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QUANTUM THEORIES ASSOCIATED TO INCREASING HILBERT SPACE FILTRATIONS AND GENERALIZED JACOBI 3–DIAGONAL RELATION

LUIGI ACCARDI* AND YUN-GANG LU

Abstract. We prove that the quantum decomposition of a classical random variable, or random field, is a very general phenomenon involving only an increasing filtration of Hilbert spaces and a family of Hermitean operators increasing by 1 the filtration. The creation, annihilation and preservation operators (CAP operators), defining the quantum decomposition of these Hermitean operators, satisfy commutation relations that generalize those of usual quantum mechanics. In fact there are two types of commutation relations (Type I and Type II). In Type I commutation relations the commutator is given by an operator–valued sesqui–linear form. The case when this operator–valued sesqui–linear form is scalar valued (multiples of the identity) characterizes the non–relativistic free Bose field and the associated commutation relations reduce to the Heisenberg ones. Type II commutation relations did not appear up to now because they are identically satisfied when the probability distribution of the random field is a product measure. In this sense they encode information on the self–interaction of the random field.

1. Introduction

The **Jacobi 3–diagonal relation** is a well known identity at the basis of the theory of orthogonal polynomials in one variable. Theorem 2.1 below shows that this relation naturally emerges in a much more general framework defined by:
– a Hilbert space $\mathcal{H}$,
– an **increasing filtration** of closed sub–spaces of $\mathcal{H}$

$$(\mathcal{H}_{n]}) \qquad ; \qquad \mathcal{H}_{n]} \subseteq \mathcal{H}_{n+1]} \qquad ; \qquad \forall n \in \mathbb{N} := \{0, 1, \dots\}$$

with corresponding orthogonal projectors

$$P_{n]} : \mathcal{H} \to \mathcal{H}_{n]} \qquad ; \qquad \forall n \in \mathbb{N}$$

– a set D and, for each $j \in D$, a linear Hermitean operator

$$Y_j = Y_j^* : \mathrm{Dom}(Y_j) \subseteq \mathcal{H} \to \mathcal{H}$$

filtration increasing of degree $+1$

$$Y_j \left(\mathrm{Dom}(Y_j) \cap \mathcal{H}_{n]}\right) \subseteq \mathcal{H}_{n+1]} \tag{1.1}$$

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Theorem 2.1 also shows that the Jacobi 3–diagonal relation is equivalent to a **canonical quantum decomposition** of the Y_j (simply quantum decomposition in the following).

The quantum decomposition leads to **commutation relations (CR)** among the operators appearing in the canonical quantum decomposition of the Y_j. **There are two types of commutation relations** that we call type I and type II CR. Type I CR have a very general root as they are naturally associated to any pair of operators a^+, b^- defined on an $\mathbb{N}$–graded Hilbert space and such that both b^-a^+ and a^+b^- are gradation–preserving. They are common to all Interacting Fock spaces (IFS) (see [8] for an introduction to this notion).

Type II CR are a **new type** of commutation relations that are specific to the case when the operators Y_j **mutually commute**. In this paper we develop the theory for general filtrations.

2. Quantum Decomposition of Filtration Increasing Hermitean Operators

Theorem 2.1. *Let $\mathcal{H}$ be a Hilbert space and $(P_{n]})$ $(n \in \mathbb{N})$ an increasing sequence of orthogonal projectors on $\mathcal{H}$. Define the sequence (P_n) of orthogonal projectors:*

$$P_n := P_{n]} - P_{n-1]} \quad ; \quad \forall n \in \mathbb{N} \tag{2.1}$$

with the notation

$$P_{-1} = P_{-1]} := 0 \tag{2.2}$$

Then the projectors in the sequence (P_n) are mutually orthogonal and

$$P_{n+k}P_{n+1]} = 0 \qquad ; \qquad \forall k \in \mathbb{N}\ ,\ k \geq 2 \tag{2.3}$$

Suppose that there exists a family $(Y_j)_{j\in D}$ (D is a set) of linear operators on $\mathcal{H}$ with the following properties:
– There exists a dense sub–space $\mathcal{D} \subseteq \mathcal{H}$ (independent of j) contained in the domain of each Y_j.
– Each Y_j is ***Hermitean on*** *$\mathcal{D}$:*

$$\langle \xi, Y_j\eta\rangle = \langle Y_j\xi, \eta\rangle \qquad ; \qquad \forall \xi, \eta \in \mathcal{D}$$

