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Quantum Treatment of Field Propagation in a Fiber near the Zero Dispersion Wavelength

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Abstract

In this report, we present a quantum theory describing the propagation of the electromagnetic radiation in a fiber in the presence of the third order dispersion coefficient. We obtained the quantum photon-polariton field, hence, we provide herein a coupled set of operator forms for the corresponding nonlinear Schrodinger equations when the third order dispersion coefficient is included. Coupled stochastic nonlinear Schrodinger equations were obtained by applying a positive P-representation that governs the propagation and interaction of quantum solitons in the presence of the third-order dispersion term. Finally, to reduce the fluctuations near solitons in the first approximation, we developed coupled stochastic linear equations.

I. Introduction

With the invention of lasers [1] and the development of optical fibers [2,3] in the second half of the 20th century, a new era in communication systems development began. For the last three decades of the past century, advancement in the optical communications field was gradual [4], although it had accelerated by the turn of the century [4,5]. Progress in the design and fabrication of various types of optical fibers led to an increase in bit rate and enabled engineering near the zero dispersion wavelength (ZDW) [2,6,7].

In modern optical communication systems, the third-order dispersion coefficient plays an important role in data transfer, especially when the wavelength of the laser is close to the ZDW of the optical fiber [4]. In this case, the bit rate B for the propagation of the laser pulse through the optical fiber with length L can be estimated as $BL < |S|^{-1}(\Delta\lambda)^{-2}$, where $S(\propto \beta_3)$ and $\Delta\lambda$ are, respectively, the third order dispersion coefficient and the laser line width [4]. In addition to diverse optical fiber communication systems design, different techniques have been applied to increase the bit rate [8-10]. It should be noted that the usage of optical fibers is not limited to communication applications; for instance, they are also used in high-quality sensing [11,12].

As in most physical systems, there are different sources of noise in optical communication systems, and some of the noise originates from quantum sources [13,16]. There has been extreme interest in noise reduction to develop the optical communication systems further. Since the reduction of noise

from quantum origins as well as the increase in data transmission security reached their limiting levels by the end of the last century [17], interest in the quantum treatment of fiber communication technology has been increased further [18]. For the first time in 1993, Drummond et al. studied quantum soliton experimentally [19]. Since then, many works have been published on quantum solitons, but the third-order dispersion coefficient is not included [20-22]. That is the case for the work by Carter [23], who presented a quantum theory for the pulse propagation along an optical fiber taking into account several Hamiltonians, thereby, extending the previous quantum theories for pulse propagation through an optical fiber. Many researchers have investigated the generation of quantum states by optical fibers and used them for diverse applications, including quantum non-demolition experiments [24], generation of Schrodinger's cat states [25] and parametric down conversion [26].

Recently, many research groups have studied single-photon and entangled photon pair sources generated in optical fibers [27-29]. However, to our knowledge, there has been no quantum treatment of optical communications for pulse propagation in optical fibers incorporating the third-order dispersion coefficient [23,30,31]. Due to the important role of this term in data transfer, we quantum-mechanically developed a description of pulse propagation in the presence of third-order dispersion in this study.

Employing the dual potential [31], we calculated and quantized the Hamiltonian for the three-dimensional field propagation in a dielectric with a combination of inhomogeneity, dispersion and nonlinearity in the presence of the third-order dispersion term [32]. To help the reader, the result of the work are provided briefly in Appendix A. In this paper, making use of a dual potential and Hamiltonian, equation (A-5), we will present the derivation of the equation governing pulse propagation along an optical fiber in the presence of the third-order dispersion coefficient.

II. Quantization of Field Propagation in Fiber

In this section, the application of a method established earlier [32] to field propagation along a fiber, i.e., a cylindrical optical waveguide, is presented. To be self-contained, we provide the necessary preceding formulae Appendix A. This field is assumed to be a plane wave, polarized in the xy-plane, and propagating in the z-direction. Additionally, the single transverse mode ν_0, which refers to a wave of single frequency ω, is assumed, while the longitudinal mode components have discrete wave numbers k ranging from $-k_{max}$ to k_{max}. For a description of transverse modes of a cylindrical cavity, we refer the reader to Appendix B. The dual potential is defined as:

$$\Lambda(t,\mathbf{x}) = \sum_{v=-N}^{v=N} \Lambda^v(t,\mathbf{x}) \tag{1}$$

where

$$\Lambda^v(t,\mathbf{x}) = \Gamma_{\perp}^v e^{in\varphi} \Lambda^v(t,z), \tag{2}$$

is a solution to the wave equation (equation (A-9)) in cylindrical coordinates and $\Gamma_{\perp}^v$ is defined by equation (B-7). Hence, the magnetic field and the electric displacement vector are, respectively, reduced to:

$$\mathbf{B}^v(t,\mathbf{x}) = \mu\dot{\Lambda}^v(t,\mathbf{x}) = \mu\Gamma_{\perp}^v e^{in\varphi}\dot{\Lambda}^v(t,z), \tag{3}$$

and

$$\mathbf{D}^v(t,\mathbf{x}) = \nabla\times\Lambda^v(t,\mathbf{x}) = \Gamma_{\perp}^v\times\hat{\mathbf{z}}e^{in\varphi}\partial_z\Lambda^v(t,z) + \hat{\mathbf{z}}\alpha^v e^{in\varphi}\Lambda^v(t,z). \tag{4}$$

where α^v is defined by equation (B-10). The functions $f^v(\rho)$ and $g^v(\rho)$ are the solutions to the Bessel equation (equation (B-5)). Hence, one could assume the solution to $g^v(\rho)$ to be:

$$g^v(\rho) = \frac{1}{\sqrt{F^v}}J_n(k_\rho^m\rho), \tag{5}$$

where F^v is a normalization factor.

The gauge condition (equation A-8) for the dual potential, equation (2), is:

$$\nabla\cdot\Lambda(t,\mathbf{x}) = \frac{1}{\rho}\left(\rho\frac{dg^v(\rho)}{d\rho} + g^v(\rho) + inf^v(\rho)\right)e^{in\varphi}\Lambda(t,z) = 0 \rightarrow \rho\frac{dg^v(\rho)}{d\rho} + g^v(\rho) + inf^v(\rho) = 0 \tag{6}$$

and enables us to find $f^v(\rho)$ as:

$$f^v(\rho) = i(\frac{n+1}{n\sqrt{F^v}})J_n(k_\rho^m\rho) - \frac{i}{n\sqrt{F^v}}J_{n+1}(k_\rho^m\rho) \tag{7}$$

which is also a solution to equation (B-5). In order to find F^v, the condition:

$$\int_0^\infty d\rho\int_0^{2\pi} d\varphi\rho\left(f^{-v}(\rho)f^v(\rho) + g^{-v}(\rho)g^v(\rho)\right) = 1 \tag{8}$$

should be satisfied.

