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Quantum Theory of Gravitation vs. Classical Theory*)

——*Fourth-Order Potential*——

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The perihelion-motion of Mercury depends on the fourth-order potential in quantum field theory; it is a "Lamb shift". In spite of the unrenormalizability of the theory, we have extracted a finite and physically meaningful quantity, a fourth-order potential, from fourth-order graphs. We have also discussed briefly renormalization of the Newtonian potential in the fourth-order perturbation.

The Hamiltonian obtained is the same as the classical one and so it cannot explain the Dicke-Goldenberg experiment.

We have calculated fourth-order potential also in Q.E.D.

§ 1. Introduction

In constructing a quantum theory of gravitation there are two main problems: One is to formulate it *consistently* and the other is to describe gravitational phenomena *correctly* by it.

The former includes, for example, maintaining unitarity, Lorentz invariance and gauge invariance, and the removal of divergences. In the covariant formalism[1),2),3)] which we use in this paper this problem has been resolved except for the removal of divergences. Since the quantum theory of gravitation is unrenormalizable by standard criteria, almost nothing is known about how to remove the divergences.

Concerning the latter problem, before predicting new effects quantatively we should first describe correctly classical phenomena including three famous tests. Here by classical phenomena is meant the phenomena which we can describe correctly even in the limit $h \to 0$. In this paper we want to discuss particularly the perihelion motion of Mercury in the framework of quantum theory, since the other two tests have been much discussed elsewhere and can be easily explained in the framework of quantum theory.[4)]

As will be discussed in § 2, the perihelion motion depends on fourth-order gravitational potential. In other words it is a kind of "Lamb shift" and in the case of Mercury the magnitude of the energy shift $\Delta E/E$ is of the order 10^{-8}.

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*) Some of the material of this article has been discussed briefly in a recent letter [Y. Iwasaki, Lett. Nuovo Cim. 1 (1971), 783]. We will repeat a few points here, in order that the present article be completely self-contained.

So it is not obvious whether quantum theory predicts it to be the same as c-number theory. Moreover it is not clear whether we can indeed obtain finite physically meaningful quantities from fourth-order Feynman diagrams, since the quantum theory of gravitation is unrenormalizable as stated above. It will be shown that in spite of the unrenormalizability we can obtain a finite physically meaningful potential. We will also discuss briefly the renormalization of the Newtonian potential by the fourth-order perturbation.

On the other hand it was believed for a long time that the experimental value and the theoretical value resulting from Einstein's theory for the perihelion motion of Mercury are in excellent agreement. However, Dicke and Goldenberg[5] have observed the oblateness of the Sun and have concluded that if we take into account the quadrupole moment of the Sun, Einstein's theory predicts 8% excess in the value of the perihelion motion of Mercury. Therefore we want to discuss also the question of whether or not we can explain the value obtained from the Dicke-Goldenberg experiment in quantum theory.

Since our primary interest in this paper is whether the quantum theory of gravitation gives *in principle* the same result for the perihelion motion as classical theory, we will deal with the two-body problem between scalar particles and between Dirac particles. If quantum theory gives the same results as classical theory for both cases, it is probable that quantum theory gives *in general* the same result. At the present stage, however, we have no rigorous proof that it gives *in general* the same result. We will briefly discuss the case of celestial bodies in § 4.

Even if there were to exist a formal proof that quantum theory gives the same result, it would be another matter to show that the perturbation calculation really gives the same result on account of the existence of a closed loop: The Ward identity, for example, for the axial-vector current does not hold in perturbation theory.[6]

One of the problems we have discussed above is whether quantum theory predicts the same fourth-order potential which does not contain Planck's constant h as classical theory. Since, at present, we have no prescription for what cases it does predict the same result, we must examine by calculation whether it does or not in other cases. In Q.E.D., for example, this fourth-order potential means that of the type $e^4/c^2r^2\cdot 1/m$. In classical theory the potential is rigorously given by the retarded coulomb potential and does not contain terms of the type $e^4/c^2r^2 \times 1/m$. Whether fourth-order diagrams in Q.E.D. predict a zero potential of the type $e^4/c^2r^2\cdot 1/m$ is the question which we are discussing. A discussion of this "Correspondence Principle" is also given in § 2.

We will perform the detailed calculation in the case of gravitation taking the scalar field as the matter field in § 3. The proof of one equation in § 3 is given in Appendix 1. The calculation in the case of gravitation taking the Dirac field as the matter field is briefly given in Appendix 2. The calculation in the cases of Q.E.D., is briefly given in Appendix 3. Section 4 will be devoted to conclusions and discussion.

§ 2. Perihelion motion and fourth-order gravitational potential

Correspondence principle in quantum field theory

The motion of the perihelion of a planet is described in Einstein's theory of gravitation by the Hamiltonian

$$H=\frac{p^2}{2m}-\frac{p^4}{8c^2m^3}-\frac{kmM}{r}-\frac{3kMp^2}{2c^2mr}+\frac{k^2mM^2}{2c^2r^2} \tag{2·1}$$

if we choose harmonic coordinates[7] and expand the Hamiltonian up to the order c^{-2}. The extension of Eq. (2·1) to the two-body problem is given by[8]

$$H=\frac{1}{2}\left(\frac{p_1^2}{m_1}+\frac{p_2^2}{m_2}\right)-\frac{1}{8c^2}\left(\frac{p_1^4}{m_1^3}+\frac{p_2^4}{m_2^3}\right)-\frac{km_1m_2}{r}$$
$$-\frac{km_1m_2}{2c^2r}\left[3\left(\left(\frac{p_1}{m_1}\right)^2+\left(\frac{p_2}{m_2}\right)^2\right)-\frac{7\boldsymbol{p}_1\boldsymbol{p}_2}{m_1m_2}-\frac{(\boldsymbol{p}_1\boldsymbol{r})(\boldsymbol{p}_2\boldsymbol{r})}{m_1m_2r^2}\right]+\frac{k^2m_1m_2(m_1+m_2)}{2c^2r^2} \tag{2·2}$$

from which we can obtain the Einstein-Infeld-Hoffmann equation. In this paper we want to discuss the question whether Hamiltonians (2·1) and (2·2) are the same in quantum theory.

To obtain the value θ of the ratio of the perihelion-motion using the Hamiltonian, we regard

$$H_0=\frac{p^2}{2m}-\frac{kmM}{r} \tag{2·3}$$

as a unperturbed Hamiltonian and

$$H_I=-\frac{p^4}{8m^3c^2}-\frac{3kMp^2}{2c^2mr}+\frac{k^2mM^2}{2c^2r^2} \tag{2·4}$$

as a perturbation. Then the three terms of Eq. (2·4) are the same order, because $p^2/2m\sim\varepsilon+kmM/r$, where ε is the unperturbed energy. Thus Eq. (2·1) is essentially the same as

$$H'=\frac{p^2}{2m}-\frac{k'mM}{r}-3\frac{k^2mM^2}{c^2r^2} \tag{2·5}$$

for the purpose of calculating the ratio of the perihelion-motion. From Eq. (2·5) we can easily obtain

$$\theta_{\mathrm{Einst}}=\frac{6\pi kM}{c^2a(1-e^2)}, \tag{2·6}$$

where a is the semi-major diameter and e is the eccentricity of the orbit. If the

Hamiltonian is

$$H=\frac{p^2}{2m}-\frac{p^4}{8c^2m^3}-\frac{kmM}{r}+\lambda_1\frac{kMp^2}{c^2mr}+\lambda_2\frac{k^2mM^2}{c^2r^2} \tag{2·7}$$

instead of Eq. (2·1), then the value θ is given by

$$\theta=\tfrac{1}{3}(\tfrac{1}{2}-2\lambda_1-\lambda_2)\theta_{\mathrm{E\,inst}}\,. \tag{2·8}$$

The essential point in the previous paragraph is that the term quadratic in k is the same order as the velocity-dependent k-linear term. In quantum theory the potential linear in k corresponds to the second order graph, while the potential quadratic in k corresponds to the fourth-order potential.

Here we want to point out that there seems to exist an erroneous belief*),**) that only tree diagrams contribute to the classical process. Contrary to this belief, the quadratic term in k corresponds to fourth-order diagrams each of which contains a closed loop; it is a "radiative correction" term. Since the quantum theory of gravitation is unrenormalizable by standard criteria, almost nothing is known[8] about how to extract finite and physically meaningful radiative corrections from the results in higher order. We will extract a meaningful term as fourth-order potential.

Although usually the effect of the perturbation is described by the value of the perihelion-motion, for this value can be obtained from the observation, we can estimate the effect by the energy shift as the same as in Q.E.D. Then this energy shift $\Delta E/E$ is of the order 10^{-8} which is the order v^2/c^2.

