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Quantum Theory of Optical Feedback via Homodyne Detection

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We present a quantum theory of feedback in which the homodyne photocurrent alters the dynamics of the source cavity. To the nonlinear stochastic (Ito) evolution of the conditioned system state we add a feedback term linear in the instantaneous stochastic (Stratonovich) photocurrent. Averaging over the photocurrent gives a feedback master equation which has the desired driftlike term, plus a diffusionlike term. We apply the model to phase locking a regularly pumped laser, and show that under ideal conditions the noise spectra of the output light exhibit perfect squeezing on resonance.

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Electrical engineers have long used feedback to prevent noise from rendering a system unstable, and these techniques are now put to use in optical systems, for example, to stabilize the phase or intensity of a laser oscillator. In the past decade, the noise control in optical systems has entered the quantum domain, attenuating noise as far as possible within the constraints of Heisenberg's uncertainty principle. This has generally been achieved not by feedback, but by exploiting the intrinsic nonlinearities of certain media. In this way, a variety of optical states have been produced which lead to sub-shot-noise photocurrents [1]. Until recently [2], the noise reducing capabilities of feedback in quantum optical systems have not been explored. In this Letter we rigorously derive a master equation which describes quantum-limited feedback of a homodyne current to control an optical cavity.

Feedback entails a measurement step which necessarily introduces noise into the feedback process even if all classical noise sources are eliminated. In our theory, this is manifest as a diffusionlike term in the master equation. This suggests a possible fluctuation-feedback theorem, analogous to the fluctuation-dissipation theorem. The desired feedback term is a nonlinear, nonunitary, driftlike term. We show that if the action of the feedback is classical (e.g., driving, detuning, or damping), the feedback terms cannot produce nonclassical field states. However, unlike previous treatments of quantum-limited feedback in traveling waves [3], feedback control of cavity systems enables the feedback to be nonclassical, or to be combined with nonlinear intracavity elements. This opens up the possibility of new types of nonlinear and nonclassical field evolution. In this Letter we show that phase-locking feedback of a regularly pumped laser enables perfect squeezing in the free output of the laser.

Consider an optical cavity with Liouville superoperator $\mathcal{L}$ which includes damping to the external continuum of vacuum modes via an output mirror. We measure time in inverse units of the cavity linewidth so that the damping master equation is $\dot{\rho} = a\rho a^\dagger - \frac{1}{2}a^\dagger a\rho - \frac{1}{2}\rho a^\dagger a$, where ρ is the state matrix. This ρ represents our knowledge of the state of the system given that we know nothing of the state of the bath, which is traced over in deriving the master equation. However, in practice it is not uncommon to have knowledge of the state of the bath from measurements on the field leaving the cavity. In this case, the system must be described by a conditioned state matrix ρ_c, conditioned on the results of the measurement. The basic process by which this conditioned state is constructed is that, if b is the annihilation operator for the output field at a photodetector, then if a photodetection takes place, the conditioned state of the system changes via $\rho_c(t^+) = b\rho_c(t)b^\dagger/P_c(t)$ where $P_c(t) = \mathrm{Tr}[b\rho_c(t)b^\dagger]$ is the probability per unit time for the photodetection to occur. For simple photodetection, this leads to quantum jumps in the conditioned system state [4], but more complicated measurement schemes lead to different behavior. For example, homodyne measurements of the $\hat{\mathrm{x}}$ quadrature of the output field yields the following photocurrent [5]:

$$I_c(t) = \eta\langle a + a^\dagger\rangle_c(t) + \sqrt{\eta}\xi(t), \tag{1}$$

where η is the efficiency of the detection system, $\xi(t)$ is real delta-correlated noise [6] arising from the local oscillator, and $\langle a + a^\dagger\rangle_c(t) = \mathrm{Tr}[(a + a^\dagger)\rho_c(t)]$. We have shown previously [5] that the conditioned system state then obeys the following Ito stochastic, nonlinear equation:

$$\dot{\rho}_c^{(\mathrm{I})} = [\mathcal{L} + \sqrt{\eta}\xi(t)\mathcal{H}]\rho_c, \tag{2}$$

where $\mathcal{H}$ is defined by

$$\mathcal{H}\rho = a\rho + \rho a^\dagger - \mathrm{Tr}(a\rho + \rho a^\dagger)\rho. \tag{3}$$

If we ignore the result of the measurements by averaging over $\xi(t)$, then Eq. (2) reduces to the standard master equation for the ensemble average $\rho = E[\rho_c]$.