It is assumed that the response tensor for the medium is homogenous and isotropic $\left(\eta_v(\mathbf{x}) = \eta_v\right)$ and that the first non-zero nonlinear term corresponds to $\eta^{(3)}$ (centro-symmetric media). The Lagrangian (equation (A-18)) and the Hamiltonian (equation (A-19)) can be, respectively, simplified to:

$$L_0 = \frac{1}{2}\sum_{v=-N}^{v=N}\int\Big\{\Big[-\eta_v\big[\partial_z\Lambda^{-v}(t,z)\partial_z\Lambda^v(t,z)+Y^v\Lambda^{-v}(t,z)\Lambda^v(t,z)\big]$$

$$+\frac{i}{2}\eta_v'\big[\partial_z\Lambda^v(t,z)\partial_z\dot\Lambda^{-v}(t,z)+Y^v\Lambda^v(t,z)\dot\Lambda^{-v}(t,z)-\partial_z\Lambda^{-v}(t,z)\partial_z\dot\Lambda^v(t,z)-Y^v\Lambda^{-v}(t,z)\dot\Lambda^v(t,z)\big]$$

$$+\frac{i}{3}\eta_v'''\big[\partial_z\ddot\Lambda^{-v}(t,z)\partial_z\dot\Lambda^v(t,z)+Y^v\ddot\Lambda^{-v}(t,z)\dot\Lambda^v(t,z)-\partial_z\ddot\Lambda^v(t,z)\partial_z\dot\Lambda^{-v}(t,z)-Y^v\ddot\Lambda^v(t,z)\dot\Lambda^{-v}(t,z)\big] \qquad (9)$$

$$+\frac{i}{6}\eta_v'''\big[\partial_z\Lambda^v(t,z)\partial_z\ddot\Lambda^{-v}(t,z)+Y^v\Lambda^v(t,z)\ddot\Lambda^{-v}(t,z)-\partial_z\Lambda^{-v}(t,z)\partial_z\ddot\Lambda^v(t,z)-Y^v\Lambda^{-v}(t,z)\ddot\Lambda^v(t,z)\big]$$

$$-\frac{1}{2}\eta_v''\big[\partial_z\dot\Lambda^{-v}(t,z)\partial_z\dot\Lambda^v(t,z)+Y^v\dot\Lambda^{-v}(t,z)\dot\Lambda^v(t,z)\big]\Big]+\mu\big(\dot\Lambda^{-v}(t,z)\dot\Lambda^v(t,z)\big)\Big\}dz$$

and

$$H_0 = \frac{1}{2}\sum_{v=-N}^{v=N}\int\Big\{\Big[-\eta_v\big(\partial_z\Lambda^{-v}(t,z)\partial_z\Lambda^v(t,z)+Y^v\Lambda^{-v}(t,z)\Lambda^v(t,z)\big)$$

$$+\frac{i}{6}\eta_v'''\big(\partial_z\ddot\Lambda^{-v}(t,z)\partial_z\Lambda^v(t,z)+Y^v\ddot\Lambda^{-v}(t,z)\Lambda^v(t,z)-\partial_z\ddot\Lambda^v(t,z)\partial_z\Lambda^{-v}(t,z)-Y^v\ddot\Lambda^v(t,z)\Lambda^v(t,z)\big) \qquad (10)$$

$$-\frac{1}{2}\eta_v''\big(\partial_z\dot\Lambda^{-v}(t,z)\partial_z\dot\Lambda^v(t,z)+Y^v\dot\Lambda^{-v}(t,z)\dot\Lambda^v(t,z)\big)\Big]+\mu\big(\dot\Lambda^{-v}(t,z)\dot\Lambda^v(t,z)\big)\Big\}dz$$

where

$$Y^v = 2\pi\int_0^\infty\left[\frac{1}{\rho}\big((d\rho f^v/d\rho)(d\rho f^{-v}/d\rho)+n^2g^vg^{-v}+ing^{-v}(d\rho f^v/d\rho)-ing^v(d\rho f^{-v}/d\rho)\big)\right]d\rho . \qquad (11)$$

Let us define the scalar field $\Lambda^v(t,z)$ in terms of the canonical coordinates $\lambda_{k_z}^v(t)$ as $\Lambda^v(t,z)=\left(1/\sqrt{L}\right)\sum_{k_z}\lambda_{k_z}^v(t)e^{ik_zz}$. For simplicity, we rename k_z by k. Therefore, the Lagrangian (9) for the field propagation along the fiber can be written as:

$$L_0 = \frac{1}{2}\sum_{v,k}Q_k^v\left[-\lambda_{-k}^{-v}\lambda_k^v\eta_v-\frac{i}{2}\big(\lambda_{-k}^{-v}\dot\lambda_k^v-\dot\lambda_{-k}^{-v}\lambda_k^v\big)\eta_v'+M_k^v\dot\lambda_{-k}^{-v}\dot\lambda_k^v+\frac{i}{3}\big(\ddot\lambda_{-k}^{-v}\dot\lambda_k^v-\dot\lambda_{-k}^{-v}\ddot\lambda_k^v\big)\eta_v'''+\frac{i}{6}\big(\ddot\lambda_{-k}^{-v}\lambda_k^v-\lambda_{-k}^{-v}\ddot\lambda_k^v\big)\eta_v'''\right] \qquad (12)$$

where Q_k^v is defined as:

$$Q_k^v = (k_z^2+Y^v)=(k^2+Y^v) \qquad (13)$$

and $M_k^v = \mu(Q_k^v)^{-1}-\eta_v''/2$. As noted earlier, a single transverse mode v_0 ($v_0=(n_0,m_0)$ for mode numbers n_0 and m_0) will be considered; then, the canonical momenta associated with $\lambda_k\equiv\lambda_k^{v_0}$ and $\lambda_k^\dagger\equiv\lambda_{-k}^{-v_0}$ are:

$$\pi_k = \frac{1}{12}\Big[-3i\eta_{v_0}'\lambda_k^\dagger+6M_k^{v_0}\dot\lambda_k^\dagger+2i\eta_{v_0}'''\ddot\lambda_k^\dagger\Big]Q_k^{v_0} \qquad (14)$$

and

$$\pi_k^\dagger = \frac{1}{12}\Big[3i\eta_{v_0}'\lambda_k+6M_k^{v_0}\dot\lambda_k-2i\eta_{v_0}'''\ddot\lambda_k\Big]Q_k^{v_0} , \qquad (15)$$

respectively. Similarly, one can find that the linear part of the Hamiltonian, H_0, for field propagation along the optical fiber is:

$$H_0 = \sum_{k_z} \frac{Q_k^{v_0}}{M_k^{v_0}} \left[\left(\frac{\pi_k}{Q_k^{v_0}} + i\frac{\eta'_{v_0}}{2}\lambda_{k_z}^\dagger \right) \left(\frac{\pi_k^\dagger}{Q_k^{v_0}} - i\frac{\eta'_{v_0}}{2}\lambda_k \right) + \eta_{v_0} M_k^{v_0} \lambda_k^\dagger \lambda_k + i\frac{\eta'''_{v_0}}{3} \left(\frac{\pi_k}{Q_k^{v_0}} + i\frac{\eta'_{v_0}}{2}\lambda_k^\dagger \right) \ddot{\lambda}_k \right.$$

$$\left. + \frac{i\eta'''_{v_0} M_k^{v_0}}{12} \left(\lambda_k^\dagger \dddot{\lambda}_k - \dddot{\lambda}_k^\dagger \lambda_k \right) - i\frac{\eta'''_{v_0}}{3}\ddot{\lambda}_k^\dagger \left(\frac{\pi_k^\dagger}{Q_k^{v_0}} - i\frac{\eta'_{v_0}}{2}\lambda_k \right) + \left(\frac{\eta'''_{v_0}}{3} \right)^2 \ddot{\lambda}_k^\dagger \ddot{\lambda}_k \right]. \tag{16}$$