On the other hand there seems to exist an argument that the quantum theory should coincide with the classical theory in the classical limit because both are invariant with respect to the general coordinate transformation. As has been pointed out by Fock,[7] however, the invariance with respect to general coordinate transformation is not a strong constraint. If we choose harmonic coordinates, there remains only Lorentz invariance. We can indeed prove that the equation of motion derived from Eq. (2·2) is Lorentz covariant up to the approximation considered above *irrespective of the numerical factor of the last term of Eq. (2·2).*

We want to emphasize that by classical theory is meant only "c-number" theory here which is not completely established by classical observations except for the Newtonian potential. If we assume that quantum field theory is more fundamental than c-number theory, then there is a possibility, in principle, that quantum field theory describes correctly experiment and that c-number theory

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*) E. Corinaldesi[9] said that he could obtain the Einstein-Infeld-Hoffmann equation only from the second order graph. His calculations are incorrect. We obtain using his linear theory $\ddot{\boldsymbol{x}}=k^2m_2\boldsymbol{x}/c^2r^4\cdot(3m_2+4m_1)$ instead of his equation (28) $\ddot{\boldsymbol{x}}=k^2m_2\boldsymbol{x}/c^2r^4\cdot(4m_2+5m_1)$. This paper has been quoted by several authors.

**) The statement that only tree diagrams contribute to classical process is given in several places, e.g., in Ref. 2).

does not.

This is quite different from the situation in quantum mechanics. There a potential is given and under the potential the motion of a particle is the same as in classical mechanics in the limit $h \to 0$. In short the Correspondence Principle holds in quantum mechanics and should do since classical mechanics is well established in a given potential.

Here, however, the situation is quite different. We are calculating a potential, the type and the strength of which is not well established by experiment. So there is no reason *a priori* from either experiment or theory to expect that the perihelion advance be predicted to be the same in the two approaches.

§ 3. Calculation of the potential*)

Let us calculate explicitly the fourth-order potential from quantum theory in order to answer the questions mentioned in § 2 in the following steps: (i) We construct the Lagrangian; (ii) we calculate the fourth-order S-matrix in the momentum representation; (iii) we integrate over the energy variable of a closed loop by means of the contour method; (iv) we expand the result in terms of the inverse of the two masses m_1 and m_2; (v) by Fourier transformation we obtain the r-representation; (vi) we finally extract the potential of the form k^2m^3/c^2r^2. From the resulting potential we must subtract the second Born term using an expansion up to the order c^{-2}.

Perhaps it is appropriate to remark here the following: In general we cannot define a potential uniquely from the S-matrix but we can from a Green-function; in general we cannot use the Feyman gauge in bound state problem but we must use the Coulomb gauge in Q.E.D. The fourth-order potential to be obtained is, however, of the form k^2m^3/c^2r^2 which is independent of momentum and so we can obtain uniquely the fourth-order potential from the fourth-order S-matrix. On the other hand when we extract the second-order momentum-dependent potential, we must take note of the above remark. Later we will show how to obtain it.

As we have stated in Introduction, we calculate the potential in the cases of the scalar field and the Dirac field. Hereafter we will discuss mainly the calculation for the case of the scalar field, since the calculation is similar in the case of the Dirac field (see, Appendix 2).

3.1 *Lagrangian*

We start with the general coordinate transformation invariant Lagrangian density

*) We put $c=1$ and $h=1$ in the calculation and write explicitly c and h only in the resulting potential, since we can easily count the powers of c and h in the results by means of a dimension analysis.

$$\mathcal{L}=-\frac{1}{16\pi k}\sqrt{g}R-\frac{1}{2}\sqrt{g}(\partial_\mu\phi\partial_\nu\phi g^{\mu\nu}+m^2\phi^2), \tag{3·1}$$

where $g=-\det(g_{\mu\nu})$ and R is the scalar curvature, and expand it in powers of the gravitation constant by substituting[2),3)]

$$g_{\mu\nu}=\delta_{\mu\nu}+\kappa h_{\mu\nu}, \qquad \text{(expansion } F\text{)} \tag{3·2}$$

where $h_{\mu\nu}$ is the field of the graviton and $\kappa^2=32\pi k$.

There are infinitely many ways at defining the field of the graviton, for example,

$$g^m g_{\mu\nu}=\delta_{\mu\nu}+\kappa h^{(m)}_{\mu\nu} \tag{3·3}$$

or

$$g^n g^{\mu\nu}=\delta_{\mu\nu}+\kappa h_{\mu\nu(n)}; \tag{3·4}$$

among these we choose the type of (3·2) for the sake of convenience. They are all equivalent because of the S-matrix equivalence theorem. We calculate also the case[1)]

$$\sqrt{g}g^{\mu\nu}=\delta_{\mu\nu}+\kappa h_{\mu\nu} \qquad \text{(expansion } G\text{)} \tag{3·5}$$

to check the calculations.

The method for obtaining such a Lagrangian from the viewpoint of particle physics will be presented in detail in a separate paper.[10)]

Then we write down explicitly the Lagrangian up to the order which is necessary to calculate the fourth-order potential. In the case of expansion F we obtain

$$L(\varphi^2)=-\tfrac{1}{2}(\partial_\mu\varphi\partial_\mu\varphi+m^2\varphi^2), \tag{3·6}$$

$$L(h^2)=-\tfrac{1}{2}\{h_{\rho\lambda,\sigma}h_{\rho\lambda,\sigma}-2h_{\sigma\lambda,\rho}h_{\sigma\rho,\lambda}+2h_{\lambda\lambda,\rho}h_{\sigma\rho,\sigma}-h_{\lambda\lambda,\rho}h_{\sigma\sigma,\rho}\}, \tag{3·7}$$

$$L(h\varphi^2)=\frac{\kappa}{2}\left[h_{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi-\frac{1}{2}h_{\rho\rho}(\partial_\lambda\varphi\partial_\lambda\varphi+m^2\varphi^2)\right], \tag{3·8}$$

$$\begin{aligned}L(h^3)=\frac{\kappa}{2}\Big[&h_{\tau\tau}\Big\{h_{\sigma\lambda,\rho}h_{\sigma\rho,\lambda}-\frac{1}{2}h_{\rho\lambda,\sigma}h_{\rho\lambda,\sigma}-h_{\lambda\lambda,\rho}h_{\sigma\rho,\sigma}+\frac{1}{2}h_{\lambda\lambda,\rho}h_{\sigma\sigma,\rho}\Big\}\\&+h_{\mu\nu}\{h_{\rho\lambda,\mu}h_{\rho\lambda,\nu}-2h_{\mu\lambda,\rho}h_{\nu\rho,\lambda}+2h_{\mu\lambda,\rho}h_{\nu\lambda,\rho}+2h_{\lambda\lambda,\rho}h_{\mu\rho,\nu}\\&-2h_{\lambda\lambda,\rho}h_{\mu\nu,\rho}+2h_{\lambda\mu,\lambda}h_{\rho\rho,\nu}-2h_{\lambda\lambda,\mu}h_{\rho\rho,\nu}+2h_{\lambda\rho,\lambda}h_{\mu\nu,\rho}-4h_{\lambda\mu,\rho}h_{\lambda\rho,\nu}\}\Big],\end{aligned} \tag{3·9}$$

$$\begin{aligned}L(h^2\varphi^2)=-\frac{\kappa^2}{2}\Big[&h_{\mu\rho}h_{\nu\rho}\partial_\mu\varphi\partial_\nu\varphi-\frac{1}{2}h_{\rho\rho}h_{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\\&+\left(\frac{1}{8}h_{\rho\rho}h_{\lambda\lambda}-\frac{1}{4}h_{\rho\lambda}h_{\rho\lambda}\right)(\partial_\rho\varphi\partial_\rho\varphi+m^2\varphi^2)\Big],\end{aligned} \tag{3·10}$$

where $h_{\alpha\beta,\gamma} \equiv \partial h_{\alpha\beta}/\partial x_\gamma$ and $L(h^m\phi^n)$ means the part of the Lagrangian with $h_{\mu\nu}$ raised to the power m and ϕ to the power n $(n=0$ or $2)$.

3.2 *S-matrix and potential*

First we mention the method for obtaining the S-matrix from a Lagrangian. In Q.E.D. it is

$$S = T \exp\left(i \int L_I d^4x\right), \tag{3·11}$$

where T is the time ordered operator and L_I is the interaction Lagrangian.

In the quantum theory of gravity the answer is not so simple as (3·11) and it is not even

$$S = T^* \exp\left(i \int L_I d^4x\right), \tag{3·12}$$

where T^* is the time ordered operator which commutes with the derivative of the L_I.