– For each $j \in D$ and $n \in \mathbb{N}$, $P_{n]}\mathcal{D}$ is contained in the domain of each Y_j and

$$Y_jP_{n]}\mathcal{D} \subseteq P_{n+1]}\mathcal{H} \tag{2.4}$$

Then:
(i) The identity

$$P_mY_jP_n = 0 \quad \text{if} \quad m \notin \{n-1, n, n+1\} \tag{2.5}$$

holds on $\mathcal{D}$.
(ii) If, in addition the limit

$$\lim_{n\to\infty} P_{n]} = 1_{\mathcal{H}} \tag{2.6}$$

exists on $\mathcal{D}$ for some topology on $\mathcal{B}(\mathcal{H})$ then (P_n) is a partition of the identity in the same topology and

$$Y_jP_n = P_{n+1}Y_jP_n + P_nY_jP_n + P_{n-1}Y_jP_n \quad (\text{on } \mathcal{D}) \tag{2.7}$$

(iii) If (i) and (ii) hold, denoting

$$(\mathcal{H}, \langle\ \cdot\ ,\ \cdot\ \rangle) = \bigoplus_{n\in\mathbb{N}} (\mathcal{H}_n, \langle\ \cdot\ ,\ \cdot\ \rangle_n) \tag{2.8}$$

$$(\mathcal{D}, \langle\ \cdot\ ,\ \cdot\ \rangle) = \bigoplus_{n\in\mathbb{N}} (\mathcal{D}_n, \langle\ \cdot\ ,\ \cdot\ \rangle_n) \tag{2.9}$$

the ***orthogonal gradations*** *induced by* (P_n)*, one has*

$$\mathcal{H}_{N]} = \bigoplus_{n\in\{0,1,\dots,N\}} (\mathcal{H}_n, \langle\ \cdot\ ,\ \cdot\ \rangle_n) \quad ; \quad \forall N \in \mathbb{N} \tag{2.10}$$

$$\mathcal{D}_{N]} = \bigoplus_{n\in\{0,1,\dots,N\}} (\mathcal{D}_n, \langle\ \cdot\ ,\ \cdot\ \rangle_n) \quad ; \quad \forall N \in \mathbb{N} \tag{2.11}$$

and (2.7) *is equivalent to*

$$Y_j = \sum_{n\in\mathbb{N}} P_{n+1} Y_j P_n + \sum_{n\in\mathbb{N}} P_n Y_j P_n + \sum_{n\in\mathbb{N}} P_{n-1} Y_j P_n \quad (\textit{on } \mathcal{D}) \tag{2.12}$$

where each of the series on the right hand side is convergent in the strongly finite sense on

$$\mathcal{D}^{(w)} := \textit{lin. span}\{\mathcal{D}_n \ :\ n \in \mathbb{N}\} \tag{2.13}$$

(iv) If (i), (ii) and (iii) hold and condition (2.4) *is strengthened as follows*

$$\sum_{j\in D} Y_j \mathcal{D}_{n]} \ +\ \mathcal{H}_{n]} \quad \textit{is dense in } \ \mathcal{H}_{n+1]} \qquad ; \qquad \forall n \in \mathbb{N} \tag{2.14}$$

where $\sum$ *and* $+$ *denote vector space sum, then*

$$P_n = 0 \textit{ for some } n \in \mathbb{N} \Rightarrow P_N = 0 \quad , \quad \forall N \geq n \tag{2.15}$$

Proof. (i) The P_n are mutually orthogonal because if $m \neq n$, assuming without loss of generality that $m < n$, one has

$$P_m P_n = (P_{m]} - P_{m-1]})(P_{n]} - P_{n-1]}) = P_{m]}P_{n]} - P_{m]}P_{n-1]} - P_{m-1]}P_{n]} + P_{m-1]}P_{n-1]}$$
$$= P_{m]} - P_{m]} - P_{m-1]} + P_{m-1]} = 0$$

Moreover, since $(P_{n]})$ is increasing, if $n + k - 1 \geq n + 1$, it follows that

$$P_{n+k} P_{n+1]} = (P_{n+k]} - P_{n+k-1]}) P_{n+1]}$$
$$= P_{n+k]} P_{n+1]} - P_{n+k-1]} P_{n+1]} = P_{n+1]} - P_{n+1]} = 0$$

This proves (2.3). From this one deduces that, if $m \geq n + 2$, then (2.4) implies that, for each $j \in D$, $P_m Y_j P_n = P_m P_{n+1]} Y_j P_n = 0$. Hence by Hermiteanity

$$0 = (P_m Y_j P_n)^* = P_n Y_j P_m \quad ; \quad \forall m \geq n + 2 \iff n \leq m - 2$$

Thus $P_m Y_j P_n = 0$ if $m \notin \{n-1, n, n+1\}$ and this proves (2.5).