In this model, the field is quantized when operators λ_k and π_k obey the following commutation relation:

$$\left[\hat{\lambda}_k, \hat{\pi}_{k'} \right] = i\hbar\delta_{kk'}. \tag{17}$$

An annihilation operator is defined as:

$$\hat{a}_k = \frac{1}{\sqrt{2\hbar}} \left(A_k \hat{\lambda}_k + i(A_k^*)^{-1}\hat{\pi}_k^\dagger \right), \tag{18}$$

where A_k is a complex number. Therefore, the creation operator is:

$$\hat{a}_k^\dagger = \frac{1}{\sqrt{2\hbar}} \left(A_k^* \hat{\lambda}_k^\dagger - i(A_k)^{-1}\hat{\pi}_k^\dagger \right). \tag{19}$$

One can write the linear Hamiltonian as an expansion in terms of the creation and annihilation operators:

$$\hat{H}_0 = \hbar\sum_k \omega(k)\hat{a}_k^\dagger\hat{a}_k, \tag{20}$$

where the equation of motion for the annihilation operators is $\hat{\dot{a}}_k = -i\omega(k)\hat{a}_k$. Making use of the equations (A-17) and (A-10) by equating two forms of the Hamiltonian (equations (16) and (20)), A_k is given as:

$$\left| A_k \right|^4 = \left(Q_k^{v_0} \right)^2 \left[\eta_{v_0} M_k^{v_0} + \frac{1}{4}\left(\eta'_{v_0} \right)^2 + \frac{1}{3}\omega^2\eta'_{v_0} \cdot \eta'''_{v_0} - \frac{1}{6}\omega^3 \left(\eta'''_{v_0} \right) M_k^{v_0} + \frac{1}{9}\omega^4 \left(\eta'''_{v_0} \right)^2 \right] \tag{21}$$

and $\omega(k)$ is the solution to the equation:

$$\omega(k) = \left[A_k + \left(\frac{1}{2}\eta'_{v_0} + \frac{1}{3}\eta'''_{v_0}\omega^2(k) \right)\left(A_k^* \right)^{-1} \right]\left(M_k^{v_0} \right)^{-1} A_k^*. \tag{22}$$

We also have

$$\left| A_k \right|^2 = Q_{k_z}^{v_0} \left[\frac{1}{2}\eta'_{v_0} + \frac{1}{3}\eta'''_{v_0}\omega^2 \right]. \tag{23}$$

The formalism introduced here can describe the propagation of quantum fields in nonlinear dispersive media, specifically in optical fibers. Assuming the field wavenumber and frequency to be near k_0 and $\omega = \omega(k_0)$, respectively, the slowly varying quantum photon-polariton field is defined as:

$$\psi(z,t) = \frac{1}{\sqrt{L}}\sum_k e^{i(k-k_0)z}\hat{a}_k(t) \tag{24}$$

to describe the field propagation along a fiber with nonlinearity $\chi^{(3)}$.

By ignoring the smearing effect for all practical purposes, the commutation relation for these fields can be expressed as:

$$\left[\psi(z,t),\psi^{\dagger}(z',t)\right]=\delta(z-z'),\tag{25}$$

which is a simple case of the general form:

$$\left[\psi(z,t),\psi^{\dagger}(z',t')\right]=\delta(z-z')\delta(t-t'),\tag{26}$$

also applicable for temporally ultrashort fields. Inverting the relation between $\hat{a}_k$ and $\psi(z,t)$ yields:

$$\hat{a}_k(t)=\frac{1}{\sqrt{L}}\int\psi(z,t)e^{-i(k-k_0)z}dz.\tag{27}$$

One can insert equation (27) into equation (20) to express the Hamiltonian in terms of $\psi(z,t)$. Thus, the linear part of Hamiltonian can be written as:

$$\hat{H}_0=\hbar\sum_k\omega(k)\hat{a}_k^{\dagger}\hat{a}_k=\frac{\hbar}{L}\int dz\int dz'\left(\sum_k\omega(k)e^{i(k-k_0)(z-z')}\right)\psi^{\dagger}(z,t)\psi(z',t).\tag{28}$$

Expanding $\omega(k)$ near k_0 as:

$$\omega(k)=\omega(k_0)+(k-k_0)\omega'+\frac{1}{2}(k-k_0)^2\omega''++\frac{1}{6}(k-k_0)^3\omega'''+\cdots\tag{29}$$

where ω', ω'' and ω''' are the group velocity and the first and second derivatives of group velocity, respectively, the expression in parenthesis in equation (28) becomes:

$$\frac{1}{L}\sum_k\omega(k)e^{i(k-k_0)(z-z')}\cong\frac{1}{L}\sum_k\left[\omega(k_0)+\omega'(k-k_0)+\frac{1}{2}\omega''(k-k_0)^2+\frac{1}{6}\omega'''(k-k_0)^3\right]e^{i(k-k_0)(z-z')}\tag{30}$$

where the operator form is taken as:

$$\frac{1}{L}\sum_k\omega(k)e^{i(k-k_0)(z-z')}\cong\frac{1}{L}\sum_k\left[\omega(k_0)+\frac{i}{2}\omega'(\partial_{z'}-\partial_z)+\frac{1}{2}\omega''\partial_z\partial_{z'}+\frac{i}{12}\omega'''\left(\partial_z(\partial_{z'})^2-(\partial_z)^2\partial_{z'}\right)\right]e^{i(k-k_0)(z-z')}$$
$$\cong\left[\omega(k_0)+\frac{i}{2}\omega'(\partial_{z'}-\partial_z)+\frac{1}{2}\omega''\partial_z\partial_{z'}+\frac{i}{12}\omega'''\left(\partial_z(\partial_{z'})^2-(\partial_z)^2\partial_{z'}\right)\right]\delta(z-z').\tag{31}$$

The linear part of Hamiltonian is, therefore, given by:

$$\hat{H}_0=\hbar\int\left[\omega(k_0)\hat{\psi}^{\dagger}\hat{\psi}+\frac{i}{2}\omega'\left(\hat{\psi}^{\dagger}\frac{\partial\hat{\psi}}{\partial z}-\hat{\psi}\frac{\partial\hat{\psi}^{\dagger}}{\partial z}\right)+\frac{1}{2}\omega''\frac{\partial\hat{\psi}^{\dagger}}{\partial z}\frac{\partial\hat{\psi}}{\partial z}+\frac{i}{12}\omega'''\left(\frac{\partial\hat{\psi}^{\dagger}}{\partial z}\left(\frac{\partial^2\hat{\psi}}{\partial z^2}\right)-\left(\frac{\partial^2\hat{\psi}^{\dagger}}{\partial z^2}\right)\frac{\partial\hat{\psi}}{\partial z}\right)\right]dz,\tag{32}$$

where the argument of the photon-polariton field has been eliminated for simplicity. The nonlinear part of the Hamiltonian is given by:

$$\hat{H}^{NL}=\frac{1}{4}\int d^3\mathbf{x}\eta^{(3)}\cdot\left(\hat{\mathbf{D}}(\mathbf{x},t)\right)^4,\tag{33}$$

where the displacement vector is defined as:

$$\hat{\mathbf{D}}(\mathbf{x},t)=\hat{\mathbf{D}}^{v_0}(\mathbf{x},t)+\hat{\mathbf{D}}^{-v_0}(\mathbf{x},t),\tag{34}$$

for

$$\hat{\mathbf{D}}^{v_0}(\mathbf{x}, t) = \sqrt{\frac{\hbar}{2}} \sum_k \frac{1}{A_k L} \left[ik \mathbf{\Gamma}_\perp^{v_0} \times \hat{\mathbf{z}} + \hat{\mathbf{z}} \alpha^{v_0} \right] e^{in\varphi} \int dz' \left(e^{ik(z-z') + ik_0 z'} \hat{\psi}(z') \right) \tag{35}$$

and the functions $\mathbf{\Gamma}_\perp^{v_0}$ and α^{v_0} are defined in Appendix C. Therefore, the nonlinear part of the Hamiltonian can be approximated as:

$$\hat{H}^{NL} \cong \frac{\eta^{(3)}}{4} \left(\sqrt{\frac{\hbar}{2}} \frac{1}{A_{k_0}} \right)^4 \int \rho d\rho d\varphi dz \left[\left(ik \mathbf{\Gamma}_\perp^{v_0} \times \hat{\mathbf{z}} + \hat{\mathbf{z}} \alpha^{v_0} \right) e^{i(n\varphi + k_0 z)} \hat{\psi} - \left(-ik \mathbf{\Gamma}_\perp^{-v_0} \times \hat{\mathbf{z}} + \hat{\mathbf{z}} \alpha^{-v_0} \right) e^{-i(n\varphi + k_0 z)} \hat{\psi}^\dagger \right]^4 . \tag{36}$$

Hence, keeping the slowly varying terms, one obtains:

$$\hat{H}^{NL} \cong \frac{3\hbar^2 \eta^{(3)}}{8} \left(\frac{1}{A_{k_0}} \right)^4 \theta^{v_0} \int dz \left(\hat{\psi}^\dagger \right)^2 \hat{\psi}^2 . \tag{37}$$

where θ^{v_0} is defined in Appendix C. It should be noted that the quantities F^{v_0}, Y^{v_0} and θ^{v_0} can be obtained by using prevalent computational software (e.g. Mathematica). There are terms proportional to $\hat{\psi}^\dagger \hat{\psi}$ as well as terms without any order of $\hat{\psi}^\dagger$ and/or $\hat{\psi}$ in the nonlinear part of the Hamiltonian (equation (36)), which we herein called simple terms. They are called simple terms because they can be interpreted similarly to the first term of the linear Hamiltonian (equation (36)); therefore, they do not contribute to the nonlinear terms. This point will be discussed further, as the total Hamiltonian is incorporated into the interaction picture later.

The total Hamiltonian and field operator can be conveyed into the interaction picture as:

$$\hat{H}_I = e^{-i\hat{h}t/\hbar} \hat{H} e^{i\hat{h}t/\hbar} \tag{38}$$

and

$$\hat{\psi}_I = e^{-i\hat{h}t/\hbar} \hat{\psi} e^{i\hat{h}t/\hbar} \tag{39}$$

where $\hat{h}$ is defined as:

$$\hat{h} = \hbar \int dz \, \omega(k_0) \hat{\psi}^\dagger \hat{\psi} . \tag{40}$$

We also define a new frame that moves at the group velocity ($Z = z - \omega' t$). One can obtain the following equation of motion in this frame from the interaction Hamiltonian:

$$i \frac{\partial \hat{\psi}_I}{\partial t} = -\frac{\omega''}{2} \frac{\partial^2 \hat{\psi}_I}{\partial Z^2} - i \frac{\omega'''}{6} \frac{\partial^3 \hat{\psi}_I}{\partial Z^3} + g \hat{\psi}_I^\dagger \hat{\psi}_I^2 , \tag{41}$$

where $g = (3\hbar^2 \eta^{(3)}/4)(\theta^{v_0})^2 (k_0/A_{k_0})^4$. This equation is an operator form of the nonlinear Schrodinger equation, taking the third-order dispersion term into account. There is another operator equation, the equation of motion for $\hat{\psi}_I^\dagger$, which is coupled to equation (41) by transforming $\hat{\psi}_I \rightarrow \hat{\psi}_I^\dagger$ and $i \rightarrow -i$.

Depending on the fiber characteristics, the signs of g (nonlinear parameter) and ω'' (group velocity dispersion) can be positive or negative. Taking into account their signs, one can form the quantum

solitons and study their interactions in the presence of the third-order dispersion term. Note that the fourth order term in equation (37) leads to the last term in equation (41). The terms of the order $\hat{\psi}^{\dagger}\hat{\psi}$ in equation (37), which have been omitted, would lead to a linear term and have no nonlinear effect on the operator form of the nonlinear Schrodinger equation, equation (41) or equivalently equation (42). These terms just change $\omega(k_0)$ in equation (32), so the nonlinear term of equations (41) and (42) are not affected by the simple terms.

To solve the coupled operator equation, equation (41), one can rewrite it as common partial differential equations. Therefore, we take into account the results of the positive P-representation [33,34]. The Glauber-Sudarshan P-representation [35,36] is not used, here, as it leads to a Fokker-Planck equation with non-positive definite diffusion coefficients. By using the positive P-representation in the master equation, one can arrive at the coupled stochastic partial differential equation [23,33.34,37]:

$$\frac{\partial}{\partial z}\Psi_I(T,z) = \left(-\frac{i\omega''}{2\omega'^3}\frac{\partial^2}{\partial T^2} + \frac{\omega'''}{6\omega'^4}\frac{\partial^3}{\partial T^3}\right)\Psi_I(T,z) - i\left(g/\omega'\right)\Psi_I^{+}(T,z)\Psi_I^{2}(T,z) + \left(ig/\omega'\right)^{1/2}\xi(T,z)\Psi_I(T,z) \tag{42a}$$

and

$$\frac{\partial}{\partial z}\Psi_I^{+}(T,z) = \left(\frac{i\omega''}{2\omega'^3}\frac{\partial^2}{\partial T^2} + \frac{\omega'''}{6\omega'^4}\frac{\partial^3}{\partial T^3}\right)\Psi_I^{+}(T,z) + i\left(g/\omega'\right)\Psi_I(T,z)\Psi_I^{+2}(T,z) + \left(-ig/\omega'\right)^{1/2}\xi^{+}(T,z)\Psi_I^{+}(T,z) \tag{42b}$$

for the functions $\Psi_I(T,z)$ and $\Psi_I^{+}(T,z)$, respectively, which are related to the creation and annihilation operators. The fields $\xi(T,z)$ and $\xi^{+}(T,z)$ are Gaussian stochastic fields with correlation relations:

$$\left\langle\xi(T_1,z_1)\xi(T_2,z_2)\right\rangle = \delta(z_1 - z_2)\delta(T_1 - T_2), \tag{43}$$

$$\left\langle\xi^{+}(T_1,z_1)\xi^{+}(T_2,z_2)\right\rangle = \delta(z_1 - z_2)\delta(T_1 - T_2), \tag{44}$$

and

$$\left\langle\xi(T_1,z_1)\xi^{+}(T_2,z_2)\right\rangle = 0. \tag{45}$$

The origin of stochastic fields, equations (42), comes from the positive P-representation. Making use of the positive P-representation, the Fokker-Planck equation with positive semi-definite diffusion coefficients amounts to an equivalent Ito stochastic differential equation, equations (42), [18,23,37-39]. $\Psi_I(T,z)$ and $\Psi_I^{+}(T,z)$ are not complex conjugates of one another except in the mean case [38,39].

