Quark sector of Coulomb gauge Quantum Chromodynamics

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Carina Popovici
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Dekan: Prof. Dr. Wolfgang Rosenstiel
1. Berichterstatter: Prof. Dr. Hugo Reinhardt
2. Berichterstatter: Prof. Dr. Werner Vogelsang

To my niece Carina-Elena

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Zusammenfassung

Quantenchromodynamik [QCD] wird heute als die Theorie betrachtet, welche die Starke Wechselwirkung zwischen den fundamentalen Konstituenten der Hadronen, den Quarks und Gluonen, korrekt beschreibt [1–5]. Bei großen Energien (kleinen Abständen) verschwindet die Kopplungskonstante – ein Phänomen, das gemeinhin als *asymptotische Freiheit* bezeichnet wird. In dem entsprechenden Impulsbereich wurde die Störungstheorie angewendet und in tiefinelastischen Streuexperimenten erfolgreich überprüft. Für mittlere und kleine Impulse hingegen wird die Kopplungskonstante so groß, dass die Störungstheorie nicht mehr angewendet werden kann. Daher müssen andere Methoden angewendet werden, um das *Confinement* zu erklären, also das Phänomen, dass im Experiment nur farblose Hadronenzustände beobachtet werden. In dieser Dissertation wird der Quark-Sektor der QCD in Coulomb-Eichung mit Hilfe der Dyson-Schwinger-Gleichungen untersucht. In diesem Rahmen nutzen wir verschiedene Strategien, um die Eigenschaften der QCD sowohl bei großen als auch bei kleinen Impulsen zu studieren.

Im ersten Kapitel erinnern wir an einige grundsätzliche Eigenschaften der QCD. Ausgehend von der Lagrangefunktion der QCD präsentieren wir die Eichfixierung und motivieren unsere Wahl der Coulomb-Eichung. Verschiedene Aspekte des Confinements werden diskutiert, insbesondere der Gribov-Zwanziger-Confinement-Mechanismus und seine Relevanz in Coulomb-Eichung.

Das zweite Kapitel beschäftigt sich mit der grundsätzlichen Ableitung der Dyson-Schwingen-Gleichungen. Funktionale Methoden werden eingeführt und der Quark-Propagator sowie die Quark-Beiträge zur Gluon-Zweipunktfunktion und zur Quark-Gluon-Vertexfunktion werden formal abgeleitet.

Im dritten Kapitel werden diese Funktionen in 1-Loop-Störungstheorie ausgearbeitet. Um die in den Gleichungen auftretenden nicht-kovarianten Loop-Integrale in Coulomb-Eichung zu behandeln, wird eine neue Methode basierend auf Differentialgleichungen und partieller Integration entwickelt. Physikalische Resultate werden verifiziert, so z.B. die Gltigkeit der analytischen Fortsetzung zwischen Minkowski- und Euklidischer Raum-Zeit und die Renormierung der Quarkmasse. Des weiteren wird der Quark Beitrag zum 1-Loop-Koeffizient der β-Funktion berechnet.

Das vierte Kapitel ist der Slavnov-Taylor-Identität der Quark-Gluon-Vertexfunktion gewidmet. Insbesondere wird das Auftreten des sogenannten Quark-Geist-Streukerns untersucht.

In Kapitel 5 nähern wir uns dem Confinement-Problem durch Betrachtung des Grenzfalls schwerer Quarks. In diesem Limes nutzen wir den (vollständig nicht-störungstheoretischen) funktionalen Formalismus kombiniert mit einer Entwicklung nach Potenzen des Inversen der schweren Quarkmasse. Durch Einschränkung auf die führende Ordnung in dieser Massen-Entwicklung leiten wir eine strenge analytische Lösung für den Propagator der schweren Quarks ab. Anschließend nutzen wir die Gleichungen für gebundene Zustände von Mesonen und Baryonen, um das linear wachsende Potential abzuleiten, das Quark-Confinement erklärt.

Kapitel 6 behandelt die Vier-Punkt Greenschen Funktionen der Theorie. Ausgehend vom in Kapitel 2 eingeführten Funktionalformalismus werden diese Funktionen explizit abgeleitet und ihre Beziehung zu den Gleichungen gebundener Zustände aus dem vorangegangenen Kapitel diskutiert.

