with the following two constraints:

$$G_{mn} \psi_n = 0 \tag{3.5}$$

$$u_n \psi_n + c c = 0. \tag{3.6}$$

In addition, the evolution operator  $\hat{L}$  must be tested against the master constraint (2.10). Assuming that the coefficients  $L_{mnr s}$  are given, we substitute the transpose operator (3.4) into (2.10) yielding

$$G_{mn} - g\psi_m \psi_n^* + u_m \psi_n^* + \psi_m^* u_n = L_{mnr s} \psi_r \psi_s^*. \tag{3.7}$$

By simple algebra, it can be seen that, for given  $L_{mnr s}$ , equations (3.5)-(3.7) possess a unique solution expressed by the frictional Hamiltonian and the transition rate matrix introduced for Markovian open quantum systems [24, 25]:

$$u_n = -i H_{nr} \psi_r \tag{3.8}$$

$$G_{mn} = W_{mn}. \tag{3.9}$$

The non-linear frictional Hamiltonian  $H_{mn}$  and the positive semidefinite transition rate matrix  $W_{mn}$  is defined in the appendix by expressions (A5) and (A6). Using solution (3.8) and (3.9) we can rewrite the FPE (3.1) of the continuous QSP as follows:

$$\dot{p} = [\partial_m^* \partial_n W_{mn} + \frac{1}{2}(\partial_n w \psi_n + c c) + i(\partial_n H_{nr} \psi_r - c c)] p \tag{3.10}$$

where  $w = W_{rr}$ .

In this notation it is obvious that the diffusion of the state vector is ruled by the transition rate matrix  $W_{mn}$  while the second term on the RHS corresponds to a 'Hamiltonian-like' (but non-unitary) drift of the quantum state.

Let us summarise the main result of this section. Once the coefficients  $L_{mnr s}$  of the master equation/constraint (2.9) and (2.10) have been fixed the ansatz (3.1) for the evolution of the corresponding continuous QSP yields the unique FPE (3.10).

Sometimes it is convenient to introduce new variables:

$$x_n \equiv |\psi_n|^2 \quad \theta_n = \arg(\psi_n) \tag{3.11}$$

and the corresponding volume element  $\Pi_{n=1}^N(dx_n d\theta_n)$ . Straightforward and not too lengthy calculations lead to the following form of the FPE (3.10) in the new coordinates (3.11):

$$\begin{aligned} \dot{p} = & \sum_{m,n=1}^N \left\{ \partial_m^{(x)} \partial_n^{(x)} (x_m x_n)^{1/2} + \frac{1}{4} (x_m x_n)^{-1/2} \partial_m^{(\theta)} \partial_n^{(\theta)} \right. \\ & \left. + \frac{1}{2} i \left[ \partial_m^{(x)} \partial_n^{(\theta)} \left( \frac{x_m}{x_n} \right)^{1/2} - \partial_n^{(x)} \partial_m^{(\theta)} \left( \frac{x_n}{x_m} \right)^{1/2} \right] \right\} \exp[i(\theta_n - \theta_m)] W_{mn} p \\ & - \sum_{n=1}^N \partial_n^{(x)} L_{nn} p - \frac{1}{2} \sum_{n,m=1}^N \left( \frac{x_m}{x_n} \right)^{1/2} \partial_n^{(\theta)} \{ \exp[i(\theta_n - \theta_m)] H_{mn} + c c \} p \end{aligned} \tag{3.12}$$

(using the notations  $\partial_n^{(x)} \equiv \partial/\partial x_n$  and  $\partial_n^{(\theta)} \equiv \partial/\partial \theta_n$ ).

Let us recall that distribution  $p$  is concentrated permanently on the subspace of the normalised states (cf (2.2c)), i.e. function  $p$  contains the singular factor  $\delta(1 - \sum_{n=1}^N x_n)$ .