Now consider instantaneously feeding back the homodyne photocurrent to change the system dynamics. It is reasonable to assume that the strength of the feedback is linear in the photocurrent, since higher powers of (1) are ill defined because of the white noise term. The feedback term can thus be written

$$[\dot{\rho}_c]_{\mathrm{fb}} = [\langle a + a^\dagger\rangle_c(t) + \xi(t)/\sqrt{\eta}]\mathcal{K}\rho_c\,, \tag{4}$$

where $\mathcal{K}$ is an arbitrary Liouville superoperator with the proviso that in general $-\mathcal{K}$ must also be a valid Liouville superoperator. The question now arises of whether to treat Eq. (4) as an Ito or Stratonovich equation [6]. Equation (2) was derived using the Ito stochastic calculus, but Eq. (4) has been postulated and so its interpretation is open. We will show that the only consistent interpretation of (4) is as a Stratonovich equation. To reconcile this with the Ito equation (2), we first put the latter into Stratonovich form:

$$\dot{\rho}_c^{(\mathrm{S})} = \left[\mathcal{L} + \sqrt{\eta}\xi(t)\mathcal{H} - \tfrac{1}{2}\eta\mathcal{H}^2\right]\rho_c. \quad (5)$$

Adding Eq. (4) gives

$$\dot{\rho}_c^{(\mathrm{S})} = [\,\mathcal{L} - \tfrac{1}{2}\eta\mathcal{H}^2 + \langle a + a^\dagger\rangle_c(t)\mathcal{K} + \sqrt{\eta}\xi(t)(\mathcal{H} + \eta^{-1}\mathcal{K})]\rho_c. \quad (6)$$

Converting this back to Ito form and using the definition (3) of $\mathcal{H}$ gives

$$\dot{\rho}_c^{(\mathrm{I})} = \mathcal{L}\rho_c + \mathcal{K}(a\rho_c + \rho_c a^\dagger) + \frac{1}{2\eta}\mathcal{K}^2\rho_c + \sqrt{\eta}\xi(t)(\mathcal{H} + \eta^{-1}\mathcal{K})\rho_c. \quad (7)$$

Here we have used the superoperator ordering $\mathcal{KH}$, rather than a symmetric product because this is necessary to give a trace preserving master equation, as will be seen shortly. This procedure can be justified by considering time-delayed feedback, and then letting the time delay go to zero. The principle advantage of the Ito calculus is that $\xi(t)$ is independent of $\rho_c(t)$, so that we can easily take the ensemble average of Eq. (7) to obtain the following master equation for the nonconditioned state matrix $\rho = E[\rho_c]$:

$$\dot{\rho} = \mathcal{L}\rho + \mathcal{K}(a\rho + \rho a^\dagger) + \frac{1}{2\eta}\mathcal{K}^2\rho. \quad (8)$$

This is our general master equation for homodyne-mediated feedback.

An important point to note about Eq. (8) is that it is linear. That is to say, terms like $\langle a + a^\dagger\rangle_c(t)\rho_c(t)$ have been eliminated. This is essential since the fundamentals of probability theory imply that the generator of motion for any complete statistical representation of a system must be linear [7]. This fact allows us to accept Eq. (8) and to reject the Ito interpretation of Eq. (4) which would lead to the following equation:

$$\dot{\rho}_c^{(\mathrm{I})} = \left[\mathcal{L} + \sqrt{\eta}\xi(t)(\mathcal{H} + \eta^{-1}\mathcal{K}) + \langle a + a^\dagger\rangle_c(t)\mathcal{K}\right]\rho_c. \quad (9)$$

By inspection, this equation would not give a valid master equation if one were to attempt to average over $\xi(t)$. This result is not surprising from physical intuition, because the necessary delay in any feedback loop would invalidate the Ito assumption that the noise in the feedback current is independent of the state of the system at the time at which it takes effect.

