# Quantum tunneling in a Kerr medium with parametric pumping

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(Received 16 March 1993)

A quantum optical model with a classical phase space exhibiting nonlinear oscillations around two elliptic fixed points is investigated. The quantum system is found to display coherent tunneling between near coherent states of opposite phase centered at the classical fixed points.

PACS number(s): 42.50.Lc, 42.65.Ky, 42.65.Re

## I. INTRODUCTION

Quantum-mechanical tunneling through the barrier of a double-well potential is a standard model for exhibiting this typical quantum feature of a nonlinear system. The model has a long history, appearing in the earliest quantum-mechanical literature [1]. A realization of the model in a Josephson junction has been extensively discussed, as a means of producing superpositions of macroscopically distinct states [2]. Quite recently the model (with the addition of a periodic driving field) has been discussed in the context of contrasting classical and quantum nonlinear dynamics when the classical description exhibits chaos [3,4].

In this paper we discuss a simple quantum optical model which is dynamically equivalent to a double-well-potential model. Very recently Di Filippo *et al.* [5] have considered an equivalent model in the context of a parametrically driven anharmonic oscillator with possible application to high-precision measurements on trapped ions. The model involves the interaction of a single mode of the field with both an intensity-dependent refractive index and a parametric nonlinearity. The system is driven externally entirely by the parametric pump field. Of course it is unlikely that any single material would exhibit both nonlinearities of the required strength. However, a single cavity could contain materials with the appropriate nonlinearity. In that case one would arrange for the parametric pump, however, not to see a cavity. The classical phase-space orbits of this model are in fact ovals of Casinni, and for this reason we will refer to the device as a "Cassinian oscillator." In this Brief Report the classical and quantum descriptions are given and contrasted. In particular we show how quantum tunneling in this model leads to the production of a quantum superposition of two nearly coherent states of the electromagnetic field. We also discuss the observational consequences of such a superposition state.

## II. THE CASSINIAN OSCILLATOR

The Hamiltonian of the model (in the interaction picture) is

$$H = \frac{\hbar\chi}{2}\left(a^{\dagger}a\right)^{2} - \frac{\hbar\kappa}{2}\left(a^{2} + a^{\dagger 2}\right) \tag{1}$$

where $a$ is the annihilation operator for the cavity field, $\chi$ is proportional to the third-order nonlinear susceptibility, and $\kappa$ is proportional to the product of the amplitude of the parametric pump field and the second-order susceptibility. The Heisenberg equation of motion for $a$ is

$$\frac{da}{dt} = -i\frac{\chi}{2}\left(aa^{\dagger}a + a^{\dagger}a^{2}\right) + i\kappa a^{\dagger}. \tag{2}$$

We define the corresponding classical model by replacing the operators $a$ and $a^{\dagger}$ in the equations of motion, by the commuting $c$ numbers $\alpha$ and $\alpha^{*}$, respectively. In terms of canonical quadrature phase amplitudes defined by $X_1 = \frac{1}{2}(\alpha + \alpha^{*})$, $X_2 = \frac{-i}{2}(\alpha - \alpha^{*})$ , the resulting equations of motion are

$$\dot{X}_1 = 4(X_1^2 + X_2^2)X_2 + 4\mu^2 X_2, \tag{3}$$

$$\dot{X}_2 = -4(X_1^2 + X_2^2)X_1 + 4\mu^2 X_1, \tag{4}$$

where the dot implies differentiation with respect to the scaled time variable $\tau = t\chi/4$, and we have defined $\mu^2 = \frac{\kappa}{\chi}$. These equations imply a Hamiltonian of the form

$$H = (X_1^2 + X_2^2)^2 + 2\mu^2(X_2^2 - X_1^2). \tag{5}$$

The phase curves of constant energy for this model are given by the ovals of Cassini, with foci on the $X_1$ axis at $\pm\mu$ (see Fig. 1). There are three fixed points, two elliptically stable points at $\pm\mu$ and one unstable point at the origin. The separatrix occurs at $H(X_1, X_2) = 0$ and takes the form of a lemniscate. In order to find the nonlinear oscillation frequency of the model we transform to action-angle variables. This is most easily done by writing the Hamiltonian in polar coordinates using $X_1 = r\cos\theta$, $X_2 = r\sin\theta$. Inside the separatrix the action with respect to a closed orbit around one fixed point is given by

$$J(|H|) = \frac{\beta^2}{2\pi}E\left(2\theta_c, \frac{\mu^2}{\beta^2}\right) \tag{6}$$

where $E$ is the elliptic integral of the second kind [6], $\beta^4 = \mu^4 + H$, and the critical angle $\theta_c$ is defined by