**4. Discontinuous QSP**

In the case of a continuous QSP (§ 3) the quantum state is subjected to a certain Wiener process and the state vector  $\Psi$  displays therefore a continuous random path on the surface of the  $2N$ -dimensional unit sphere. In this section, however, we consider another type of QSP where, instead of diffusion, we assume discrete stochastic transitions (jumps)

$$\Psi \rightarrow \Psi^{(\alpha)} \quad \alpha = 2, 3, \dots, N \tag{4.1}$$

superposed on a causal continuous drift. Throughout this paper we suppose that  $\{\Psi; \Psi^{(\alpha)}; \alpha = 2, 3, \dots, N\}$  forms an orthonormal system. Stochastic processes with discrete transitions like those in (4.1) are called discontinuous processes [23] and consequently we shall speak about discontinuous QSP in this section. For technical reasons, we shall specify them by the adjoint evolution equation (2.5).

Let us consider an arbitrary 'test function'  $f$  of the quantum state  $\Psi$ . Given an orthonormal set  $\{\Psi; \Psi^{(\alpha)}; \alpha = 2, 3, \dots, N\}$  of states we assume that the actual quantum state  $\Psi$  can decay stochastically into a certain orthogonal state  $\Psi^{(\alpha)}$  (cf (4.1)) with the corresponding transition rate  $w^{(\alpha)}$ . If the transitions are instantaneous the adjoint evolution equation (2.5) will be of the form

$$\frac{\partial}{\partial t} \langle f \rangle = \left\langle \sum_{\alpha=2}^N w^{(\alpha)} [f(\Psi^{(\alpha)}) - f(\Psi)] \right\rangle.$$

In addition, if there is a causal drift too, then we obtain the following general form for the adjoint evolution equation of the discontinuous QSP:

$$\frac{\partial}{\partial t} \langle f \rangle \equiv \langle \hat{L}^T f \rangle = \left\langle (u_n \partial_n f + c c) + \sum_{\alpha=2}^N w^{(\alpha)} [f(\Psi^{(\alpha)}) - f(\Psi)] \right\rangle. \tag{4.2}$$

This ansatz specifies a discontinuous QSP as opposed to the continuous QSP ansatz (3.1) of the previous section. We are going to show that, by fixing the coefficients  $L_{mnr}$  of the master equation/constraint (2.9) and (2.10), one can construct a unique discontinuous QSP.

It can be shown that condition (3.6)

$$\check{u}_n \psi_n + c c = 0 \tag{4.3}$$

is now sufficient to conserve the property (2.2c) of the distribution, i.e. the normalisation of the state vector.

Furthermore, by substituting  $f = \psi_m \check{\psi}_n$  into (4.2) we obtain the master constraint (2.10) in the form

$$(u_m \check{\psi}_n + h c) + \sum_{\alpha=2}^N w^{(\alpha)} (\psi_m^{(\alpha)} \check{\psi}_n^{(\alpha)} - \psi_m \check{\psi}_n) = L_{mnr} \psi_r \check{\psi}_s. \tag{4.4}$$

(The angular brackets  $\langle \rangle$  have been omitted because of the arbitrariness of the distribution  $p$ ; cf the proof of (2.10).)

The ansatz (4.2) and the constraints (4.3) and (4.4) determine completely the corresponding discontinuous QSP because (4.3) and (4.4) have the following solution:

$$u_n = -iH_{nr}\psi_r \tag{4.5}$$

$$\sum_{\alpha=2}^N w^{(\alpha)} \psi_m^{(\alpha)} \psi_n^{(\alpha)*} = W_{mn}. \tag{4.6}$$

The frictional Hamiltonian  $H_{mn}$  and the transition rate matrix  $W_{mn}$  are defined by the expressions (A5) and (A6) respectively. If the spectrum of  $W_{mn}$  is not degenerate then its orthogonal decomposition (4.6) is unique.