Equation (8) shares much in common with a feedback master equation derived from the idealized continuous position measurement model of Caves and Milburn [8]. The first additional term in Eq. (8) is the desired feedback effect, which is nonlinear and nonunitary. The second is a diffusion term due to the inevitable introduction of noise by a quantum-limited feedback loop. The lower the efficiency of the detecting system, the larger this diffusion term becomes, as expected. Another important point to note about Eq. (8) is that, if the feedback superoperator $\mathcal{K}$ corresponds to a "classical" process (driving, detuning, or damping), then the two feedback terms give a true Fokker-Planck equation for the Glauber-Sudarshan P function. This has the significance that, unless the superoperator $\mathcal{L}$ has the ability to produce nonclassical states, then adding feedback will not reduce noise below the classical limit.

As an example, we now apply the above formalism to a simple system, a single mode laser. It can be shown [9, 10] that an ideal laser with a Poissonian pump rate of $\mu \gg 1$ rapidly evolves to a mixture of coherent states of amplitude $\sqrt{\mu}$, and thereafter its dynamics is described excellently by the following master equation:

$$\dot{\rho} = -\frac{1+\nu}{4\mu}[a^\dagger a, [a^\dagger a, \rho]], \quad (10)$$

where $\nu \geq 0$ represents the excess phase noise in the laser above the quantum limit. This master equation obviously describes classical phase diffusion, causing an initially coherent state to become eventually a mixture over all phases. To prevent this, the laser can be phase-locked to a local oscillator of (relatively) fixed phase. This is achieved by changing the frequency of the cavity by feeding back a homodyne photocurrent just as described above. To lock the phase at $\pi/2$, a positive $\hat{\mathbf{x}}$ homodyne current should result in a decreased frequency. To justify the assumption of instantaneous feedback, the time delay would have to be much less than the inverse of the cavity linewidth (that is, submicrosecond). In practice this could be achieved by changing the optical path length of the cavity, perhaps by an electro-optic modulator. For small changes in the path length, this is well modeled by the Stratonovich equation (4) with the feedback superoperator defined by

$$\mathcal{K}\rho = i\frac{\lambda}{2\sqrt{\mu}}[a^\dagger a, \rho]\,. \quad (11)$$

Substituting this into Eq. (9) gives the feedback master equation

$$\dot{\rho} = -\frac{1+\nu+\lambda^2/2\eta}{4\mu}[a^\dagger a, [a^\dagger a, \rho]] + i\frac{\lambda}{2\sqrt{\mu}}[a^\dagger a, a\rho + \rho a^\dagger]. \quad (12)$$

This equation can be solved using the Glauber-Sudarshan $P(\alpha)$ function. Since the phase space is restricted to the one-dimensional manifold $\{\alpha : \alpha = \sqrt{\mu}e^{i\phi}, 0 \le \phi \le 2\pi\}$, we need to only consider $P(\phi)$, which obeys

$$\dot{P}(\phi) = \left[-\frac{\partial}{\partial\phi}\lambda\cos\phi + \frac{1}{2}\frac{\partial^2}{\partial\phi^2}\frac{1+\nu+\lambda^2/2\eta}{2\mu}\right]P(\phi). \tag{13}$$

The stationary solution will obviously have a phase variance of order $1/\mu$. Since this is much less than 1, we are justified in ignoring the periodicity of ϕ and linearizing Eq. (13) to obtain the stationary solution

$$P(\phi) = \frac{1}{\sqrt{2\pi U_\phi}}\exp\left[-\frac{(\phi-\pi/2)^2}{2U_\phi}\right], \tag{14}$$

where

$$U_\phi = \frac{1+\nu+\lambda^2/2\eta}{4\mu\lambda} \tag{15}$$

is the normally ordered phase variance. A truer representation of the phase uncertainty is the symmetrically ordered phase variance, V_ϕ, found from the Wigner function. In this case we simply add the Wigner phase variance of a large amplitude coherent state to get $V_\phi = U_\phi + 1/4\mu$. Thus we see that the stationary state of a phase-stabilized Poissonian laser does *not* approach a coherent state. Nevertheless, the extra phase uncertainty above that of a coherent state is still on a quantum scale.