 

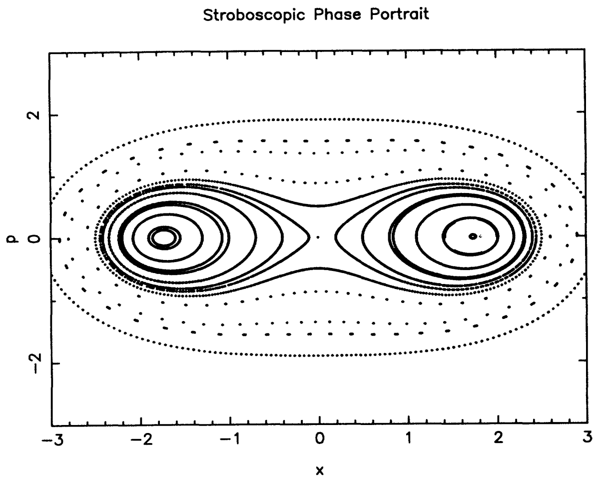


FIG. 1. The classical phase-space trajectories with $\mu^2 = 3.0$.

$$\cos 2\theta_c = \pm \frac{\sqrt{|H|}}{\mu^2}. \tag{7}$$

Outside the separatrix the action is

$$J(H) = \frac{\beta^2}{2\pi} E\left(\pi, \frac{\mu^2}{\beta^2}\right). \tag{8}$$

The nonlinear frequency, as a function of energy (which is proportional to the intensity), is then given by

$$\omega(H) = \left(\frac{\partial J}{\partial H}\right)^{-1} = \begin{cases} \dfrac{4\pi\beta^2}{F\left(\pi, \dfrac{\mu^2}{\beta^2}\right)} & \text{if } E > 0 \\[2ex] \dfrac{4\pi\beta^2}{F\left(2\theta_c, \dfrac{\mu^2}{\beta^2}\right)} & \text{if } E < 0 \end{cases} \tag{9}$$

where $F$ is the elliptic integral of the first kind [6]. In the case $E < 0$ the frequency is with respect to oscillations around one of the fixed points. In Fig. 2 we plot the nonlinear frequency as a function of the energy. As expected the frequency goes to zero quite rapidly as the separatrix is approached.

The solutions to the equations of motion are also easily found once action-angle variables are determined. We simply quote the results. The polar radius is given as an implicit function of the polar angle and energy by

$$r^4 - 2\mu^2 r^2 \cos 2\theta = E \tag{10}$$

while the polar angle $\theta$ is given by

$$\cos 2\theta(t) = \begin{cases} \mathrm{cn}\left(8\beta^2 t, \dfrac{mu^2}{\beta^2}\right) & \text{if } E > 0 \\[2ex] \mathrm{cn}\left(4\beta^2 t, \dfrac{\mu^2}{\beta^2}\right) & \text{if } E < 0 \end{cases} \tag{11}$$

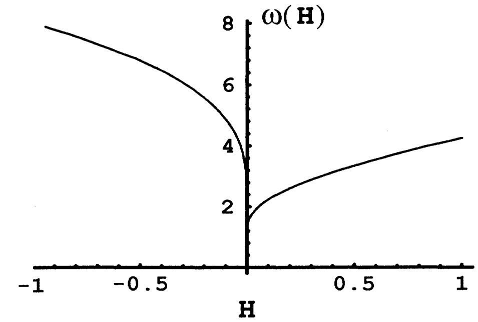


FIG. 2. The energy dependence of the nonlinear frequency with $\mu = 1.0$.

where cn is one of the Jacobi elliptic functions [6] and the quadrature phase amplitudes are found from $X_1(t) = r\cos\theta$, $X_2(t) = r\sin\theta$.