Using (4.5) and (4.6) we get the final form of the adjoint evolution equation of the discontinuous QSP:

$$\frac{\partial}{\partial t} \langle f \rangle \equiv \langle \hat{L}^T f \rangle = \left\langle \sum_{\alpha=2}^N w^{(\alpha)} [f(\Psi^{(\alpha)}) - f(\Psi)] - i(H_{nr}\psi_r \partial_n - c c) f(\Psi) \right\rangle \tag{4.7}$$

where  $\{w^{(\alpha)}\}$ ,  $\{\Psi^{(\alpha)}\}$  are the eigenvalues and the eigenvectors, respectively, of the transition rate matrix (A6);  $H_{mn}$  is the frictional Hamiltonian (A5). The operational content of (4.7) concerning the evolution of the quantum state  $\Psi$  can be summarised as follows. The quantum state  $\Psi$  evolves according to the causal non-linear Schrödinger equation (A7) except for discrete orthogonal jumps  $\Psi(t+0) = \Psi^{(\alpha)}(t)$  (4.1) occurring from time to time at random with partial  $\Psi$ -dependent transition rates  $w^{(\alpha)}$ .

From the above definition it follows that during a given infinitesimal period  $(t, t + dt)$  the probability of the jump-free (i.e. causal) evolution is  $1 - w(t)dt$ , where

$$w(t) \equiv \sum_{\alpha=2}^N w^{(\alpha)}(t) = W_{nn}(t) \tag{4.8}$$

is the total transition rate. Consequently, the *a priori* probability of continuous evolution for an arbitrarily given period  $(t_1, t_2)$  is

$$\exp\left(-\int_{t_1}^{t_2} w(t) dt\right). \tag{4.9}$$

Let us note the surprising fact that the drift term of the discontinuous QSP has turned out to be identical to that of the continuous QSP (3.10). It is also interesting that the transition rate matrix  $W_{mn}$  played the role of the diffusion matrix of the continuous QSP. Hence, each discontinuous QSP of the presented type has its unique continuous counterpart (3.10) and vice versa.

### 5. Quantum-stochastic state reduction with continuous QSP

Preceding the construction of any concrete QSP one has to fix the form of the master equation/constraint (i.e. the coefficients  $L_{mnr}$ ) for the given quantum system subjected to the measurement process. As the most simple but still instructive model for quantum measurement, one can consider the following master equation [12]:

$$\dot{\rho}_{mn} = \begin{cases} (-1/\tau)\rho_{mn} & m = n \\ 0 & m \neq n. \end{cases} \tag{5.1}$$

It is expected that, at least for certain QSP with master equation (5.1), the exponentially decaying off-diagonal elements will enforce the reduction (1.1) into a certain pointer

state:

$$\mathbf{x}(0) \equiv \{x_1(0), x_2(0), \dots, x_N(0)\} \rightarrow \mathbf{x}(T) \equiv \{0, 0, \dots, 0, 1, 0, \dots, 0\} \tag{5.2}$$

1, 2, \dots, k, \dots, N.

(Remember coordination (3.11); angles are omitted in the notation of state vectors.) According to the definition (2.7) of the density matrix, the constancy of diagonal elements corresponds to the so-called [10, 11] martingale property

$$\langle \mathbf{x}(t) \rangle = \text{constant} \tag{5.3}$$

of any QSP possessing the master equation (5.1). The martingale property assures that the probability of the *k*th outcome (5.2) of the measurement will be equal to  $x_k(0)$  which is just the quantum mechanical prediction (1.2).

In this and subsequent sections, we will construct continuous and discontinuous QSP with the same master equation (5.1) and we shall verify that both types lead, in fact, to reduction (5.2).

First of all, let us read out the coefficients (A3) by comparing (5.1) and (A1):

$$L_{mn} = (1/\tau)(\delta_{mn} - 1)\psi_m^* \psi_n. \tag{5.4}$$

(From here, we dispense with the Einstein convention for summing over double dummy indices. Besides, we apply the  $(\psi, \psi^*)$  (2.1) and the  $(\mathbf{x}, \theta)$  (3.11) coordinations together. We do this for economy of notation and it is hoped that this will not lead to misunderstanding.)