Equation (42) governs the quantum treatment of pulse propagation in an optical fiber in the presence of the third-order dispersion term $(\beta_3 = \omega'''/\omega'^4)$. To increase the bit rate in optical communication systems, a light source (a laser) is usually employed close to the ZDW

$(\beta_2 = \omega''/\omega'^3 \big|_{\omega_0} = 0$ i.e. $\omega'' = 0$) of the optical fiber, where β_3 plays an important role [4]. Thus, equations (42) provide the quantum treatment of pulse propagation in this situation.

To study the quantum noises in near the propagating solitons by using the available definitions and results related to this topic [33], the linearized fluctuation equation [33] for the propagating soliton is:

$$\frac{\partial}{\partial z} \delta\Psi(T,z) = \left(-\frac{i\omega''}{2\omega'^3} \frac{\partial^2}{\partial T^2} + \frac{\omega'''}{6\omega'^4} \frac{\partial^3}{\partial T^3} + 4ig\psi_0^2(T) \right) \delta\Psi(T,z) + 2ig\psi_0^2(T)\delta\Psi^+(T,z) + \left(ig\right)^{1/2}\psi_0(T)\xi(T,z) \quad (46)$$

where $\Psi(T,z) = \psi_0(T,z) + \delta\Psi(T,z)$ and $\psi_0(T,z) = \langle \Psi(T,z) \rangle$. Recall that the above equation together with the equation obtained by using the transformation, $\Psi \to \Psi^+$, $i \to -i$, and $\xi \to \xi^+$ form a set of coupled equations.

Here, ψ_0 can be considered as a classical solution to the generalized first-order approximation of thee nonlinear Schrodinger equation. The function ψ_0 corresponds to a classical coherent state input at z=0, which has a mathematical form of $\psi_0(T, z=0) = \sqrt{P_0}\,\text{sech}(T/T_0)$, where P_0 and T_0 are, respectively, the peak power and width of the input pulse launched into the optical fiber.

III. Simulation Results

We intend to study the significant role of β_3 in increasing the bit rates of optical communication system. Dispersion-shifted fibers (DSFs) have been designed to take advantage of high bit rates by shifting the ZDW into the 1550nm band, i.e. $\beta_2(\lambda \simeq 1550nm) = 0$. DSFs are ideal for maximizing the bit rate in optical communication systems in the 1550nm window, which is suitable for single-channel applications. However, it turns out that having zero-dispersion at 1550nm is a negative condition for dense wavelength division multiplexing (DWDM) [8,41] applications. The "absence" of chromatic dispersion enhances nonlinear impairments such as four-wave mixing (FWM) and cross-phase modulation (XPM), limiting the ultimate system performance. To counteract the role of nonlinear effects in DWDM systems and still benefit from reduced dispersion in the 1550nm region, a new generation of fibers called non-zero dispersion-shifted fibers (NZDSFs) has been developed [41]. In this type of fibers, the dispersion at 1550nm is low, but not zero, with respect to those of DSF fibers. Due to the current variety of NZDSF applications [11,41], we considered this kind of fiber in this study. In the telecommunication standardization sector of the International Telecommunication Union, the G.655 fibers are categorized as NZDSFs.

III. Simulation Results

In this research, we simulated equations (42) in the mean case [38,39], where $\Psi_I(T,z)$ and $\Psi_I^+(T,z)$ are complex conjugates of one another. We simulated an input pulse with a peak power of 10W and width of 1ps propagating through an NZDSF, whose parameters are shown in table 1. The evolution of the pulse along the fiber at the lengths L equal to 100 m and 1000 m is shown in figures 1 and 2, respectively. These figures illustrate the dispersion effects, nonlinear effects and their interaction. Figure 2 shows the interaction more clearly in the intervals which the pulse broadening is compensated for by the nonlinear effects. These simulation results agree with those presented in [42]. Both sets of results show similar pulse broadening and pulse asymmetry (due to the terms containing parameter g in equations 42 and 46), as well as the interaction between the dispersion and nonlinear effects and the generation of new frequencies due to nonlinear phenomena such as self-phase modulation, XPM and FWM.
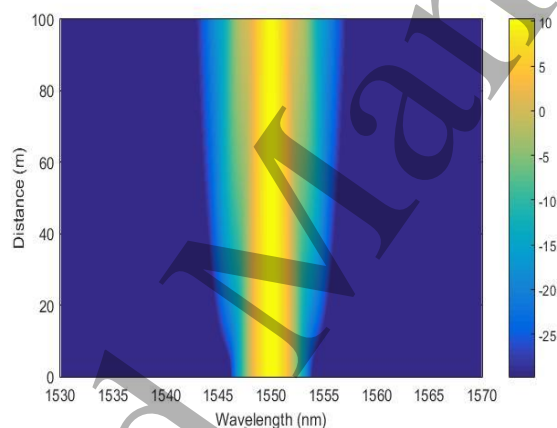

Figure 1. Pulse evolution along the optical fiber with length 100 m by launching a pulse ($\lambda = 1550$ nm) which having a peak power of 10W and a width of 1ps.
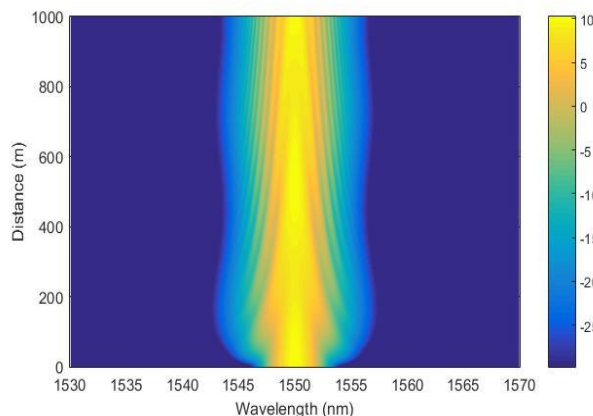
Figure 2. Pulse evolution along the 1000 m long optical fiber by launching a 100W light pulse ($\lambda = 1550$ nm) having a width of 1ps.

Table 1. Parameters for a special type of G.655 fibers (Alcatel 6912 Teralight Ultra) at λ=1550 nm.