## III. QUANTUM DYNAMICS

The quantum dynamics of the system are investigated by numerically diagonalizing the Hamiltonian matrix in the number state basis, thereby finding the energy eigenvectors and eigenfrequencies. As is expected from the symmetry of the Hamiltonian these are also parity eigenstates and so a superposition of the two states with lowest energy may be expected to be localized around one of the classical fixed points. This is easily seen by plotting the Husimi or $Q$ function for the ground state. The $Q$ function is a true phase-space probability distribution for simultaneous measurement of position and momentum [7] and is defined as the matrix elements of the density operator in the coherent-state basis $Q(\alpha) = \langle\alpha|\rho|\alpha\rangle$. Examination of the $Q$ functions of these states (see Fig. 3) shows that they appear as overlapping Gaussian functions each centered at one of the classical fixed points. So from this it might be expected that their superposition would closely approximate a coherent state. When the expansion of a coherent state centered on the classical fixed point in terms of these eigenstates is calculated this is found to be the case as over 95% of the state is contained in the contribution of these first two eigenstates. We can approximately write these first two states as

$$|+\rangle = N^+(|\alpha\rangle + |-\alpha\rangle), \tag{12}$$

$$|-\rangle = N^-(|\alpha\rangle - |-\alpha\rangle), \tag{13}$$

where $N^+$ and $N^-$ are appropriate normalizing constants. Using this the time evolution of a coherent state becomes

$$|\alpha(t)\rangle = \frac{e^{-i\omega_+ t}}{2}[(1 + e^{-i(\omega_- - \omega_+)t})|\alpha\rangle \tag{14}$$

$$+(1 - e^{-i(\omega_- - \omega_+)t})|-\alpha\rangle]. \tag{15}$$

This will tunnel from the orignal $|\alpha\rangle$ state through an intermediate superposition of the $|\alpha\rangle$ and $|-\alpha\rangle$ state to the $|-\alpha\rangle$ state and back again. Both direct calcula-

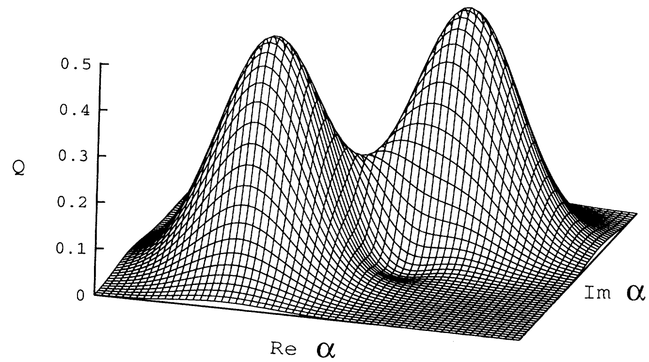


FIG. 3. The $Q$ function of the lowest-energy eigenstate, with $\chi = 4, \kappa = 12$, and $\mu^2 = 3$.

tion with the energy eigenstates and numerical solution of the Schrödinger equation confirm this. The frequency of this tunneling is given by $\Delta_s = \omega_+ - \omega_-$. Degenerate perturbation theory gives a tunneling splitting frequency of

$$\Delta_s = \chi \exp\left(-2\left|\frac{\kappa}{\chi}\right|\right). \tag{16}$$

This indicates that the tunnel time is an exponentially decreasing function of the square of the separation of the fixed points. Midway through the tunneling process the state is very close to a quantum superposition of two coherent states. In this case the probability distribution for the $X_2$ quadrature should show interference fringes. In Fig. 4 the probability distribution along the quadrature at right angles to the line connecting the two classical fixed points is plotted. Interference fringes are clearly apparent. A classical mixture of the same states would

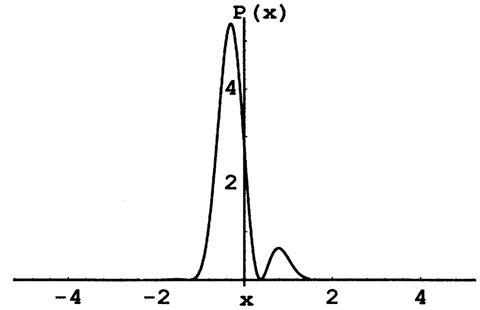


FIG. 4. The quadrature phase distribution orthogonal to the line connecting the classical fixed points, with $\mu^2 = 3$.

be very close to a Gaussian function centered on the origin.

## IV. CONCLUSION

We have shown how a standard quantum optical model may be constructed to exhibit quantum-mechanical tunneling which is completely analogous to tunneling through a double-well potential. A similar tunneling behavior would occur in the equivalent model of Di Filippo *et al.* [5]. In that realization it might be possible to observe the tunneling of a trapped ion. Of course, as is well known, such coherent tunneling is very sensitive to fluctuations and is easily destroyed. The effect of damping and noise on quantum tunneling in this model will be presented in the future. However, we expect that in the limit of strong quantum noise (that is, strong nonlinearity) the steady state will contain some degree of quantum coherence due to coherent tunneling, as is found in the optical parametric oscillator above threshold [8].

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