For later reference, we apply (5.4) for calculating the frictional Hamiltonian (A5) and the transition rate (or diffusion) matrix (A6), respectively:

$$H_{mn} = \left(\frac{i}{\tau}\right)(\delta_{mn} - 1)\psi_m^* \psi_n + \left(\frac{i}{\tau}\right) \left(1 - \sum_{r=1}^N x_r^2\right) \delta_{mn} \tag{5.5}$$

$$W_{mn} = \left(\frac{1}{\tau}\right) \left(\delta_{mn} - x_m - x_n + \sum_{r=1}^N x_r^2\right) \psi_m^* \psi_n. \tag{5.6}$$

Let us also write down the expression of the total transition rate:

$$w \equiv \sum_{n=1}^N W_{nn} = \left(\frac{1}{\tau}\right) \left(1 - \sum_{r=1}^N x_r^2\right). \tag{5.7}$$

After these preparations let us turn to our first example of a state reduction scheme corresponding to the measurement according to (5.1). Let the state vector follow the continuous QSP introduced in § 3. By substituting expressions (5.4)–(5.6) into the general form (3.12) of the evolution equation (FPE), the drift term vanishes:

$$\begin{aligned} \dot{p}(\mathbf{x}, \theta; t) &= \left(\frac{1}{\tau}\right) \sum_{m,n=1}^N [\partial_m^{(x)} \partial_n^{(x)} (x_m x_n)^{1/2} + \frac{1}{4} (x_m x_n)^{-1/2} \partial_m^{(\theta)} \partial_n^{(\theta)}] \\ &\quad \times (x_m x_n)^{1/2} \left(\delta_{mn} - x_m - x_n + \sum_{r=1}^N x_r^2\right) p(\mathbf{x}, \theta; t). \end{aligned} \tag{5.8}$$

Observe that the coefficients of this FPE do not depend on the angle variables; hence, by integrating over  $\theta$ , we arrive at a closed diffusion equation for the  $\mathbf{x}$  distribution alone:

$$\begin{aligned} \dot{p}(\mathbf{x}; t) &= \left(\frac{1}{\tau}\right) \sum_{m,n=1}^N \partial_m^{(x)} \partial_n^{(x)} (x_m x_n) \left(\delta_{mn} - x_m - x_n + \sum_{r=1}^N x_r^2\right) p(\mathbf{x}; t) \\ &\equiv \left(\frac{1}{2\tau}\right) \sum_{m,n=1}^N (\partial_m^{(x)} - \partial_n^{(x)})^2 x_m x_n \left(x_m + x_n - \sum_{r=1}^N x_r^2\right) p(\mathbf{x}; t). \end{aligned} \tag{5.9}$$



Recently, Pearle [10] has shown that this FPE follows from Gisin's dynamical reduction theory [12] which is, in fact, identical to our continuous QSP model. From (5.9) it follows that, for  $t = T = \infty$ , the Wienerian random walk of the vector  $x(t)$  leads asymptotically to one of the pointer states: all components of  $x(T)$  will vanish but, for example, the  $k$ th will approach unity; cf process (5.2). The right quantum mechanical probability of each outcome is assured by (5.3). The detailed proof has been done, for example, in [10].

In order to make the reduction time  $T$  finite, Pearle [7-11] proposed different FPE at the price that the master constraint has been neglected. Hence, Pearle's reduction model does not fit to our definition of a QSP. In our opinion, the closed evolution equation for the density matrix (i.e. for the observables) is probably important to retain. We do not see any troubles when the total reduction time  $T$  remains infinite if the characteristic time  $\tau$  of the asymptotic reduction is small.