Dispersion	7ps/(nm-km)
Effective mode area	63 μm^2
Slope of dispersion	0.052ps/(nm^2-km)
Nonlinear parameter	1.673 $W^{-1} km^{-1}$
β_2	-8.9281 (ps^2/km)
β_3	0.0993 (ps^3/km)

IV. Concluding Remarks

In this study, the Hamiltonian and the creation and annihilation operators were obtained for pulse propagation via an optical fiber in the presence of the third-order dispersion coefficient by applying the quantum theory for field propagation through a dielectric to single-dimensional pulse propagation through an optical fiber. The quantum photon-polariton field was introduced and the equations of motion for $\hat{\psi}_I$ and $\hat{\psi}_I^\dagger$ were given by expanding the frequency about ω_0. These coupled equations, given as equations (42), are the operator forms of the nonlinear Schrodinger equation and treat pulse propagation through an optical fiber quantum mechanically, when the third-order dispersion term is included. These equations enabled us to study the quantum solitons when the input wavelength was equal to the ZDW of the fiber (this condition is approximated in optical communication systems because the bit rate is usually increased in such cases) and to entangle these solitons for some applications (e.g., quantitative experiments). By applying the positive P-representation in the master equation, a coupled stochastic nonlinear partial differential equations

for $\Psi_I(T,x)$ and $\Psi_I^+(T,x)$, which generally are not the complex conjugates of one another, were finally obtained. In addition, we derived coupled linearized fluctuation equations to reduce the quantum noise around the solitons. It must be mentioned that the linearization is not valid, due to the dependence on the fiber pulse characteristics. Finally, we simulated the obtained equation in the mean case and confirmed the agreement between our simulation results and those available in the literature. It is possible to develop the quantum theory introduced here for pulse propagation in an optical fiber with a retarded medium response.

Further work on the propagation of light in an optical fiber is needed as the bandwidth of telecommunication via fiber lines increase. Today, cable television, internet communication and telephone signal are being transmitted through fiber, where shorter light pulses are needed. Pulses of pico-second or shorter length are to be transmitted through fibers which could be described quantum mechanically. More experimental and theoretical studies are needed for better use of optical fiber at these extremes. It is also important to understand the noise involved in optical fiber systems, specifically the quantum noise.

Acknowledgment

Appendix A:

The time-averaged linear dispersive energy for a monochromatic field at frequency ω is:

$$\langle H \rangle = \int \left\{ \frac{1}{2} \mathbf{E}^{*}(t,\mathbf{x}) \cdot \frac{\partial}{\partial \omega} \big(\omega \boldsymbol{\varepsilon}(\omega,\mathbf{x}) \big) \cdot \mathbf{E}(t,\mathbf{x}) + \frac{1}{2\mu} \big(\mathbf{B}^{*}(t,\mathbf{x}) \cdot \mathbf{B}(t,\mathbf{x}) \big) \right\} d^{3}\mathbf{x} , \qquad (A-1)$$

where $\mathbf{E}(t,\mathbf{x})$, $\mathbf{B}(t,\mathbf{x})$, and $\boldsymbol{\varepsilon}(\omega,\mathbf{x})$ are the electric field, the magnetic field and the permittivity of the medium, respectively. Equation (A-1) has the following equivalent form in terms of displacement fields $\mathbf{D}(t,\mathbf{x})$:

$$\langle H \rangle = \int \left\{ \frac{1}{2} \mathbf{D}^{*}(t,\mathbf{x}) \cdot \left(\eta(\omega,\mathbf{x}) - \omega \frac{\partial}{\partial \omega} \eta(\omega,\mathbf{x}) \right) \cdot \mathbf{D}(t,\mathbf{x}) + \frac{1}{2\mu} \big(\mathbf{B}^{*}(t,\mathbf{x}) \cdot \mathbf{B}(t,\mathbf{x}) \big) \right\} d^{3}\mathbf{x} , \qquad (A-2)$$

where $\eta(\omega,\mathbf{x}) = \boldsymbol{\varepsilon}^{-1}(\omega,\mathbf{x})$.

The dual potential, $\boldsymbol{\Lambda}(t,\mathbf{x})$, can be defined in terms of a $2N+1$ narrow-band dual potential $\boldsymbol{\Lambda}^{\nu}(t,\mathbf{x})$ as:

$$\boldsymbol{\Lambda}(t,\mathbf{x}) = \sum_{\nu=-N}^{\nu=N} \boldsymbol{\Lambda}^{\nu}(t,\mathbf{x}) , \qquad (A-3)$$

for a cavity or the waveguide in mode ν. The electric displacement and magnetic fields, respectively, can be written as:

$$\mathbf{D}(t,\mathbf{x}) = \nabla \times \boldsymbol{\Lambda}(t,\mathbf{x}) \qquad (A-4)$$

and

$$\mathbf{H}(t,\mathbf{x}) = \dot{\boldsymbol{\Lambda}}(t,\mathbf{x}) . \qquad (A-5)$$

These definitions of the electric and magnetic fields in terms of the dual potential readily results in Maxwell's equations:

$$\nabla \cdot \mathbf{D}(t,\mathbf{x}) = 0 \qquad (A-6)$$

and

$$\nabla \times \mathbf{H}(t,\mathbf{x}) = \partial \mathbf{D}(t,\mathbf{x}) / \partial t \qquad (A-7)$$

as the medium is charge- and current- free. The dual potential, $\boldsymbol{\Lambda}(t,\mathbf{x})$, is similar to vector potential in some respects but differs in other respects because their origins are different [31]. The magnetic vector potential could not derive both fields without the scaler electric potential, while we were able to define both electric and magnetic fields employing the dual potential, $\boldsymbol{\Lambda}(t,\mathbf{x})$. In fact, it is a mathematical tool for quantization procedure.

In order that the fields would satisfy the Maxwell's equation, $\nabla \cdot \mathbf{B}(t,\mathbf{x}) = 0$, we defined a gauge:

$$\nabla \cdot \boldsymbol{\Lambda}^{\nu}(t,\mathbf{x}) = 0 . \qquad (A-8)$$

Finally, the wave equation in a medium where $\eta(\omega) \neq \eta(\omega,\mathbf{x})$ takes the form:

$$\nabla^2 \mathbf{\Lambda}^{\nu}(\omega, \mathbf{x}) + \mu \varepsilon \omega^2 \mathbf{\Lambda}^{\nu}(\omega, \mathbf{x}) = 0, \tag{A-9}$$

making use of Maxwell's equation:

$$\nabla \times \mathbf{E}(t, \mathbf{x}) = -\partial \mathbf{B}(t, \mathbf{x})/\partial t. \tag{A-10}$$