(ii) Now suppose that (2.6) is satisfied. It is known that any increasing filtration $(P_{n]})_{n\in\mathbb{N}}$ of projectors on a pre–Hilbert space $\mathcal{K}$ converging to the identity in some topology uniquely defines, through (2.1), a partition of the identity in the same topology because by assumption $P_{N]} \to 1$ and $\sum_{k=0}^{N} P_k = P_{N]}$ due to the fact that the sum is telescopic and to the convention $P_{-1]} := 0$. Similarly, any partition of the identity of projectors on $\mathcal{H}$ defines an orthogonal gradation because, since $\sum_{n\in\mathbb{N}} P_n = 1$, one has

$$\mathcal{K} = \sum_{n\in\mathbb{N}} P_n(\mathcal{K}) =: \sum_{n\in\mathbb{N}} \mathcal{K}_n$$

and the sum is orthogonal because the P_n are mutually orthogonal. This argument, applied to $\mathcal{K} = \mathcal{H}$ and $\mathcal{K} = \mathcal{D}$ gives (2.8) and (2.9).

Condition $\sum_{n\in\mathbb{N}} P_n = 1$ together with (2.5) implies that, for each $j \in D$,

$$Y_j P_n = \sum_{m\in\mathbb{N}} P_m Y_j P_n = P_{n+1} Y_j P_n + P_n Y_j P_n + P_{n-1} Y_j P_n$$

which is (2.7).

(iii) Summing the identity (2.7) over $n \in \mathbb{N}$ and recalling (2.1) and (2.2), one finds

$$Y_j = Y_j \sum_{n=0}^{\infty} P_n = \sum_{n=0}^{\infty} P_{n+1} Y_j P_n + \sum_{n=0}^{\infty} P_n Y_j P_n + \sum_{n=0}^{\infty} P_{n-1} Y_j P_n \tag{2.16}$$

The convergence, in the strongly finite sense on $\mathcal{D}^{(w)}$, of each of the series in (2.16) follows from the fact that, by (2.13) every vector in $\mathcal{D}^{(w)}$ is a sum of vectors of the form $\xi_k \in \mathcal{D}_k$, and, since $\mathcal{D}_k \subseteq \mathcal{H}_k = P_k(\mathcal{H})$, for each such ξ_k one has

$$\sum_{n=0}^{\infty} P_{n+1} Y_j P_n \xi_k = P_{k+1} Y_j \xi_k$$

and similarly for the other two series.

(iv) Suppose condition (2.14) holds and that $P_n = 0$ for some $n \in \mathbb{N}$, then (2.1) implies that

$$P_{n]} = P_{n-1]} \iff \mathcal{H}_{n]} = \mathcal{H}_{n-1]}$$

and this implies

$$Y_j \mathcal{D}_{n]} \subseteq \mathcal{H}_{n]} = \mathcal{H}_{n-1]}$$

This and (2.14) imply that

$$\mathcal{H}_{n+1]} = \text{clos. lin. span}\left\{Y_j \mathcal{D}_{n]} \ : \ j \in D\right\} \ + \ \mathcal{H}_{n]} \subseteq \mathcal{H}_{n-1]} \ + \ \mathcal{H}_{n-1]} = \mathcal{H}_{n-1]}$$

Since the filtration is increasing,

$$\mathcal{H}_{n+1]} = \mathcal{H}_{n]} \iff P_{n+1} = P_{n]} = P_{n-1]}$$

Since $n \in \mathbb{N}$ is arbitrary, (2.15) follows by induction given (2.1). □

Corollary 2.2. *Any triple $(\mathcal{H}, (P_{n]})_{n\in\mathbb{N}}, \mathcal{D}, Y)$ satisfying all the conditions of Theorem 2.1* ***uniquely*** *defines the following objects:*
(i) The ***partition of the identity*** (P_n) *defined by* (2.1) *and consisting of orthogonal projectors.*
(ii) The ***orthogonal gradations*** (2.8), (2.9) *satisfying* (2.10), (2.11) *respectively.*
(iii) A family of operators

$$a^+_{Y_j} := \sum_{n\in\mathbb{N}} P_{n+1} Y_j P_n \quad ; \quad a^-_{Y_j} := \sum_{n\in\mathbb{N}} P_{n-1} Y_j P_n \quad ; \quad a^0_{Y_j} := \sum_{n\in\mathbb{N}} P_n Y_j P_n \tag{2.17}$$

where each of the series on the right hand side is convergent in the strongly finite sense on the space $\mathcal{D}^{(w)}$ defined by (2.13) and with the following properties:

$$(a^0_{Y_j})^* = a^0_{Y_j} \quad ; \quad (a^+_{Y_j})^* := a^-_{Y_j} \quad ; \quad \text{on } \mathcal{D} \tag{2.18}$$

$$a^+_{Y_j}(\mathcal{D}_n) \subseteq \mathcal{H}_{n+1} \quad \text{gradation increasing of degree } +1 \tag{2.19}$$

$$a^0_{Y_j}(\mathcal{D}_n) \subseteq \mathcal{H}_n \quad \text{gradation preserving of degree } 0 \tag{2.20}$$

$$a^-_{Y_j}(\mathcal{P}_n) \subseteq \mathcal{H}_{n-1} \quad \text{gradation decreasing of degree } -1 \quad ; \quad \mathcal{P}_{-1} := \{0\} \tag{2.21}$$