**6. Quantum state reduction with discontinuous QSP**

Our second example of a state reduction process will be the discontinuous counterpart of Gisin's model (5.7): the discontinuous QSP of § 4 is to be applied to the measurement process with master equation (5.1).

First of all, we rewrite some formulae of the previous section. Observe that the frictional Schrödinger equation (A7) with the Hamiltonian (5.3) does not change the angles  $\theta$ ; therefore we write it in terms of the  $(x, \theta)$  variables:

$$\dot{x}_n = \left(\frac{1}{\tau}\right) 2x_n \left(x_n - \sum_{r=1}^N x_r^2\right) \quad \dot{\theta}_n = 0. \tag{6.1}$$

For completeness, we repeat the formulae (5.6) and (5.7) of transition rates:

$$W_{mn} = \left(\frac{1}{\tau}\right) \left(\delta_{mn} - x_m - x_n + \sum_{r=1}^N x_r^2\right) \psi_m \psi_n^* \tag{6.2}$$

$$w \equiv \sum_{n=1}^N W_{nn} = \left(\frac{1}{\tau}\right) \left(1 - \sum_{r=1}^N x_r^2\right). \tag{6.3}$$

Now let us invoke and apply the specification of continuous QSP given in § 4: The quantum state  $\Psi$  satisfies the causal equation (6.1) for most of the time and, from time to time, it randomly decays to the  $\alpha$ th eigenstate  $\Psi^{(\alpha)}$  of the transition rate matrix (6.2) with rate  $w^{(\alpha)}$  identical to the corresponding eigenvalue;  $\alpha = 2, 3, \dots, N$ .

It is trivial to see that each pointer state (cf RHS of (5.2)) is a stationary solution of the above discontinuous QSP. Let us choose  $k = 1$  for concreteness, then the vector  $x \equiv (1, 0, \dots, 0)$  is a stationary solution. In fact,  $\dot{x}$  and  $w$  vanish due to (6.1) and (6.3), respectively; hence there are no causal drifts and no discrete stochastic jumps either. We shall always neglect the exceptional state ( $x_1 = x_2 = \dots = x_N = 1/N$ ); the only stationary states are the pointer states.

Furthermore, we prove that the pointer states are locally and stochastically stable stationary solutions for the given discontinuous QSP. Choosing again  $k = 1$ , let us suppose that at some moment, say at  $t = 0$ , the state is almost reduced, i.e.  $x_1(t) = 1 - \varepsilon(t)$  where  $\varepsilon(0) \ll 1$ . Invoking (6.1) and observing that  $x_n = O(\varepsilon)$  for  $n \neq 1$ , one obtains  $\varepsilon \approx -2\varepsilon/\tau$ , up to  $\varepsilon^2$  terms; consequently  $\varepsilon(t) \approx \varepsilon(0) \exp(-2t/\tau)$ . In the same approximation (6.3) yields  $w \approx 2\varepsilon/\tau$ . Thus, from the expression (4.9) with  $(t_1, t_2) \equiv (0, \infty)$ , the

probability of jump-free evolution is just equal to  $1 - \varepsilon(0)$ , if  $\varepsilon^2$  terms were neglected. Hence we conclude that (i)  $\varepsilon$  decays exponentially with a characteristic time  $\tau/2$  and (ii) the *a priori* probability of the interruption of this relaxation by discrete stochastic transitions is equal to  $\varepsilon(0)$ . In general, once the  $k$ th pointer state has been approached in the course of the discontinuous QSP, i.e.  $\varepsilon = 1 - x_k \ll 1$ , the reduction process then perfectuates with probability  $1 - \varepsilon$ . This is what we mean by local and stochastic stability of the pointer states.

We are still owing the proof that, starting from *any* given initial state  $x(0)$ , at least one pointer state is achievable with definitely positive probability. This, invoking that pointer states are the only stable stationary states, would complete the proof of reduction (5.2) in the above discontinuous QSP for *all* initial states (with one exception mentioned).