Let us expand $\eta(\omega, \mathbf{x})$ in Taylor series near the narrow-field frequency, w^n, as:

$$\eta(\omega, \mathbf{x}) = \eta_{\nu}(\omega^{\nu}, \mathbf{x}) + (\omega - \omega^{\nu})\eta_{\nu}'(\omega^{\nu}, \mathbf{x}) + \frac{1}{2}(\omega - \omega^{\nu})^2 \eta_{\nu}''(\omega^{\nu}, \mathbf{x})$$
$$+ \frac{1}{6}(\omega - \omega^{\nu})^3 \eta_{\nu}'''(\omega^{\nu}, \mathbf{x}) + O\left((\omega - \omega^{\nu})^4\right) \tag{A-11}$$

leading to:

$$\eta(\omega, \mathbf{x}) \approx \eta_{\nu}(\mathbf{x}) + \omega \eta_{\nu}'(\mathbf{x}) + \frac{1}{2}\omega^2 \eta_{\nu}''(\mathbf{x}) + \frac{1}{6}\omega^3 \eta_{\nu}'''(\mathbf{x}) + O\left((\omega - \omega^{\nu})^4\right) \tag{A-12}$$

where

$$\eta_{\nu}(\mathbf{x}) = \eta_{\nu}(\omega^{\nu}, \mathbf{x}) - \omega^{\nu} \eta_{\nu}'(\omega^{\nu}, \mathbf{x}) + \frac{1}{2}(\omega^{\nu})^2 \eta_{\nu}''(\omega^{\nu}, \mathbf{x}) - \frac{1}{6}(\omega^{\nu})^3 \eta_{\nu}'''(\omega^{\nu}, \mathbf{x}), \tag{A-13}$$

$$\eta_{\nu}'(\mathbf{x}) = \eta_{\nu}'(\omega^{\nu}, \mathbf{x}) - \omega^{\nu} \eta_{\nu}''(\omega^{\nu}, \mathbf{x}) + \frac{1}{2}(\omega^{\nu})^2 \eta_{\nu}'''(\omega^{\nu}, \mathbf{x}), \tag{A-14}$$

$$\eta_{\nu}''(\mathbf{x}) = \eta_{\nu}''(\omega^{\nu}, \mathbf{x}) - (\omega^{\nu})^2 \eta_{\nu}'''(\omega^{\nu}, \mathbf{x}), \tag{A-15}$$

and

$$\eta_{\nu}'''(\mathbf{x}) = \eta_{\nu}'''(\omega^{\nu}, \mathbf{x}). \tag{A-16}$$

The higher-order derivatives of expansion (A-11) are negligible with respect to the second- (if non-zero) and third-order terms in many optical fiber applications, making this expansion valid for nearly all applications.

The average linear dispersive energy and first nonlinear energy for centro-symmetric media is:

$$\langle H \rangle = H_0 + H_{NL} = \frac{1}{2}\int \left\{ \left[\mathbf{D}^* \cdot \left(\eta_{\nu}(\mathbf{x}) - \frac{1}{2}(\omega^{\nu})^2 \eta_{\nu}''(\mathbf{x}) - \frac{1}{3}(\omega^{\nu})^3 \eta_{\nu}'''(\mathbf{x}) \right) \cdot \mathbf{D} + (\mathbf{B}^* \cdot \mathbf{B})/2\mu \right] + \frac{1}{2}\mathbf{D} \cdot \eta^{(3)}(\mathbf{x}) \vdots \mathbf{DDD} \right\} d^3\mathbf{x}. \tag{A-17}$$

Making use of the slowly varying envelope approximation and the superposition principle over narrow-band field numbers -N to N, the Lagrangian:

$$L_0 = \frac{1}{2}\sum_{\nu=-N}^{\nu=N}\int \left\{ \left[-\left(\nabla \times \mathbf{\Lambda}^{-\nu}(t, \mathbf{x})\right) \cdot \eta_{\nu}(\mathbf{x}) \cdot \left(\nabla \times \mathbf{\Lambda}^{\nu}(t, \mathbf{x})\right) - \frac{i}{2}\left(\nabla \times \mathbf{\Lambda}^{-\nu}(t, \mathbf{x})\right) \cdot \eta_{\nu}'(\mathbf{x}) \cdot \left(\nabla \times \dot{\mathbf{\Lambda}}^{\nu}(t, \mathbf{x})\right) \right. \right.$$
$$+ \frac{i}{2}\left(\nabla \times \mathbf{\Lambda}^{\nu}(t, \mathbf{x})\right) \cdot \eta_{\nu}'(\mathbf{x}) \cdot \left(\nabla \times \dot{\mathbf{\Lambda}}^{-\nu}(t, \mathbf{x})\right) - \frac{1}{2}\left(\nabla \times \dot{\mathbf{\Lambda}}^{-\nu}(t, \mathbf{x})\right) \cdot \eta_{\nu}''(\mathbf{x}) \cdot \left(\nabla \times \dot{\mathbf{\Lambda}}^{\nu}(t, \mathbf{x})\right)$$
$$+ \frac{i}{3}\left(\nabla \times \ddot{\mathbf{\Lambda}}^{-\nu}(t, \mathbf{x})\right) \cdot \eta_{\nu}'''(\mathbf{x}) \cdot \left(\nabla \times \dot{\mathbf{\Lambda}}^{\nu}(t, \mathbf{x})\right) - \frac{i}{3}\left(\nabla \times \ddot{\mathbf{\Lambda}}^{\nu}(t, \mathbf{x})\right) \cdot \eta_{\nu}'''(\mathbf{x}) \cdot \left(\nabla \times \dot{\mathbf{\Lambda}}^{-\nu}(t, \mathbf{x})\right) \tag{A-18}$$
$$- \frac{i}{6}\left(\nabla \times \mathbf{\Lambda}^{-\nu}(t, \mathbf{x})\right) \cdot \eta_{\nu}'''(\mathbf{x}) \cdot \left(\nabla \times \ddot{\mathbf{\Lambda}}^{\nu}(t, \mathbf{x})\right) + \frac{i}{6}\left(\nabla \times \mathbf{\Lambda}^{\nu}(t, \mathbf{x})\right) \cdot \eta_{\nu}'''(\mathbf{x}) \cdot \left(\nabla \times \ddot{\mathbf{\Lambda}}^{-\nu}(t, \mathbf{x})\right) \right]$$
$$\left. + \mu\left(\dot{\mathbf{\Lambda}}^{-\nu}(t, \mathbf{x}) \cdot \dot{\mathbf{\Lambda}}^{\nu}(t, \mathbf{x})\right) \right\} d^3\mathbf{x}$$

can be introduced to obtain the linear part of the Hamiltonian for a wideband field to be:

$$
\begin{aligned}
H_0 = \frac{1}{2}\sum_{\nu=-N}^{\nu=N}\int\Bigg\{\Bigg[&-\left(\nabla\times\boldsymbol{\Lambda}^{-\nu}(t,\mathbf{x})\right)\cdot\eta_\nu(\mathbf{x})\cdot\left(\nabla\times\boldsymbol{\Lambda}^{\nu}(t,\mathbf{x})\right)-\frac{1}{2}\left(\nabla\times\dot{\boldsymbol{\Lambda}}^{-\nu}(t,\mathbf{x})\right)\cdot\eta_\nu''(\mathbf{x})\cdot\left(\nabla\times\dot{\boldsymbol{\Lambda}}^{\nu}(t,\mathbf{x})\right) \\
&+\frac{i}{6}\left(\nabla\times\dddot{\boldsymbol{\Lambda}}^{-\nu}(t,\mathbf{x})\right)\cdot\eta_\nu'''(\mathbf{x})\cdot\left(\nabla\times\boldsymbol{\Lambda}^{\nu}(t,\mathbf{x})\right)-\frac{i}{6}\left(\nabla\times\boldsymbol{\Lambda}^{-\nu}(t,\mathbf{x})\right)\cdot\eta_\nu'''(\mathbf{x})\cdot\left(\nabla\times\dddot{\boldsymbol{\Lambda}}^{\nu}(t,\mathbf{x})\right)\Bigg] \\
&+\mu\left(\dot{\boldsymbol{\Lambda}}^{-\nu}(t,\mathbf{x})\cdot\dot{\boldsymbol{\Lambda}}^{\nu}(t,\mathbf{x})\right)\Bigg\}d^3\mathbf{x}.
\end{aligned}
\tag{A-19}
$$