(iv) The decomposition

$$Y_j = a^+_{Y_j} + a^0_{Y_j} + a^-_{Y_j} \tag{2.22}$$

which is equivalent to identity (2.7)*, i.e.*

$$Y_j P_n = a^+_{Y_j} P_n + a^0_{Y_j} P_n + a^-_{Y_j} P_n \quad ; \quad n \in \mathbb{N} \tag{2.23}$$

(v) If for each $n \in \mathbb{N}$, the sub–space

$$\sum_{j\in D} a^+_{Y_j}\mathcal{D}_n \ + \ \mathcal{H}_{n]} \quad \text{is dense in } \ \mathcal{H}_{n+1]} \tag{2.24}$$

The number belonging to $\mathbb{N} \cup \{+\infty\}$ defined by:

$$N_{(\mathcal{H},(P_{n]})_{n\in\mathbb{N}},\mathcal{D},Y)} := \sup\{n \in \mathbb{N} \ : \ \mathcal{P}_k \neq \{0\} \ , \ \forall k \in \{0,\dots,n\}\} \tag{2.25}$$

is well defined and, if $N_{(\mathcal{H},(P_{n]})_{n\in\mathbb{N}},\mathcal{D},Y)} < +\infty$, is equal to

$$\min\{n \in \mathbb{N} \ : \ \mathcal{P}_n = \{0\}\} \ - 1 \tag{2.26}$$

Proof. (i) and (ii) have been proved in Theorem 2.1.
(iii) follows from the fact that, given the definition (2.17), (2.22) is simply a different way of writing the identity (2.12) proved in Theorem 2.1.

The identities (2.18) follow from the fact that, for each $n \in \mathbb{N}$ and $j \in D$,

$$(P_{n+1}Y_jP_n)^* = P_nY_jP_{n+1} \quad ; \quad (P_nY_jP_n)^* = P_nY_jP_n$$
$$(P_nY_jP_n)^* = P_nY_jP_n$$

The identities (2.19), (2.20) and (2.21) follow respectively from the identities

$$a^+_{Y_j}\big|_{\mathcal{D}_n} = P_{n+1}Y_jP_n \quad ; \quad a^-_{Y_j}\big|_{\mathcal{D}_n} = P_{n-1}Y_jP_n \quad ; \quad a^0_{Y_j}\big|_{\mathcal{D}_n} = P_nY_jP_n$$

(iv) Multiplying on the right by P_n identity (2.22), gives (2.23). Summing over n identity (2.23), one obtains (2.22). Thus the two identities are equivalent.

From (2.14) we know that $\sum_{j\in D} Y_j\mathcal{D}_{n]} \ + \ \mathcal{H}_{n]}$ is dense in $\mathcal{H}_{n+1]}$. On the other hand, (2.10) implies that, for all $n \in \mathbb{N}$

$$\sum_{j\in D} Y_j\mathcal{D}_{n]} \ + \ \mathcal{H}_{n]} = \sum_{j\in D} Y_j(\mathcal{D}_n + \mathcal{D}_{n-1]}) \ + \ \mathcal{H}_{n]}$$

$$= \sum_{j\in D} Y_j\mathcal{D}_n + \left(\sum_{j\in D} Y_j\mathcal{D}_{n-1]}\right) \ + \ \mathcal{H}_{n]} = \sum_{j\in D} Y_j\mathcal{D}_n \ + \ \mathcal{H}_{n]} \tag{2.27}$$

because $\sum_{j\in D} Y_j\mathcal{D}_{n-1]} \subseteq \mathcal{H}_{n]}$. Using (2.22), one can write (2.27) as

$$\sum_{j\in D}(a^+_{Y_j} + a^0_{Y_j} + a^-_{Y_j})\mathcal{D}_n \ + \ \mathcal{H}_{n]}$$

$$= \sum_{j\in D} a^+_{Y_j}\mathcal{D}_n + \sum_{j\in D}(a^0_{Y_j} + a^-_{Y_j})\mathcal{D}_n \ + \ \mathcal{H}_{n]} = \sum_{j\in D} a^+_{Y_j}\mathcal{D}_n \ + \ \mathcal{H}_{n]}$$

because $\sum_{j\in D}(a^0_{Y_j} + a^-_{Y_j})\mathcal{D}_n \subseteq \mathcal{H}_{n]}$. This proves (2.24).