Fortunately, we are able to show that the probability of reduction to the  $k$ th pointer state is always positive if label  $k$  belongs to the maximum initial coordinate, i.e.  $x_k(0) = \max\{x_1(0), x_2(0), \dots, x_N(0)\}$ . We are going to show that the continuous evolution pushes  $x_k(\infty)$  to unity and this (i.e. jump-free) path is realised with positive probability.

Accordingly, assume that stochastic transitions will not occur at all for  $t > 0$ . Thus  $x(t)$  is governed by the frictional Schrödinger equation (6.1) alone. Observe furthermore that  $x_k > \sum_{r=1}^N x_r^2$  provided  $x_k$  is the maximum coordinate and the exceptional state  $x_1 = x_2 = \dots = x_N = 1/N$  is excluded. Then from (6.1) we get  $d(x_k - x_r)/dt > 0$  for each  $r \neq k$  so  $x_k(t)$  stays at maximum value for  $t > 0$ , too. Hence, again from (6.1), it is seen that  $\dot{x}_k(t) > 0$  and therefore the maximum coordinate  $x_k(t)$  tends to unity in the limit  $t = \infty$ . Recall now that we have assumed jump-free causal evolution for  $t > 0$ . The expression (4.9) with  $(t_1, t_2) \equiv (0, \infty)$  yields a definitely positive probability for the considered case since  $w(t)$  is bound and, for large  $t$ , it decreases like  $\exp(-2t/\tau)$  as we showed above.

Finally, let us characterise the general flow of the state reduction in our discontinuous QSP model. The actual candidate for the outcome of the reduction is always the pointer state belonging to the maximum coordinate since the maximum coordinate tends to grow faster than the others. This tendency may change from time to time due to discrete random transitions which presumably alter the label of the maximum coordinate. We have to recall that, beside the reduction property (5.2), proper quantum mechanical probabilities of the final pointer states are also assured by the martingale property (5.3) of the QSP.

## 7. Reduction with discontinuous QSP in two-state systems

Since the exact solution of the discontinuous QSP equations would require the diagonalisation of the transition rate matrix (6.2) we are going to consider the simplest case, i.e. the two-state ( $N = 2$ ) quantum system where analytic solution is possible.

For two-state systems it is rather convenient to introduce the difference  $q = x_2 - x_1$  and to use the coordination

$$\begin{aligned} \psi_1 &= 2^{-1/2}(1 - q)^{1/2} \exp(i\theta_1) \\ \psi_2 &= 2^{-1/2}(1 + q)^{1/2} \exp(i\theta_2) \end{aligned} \quad (7.1)$$

which assures the right normalisation of the quantum state  $\Psi$ . The frictional

Schrödinger equation (6.1) then takes the following form:

$$\dot{q} = (1/\tau)q(1 - q^2) \quad \dot{\theta}_1 = \dot{\theta}_2 = 0. \tag{7.2}$$

The transition rate matrix (6.2) will consist of a single diad:  $W_{mn} = w^{(2)}\psi_m^{(2)}\psi_n^{(2)*}$  (cf (4.6)) where

$$w^{(2)} \equiv w = (1/2\tau)(1 - q^2) \tag{7.3}$$

is the total transition rate; the eigenstate  $\Psi^{(2)}$  of coordinates

$$\begin{aligned} \psi_1 &= 2^{-1/2}(1 + q)^{1/2} \exp(-i\theta_2) \\ \psi_2 &= 2^{-1/2}(1 - q)^{1/2} \exp(-i\theta_1) \end{aligned} \tag{7.4}$$

is the orthogonal to the actual quantum state  $\Psi$  (7.1).

By recalling the specification of the discontinuous QSP given in the previous section, let us apply it to the simple two-state system. Accordingly, the quantum state (7.1) of the system satisfies the causal equation (7.2) for most of the time and, from time to time, it decays at rate (7.3) to the state (7.4) which is the actual orthogonal complement state. We note immediately that such a random jump from state (7.1) to (7.4) corresponds to sign flip of  $q$ .