The fact that Eq. (3·12) is not right for the quantum theory of gravity was first pointed out by Feynman.[2] The right formula is[3]

$$S = T^* \exp\left(i \int L_I{}^* d^4x\right), \tag{3·13}$$

where

$$L_I{}^* = L_I + L_a\,, \tag{3·14}$$

and

$$L_a = -i \operatorname{Tr} \ln \{\delta_{\mu\nu} + [h_{\mu\nu}\Box + (\partial_\alpha h_{\beta\mu} - \tfrac{1}{2}\partial_\mu h_{\beta\alpha}) \times (\delta_{\sigma\alpha}\partial_\sigma\delta_{\nu\beta} + \delta_{\sigma\beta}\partial_\sigma\delta_{\nu\alpha} - \delta_{\alpha\beta}\partial_\nu)]\Box\}\,. \tag{3·15}$$

Thus the fourth-order Feynman graph consist of the usual Feynman graphs some of which are represented in Fig. 1 and an additional graph which is similar to Fig. 1(e), with fictious quanta running around a loop.

Using Eq. (3·13), the propagator for the graviton

$$\Delta_{\mu_1\mu_2,\nu_1\nu_2}(k^2) = -\frac{i}{(2\pi)^4}\cdot\frac{1}{k^2}\cdot\frac{1}{2}(\delta_{\mu_1\nu_1}\delta_{\mu_2\nu_2} + \delta_{\mu_1\nu_2}\delta_{\mu_2\nu_1} - \delta_{\mu_1\mu_2}\delta_{\nu_1\nu_2}), \tag{3·16}$$

and the propagator for the scalar particle, we can calculate the S-matrix.

On the other hand with a given potential $V(r)$ which may in general depend on the momentum, we obtain the S-matrix

$$S = 1 - 2\pi i\delta(E_i - E_f)\langle f|V|i\rangle - 2\pi i\delta(E_i - E_f)\frac{\langle f|V|n\rangle\langle n|V|i\rangle}{E_i - E_n} + \cdots\,. \tag{3·17}$$

If we denote the incoming and the outgoing momenta of particle 1 and 2 by p_1, p_2, q_1 and q_2, respectively and normalize the state vector by

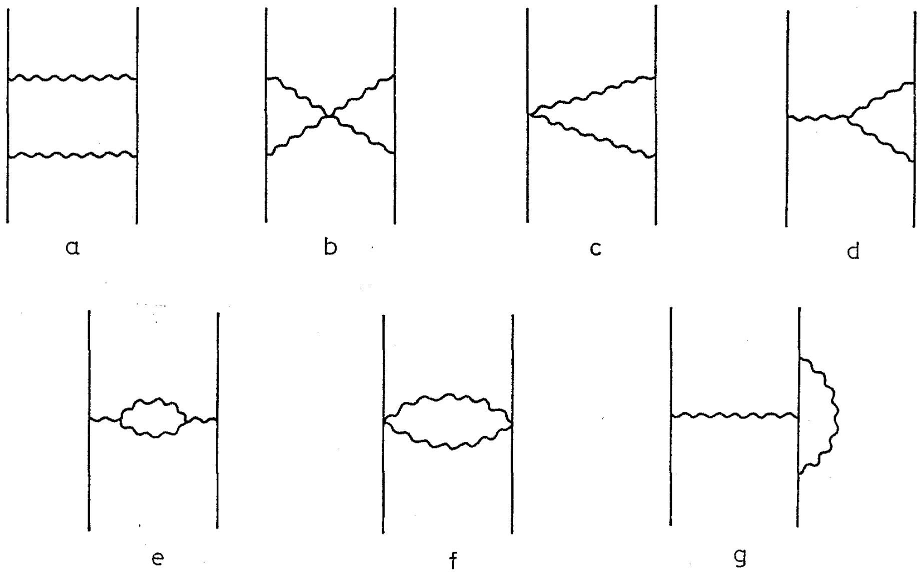

Fig. 1. Some of the fourth-order diagrams. The wavy line represents the graviton and the solid line matter.

$$(p_2q_2|p_1q_1)=(2\pi)^3\delta^{(3)}(p_1-p_2)\,(2\pi)^3\delta^{(3)}(q_1-q_2)\,, \tag{3·18}$$

then

$$(p_2q_2|V|p_1q_1)=(2\pi)^3\delta^{(3)}(p_1+q_1-p_2-q_2)\int V(r)e^{i\boldsymbol{k}\boldsymbol{r}}d^3r\,. \quad (\boldsymbol{k}=\boldsymbol{p}_1-\boldsymbol{p}_2) \tag{3·19}$$

If we write the S-matrix obtained from a Feynman graph as

$$S=1-i(2\pi)^4\delta^{(4)}(p_1+q_1-p_2-q_2)\,T(\boldsymbol{k})\,, \tag{3·20}$$

then we obtain from Eqs. (3·17) and (3·19)

$$V(r)=\frac{1}{(2\pi)^3}\int T(\boldsymbol{k})e^{-i\boldsymbol{k}\boldsymbol{r}}d^3k\,, \tag{3·21}$$

for the contribution from each graph except graph a.

To obtain the contribution from graph a, we must subtract from Eq. (3·20) the second Born term, the third term of Eq. (3·17) where V is of the order κ^2.

3. 3 *Second order potential*

In order to calculate the second order momentum-dependent potential we must take note of the remark made in the second paragraph of this section. As is stated in Appendix 3, the use of the Coulomb gauge for calculating the Green-function in Q.E.D. is equivalent to assuming

$$\frac{1}{k^2}=\frac{1}{\boldsymbol{k}^2}+\frac{1}{4m_1m_2c^2}\cdot\frac{\boldsymbol{k}\cdot(\boldsymbol{p}_1+\boldsymbol{p}_2)\,\boldsymbol{k}\cdot(\boldsymbol{q}_1+\boldsymbol{q}_2)}{(\boldsymbol{k}^2)^2}\,, \tag{3·22}$$

up to order c^{-2}.

If we use the radiation gauge also in the quantum theory of gravity, we may obtain again Eq. (3·22). Using Eq. (3·22) we arrive at the S-matrix,

$$S=-i(2\pi)^4\delta^4(\ )(-4\pi k)m_1m_2\left[\frac{1}{\boldsymbol{k}^2}\left\{1+\frac{4}{3}\left(\frac{\boldsymbol{p}_1^2+\boldsymbol{p}_2^2}{m_1^2c^2}+\frac{\boldsymbol{q}_1^2+\boldsymbol{q}_2^2}{m_2^2c^2}\right)-\frac{(\boldsymbol{p}_1+\boldsymbol{p}_2)(\boldsymbol{q}_1+\boldsymbol{q}_2)}{m_1m_2c^2}\right\}+\frac{\boldsymbol{k}\cdot(\boldsymbol{p}_1+\boldsymbol{p}_2)\,\boldsymbol{k}\cdot(\boldsymbol{q}_1+\boldsymbol{q}_2)}{4m_1m_2c^2(\boldsymbol{k}^2)^2}\right], \tag{3·23}$$

from which we can obtain using Eq. (3·21)

$$V=-\frac{km_1m_2}{r}\left[1+\frac{3}{2c^2}\left(\frac{p^2}{m_1^2}+\frac{q^2}{m_2^2}\right)-\frac{7(\boldsymbol{pq})}{2m_1m_2c^2}-\frac{(\boldsymbol{pn})(\boldsymbol{qn})}{2m_1m_2c^2}\right]. \quad \left(\boldsymbol{n}=\frac{\boldsymbol{r}}{r}\right) \tag{3·24}$$

We have used the equations

$$\frac{1}{(2\pi)^3}\int\frac{1}{\boldsymbol{k}^2}e^{-i\boldsymbol{kr}}d^3k=\frac{1}{4\pi r} \tag{3·25}$$

and

$$\frac{1}{(2\pi)^3}\int\frac{k_ik_j}{(\boldsymbol{k}^2)^2}e^{-i\boldsymbol{kr}}d^3k=\frac{1}{4\pi}\cdot\frac{1}{2r}\left(\delta_{ij}-\frac{x_ix_j}{r^2}\right). \tag{3·26}$$

Now we proceed to calculate the fourth-order potential from the diagrams (c), (d), (a), (b) and others, which amounts to starting with the simplest case and then moving on to more difficult cases.

3.4 *Contribution from graph 1(c)*

As parts of the Lagrangian $L(h\phi^2)$ and $L(h^2\phi^2)$ we take the first term of Eqs. (3·8) and (3·10), respectively, then

$$\mathcal{L}_I=\frac{\kappa}{2}h_{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi-\frac{\kappa^2}{2}h_{\mu\rho}h_{\nu\rho}\partial_\mu\varphi\partial_\nu\varphi\,. \tag{3·27}$$

If we represent momenta as Fig. 2, we obtain

$$S=-\delta^4(\ )\frac{1}{\sqrt{2p_{10}2q_{10}2p_{20}2q_{20}}}\left(\frac{\kappa}{2}\right)^2\frac{\kappa^2}{2}\int\frac{d^4l}{l^2(k-l)^2(p_3^2+m^2)}[(p_1)_\alpha(p_3)_\beta+(p_1)_\beta(p_3)_\alpha]$$
$$\times[(p_3)_\gamma(p_2)_\delta+(p_3)_\delta(p_2)_\gamma][(q_1)_\mu(q_2)_\nu+(q_1)_\nu(q_2)_\mu]\left[\frac{(\mu\rho,\alpha\beta)}{2}\cdot\frac{(\nu\rho,\gamma\delta)}{2}\times 2\right], \tag{3·28}$$

where $(\mu\rho,\alpha\beta)=\delta_{\mu\alpha}\delta_{\rho\beta}+\delta_{\mu\beta}\delta_{\rho\alpha}-\delta_{\mu\rho}\delta_{\alpha\beta}$ and $p_3=p_1+l$.