Perhaps of more interest is the phase stabilization of a sub-Poissonian pumped laser [11], in which the superoperator $\mathcal{L}$ is nonclassical. It turns out [10] that the regularity of the pump does not affect the rate of phase diffusion. Thus, the true phase variance is

$$V_\phi = \frac{1+\nu+\lambda+\lambda^2/2\eta}{4\mu\lambda}, \tag{16}$$

as before, and the photon number variance is

$$V_n = (1-r/2)\mu, \tag{17}$$

where r is a measure of the regularity of the pump, equal to 0 for a Poissonian pump and 1 for a completely regular pump. Under ideal conditions ($\nu = 0, \eta = 1$), we find a minimum of $V_\phi = (1+\sqrt{2})/4\mu$ when $\lambda = \sqrt{2}$. Assuming $r = 1$, the squeezed state has a Wigner phase-space area of

$$[V_\phi V_n]^{1/2} = [(1+\sqrt{2})/8]^{1/2} \simeq 0.5493. \tag{18}$$

This is less than 10% above the minimum of 0.5 required by Heisenberg's uncertainty relations.

The result for the output spectrum is even better. It must be remembered that the fraction θ of emitted light available as an output from the system is at most $1-\eta$, because the fraction of emitted light used in the feedback loop is at least η. Fluctuations in the output light are best represented by the noise spectra for the quadratures $\hat{\mathbf{x}}$ (which corresponds to phase in the limit of high photon numbers as here) and $\hat{\mathbf{y}}$ (which similarly corresponds to intensity). It can be shown that the zero frequency (on resonance) noise spectrum for $\hat{\mathbf{x}}$ is

$$S_x(0) = 1 + \theta[1/\eta + 2(1+\nu)/\lambda^2], \tag{19}$$

where the spectrum is normalized to equal 1 at high frequencies. Meanwhile, the noise spectrum for $\hat{\mathbf{y}}$ has a minimum of

$$S_y(0) = 1 - \theta r, \tag{20}$$

which exhibits squeezing for $r > 0$. Assuming $r = 1$, $\theta = 1-\eta$, and $\lambda \to \infty$, we find the following simple expressions for optimum output noise reduction:

$$S_x(0) = 1/\eta \ , \quad S_y(0) = \eta. \tag{21}$$

These results are characteristic of perfect squeezing on resonance in that (i) $S_x(0)S_y(0) = 1$, a minimum uncertainty relation; and (ii) arbitrary squeezing is obtainable in the $\hat{\mathbf{y}}$ quadrature, with $S_y(0) \to 0$ as $\eta \to 0$. In the other limit ($\eta \to 1$), the laser output has the noise characteristics of a coherent state simply because the fraction of output light is almost zero.

It must be emphasized that the feedback is not responsible for the nonclassicality of the light produced by the phase-locked regularly pumped laser. As we have shown, classical feedback cannot produce nonclassical light. What the feedback does achieve is to change the nonclassical laser steady state from a sub-Poissonian (but far from minimum uncertainty) state, to an almost minimum uncertainty quadrature squeezed state. This is achieved by forcing the laser phase (which is completely undefined in a free running laser) to a definite value within a small uncertainty. Without feedback, the laser output has a sub-shot-noise intensity spectrum, but the spectrum of any quadrature is greatly super-shot-noise. Locking the phase to $\pi/2$ causes the nonclassical intensity noise reduction to become nonclassical $\hat{\mathbf{y}}$ quadrature noise reduction. In the ideal case of perfect photodetection, the on-resonance noise in the other quadrature ($\hat{\mathbf{x}}$) may be as small as possible given Heisenberg's uncertainty relations. In this sense, the phase-locking feedback makes the light squeezed, even though it does not produce the nonclassicality.

In conclusion, we have solved the problem of quantum-limited optical cavity feedback for the case of instantaneous feedback of the homodyne photocurrent. The exact master equation for the feedback-controlled cavity mode exhibits a novel non-Hermitian driftlike term, and a diffusive term due to quantum noise inherent in the detection process. These terms are such that classical feedback mechanisms (altering the cavity driving, detuning or damping) cannot produce nonclassical states. However, the possibilities for nonclassical feedback or feedback coupled to nonclassical intracavity dynamics are of great interest. As a simple application of the latter, we have shown that using feedback to lock the phase of a regularly pumped laser produces perfect squeezing on resonance in the output light. This work will be expanded upon in future publications.

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