Let us investigate the main characteristics of the state reduction in the two-state system. It is obvious that the two pointer states are represented by the condition  $q = \pm 1$ . Therefore, we are going to prove that, according to our discontinuous QSP,  $|q(t)|$  tends to unity in the limit  $t = \infty$ . Really, for the absolute value of  $q$  the causal equation (7.2) holds for all times independently of possible sign flips of  $q$ . This differential equation can be integrated and one obtains:

$$\begin{aligned} |q(t)| &= [1 + (q(0)^{-2} - 1) \exp(-2t/\tau)]^{-1/2} \\ &\approx 1 - \frac{1}{2}(q(0)^{-2} - 1) \exp(-2t/\tau) \quad \text{for } t \gg \tau. \end{aligned} \tag{7.5}$$

In addition, the above solution yields an analytic solution for the time-dependent rate (7.3) of discrete orthogonal transitions (i.e. of  $q \rightarrow -q$ ,  $\theta_1 \rightarrow -\theta_2$ ,  $\theta_2 \rightarrow -\theta_1$ ):

$$w(t) = (1/2\tau)(1 - q(t)^2) \approx (1/2\tau)(q(0)^{-2} - 1) \exp(-2t/\tau) \quad \text{for } t \gg \tau. \tag{7.6}$$

Hence, (7.5) shows that  $q(t) \approx \pm 1$  if  $t \gg \tau$ ; in other words, either the  $(x_1 = 1, x_2 = 0)$  or the  $(x_1 = 0, x_2 = 1)$  pointer state is approached depending on the total number of flips  $q \rightarrow -q$  that have hitherto occurred. Assume, e.g., that for a given time  $t \gg \tau$  the quantum state has almost been reduced to the first pointer state  $(x_1 = 1, x_2 = 0)$ . Then from (4.9) and (7.6) we see that the total probability of further flips is  $1 - \exp(-\int_t^\infty w(t') dt') \approx \exp(-2t/\tau)$ , i.e. additional flips are very unlikely if  $t$  is large enough. Nevertheless, their eternal potentiality is in line with the delicate experience gained from other dynamical reduction models, namely that the convergence of reduction is always especially weak and is highly conditional.

For completeness of this section let us recopy the two independent equations coming from master equation (5.1) in the  $N = 2$  case:

$$\dot{\rho}_{11} = 0 \quad \dot{\rho}_{12} = -(1/\tau)\rho_{12}. \tag{7.7}$$

Although these equations are obviously satisfied by our QSP due to its construction, nevertheless we are going to prove them by direct calculations as well.

Recalling the definition (2.7) of the density matrix we write the components  $\rho_{11}$  and  $\rho_{12}$  in the coordination (7.1) as follows:

$$\rho_{11} = \frac{1}{2}(1 - \langle q \rangle) \quad \rho_{12} = \frac{1}{2}\langle (1 - q^2) \exp[i(\theta_1 - \theta_2)] \rangle. \tag{7.8}$$

Apply the specification of our QSP. During an infinitesimal period  $dt$ ,  $q$  will acquire a causal change  $dq_{\text{caus}} = \tau^{-1}q(1 - q^2) dt$  due to (7.2) and, with probability  $w dt$ ,  $q$  may have a discrete stochastic change  $\Delta q_{\text{stoch}} = -2q$  due to a sign flip. Thus, for the expectation value of  $dq$  one obtains

$$\langle dq \rangle = dq_{\text{caus}} + w dt \Delta q_{\text{stoch}} = (1/\tau)q(1 - q^2) dt - 2qw dt \tag{7.9}$$

which yields zero with  $w$  substituted by (7.3). Consequently,  $\langle q \rangle$  has turned to be constant which means the constancy of  $\rho_{11}$  as well (martingale property).