Expanding the S-matrix in terms of the inverse of the two masses m_1 and m_2, we find the leading term is of order m^3. So it is possible to put $\alpha=\beta=\gamma=\delta=\mu=\nu=4$ and $(p_1)_4=im_1$, etc. Then we integrate over the energy variable l_0 by means of the contour method and obtain

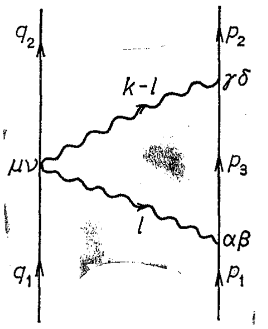

Fig. 2. Details of graph 1(c).

$$\int \frac{d^4 l}{[l^2-i\varepsilon][(k-l)^2-i\varepsilon][(p+l)^2+m_1^2-i\varepsilon]}$$
$$=2\pi i\frac{1}{4m_1}\int\frac{d^3 l}{\boldsymbol{l}^2(\boldsymbol{k}-\boldsymbol{l})^2}+O\left(\frac{1}{m_1^2}\right). \quad (3\cdot 29)$$

Now the S-matrix becomes

$$S=-i(2\pi)^4\delta^4(\ )\left[-\frac{m_1^2 m_2}{2}\left(\frac{\kappa}{2}\right)^4\frac{1}{(2\pi)^3}\right.$$
$$\left.\times\int\frac{1}{\boldsymbol{l}^2(\boldsymbol{k}-\boldsymbol{l})^2}d^3 l\right]+O(m^2). \quad (3\cdot 30)$$

Using

$$\frac{1}{(2\pi)^6}\iint\frac{1}{\boldsymbol{l}^2(\boldsymbol{k}-\boldsymbol{l}^2)^2}d^3 l e^{-i\boldsymbol{k}\boldsymbol{r}}d^3 k=\frac{1}{(4\pi)^2}\cdot\frac{1}{r^2} \quad (3\cdot 31)$$

and $\kappa^2=32\pi k$, we obtain

$$V=-\frac{2k^2 m_1^2 m_2}{c^2 r^2}. \quad (3\cdot 32)$$

Similarly we obtain as a contribution from the second term of Eq. (3·10) and the first term of Eq. (3·8) $V=-2k^2 m_1^2 m^2/r^2$. Since there are no other contributions, we finally obtain as a contribution from graph 1(c) after symmetrizing m_1 and m_2

$$V=-4k^2\frac{m_1 m_2(m_1+m_2)}{c^2 r^2}. \quad (3\cdot 33)$$

3.5 *Contribution from graph 1(d)*

As an example we take

$$\mathcal{L}_I=\frac{\kappa}{2}h_{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi+\frac{\kappa}{2}h_{\mu\nu}h_{\rho\lambda,\mu}h_{\rho\lambda,\nu}, \quad (3\cdot 34)$$

and denote the momenta as in Fig. 3, then

$$S=-\delta^4(\ )\frac{1}{\sqrt{2p_{10}2q_{10}2p_{20}2q_{20}}}\left(\frac{\kappa}{2}\right)^4\frac{1}{k_1^2}\int\frac{d^4 k_2}{k_2^2 k_3^2(p_3^2+m_1^2)}(p_{1\alpha}p_{3\beta}+p_{1\beta}p_{3\alpha})$$
$$\times(p_{3\gamma}p_{2\delta}+p_{3\delta}p_{2\gamma})(q_{1\varepsilon}q_{2\kappa}+q_{1\kappa}q_{2\varepsilon})\frac{(\alpha\beta,\rho\lambda)}{2}\frac{(\gamma\delta,\rho\lambda)}{2}\frac{(\varepsilon\kappa,\mu\nu)}{2}(k_2)_\mu(k_3)_\nu. \quad (3\cdot 35)$$

Changing k_1, k_2 and k_3 cyclicly, we obtain six similar expressions in total. If Eq. (3·35) contains a term like $(k_2 p)(k_3 p)\cdots$, then there are also terms like $(k_3 p)(k_1 p)\cdots$ and $(k_1 p)(k_2 p)\cdots$.

Since the leading term is proportional to m^3 as in the case of graph 1(c), we may put $\alpha=\beta=\gamma=\delta=\varepsilon=\kappa=4$ and $(p_1)_4=im_1$, etc. In doing so, the term like $(k_2 p)(k_3 p)\cdots$ becomes $(k_2)_0(k_3)_0 m^2$. As $k_{10}=0$ in the c.m. system, we may put $k_{10}=0$, because we are calculating momentum-independent potential. Then $k_{30}=k_{10}+k_{20}\to k_{20}$, $k_{10}k_{20}\to 0$, $k_{10}k_{30}\to 0$, $k_{20}k_{30}\to k_{20}k_{20}$. On the other hand

$$\int\frac{l_0^2 dl_0 d^3l}{(\boldsymbol{l}^2-l_0^2-i\varepsilon)\,[(\boldsymbol{k}-\boldsymbol{l})^2-(k_0-l_0)^2-i\varepsilon]\,[(\boldsymbol{p}+\boldsymbol{l})^2-(p_0+l_0)^2+m^2-i\varepsilon]}=O\left(\frac{1}{m}\right). \tag{3·36}$$

Thus the term like $(k_2p)\,(k_3p)\cdots$ does not contribute to the fourth-order potential, but the term like $(k_2k_3)\,(pp)\,(pq)\cdots$ does.

As the term $k_{20}k_{30}$ does not contribute to the potential, we may put $(k_2k_3)\to\boldsymbol{k}_2\boldsymbol{k}_3$. Changing k_1, k_2 and k_3 cyclicly we obtain $2(\boldsymbol{k}_2\boldsymbol{k}_3+\boldsymbol{k}_3\boldsymbol{k}_1+\boldsymbol{k}_1\boldsymbol{k}_2)=-(\boldsymbol{k}_1^2+\boldsymbol{k}_2^2+\boldsymbol{k}_3^2)$.

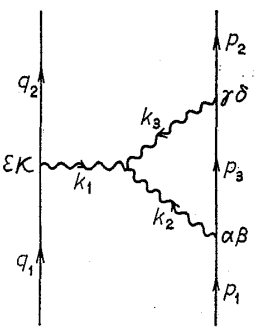

Fig. 3. Details of graph 1(d).

First we discuss the term $\boldsymbol{k}_2^2$. In this case the integral becomes

$$\int\frac{\boldsymbol{k}_2^2 d^3k_3 d(k_3)_0}{k_2^2k_3^2[(p_2-k_3)^2+m_1^2]}\sim\frac{1}{m_1}\int\frac{d^3l}{\boldsymbol{l}^2}, \tag{3·37}$$

and so the S-matrix is given by

$$S\sim\kappa^4 m_1^2 m_2\frac{1}{\boldsymbol{k}^2}\int\frac{d^3l}{\boldsymbol{l}^2} \tag{3·38}$$

to order m^3. The contribution from the term $\boldsymbol{k}_3^2$ is the same as the term $\boldsymbol{k}_2^2$.

Since the integral

$$I=\int\frac{d^3l}{\boldsymbol{l}^2} \tag{3·39}$$

is linearly divergent and is independent of $\boldsymbol{k}$, we cut off the integral and express it by a constant Λ. As the momentum dependence is given by $1/\boldsymbol{k}^2$, the corresponding potential is given by

$$V\sim\frac{k^2m_1^2m_2\Lambda}{r}=k\frac{m_1m_2}{r}(km_1\Lambda), \tag{3·40}$$

which is of the same type as the Newtonian potential. So we must renormalize it, but we cannot renormalize it as we did the gravitational constant, for it depends on m_1. The unique solution is to renormalize it as a mass renormalization. Then there is the problem of whether this renormalization is consistent with the renormalization of the self-energy graph. This problem will be discussed in a separate paper (it can be solved formally by using the Ward identity).

The term $\boldsymbol{k}_1^2$ is cancelled out by $1/k_1^2$ and the S-matrix becomes the same as graph 1(c) as follows:

$$S=-i(2\pi)^4\delta^4(\ )\left[-\frac{1}{4}\left(\frac{\kappa}{2}\right)^4 m_1^2m_2\frac{1}{(2\pi)^3}\int\frac{d^3l}{\boldsymbol{k}^2(\boldsymbol{l}-\boldsymbol{k})^2}\right], \tag{3·41}$$

and

$$V=-\frac{k^2m_1^2m_2}{r^2}. \tag{3·42}$$

For each term of the 13 terms of Eq. (3·9) we obtain similarly the corresponding potential. If we number them 1 to 13 from left to right, we obtain the result in Table I.