To prove the uniqueness of the decomposition (2.22), suppose that

$$Y_j = b^+_{Y_j} + b^0_{Y_j} + b^-_{Y_j}$$

is another decomposition **corresponding to the same gradation**. Then

$$0 = (a^+_{Y_j} - b^+_{Y_j}) + (a^0_{Y_j} - b^0_{Y_j}) + (a^-_{Y_j} - b^0_{Y_j})$$

Since the 3 operators in parentheses have mutually orthogonal ranges, their sum being zero is equivalent to each of them being zero separately. Thus the decomposition (2.22) is unique.

(v) Suppose that the sup in (2.25) is not $+\infty$. Then there is a $k \in \mathbb{N}$ such that $\mathcal{P}_k = \{0\}$ and this is equivalent to $P_k = 0$. Denoting $N_{(\mathcal{D},(P_{n]})_{n\in\mathbb{N}},\mathcal{D},Y)} + 1$ $(\leq k)$ the minimum k with this property, (2.15) implies that $N_{(\mathcal{D},(P_{n]})_{n\in\mathbb{N}},\mathcal{D},Y)}$ satisfies (2.26). □

Definition 2.3. The decomposition (2.22) is called the *quantum decomposition* associated to the triple $(\mathcal{H}, (P_{n]})_{n\in\mathbb{N}}, (Y_j, \mathcal{D}))_{j\in D}$ (or simply the quantum decomposition of Y_j).

The operators $a^+_{Y_j}$, $a^-_{Y_j}$, $a^0_{Y_j}$ appearing in the quantum decomposition (2.22) of Y_j are called respectively the *creation*, *annihilation*, *preservation (CAP)* operators associated to the triple $(\mathcal{H}, (P_{n]})_{n\in\mathbb{N}}, (Y_j, \mathcal{D}))_{j\in D}$.

The identity (2.22) is called the *symmetric* 3*–diagonal relation* associated to the triple $(\mathcal{H}, (P_{n]})_{n\in\mathbb{N}}, (Y_j, \mathcal{D}))_{j\in D}$.

3. Commutation Relations

Without loss of generality one can assume that the family $(Y_j)_{j\in D}$ is linearly independent. In the following this property will be assumed.

Denote V_Y the real linear span of the family $(Y_j)_{j\in D}$. If V is any real vector space isomorphic to V_Y and $e \equiv (e_j)_{j\in D}$ is a linear basis of V, the assignment of $(Y_j)_{j\in D}$ is equivalent to the assignment of a map

$$Y : v = \sum_{j\in D} v_j e_j \to Y_v = \sum_{j\in D} v_j Y_j$$

(with almost all the v_j equal to zero). The quantum decomposition (2.22) then guarantees the existence of three families of maps

$$a^+_j \,,\, a^0_j \,,\, a^-_j : V \to \text{ linear adjointable operators } \mathcal{D} \to \mathcal{H} \tag{3.1}$$

of which the a^+_j and a^0_j are complex linear and a^-_j complex anti–linear.

In the following we assume that products of the form $a^{\varepsilon_j}_j a^{\varepsilon_k}_k$ are well defined on $\mathcal{D}$ for each $j \in D$ and all $\varepsilon_j, \varepsilon_j \in \{+, -, 0\}$ (this is automatic if $|D| < +\infty$).

Moreover, we extend the CAP operators (3.1) to the complexification of V denoted $V_{\mathbb{C}}$ through the prescription that the maps a_j^+ and a_j^0 are complex linear. Therefore, a_v^- depends **anti–linearly** on $v \in V_{\mathbb{C}}$.

Finally, we assume that

$$\mathcal{H}_0 = \mathbb{C} \cdot \Phi_0 \qquad ; \qquad \|\Phi_0\| = 1$$

where the unit vector Φ_0, called **vacuum vector**, satisfies

$$a_v^- \Phi_0 = 0 \qquad \text{(Fock prescription)}$$

3.1. Type I commutation relations. Since the map $u \in V_{\mathbb{C}} \to a_u^+$ is **linear** in u, from

$$[a_v^-, a_u^+] = a_v^- a_u^+ - a_u^+ a_v^- =: \partial\Omega(v, u) \tag{3.2}$$

and the linearity (resp. anti–linearity) of the map $u \mapsto a_u^+$ (resp. $v \mapsto a_v^-$) it follows that, denoting $\mathcal{L}(\mathcal{D}, \mathcal{H})$ the space of linear maps $\mathcal{D} \to \mathcal{H}$,

$$(v, u) \in V_{\mathbb{C}} \times V_{\mathbb{C}} \to \partial\Omega(v, u) \in \mathcal{L}(\mathcal{D}, \mathcal{H})$$

is a **sesqui–linear** map from $V_{\mathbb{C}} \times V_{\mathbb{C}}$ to $\mathcal{L}(\mathcal{D}, \mathcal{H})$. Since both $a_v^- a_u^+$ and $a_u^+ a_v^-$ are gradation preserving, the same is true for each $\partial\Omega(v, u)$.