It should be added that one can define different forms for the Lagrangian to derive the equations of motion, the Maxwell's equations. The correct Lagrangian is the one that enables the derived Hamiltonian to lead to the classical energy and the equations of motion. The Lagrangian is defined by resemblance with the terms in equations (A-11) or (A-12). The first (fourth) term in the Lagrangian in equation (A-18) corresponds to the first (third) term in equation (A-12) and did not require symmetrization as it is symmetric. Hence, the first and the forth terms in the Lagrangian describe the effect of the zero order permittivity (or dispersionless part of permittivity) and the dispersion of the media, respectively. The second (fourth) term in equation (A-12) led us to write the second (seventh) and the third (eighth) terms in equation (A-18), where symmetrization was needed. One of these terms describes the group velocity and the other one is the dispersion slope. The resulting linear part of the Hamiltonian has terms similar to the Lagrangian, except that the group velocity term, corresponding to $\eta_\nu'(\mathbf{x})$ in equation (A-17), is absent as this term is not present in $\eta(\omega,\mathbf{x})-\omega\partial\eta(\omega,\mathbf{x})/\partial\omega$. The fifth and the sixth term in the Lagrangian were needed to arrive at the correct equations of motion, which are similar to the slope of the dispersion. These two terms are not present in the Hamiltonian either. Finally, the last term of the Lagrangian and the Hamiltonian are the only term related to the magnetic field as permeability is considered dispersionless. The last term in the Hamiltonian is the part of energy related to the magnetic field.

Appendix B:

The derivation assumes a cylindrical waveguide whose axis lies along the z-axis, an example of which is an optical fiber. Maxwell's equations in the frequency domain lead to the wave equation (equation (A-9)) for the dual potential as:

$$
\nabla^2\boldsymbol{\Lambda}^{\nu}(\omega,\mathbf{x})+\mu\tilde{\varepsilon}\omega^2\boldsymbol{\Lambda}^{\nu}(\omega,\mathbf{x})=0
\tag{B-1}
$$

where $\tilde{\varepsilon}$ is the permittivity of the medium and the relation:

$$
\mu\tilde{\boldsymbol{\varepsilon}}\omega^2\equiv\mu\begin{pmatrix}\varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3\end{pmatrix}\omega^2=k^2\mathbf{1},
\tag{B-2}
$$

holds where the three εs are also assumed to be equal. Wave equation (B-1) could be written in cylindrical coordinates as:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Lambda_j^\nu}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Lambda_j^\nu}{\partial\varphi^2}+\frac{\partial^2\Lambda_j^\nu}{\partial z^2}+k^2\Lambda_j^\nu = 0 \tag{B-3}$$

where Λ_j^ν is any component of $\mathbf{\Lambda}^\nu(\omega,\mathbf{x})$ in the cylindrical coordinates. The general solution to equation (B-3) is:

$$\Lambda_j^\nu(\rho,\varphi,z) = F(k_\rho\rho)e^{in\varphi}Z(z) \tag{B-4}$$

where the function $Z(z)$ is the solution to the differential equation $\partial^2 Z/\partial z^2 + k_z^2 Z = 0$ and $k^2 = k_\rho^2 + k_z^2$. The function $F(k_\rho\rho)$ obeys the relation:

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dF(k_\rho\rho)}{d\rho}\right)+\left(k_\rho^2-\frac{n^2}{\rho^2}\right)F(k_\rho\rho) = 0 \tag{B-5}$$

where the solutions are Bessel functions. The boundary conditions lead to the quantized values for k_ρ, $k_\rho^{(m)}$ as the solutions to equations such as $h(k_\rho a) = 0$. Each transverse mode, ν, is defined by n and m and mode -ν correspond to -n and m. Therefore, the dual potential can be defined as:

$$\mathbf{\Lambda}^\nu(t,\mathbf{x}) = \mathbf{\Gamma}_\perp^\nu e^{in\varphi}\Lambda^\nu(t,z) \tag{B-6}$$

where

$$\mathbf{\Gamma}_\perp^\nu = \hat{\boldsymbol{\rho}}g^\nu(\rho)+\hat{\boldsymbol{\varphi}}f^\nu(\rho) \tag{B-7}$$

and $f^\nu(\rho)$ and $g^\nu(\rho)$ are the solutions to equation (B-5). In the present work, the displacement vector and the magnetic fields, respectively, were:

$$\mathbf{D}^\nu(t,\mathbf{x}) = e^{in\varphi}\partial_z\Lambda^\nu(t,z)\mathbf{\Gamma}_\perp^\nu\times\hat{\mathbf{z}}+\alpha^\nu e^{in\varphi}\Lambda^\nu(t,z)\hat{\mathbf{z}} \tag{B-8}$$

and

$$\mathbf{B}^\nu(t,\mathbf{x}) = e^{in\varphi}\dot{\Lambda}^\nu(t,z)\mathbf{\Gamma}_\perp^\nu. \tag{B-9}$$

where

$$\alpha^\nu = \frac{1}{\rho}\Big[\partial_\rho\rho f^\nu(\rho)-ing^\nu(\rho)\Big]. \tag{B-10}$$

Appendix C:

To obtain the nonlinear part of the Hamiltonian, we applied the definition of the Dirac delta function by assuming that k and A_k could be approximated as k_0 and A_{k0}, respectively. The

quantities $\boldsymbol{\Gamma}_\perp^{\nu_0}$ and α^{ν_0} are given by equations (B-7) and (B-10) where $\nu = \nu_0 = (n_0, m_0)$. To calculate the Hamiltonian (equation (33)), one should note that θ^{ν_0} is defined as

$$\theta^{\nu_0} = \frac{1}{3}\upsilon^{\nu_0} + \frac{2}{3}\vartheta^{\nu_0} \tag{C-1}$$

where

$$\upsilon^{\nu_0} = \int_0^a \int_0^{2\pi} \left(\alpha^{\nu_0}\alpha^{\nu_0} - k_0^2\boldsymbol{\Gamma}_\perp^{\nu_0}\cdot\boldsymbol{\Gamma}_\perp^{\nu_0}\right)\left(\alpha^{-\nu_0}\alpha^{-\nu_0} - k_0^2\boldsymbol{\Gamma}_\perp^{-\nu_0}\cdot\boldsymbol{\Gamma}_\perp^{-\nu_0}\right)\rho\,d\rho\,d\varphi , \tag{C-2}$$

$$\vartheta^{\nu_0} = \int_0^a \int_0^{2\pi} \left[k_0^2\boldsymbol{\Gamma}_\perp^{\nu_0}\cdot\boldsymbol{\Gamma}_\perp^{-\nu_0} + \alpha^{\nu_0}\alpha^{-\nu_0}\right]^2\rho\,d\rho\,d\varphi \tag{C-3}$$

and the modes ν_0 and $-\nu_0$ are indicated by (n_0 and m_0) and ($-n_0$ and m_0), respectively.

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