Table I. The coefficient of the potential $k^2m_1m_2(m_1+m_2)/c^2r^2$ from each term of graph 1(c). The number is ordered according to Eq. (3·9).

number	1	2	3	4	5	6	7	8	9	10	11	12	13	total
coefficient	−1/2	+1	+1	−1	−1	+1/2	−1	−1	4	−1	+1	−2	+1	+1

Thus we finally obtain as the contribution from graph 1(d)

$$V=\frac{k^2m_1m_2(m_1+m_2)}{c^2r^2}. \tag{3·43}$$

3.6 *Contribution from graph 1(a)*

The S-matrix of Fig. 4 is given by

$$S=\delta^4(\ )\frac{1}{\sqrt{2p_{10}2q_{10}2p_{20}2q_{20}}}\left(\frac{\kappa}{2}\right)^4$$
$$\times\int\frac{d^4l\, M(p_1,p_1+l;q_1,q_1-l)M(p_1+l,p_2;q_1-l,q_2)}{l^2(k-l)^2[(p_1+l)^2+m_1^2][(q_1-l)^2+m_2^2]}, \tag{3·44}$$

where

$$M(p_1,p_2;q_1,q_2)=2\{(p_1q_1)(p_2q_2)+(p_1q_2)(p_2q_1)-(p_1p_2)(q_1q_2)$$
$$-(p_1p_2)m_2^2-(q_1q_2)m_1^2-2m_1^2m_2^2\}. \tag{3·45}$$

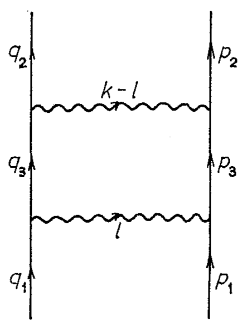

Fig. 4. Details of graph 1(a).

Hereafter we calculate the S-matrix in the c.m. system and represent $\boldsymbol{p}_1$ simply by $\boldsymbol{p}$. We can do this without losing generality, because we are calculating the potential which is independent of the momentum.

The leading term of Eq. (3·44) is of order m^5 which is cancelled out by the second Born term, while the next term is m^4 which is cancelled by the contribution from graph 1(b). So the remaining term is m^3 which we want to calculate.

Then we expand $M\times M$ in Eq. (3·44) in terms of c^{-1} up to the order c^{-2} as follows:

$$M(p_1,p_1+l;q_1,q_1-l)M(p_1+l,p_2;q_1-l,q_2)=4m_1^4m_2^4N, \tag{3·46}$$

where

$$N=1+\frac{4\boldsymbol{p}^2}{m_1{}^2c^2}+\frac{4\boldsymbol{p}^2}{m_2{}^2c^2}+\frac{4l_0}{m_1c}-\frac{4l_0}{m_2c}+\frac{(2\boldsymbol{p}+\boldsymbol{l})^2}{m_1m_2c^2}+\frac{(\boldsymbol{p}_1+\boldsymbol{p}_2+\boldsymbol{l})^2}{m_1m_2c^2}. \qquad (3\cdot47)$$

If we assume in general

$$N=1+n_1\frac{\boldsymbol{p}^2}{c^2}\left(\frac{1}{m_1{}^2}+\frac{1}{m_2{}^2}\right)+n_2\frac{(2\boldsymbol{p}+\boldsymbol{l})^2+(2\boldsymbol{p}+\boldsymbol{l}+\boldsymbol{k})^2}{4m_1m_2c^2}+n_3\left(\frac{l_0}{m_1c}-\frac{l_0}{m_2c}\right)$$
$$+n_4\frac{(\boldsymbol{p}+\boldsymbol{l})^2-\boldsymbol{p}^2}{c^2}\left(\frac{1}{m_1{}^2}+\frac{1}{m_2{}^2}\right)+n_5\frac{\boldsymbol{l}^2+(\boldsymbol{k}-\boldsymbol{l})^2}{c^2}\left(\frac{1}{m_1{}^2}+\frac{1}{m_2{}^2}\right) \qquad (3\cdot48)$$

and put $km_1m_2=\lambda$, then we obtain

$$V=\frac{\lambda^2}{c^2r^2}\left[(-1/2+n_1-n_3/2-2n_4)\left(\frac{1}{m_1}+\frac{1}{m_2}\right)+2(-n_1+n_3+2n_4)\frac{1}{m_1+m_2}\right]. \qquad (3\cdot49)$$

The proof of Eq. (3·49) is given in Appendix 1.

As in this case $n_1=4$, $n_2=1$, $n_3=4$ and $n_4=n_5=0$, the contribution from graph 1(a) is given by

$$V=\frac{3}{2}\frac{k^2m_1m_2(m_1+m_2)}{c^2r^2}. \qquad (3\cdot50)$$

3.7 *Contribution from graph 1(b)*

The S-matrix of Fig. 5 is

$$S=\delta^4(\ )\frac{1}{\sqrt{2p_{10}2q_{10}2p_{20}2q_{20}}}\left(\frac{\kappa}{2}\right)^4$$
$$\times\int\frac{d^4l\ M(p,p+l;q+l-k,q-k)M(p+l,p+k;q,q+l-k)}{l^2(k-l)^2[(p+l)^2+m_1{}^2][(q+l-k)^2+m_2{}^2]}, \qquad (3\cdot51)$$

where M is defined by Eq. (3·45). The poles of l_0 are located at

$$l_0=\pm|\boldsymbol{l}|\mp i\varepsilon,\qquad \pm|\boldsymbol{k}-\boldsymbol{l}|\mp i\varepsilon,\qquad -p_0\pm\sqrt{(\boldsymbol{p}+\boldsymbol{l})^2+m_1{}^2}\mp i\varepsilon$$

and

$$-q_0+k_0\pm\sqrt{(\boldsymbol{q}+\boldsymbol{l}-\boldsymbol{k})^2+m_2{}^2}\mp i\varepsilon,$$

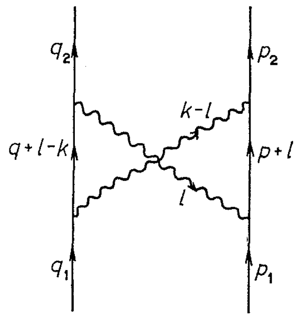

Fig. 5. Details of graph 1(b).

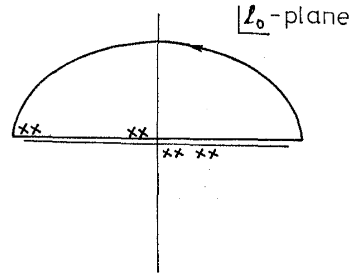

Fig. 6. The contour of the l_0 integration of graph 1(b).

and we take the contour shown in Fig. 6. Then the two poles of $-p_0-\sqrt{(\boldsymbol{p}+\boldsymbol{k})^2+m_1^2}+i\varepsilon$ and $-q_0+k_0-\sqrt{(\boldsymbol{q}+\boldsymbol{l}-\boldsymbol{k})^2+m_2^2}+i\varepsilon$ do not contribute to the fourth-order potential considered here, because the poles are of the order m^1. There is a contribution only from the poles, $-|\boldsymbol{l}|+i\varepsilon$ and $-|\boldsymbol{k}-\boldsymbol{l}|+i\varepsilon$, and so it is sufficient to expand N up to the order c^{-1}. Here N is defined by Eq. (3·46) and is given by

$$N=1+n_3\left(\frac{l_0}{m_1c}+\frac{l_0}{m_2c}\right) \tag{3·52}$$

for this graph, if N is given by Eq. (3·48) for the graph of Fig. 3.

Now the leading term of the contribution from the S-matrix (3·51) is of order m^4 and is cancelled out by the contribution from Fig. 4. The term of order m^3 is given by

$$V=\frac{\lambda^2}{c^2r^2}\cdot\frac{n_3}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right), \tag{3·53}$$

where $\lambda=km_1m_2$ and $n_3=4$ in this case. Thus the contribution from graph 1(b) is given by

$$V=\frac{2k^2m_1m_2(m_1+m_2)}{c^2r^2}. \tag{3·54}$$

3.8 *Contribution from all graphs*

By estimating the order of the mass, we can easily show that there are no contribution of the type k^2m^3/c^2r^2 from other graphs, including the graph of the fictitious quanta. So the total contribution from all the fourth-order graphs to the fourth-order potential is

$$V=\frac{1}{2}\cdot\frac{k^2m_1m_2(m_1+m_2)}{c^2r^2}, \tag{3·55}$$

adding Eqs. (3·33), (3·43), (3·50) and (3·54).