Putting $v = u$ and taking vacuum expectation value of (3.2), one finds

$$\langle \Phi_0, \partial\Omega(v, v)\Phi_0 \rangle = \langle \Phi_0, (a_v^- a_v^+ - a_v^+ a_v^-)\Phi_0 \rangle = \|a_v^+ \Phi_0\|^2 \geq 0$$

i.e.

$$\langle \Phi_0, \partial\Omega(v, v)\Phi_0 \rangle \geq 0 \tag{3.3}$$

This does not imply that $\partial\Omega(\,\cdot\,,\,\cdot\,)$ is positive definite. There are counter–examples even when $V_{\mathbb{C}}$ is 1–dimensional. However, **in one important particular case it does**! (see section 4 below).

3.2. Type II commutation relations. The following theorem was proved in [1]; we give here a slightly modified proof for completeness.

Theorem 3.1. *Let it be given:*
– a pre–Hilbert space H;
– an orthogonal gradation of H:

$$H = \bigoplus_{n \in \mathbb{N}} H_n;$$

– a family of operators $a_j^{\pm} : H_n \to H_{n\pm1}$, $a_j^0 : H_n \to H_n$, ($j \in \{1, \ldots, d\}$)

$$a_j^0 = (a_j^0)^* \qquad ; \qquad a_j^- = (a_j^+)^* \qquad ; \qquad j \in \{1, \cdots, d\}.$$

defined on a common dense domain $\mathcal{D}$ and such that the products of the form $a_j^{\varepsilon_j} a_k^{\varepsilon_k}$ are well defined on $\mathcal{D}$. Define the operators Y_j ($j \in \{1, \cdots, d\}$) on H by

$$Y_j := a_j^+ + a_j^0 + a_j^- , \qquad j \in \{1, \cdots, d\}. \tag{3.4}$$

Then the decomposition (3.4) is unique and the operators Y_j ***commute on*** *$\mathcal{D}$ if and only if the operators a_j^+, a_j^0, a_j^- satisfy the following commutation relations on the same domain: for all $j, k \in \{1, \cdots, d\}$ such that $j < k$*

$$[a_j^+, a_k^+] = 0 \tag{3.5}$$

$$[a_j^+, a_k^-] + [a_j^0, a_k^0] + [a_j^-, a_k^+] = 0 \tag{3.6}$$

$$[a_j^+, a_k^0] + [a_j^0, a_k^+] = 0 \tag{3.7}$$

Proof. Given (3.4), for each $j,\ k \in \{1, \cdots, d\}$, one has on $\mathcal{D}$:

$$\begin{aligned} 0 = [Y_j, Y_k] = [(a_j^+ + a_j^0 + a_j^-), (a_k^+ + a_k^0 + a_k^-)] = \\ = [a_j^+, a_k^+] \\ +[a_j^+, a_k^0] + [a_j^0, a_k^+] \\ +[a_j^+, a_k^-] + [a_j^0, a_k^0] + [a_j^-, a_k^+] \\ +[a_j^0, a_k^-] + [a_j^-, a_k^0] \\ +[a_j^-, a_k^-] \end{aligned} \tag{3.8}$$

The relations (3.5), (3.6) and (3.7) are identically satisfied for $j = k$ and, exchanging the roles of j and k, their left hand sides are transformed into their opposite. Therefore it is sufficient to check the validity of these relations for all $j, k \in \{1, \cdots, d\}$ such that $j < k$.

Starting with the first row, which has degree $+2$, each of the 5 rows in the right hand side of (3.8) contains operators of the same degree which is equal to the degree of the operators in the preceding row -1. The mutual orthogonality of the H_k then implies that operators on different rows are separately equal to zero.

Since the 5–th row is the adjoint of the first one and the 4–th row is equivalent to the adjoint of the second one, the vanishing of all the rows is equivalent to (3.5), (3.6), (3.7) for all $j, k \in \{1, \cdots, d\}$.