So we are left with the verification of the second of equations (7.8), from which we can see that the random jumps do not affect the value of  $\rho_{12}$ ; its evolution is thus governed by the causal change of  $q^2(t)$  taken from (7.5). This yields just the exponential decay  $\rho_{12}(t) = \rho_{12}(0) \exp(-2t/\tau)$  as is expected from the master equation (7.7).

**8. Conclusion**

Our paper has been motivated by the alternative philosophy which attributes physical reality to the quantum state as opposed to more common views suggesting that the state vector is a pure mathematical abstraction. Consequently, the stochastic reduction of the quantum state during a given measurement process has to be taken as a real process, actually a stochastic process. We developed a theory of stationary Markovian quantum stochastic processes. It recovers the well known continuous (Wienerian) QSP and, consequently, Gisin’s reduction model as well as discontinuous QSP. We pointed out that each continuous QSP has its natural discontinuous counterpart and we constructed the discontinuous counterpart of the Gisin model.

By now, it is not possible to tell which one is the true QSP because they are, by construction, physically equivalent in the context of the present theory. However, as an outlook to the future, one might imagine that a certain, still unknown, physical effect will offer new observables in addition to the density matrix elements and then the ambiguity of the wavefunction during a measurement will be resolved; the QSP will be verifiable. In the present paper we have tried to stress that one has to discuss all possible reducing QSP on an equal footing.

**Appendix**

The coefficients  $L_{mnr s}$  of the master equation

$$\dot{\rho}_{mn} = L_{mnr s} \rho_{rs} \tag{A1}$$

must ensure the normalisation

$$\rho_{nn} = 1 \tag{A2}$$

and the Hermiticity and the positive semidefiniteness of the density matrix  $\{\rho_{mn}\}$ . For a given state vector, by introducing the notation

$$L_{mn} = L_{mnr s} \psi_r \psi_s^* \tag{A3}$$

the necessary and sufficient set of conditions is the following [21]:

$$L_{mn} = \check{L}_{nm} \quad (\text{A4a})$$

$$L_{nn} = 0 \quad (\text{A4b})$$

$$L_{mn} \check{\varphi}_m \psi_n \leq 0 \quad (\text{A4c})$$

$$L_{mn} \check{\varphi}_m \varphi_n \geq 0 \quad \text{if } \check{\varphi}_n \psi_n = 0. \quad (\text{A4d})$$

It is worthwhile to introduce [24, 25] the so-called frictional Hamiltonian:

$$H_{mn} \equiv i(L_{mn} - \delta_{mn} L_{rs} \check{\psi}_r \psi_s) + \mu \delta_{mn} \quad (\text{A5})$$

and the Hermitian positive semidefinite transition rate (or diffusion) matrix:

$$W_{mn} \equiv L_{mn} - (L_{mr} \psi_r \check{\psi}_n + \text{HC}) + L_{rs} \check{\psi}_r \psi_s \psi_m \check{\psi}_n. \quad (\text{A6})$$

(It should be noticed that  $\mu$  in (A5) is an arbitrary real function of state  $\Psi$ . We shall set  $\mu \equiv 0$ ; this can always be achieved by a proper gauge transformation of the phases of the states  $\Psi$ .)

The Schrödinger equation

$$\dot{\psi}_n = -i H_{nr} \psi_r = L_{nr} \psi_r - L_{rs} \check{\psi}_r \psi_s \psi_n \quad (\text{A7})$$

with Hamiltonian (A5) is non-linear but conserves the norm of the wavefunction.

It can be shown by direct substitution that the transition rate matrix (A6) possesses trivial degeneracy:

$$W_{mn} \psi_n = 0. \quad (\text{A8})$$

*Note added in proof.* After the completion of the present work, a paper by K E Eriksson (1987 *Phys. Scr.* **36** 870) on a similar subject was brought to my attention.

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