It is worth mentioning that the terms quadratic in k and trilinear in m are given exactly by Eq. (3·55); the terms such as $k^2m_1^2m_2^2/c^2r^2\cdot 1/(m_1+m_2)$ do not appear because $-n_1+n_3+2n_4=0$ and also the terms such as $k^2m^3/c^2r^2\times f(h/pr)$ do not appear, where f is an arbitrary function other than a constant.

Combining the second-order potential (3·24) and the kinetic energy with the fourth-order potential (3·55), we finally obtain the Hamiltonian

$$H=\frac{p_1^2}{2m_1}+\frac{p_2^2}{2m_2}-\frac{1}{8c^2}\left(\frac{p_1^4}{m_1^3}+\frac{p_2^4}{m_2^3}\right)-\frac{km_1m_2}{r}$$
$$-\frac{km_1m_2}{2c^2r}\left[3\left(\left(\frac{p_1}{m_1}\right)^2+\left(\frac{p_2}{m_2}\right)^2\right)-\frac{7\boldsymbol{p}_1\boldsymbol{p}_2}{m_1m_2}-\frac{(\boldsymbol{p}_1\boldsymbol{r})(\boldsymbol{p}_2\boldsymbol{r})}{m_1m_2r^2}\right]+\frac{k^2m_1m_2(m_1+m_2)}{2c^2r^2}, \tag{3·56}$$

which is exactly the same as Eq. (2·2). Thus using the quantum field theory of gravitation we have obtained the two-body Hamiltonian up to order c^{-2}, which is the same as the classical one.

§ 4. Conclusions and discussion

On the basis of quantum gravitational field theory we have derived the two-body Hamiltonian, Eq. (3·56), for scalar particles using the Lagrangians (3·6) $\sim$(3·10).

First we want to emphasize that we are able to extract a finite and physically meaningful radiative correction in spite of unrenormalizability of the theory: As we have mentioned in § 2, the last term of the Hamiltonian can be understood as a radiative correction in our approach. There are no ultraviolet or infrared divergences in the quantities calculated. This is to be expected; the former should not exist if the theory works, since the fourth-order diagrams are the lowest order which can contribute to the potential of the type r^{-2}, and the latter also should not exist to order c^{-2} since the form of radiation of gravitational waves contains a factor c^{-5}. The ultraviolet divergence appears in the quantity which corresponds to the potential of the type r^{-1}. This is the same type as the Newtonian potential and so we must renormalize it. The consistency of renormalization will be discussed in a separate paper.

Secondly we want to point out that the derived Hamiltonian is exactly the same as the classical one, Eq. (2·1). This result does not depend on the expansion F or G as indeed it should not. We have also performed similar calculations in the case of the Dirac field and have obtained the same spin-averaged Hamiltonian. (These results including others which will be discussed later are shown in Tables II and III.)

The fact that we have obtained the same Hamiltonians both in the cases of scalar and Dirac fields suggests that *in general* quantum theory gives the same Hamiltonian in any cases. At present, however, we have no rigorous proof of this. The important point is whether there is a possibility that quantum theory

Table II. The contribution from graph 1(a). $n_1 \sim n_5$ are defined in Eq. (3·48). $(-n_1+n_3+2n_4)$ and $(1/2+n_1-n_3/2-2n_4)$ are the coefficients of the potential $k^2m_1^2m_2^2/c^2r^2\times 1/(m_1+m_2)$ and $k^2m_1m_2(m_1+m_2)/c^2r^2$, respectively. We put $\lambda=0$ in the case of gravity with Dirac matter field.

		n_1	n_2	n_3	n_4	n_5	$-n_1+n_3+2n_4$	$-1/2+n_1-n_3/2-2n_4$
gravity (tensor)	scalar	4	4	4	0	0	0	3/2
	Dirac	4	4	3/2	5/4	−3/8	0	1/4
Q.E.D.	scalar	1	1	1	0	0	0	0
	Dirac	1	1	1/2	1/4	−1/8	0	−1/4

Table III. The contributions from graphs to the potential. Each box represents the numerical factor of the potential of the form of $m_1m_2(m_1+m_2)k^2/c^2r^2$. We enumerate in the column a' the value which we subtract the second-Born term from the contribution from graph a. The symbol $\times$ denotes that there are no corresponding graphs.

			a'	b	c	d	total
Q.G.D.	scalar	F	3/2	2	-4	1	1/2
		G	3/2	2	-1	-2	1/2
	Dirac	F	$1/4+\lambda$	$3/4+\lambda$	$-3/2-2\lambda$	1	1/2
		G	$1/4+\lambda$	$3/4+\lambda$	$+3/2-2\lambda$	-2	1/2
Q.E.D.	scalar		0	1/2	$-1/2$	$\times$	0
	Dirac		$-1/4$	$+1/4$	$\times$	$\times$	0

does predict a different Hamiltonian in other cases. We want to conjecture that there is no such possibility.

If there were to exist differences between classical and quantum theory, then the choice of a theory and a model would be quite significant in applying the results to a real problem. To complete the argument here, however, we discuss the perihelion-motion of Mercury using the results obtained. There are two approaches; in one we regard the Sun and Mercury as structure-less bodies as in the Newtonian mechanics and treat them as scalar particles, and in the other we regard them as composed of many elementary particles (e.g. protons, neutrons, electrons and mesons) and calculate the potential as a sum of the potentials between elementary particles. The former approach is not so strange as it seems to be at first sight, since the absolute mass and the absolute size of the stellar object do not matter but only the relative quantities.

The latter approach is, however, more natural to elementary particle physics. In this approach we should really take account of the fact that elementary particles in stellar objects or nuclei are not free. For example, the magnetic moments of the proton and the neutron in nuclei are not the same as those of free particles because of Fermi statistics. The total electric charge, however, is not changed in nuclei. The difference between the electric charge and the magnetic moment is that the former is conserved, while the latter is not.

In this paper we are discussing a quantity related to the conserved energy-momentum tensor. So we may expect that the situation is similar to that concerning electric charge; since the electric potential between nuclei is the same as that in the case where we treat the nucleus as a unity having the total charge, the gravitational potential between stellar objects may be the same as that in the case where we treat the stellar object as a unity having the total mass.

After this work was completed, Hiida and Kikugawa[11] have calculated the potential as a sum of the potentials between the elementary particles, assuming that we can treat them as free particles. They have clarified the importance of the three-body potential in their approach and adding the three-body potentials to the fourth-order potential, they have obtained the same Hamiltonian as ours. We want to comment that even in their approach the finiteness of the fourth-order potential is very important.

Now we proceed to discuss the Dicke-Goldenberg experiment. As we have obtained the same result as the classical theory, we cannot explain the experiment at least in the tensor theory. Several alternatives[12),13),14)] have been proposed to Einstein's gravitational theory. *If* the Dicke-Goldenberg experiment is correct, an alternative, say, the Brans-Dicke theory is better than Einstein's theory at explaining the motion of the perihelion of Mercury. We have pointed out, however, in a previous paper[15)] a difficulty in quantizing of a scalar-tensor theory. *Thus we cannot explain the experiment on our fundamental assumption that quantum theory is able to describe correctly classical processes in the classical limit.* This is a serious problem.

We have discussed Q.E.D. in Appendix 3. The fourth-order potential of the type $e^4/c^2r^2\cdot 1/m$ does not exist as in classical theory. This fact has not been clearly stated elsewhere.

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Appendix 1

Proof of Eq. (3·49)

We assume that the S-matrix is given by

$$S=\delta^4(\ )\frac{(16\pi\lambda)^2}{\sqrt{2p_{10}2q_{10}2p_{20}2q_{20}}}\int\frac{d^4l\ N}{l^2(k-l)^2[(p_1+l)^2+m_1^2][(q_1-l)^2+m_2^2]}, \tag{A·1·1}$$

where

$$N=1+n_1\frac{p^2}{c^2}\left(\frac{1}{m_1^2}+\frac{1}{m_2^2}\right)+n_2\frac{(2\boldsymbol{p}+\boldsymbol{l})^2+(2\boldsymbol{p}+\boldsymbol{l}+\boldsymbol{k})^2}{4m_1m_2c^2}+n_3\left(\frac{l_0}{m_1c}-\frac{l_0}{m_2c}\right)$$
$$+n_4\frac{(\boldsymbol{p}+\boldsymbol{l})^2-\boldsymbol{p}^2}{c^2}\left(\frac{1}{m_1^2}+\frac{1}{m_2^2}\right)+n_5\frac{\boldsymbol{l}^2+(\boldsymbol{k}-\boldsymbol{l})^2}{c^2}\left(\frac{1}{m_1^2}+\frac{1}{m_2^2}\right). \tag{A·1·2}$$

Before integrating over the energy variable l_0, we represent the integrand by a quotient N/D, where $1/D$ is common to all interactions (e.g. Q.E.D. and quantum theory of gravity) and all types of fields (e.g. scalar field and Dirac field) and N depends on the type of interaction and the choice of field. We discuss separately the contributions from N and $1/D$.