The uniqueness of the decomposition (3.4) is established as in the proof of Corollary 2.2. □

4. The Free Bose Field and Usual Quantum Mechanics

In the notations of section (3.1) let us consider the simplest gradation preserving operators i.e. the **multiples of identity**. If the operator valued sesqui–linear form $\partial\Omega(v, u) := [a_v^-, a_u^+]$ (see (3.2)) has this property, then its structure is

$$\partial\Omega(v, u) = \partial\omega(v, u) \cdot 1$$

with $\partial\omega(v, u) \in \mathbb{C}$. In this case the Type I CR (3.2) become:

$$\partial\Omega(v, u) = [a_v^-, a_u^+] = \partial\omega(v, u) \cdot 1 \qquad ; \qquad v, u \in \mathcal{D} \subseteq V_{\mathbb{C}} \tag{4.1}$$

and the positivity of the vacuum expectation value is strengthened into

$$0 \leq \|a_v^+\xi\|^2 = \langle \xi, \partial\omega(v, v)\xi \rangle = \partial\omega(v, v)\|\xi\|^2 \qquad ; \qquad \forall \xi \in \mathcal{D}$$

which is equivalent to the positive definiteness, on $\mathcal{D}$, of the sesqui–linear form

$$\langle v, u \rangle_{V_{\mathbb{C}}} := \partial\omega(v, u) \tag{4.2}$$

In other words, it defines a **semi–scalar product** $\langle \ \cdot\ ,\ \cdot\ \rangle_{V_{\mathbb{C}}}$ **on** $V_{\mathbb{C}}$.

Definition 4.1. The *free Fock Boson field over the Hilbert space* $(K, \langle\ \cdot\ ,\ \cdot\ \rangle_K)$ is the interacting Fock space on K

$$(H\ ,\ \langle\ \cdot\ ,\ \cdot\ \rangle) := \bigoplus_{n\in\mathbb{N}} (H_n, \langle\ \cdot\ ,\ \cdot\ \rangle_n) \tag{4.3}$$

with creator map

$$a^+\ :\ K \to \mathcal{L}_a\left((H\ ,\ \langle\ \cdot\ ,\ \cdot\ \rangle)\right)$$

characterized by the properties ($\widehat{\otimes}$ denoting symmetric tensor product):

$$(H_1, \langle\ \cdot\ ,\ \cdot\ \rangle_1) = (K, \langle\ \cdot\ ,\ \cdot\ \rangle_K) \quad , \quad H_n = K^{\widehat{\otimes} n} \quad , \quad \langle\ \cdot\ ,\ \cdot\ \rangle_n = \langle\ \cdot\ ,\ \cdot\ \rangle_K^{\widehat{\otimes} n} \tag{4.4}$$

$$[a_u, a_v^+] = \langle u, v\rangle_K\ ,\ \forall u, v \in K\ ;\ [a_u^+, a_v^+] = [a_u, a_v] = 0 \quad ; \quad a_v = (a_v^+)^* \tag{4.5}$$

Remark 4.2. The following theorem proves that the free Fock Boson field over the Hilbert space K exists and is uniquely defined, up to IFS isomorphism, by properties (4.4) and (4.5).

Theorem 4.3. *Let V be a real vector space and let $(v, u) \in V_{\mathbb{C}} \times V_{\mathbb{C}} \mapsto \partial\omega(v, u)$ be a semi–scalar product on the complexification of V, $V_{\mathbb{C}} \equiv V \dot{+} iV$, taking real values on $V \times V$ and not identically equal to zero.*

For any symmetric IFS over $V_{\mathbb{C}}$

$$\Gamma(V_{\mathbb{C}}) := \left(\bigoplus_{n\in\mathbb{N}} (V_{\mathbb{C}}^{\widehat{\otimes} n}\ ,\ \langle\ \cdot\ ,\ \cdot\ \rangle_n), a^+, \Phi_0\right)$$

the following statements are equivalent.
(i) The quantum field $v \in V_{\mathbb{C}} \mapsto a_v^{\pm}$ satisfies the CR:

$$\partial\Omega(v, u) := [a_v^-, a_u^+] = \partial\omega(v, u) \cdot 1 \qquad ; \qquad \forall u, v \in V_{\mathbb{C}} \tag{4.6}$$

(ii) The quantum field $v \in V_{\mathbb{C}} \mapsto a_v^{\pm}$ is the ***free Fock Boson field*** *over the Hilbert space $(V_{\mathbb{C},\partial\omega}\ ,\ \langle\ \cdot\ ,\ \cdot\ \rangle_{\partial\omega})$, obtained by completing $V_{\mathbb{C}}$ with the semi–scalar product $\partial\omega$.*
(iii) The quantum field in item (i) defines, through $u \in V \mapsto a_u^- + a_u^+$, the ***canonical quantum decomposition of the standard unit classical Gaussian field*** *on the real Hilbert space $(V_{\partial\omega}\ ,\ \langle\ \cdot\ ,\ \cdot\ \rangle_{\partial\omega_V})$, obtained by completing V with the semi–scalar product $\partial\omega_V$ where $\partial\omega_V$ is the restriction of $\partial\omega$ on V.*

Proof. That (i) $\Rightarrow$ (ii) follows from the definition of free Fock Boson field over the Hilbert space $(V_{\mathbb{C},\partial\omega}\ ,\ \langle\ \cdot\ ,\ \cdot\ \rangle_{\partial\omega})$.

That (ii) $\Rightarrow$ (iii) is a well known fact.

To prove that (iii) $\Rightarrow$ (i), suppose that $X : v \in V \mapsto X_v$ is the standard unit classical Gaussian field on the real Hilbert space $(V_{\partial\omega}\ ,\ \langle\ \cdot\ ,\ \cdot\ \rangle_{\partial\omega_V})$.

Notice that, denoting $L^2(X)$ the L^2–space of the process X, Gaussianity implies continuity of the map $X : v \in V \mapsto X_v \in L^2(X)$ for the norm topology on both spaces.

This, combined with linearity, implies that the classical field $v \in V \mapsto X_v$ is uniquely determined by the sequence (X_{e_j}), where $e \equiv (e_j)_{j\in D_0}$ is an ortho–normal basis of $(V_{\partial\omega}\ ,\ \langle\ \cdot\ ,\ \cdot\ \rangle_{\partial\omega_V})$.

For each $j \in D_0$, $\|e_j\| = 1$, hence each X_{e_j} standard Gaussian random variable. The orthogonality of the e_j implies that (X_{e_j}) is a sequence of independent identically distributed classical standard Gaussian random variables.

This implies that the L^2–space of the process can be identified with the space $L^2_{\mathbb{C}}\left(\prod_{\mathbb{N}} \mathbb{C} \ , \ \prod_{\mathbb{N}} \gamma\right)$ where γ is the standard Gaussian measure with unit variance. In its turn $L^2_{\mathbb{C}}\left(\prod_{\mathbb{N}} \mathbb{C} \ , \ \prod_{\mathbb{N}} \gamma\right)$ is canonically isomorphic to a tensor product of $|D_0|$ copies of $L^2_{\mathbb{C}}(\mathbb{R} \ , \ \gamma)$ (if $|D_0| = +\infty$ the tensor product is the incomplete tensor product in the sense of von Neumann with respect to the constant function equal to 1) and the distribution of X becomes a tensor product of standard Gaussians on $\mathbb{R}$. Identifying each X_{e_j} with the operator of multiplication by X_{e_j} in $L^2_{\mathbb{C}}(\mathbb{C} \ , \ \gamma)$, the quantum decomposition of the field X is uniquely determined by the quantum decomposition of the classical random variables

$$X_{e_j} = a^+_{e_j} + a^-_{e_j} \qquad ; \qquad j \in D_0$$

where the preservation operator $a^0_{e_j}$ is absent because the standard Gaussian is symmetric (see [7]). Moreover, since the measure defining $L^2_{\mathbb{C}}\left(\prod_{\mathbb{N}} \mathbb{C} \ , \ \prod_{\mathbb{N}} \gamma\right)$ is a product measure, we know from cite[AcKuoSta09-IFS] that the CAP operators a^{ε}_j ($\varepsilon \in \{+, -\}$) associated to different indices commute, i. e.

$$[a^-_{e_i}, a^+_{e_j}] = \delta_{i,j}(\omega_{\Lambda_j+1} - \omega_{\Lambda_j}) =: \delta_{i,j}\partial\omega_{\Lambda_j} \tag{4.7}$$

where (ω_n) is the principal Jacobi sequence of the standard Gaussian on $\mathbb{R}$ and Λ_j is the number operator on $L^2(X_{e_j})$. But it is known that the principal Jacobi sequence of the standard Gaussian measure on $\mathbb{R}$ is given by $\omega_n = n$ for all n. With this choice of the ω_n, for any $j \in D_0$, $\omega_{\Lambda_j+1} - \omega_{\Lambda_j} = \Lambda_j + 1 - \Lambda_j = 1$ and (4.7) becomes

$$[a^-_{e_i}, a^+_{e_j}] = \delta_{i,j} \cdot 1$$

Given the definition of the e_j, this is equivalent to (4.6). □

Remark 4.4. If $V_{\mathbb{C}} = L^2_{\mathbb{C}}(\mathbb{R}^d)$, (4.6) becomes

$$[a^-_v, a^+_u] = \langle v, u\rangle_{V_{\mathbb{C}}} \cdot 1 \quad ; \quad v, u \in L^2_{\mathbb{C}}(\mathbb{R}^d) \tag{4.8}$$

and one recognizes in (4.8) the first relation defining the Heisenberg commutation relations for the free non–relativistic Boson field on $L^2_{\mathbb{C}}(\mathbb{R}^d)$. The relation,

$$[a^+_v, a^+_u] = 0 \quad ; \quad v, u \in L^2_{\mathbb{C}}(\mathbb{R}^d) \tag{4.9}$$

follows, in a much more general framework, from of the Type II commutation relations (see (3.5)).

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