The positions of the l_0-poles are $\pm|\boldsymbol{l}|\mp i\varepsilon$, $\pm|\boldsymbol{l}-\boldsymbol{k}|\mp i\varepsilon$, $-p_{10}\pm\sqrt{(\boldsymbol{p}_1+\boldsymbol{k})^2+m_1^2}\mp i\varepsilon$ and $q_{10}\pm\sqrt{(\boldsymbol{q}_1-\boldsymbol{k})^2+m_2^2}\mp i\varepsilon$, and we take the contour indicated in Fig. 7.

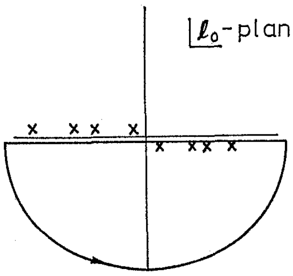

Fig. 7. The contour of the l_0 integration of graph 1(a).

A. 1. 1 *Contribution from 1/D*

1° Contribution from the pole $|\boldsymbol{l}|-i\varepsilon$

We find that the contribution is zero, after integrating over the momentum variables.

2° Contribution from the pole $|\boldsymbol{k}-\boldsymbol{l}|-i\varepsilon$

It is zero as in the case of 1°.

3° Contribution from the pole $-p_0+\sqrt{(p+k)^2+m_1^2}-i\varepsilon$

$$T(\boldsymbol{k})=(4\pi k)^2\frac{m_1m_2(m_1^2+m_1m_2+m_2^2)}{m_1+m_2}\frac{p^2}{(2\pi)^3}\int\frac{d^3l}{\boldsymbol{l}^2(\boldsymbol{k}-\boldsymbol{l})^2[(\boldsymbol{p}+\boldsymbol{l})^2-\boldsymbol{p}^2]}. \tag{A·1·3}$$

To obtain this result we must expand $1/\sqrt{2p_{10}\cdots}$ in terms of c^{-1} up to the order c^{-2}.

4° Contribution from the pole $q_0+\sqrt{(\boldsymbol{q}-\boldsymbol{k})^2+m_2^2}$

It is zero, for $q_0+\sqrt{(\boldsymbol{q}-\boldsymbol{k})^2+m_2^2}\sim 2m_2$.

So we obtain Eq. (A·1·3) as the contribution from $1/D$. Then we must subtract the second Born term from Eq. (A·1·3). Since the first-Born term corresponding to $1/D$ is

$$\frac{1}{|\boldsymbol{k}|^2}\left(1+\frac{\boldsymbol{k}\cdot(\boldsymbol{p}_1+\boldsymbol{p}_2)\boldsymbol{k}\cdot(\boldsymbol{q}_1+\boldsymbol{q}_2)}{4m_1m_2c^2|\boldsymbol{k}|^2}-\frac{\boldsymbol{p}_1^2+\boldsymbol{p}_2^2}{4m_1^2c^2}-\frac{\boldsymbol{q}_1^2+\boldsymbol{q}_2^2}{4m_2^2c^2}\right), \tag{A·1·4}$$

the second Born term is given by

$$T(\boldsymbol{k})=(4\pi k)^2\frac{m_1m_2(m_1^2+m_1m_2+m_2^2)}{m_1+m_2}\frac{p^2}{(2\pi)^3}\int\frac{d^3l}{\boldsymbol{l}^2(\boldsymbol{k}-\boldsymbol{l})^2[(\boldsymbol{p}+\boldsymbol{l})^2-\boldsymbol{p}^2]}$$
$$+\frac{1}{2}m_1m_2(m_1+m_2)\frac{(4\pi k)^2}{(2\pi)^3}\int\frac{d^3l}{\boldsymbol{l}^2(\boldsymbol{k}-\boldsymbol{l})^2}, \tag{A·1·5}$$

using Eq. (3·17).

Subtracting Eq. (A·1·5) from Eq. (A·1·3) we obtain

$$T(\boldsymbol{k})=-\frac{1}{2}m_1m_2(m_1+m_2)\frac{(4\pi k)^2}{(2\pi)^3}\int\frac{d^3l}{\boldsymbol{l}^2(\boldsymbol{k}-\boldsymbol{l})^2}. \tag{A·1·6}$$

Thus we finally obtain

$$V=-\frac{1}{2}m_1m_2(m_1+m_2)\frac{k^2}{c^2r^2} \tag{A·1·7}$$

as the contribution from $1/D$.

This result can be stated as follows: If the second-order static potential has the form of λ/r, then the fourth-order potential contributed from $1/D$ is

$$V=-\frac{1}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\frac{\lambda^2}{c^2r^2}. \tag{A·1·8}$$

A. 1. 2 *Contribution from N*

If N is given by (A. 1. 2), then the corresponding term N of the second Born term is

$$N_{\text{2nd}}=1+\frac{n_1}{2}\cdot\frac{\boldsymbol{p}^2+(\boldsymbol{p}+\boldsymbol{l})^2}{c^2}\left(\frac{1}{m_1^2}+\frac{1}{m_2^2}\right)$$
$$+n_2\frac{(2\boldsymbol{p}+\boldsymbol{l})^2+(2\boldsymbol{p}+\boldsymbol{l}+\boldsymbol{k})^2}{4m_1m_2c^2}+n_5\frac{\boldsymbol{l}^2+(\boldsymbol{k}-\boldsymbol{l})^2}{c^2}\left(\frac{1}{m_1^2}+\frac{1}{m_2^2}\right). \tag{A·1·9}$$

We can indeed check the form of Eq. (A·1·9) by calculation in all cases considered in this paper. Then by subtracting Eq. (A·1·9) from Eq. (A·1·2) we obtain

$$N-N_{\text{2nd}}=(-n_1/2+n_4)\left[(\boldsymbol{p}+\boldsymbol{l})^2-\boldsymbol{p}^2\right]\left(\frac{1}{m_1^2c^2}+\frac{1}{m_2^2c^2}\right)+n_3\frac{l_0}{c}\left(\frac{1}{m_1}-\frac{1}{m_2}\right). \tag{A·1·10}$$

As the contribution from the poles $l_0=|\boldsymbol{l}|-i\varepsilon$ and $l_0=|\boldsymbol{l}-\boldsymbol{k}|-i\varepsilon$, we obtain

$$V=\frac{\lambda^2}{c^2r^2}\left[-\frac{n_3}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)+2n_3\frac{1}{m_1+m_2}\right], \tag{A·1·11}$$

and as the contribution from the pole $l_0=-p_0+\sqrt{(\boldsymbol{p}+\boldsymbol{l})^2+m_1^2}-i\varepsilon$

$$V=\frac{\lambda^2}{c^2r^2}\left[(n_1-2n_4)\left(\frac{1}{m_1}+\frac{1}{m_2}\right)+2(-n_1+2n_4)\frac{1}{m_1+m_2}\right]. \tag{A·1·12}$$

Since there is no contribution from the pole $k_0=q_0+\sqrt{(\boldsymbol{p}+\boldsymbol{l})^2+m_2^2}$, we obtain as the contribution from N

$$V=\frac{\lambda^2}{c^2r^2}\left[(n_1-n_3/2-2n_4)\left(\frac{1}{m_1}+\frac{1}{m_2}\right)+2(-n_1+n_3+2n_4)\frac{1}{m_1+m_2}\right]. \tag{A·1·13}$$

A. 1. 3 *Contribution from N and 1/D*

Combining Eqs. (A·1·8) and (A·1·13), we obtain

$$V=\frac{\lambda^2}{c^2r^2}\left[(-1/2+n_1-n_3/2-2n_4)\left(\frac{1}{m_1}+\frac{1}{m_2}\right)+2(-n_1+n_3+2n_4)\frac{1}{m_1+m_2}\right]. \tag{A·1·14}$$

Appendix 2

Gravitational potential in the case of the Dirac field (spin averaged one)

Since the method of calculating the potential is the same as in the case of the scalar field, we will only sketch the outline of the calculation and state the results.

1° Lagrangian

The parts of the free graviton Lagrangian and the three graviton interaction Lagrangian are given by Eqs. (3·7) and (3·9), respectively. Equations (3·6), (3·8) and (3·10) are replaced by

$$L(\bar{\psi}\psi) = -\bar{\psi}[\langle\!\langle \gamma_\rho \partial_\rho \rangle\!\rangle + m]\psi, \tag{A·2·1}$$

$$L(h\bar{\psi}\psi) = \frac{\kappa}{2} h_{\mu\nu}\bar{\psi}[\langle\!\langle \gamma_\mu \partial_\nu \rangle\!\rangle - \lambda\delta_{\mu\nu}(\langle\!\langle \gamma_\lambda \partial_\lambda \rangle\!\rangle + m)]\psi \tag{A·2·2}$$

and

$$L(h^2\bar{\psi}\psi) = -\frac{\kappa^2}{32}\Big[3\bar{\psi}\langle\!\langle \gamma_\rho \partial_\lambda \rangle\!\rangle \psi h_{\rho\sigma}h_{\lambda\sigma} - \frac{1}{2}\bar{\psi}(\gamma_\mu\gamma_\nu\gamma_\lambda - \gamma_\lambda\gamma_\nu\gamma_\mu)\psi h_{\mu\rho}h_{\nu\rho,\lambda}$$
$$- 2\lambda\bar{\psi}\langle\!\langle \gamma_\rho \partial_\lambda \rangle\!\rangle \psi h_{\rho\lambda}h_{\sigma\sigma}\Big], \tag{A·2·3}$$

respectively. Here $\langle\!\langle \gamma_\rho \partial_\lambda \rangle\!\rangle = 1/4(\gamma_\rho \vec{\partial}_\lambda - \overleftarrow{\partial}_\lambda \gamma_\rho + \gamma_\lambda \vec{\partial}_\rho - \overleftarrow{\partial}_\rho \gamma_\lambda)$ and λ is an arbitrary constant. We obtained Eqs. (A·2·2) and (A·2·3) by the method of Wyss.[16] The Lagrangian is given uniquely except for the λ dependence.

2° Second-order potential

$$V = -\frac{km_1m_2}{r}\Big[1 + \frac{3}{2}\Big(\frac{\boldsymbol{p}^2}{m_1^2c^2} + \frac{\boldsymbol{q}^2}{m_2^2c^2}\Big) - \frac{7\boldsymbol{pq}}{2m_1m_2c^2} - \frac{(\boldsymbol{pr})(\boldsymbol{qr})}{2m_1m_2c^2r^2}\Big]$$
$$+ \frac{3\pi\hbar^2}{2c^2}\Big(\frac{1}{m_1} + \frac{1}{m_2}\Big)\delta^{(3)}(r), \tag{A·2·4}$$

which is the same as Eq. (3·24) except for the last δ-function term.

3° Fourth-order potential

3a° Contribution from 1(a)

If $\lambda = 0$, then $n_1 = 4$, $n_2 = 4$, $n_3 = 3/2$, $n_4 = 5/4$ and $n_5 = -3/8$, where $n_1 \sim n_5$ are defined in Eq. (3·48). From Eq. (3·49) we obtain

$$V = \frac{1}{4}\frac{k^2m_1m_2(m_1+m_2)}{c^2r^2}. \tag{A·2·5}$$

Taking into account the λ-dependence we obtain

$$V = (1/4 + \lambda)\frac{k^2m_1m_2(m_1+m_2)}{c^2r^2}. \tag{A·2·6}$$

3b° Contribution from 1(b)

As $n_3=3/2$ at $\lambda=0$, we obtain from Eq. (3·53)

$$V=\frac{3}{4}\frac{k^2m_1m_2}{c^2r^2}(m_1+m_2). \qquad (\mathrm{A}\cdot 2\cdot 7)$$

Taking into account the λ-dependence we obtain

$$V=(3/4+\lambda)\frac{k^2m_1m_2}{c^2r^2}(m_1+m_2). \qquad (\mathrm{A}\cdot 2\cdot 8)$$

3c° Contribution from 1(c)

After similar calculation to that of § 3. 4, we obtain

$$V=(-3/2-2\lambda)\frac{k^2m_1m_2}{c^2r^2}(m_1+m_2). \qquad (\mathrm{A}\cdot 2\cdot 9)$$

3d° Contribution from 1(d)

After calculation similar to that of § 3. 5 we obtain

$$V=\frac{k^2m_1m_2}{c^2r^2}(m_1+m_2). \qquad (\mathrm{A}\cdot 2\cdot 10)$$

Combining Eqs. (A·2·6), (A·2·8), (A·2·9) and (A·2·10) we obtain finally the fourth-order potential

$$V=\frac{1}{2}\cdot\frac{k^2m_1m_2}{c^2r^2}(m_1+m_2). \qquad (\mathrm{A}\cdot 2\cdot 11)$$

It is perhaps worth mentioning that the final result (A·2·11) does not depend on λ.

Appendix 3

The potential in the case of Q.E.D.

Since the method of calculating the potential is the same as in the case of quantum theory of gravitation, we will only sketch the process of the calculation and state the results. We calculate the potential both in the cases of the scalar field and the Dirac field of the matter.

A. 3. 1 *The case of the scalar field*

1° Second-order potential

Taking note of the remark made in the second paragraph in the § 3 in the text, we calculate the off-the-mass-shell S-matrix in which we do not use the mass relation $p^2=-m^2$ and use the Coulomb gauge. After the Fourier transformation (3·21) we use the mass relation. This procedure is the same as using

$$\frac{1}{k^2}=\frac{1}{\boldsymbol{k}^2}+\frac{\boldsymbol{k}\cdot(\boldsymbol{p}_1+\boldsymbol{p}_2)\boldsymbol{k}\cdot(\boldsymbol{q}_1+\boldsymbol{q}_2)}{4m_1m_2c^2(\boldsymbol{k}^2)^2} \qquad (\mathrm{A}\cdot 3\cdot 1)$$

up to the order c^{-2}, throughout the calculation.[17)]

Then the S-matrix of the second-order is up to the order c^{-2}

$$S=-i(2\pi)^4\delta^4(\ )e^2\left[\frac{1}{\boldsymbol{k}^2}+\frac{\boldsymbol{k}\cdot(\boldsymbol{p}_1+\boldsymbol{p}_2)\boldsymbol{k}\cdot(\boldsymbol{q}_1+\boldsymbol{q}_2)}{4m_1m_2c^2(\boldsymbol{k}^2)^2}-\frac{(\boldsymbol{p}_1+\boldsymbol{p}_2)(\boldsymbol{q}_1+\boldsymbol{q}_2)}{4m_1m_2c^2\boldsymbol{k}^2}\right] \tag{A·3·2}$$

from which we obtain

$$V=\frac{e^2}{4\pi}\left(\frac{1}{r}-\frac{1}{2m_1m_2c^2r}\left[(\boldsymbol{pq})+(\boldsymbol{pn})(\boldsymbol{qn})\right]\right);\quad \left(\boldsymbol{n}=\frac{\boldsymbol{r}}{r}\right). \tag{A·3·3}$$

2° Fourth-order potential

As there is not 1(d) graph, there are contributions only from 1(a), 1(b) and 1(c).

2a° Contribution form 1(a)

As $n_1=1$, $n_2=1$, $n_3=1$, $n_4=n_5=0$, where $n_1\sim n_5$ are defined in Eq. (3·48), we obtain from Eq. (3·49)

$$V=0\,. \tag{A·3·4}$$

2b° Contribution from 1(b)

As $n_3=1$, from Eq. (3·53)

$$V=\frac{1}{2}\frac{e^4}{c^2r^2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right). \tag{A·3·5}$$

2c° Contribution from 1(c)

After calculation similar to that of § 3. 4, we obtain

$$V=-\frac{1}{2}\frac{e^4}{c^2r^2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right). \tag{A·3·6}$$

Combining Eqs. (A·3·5) and (A·3·6) we obtain as the total fourth-order potential

$$V=0\,. \tag{A·3·7}$$

A. 3. 2 *The case of the Dirac field* (*spin-averaged potential*)

1° Second-order potential

$$V=\frac{e^2}{4\pi}\left\{\frac{1}{r}-\frac{1}{2m_1m_2c^2r}\left[(\boldsymbol{pq})+(\boldsymbol{pn})(\boldsymbol{qn})\right]\right\}-\frac{e^2\hbar^2}{8c^2}\left(\frac{1}{m_1{}^2}+\frac{1}{m_2{}^2}\right)\delta^{(3)}(\boldsymbol{r}), \tag{A·3·8}$$

which is the same as Eq. (A·3·3) except for the last δ-function term.

2° Fourth-order potential

As there are no (1. c) and (1. d) graph, there are contributions only from 1(a) and 1(b).

2a° Contribution from 1(a)

As $n_1=1$, $n_2=1$, $n_3=1/2$, $n_4=1/4$ and $n_5=-1/8$, we obtain from Eq. (3·49)

$$V=-\frac{1}{4}\frac{e^4}{c^2r^2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right). \qquad (\mathrm{A}\cdot3\cdot9)$$

2b° Contribution from 1(b)

As $n_3=1/2$, we obtain from Eq. (3·53)

$$V=\frac{1}{4}\frac{e^4}{c^2r^2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right). \qquad (\mathrm{A}\cdot3\cdot10)$$

Combining Eqs. (A·3·9) and (A·3·10) we obtain

$$V=0\,. \qquad (\mathrm{A}\cdot3\cdot11)$$

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