Kapitel 7 enthält die Zusammenfassung und das Fazit. Es folgen die Anhänge in denen unter anderem die in Kapitel 3 abgeleiteten nicht-kovarianten Integrale explizit überprüft werden und auch einige in der Arbeit benötigte Zwei- und Drei-Punkt-Integrale berechnet sind.

Abstract

Quantum Chromodynamics [QCD] is widely believed to be the correct theory of strong interactions between the fundamental constituents of the hadrons, the quarks and gluons [1–5]. At high energies (small distances), the coupling between quarks and gluons tends to zero, a phenomenon known as *asymptotic freedom*. In this momentum region, perturbation theory has been applied and successfully tested in deep inelastic processes. At intermediate and low momenta though, the coupling constant becomes strong enough to invalidate perturbation theory. Different methods must be employed to investigate *color confinement*, i.e. the phenomenon that only colorless hadronic states are observed in the experiment. This thesis deals with the quark sector of Coulomb gauge QCD and the method we employ is the Dyson–Schwinger equations. In this framework, we utilize different strategies to explore both the large and small momentum properties of QCD.

In the first chapter we review some basic properties of strong QCD. Starting with the QCD Lagrangian, the gauge fixing procedure is presented, and our choice of using Coulomb gauge is motivated. Certain aspects of confinement are discussed, in particular the Gribov-Zwanziger confinement mechanism and its relevance in Coulomb gauge.

The second chapter is concerned with the formal derivation of the Dyson–Schwinger equations. Functional methods are introduced and the quark propagator, along with the quark contribution to the gluon two-point functions and the quark-gluon vertex functions are formally derived.

In the third chapter, these functions are examined at one-loop perturbative level. To handle the Coulomb gauge noncovariant loop integrals entering the equations, a new method based on differential equations and integration by parts technique is developed. Physical results are verified, such as the validity of the analytic continuation between Minkowski and Euclidian space, and the quark mass renormalization. The quark contribution to one-loop coefficient of the β-function is also calculated.

Chapter 4 is devoted to the Slavnov–Taylor identity for the quark-gluon vertices. In particular, the appearance of the so-called quark-ghost scattering kernels is explored.

In Chapter 5, we address the problem of confinement by restricting ourselves to the heavy quark sector of the theory. In this limit, we employ the (full nonperturbative) functional formalism, combined with an expansion in the inverse of the heavy quark mass. Restricting to the leading order in the mass expansion, we derive an exact, analytical solution for the heavy quark propagator. We then consider the bound state equations for mesons and baryons and use them to derive the linearly rising potential which confines the quarks.

Chapter 6 is devoted to the four-point Greens functions of the theory. Based on the functional formalism introduced in Chapter 2, these functions are explicitly derived and their connection to the bound state equations from the previous chapter is discussed.

Chapter 7 includes the summary and conclusions. It is followed by appendices where, among others, the noncovariant integrals derived in Chapter 3 are checked explicitly, and also some standard two- and three-point integrals needed in this work are evaluated.

Contents

Chapter 1.

QCD in Coulomb gauge

A deep understanding of QCD requires a whole toolbox of theoretical methods: analytical perturbative methods for weak coupling, numerical lattice gauge theory, along with the canonical approach [6], and functional methods [2]. The latter two methods have the advantage that they are not restricted to weak coupling and can still be treated analytically. The functional methods most commonly used are the renormalization group equations (see, for example, Refs. [7, 8] for reviews) and the Dyson–Schwinger equations of motion for the Green's functions of the theory [9–11]. In this thesis, we employ the functional equation techniques and derive the Dyson–Schwinger equations.

Since QCD is a non-abelian gauge theory, within the functional approach considered here it is necessary to fix the gauge. Thus, after briefly introducing the QCD Lagrangian, we will present the gauge fixing procedure and the problems related to it, with emphasis on Coulomb gauge. We will then motivate why among various gauges, Coulomb gauge has the advantage that it is "physical": after converting to first order formalism, we will show that the number of dynamical variables reduces to the number of the physical degrees of freedom. Further, the Gribov-Zwanziger confinement scenario [12, 13] will be introduced and its relevance for Coulomb gauge will be put forward. The alternative approaches to QCD in Coulomb gauge will be also reviewed, and in particular, the results obtained within the Hamiltonian formalism will be outlined.

Although Coulomb gauge seems to be more efficient in identifying the physical degrees of freedom, noncovariance introduces severe technical difficulties[1] and moreover, the problems related to renormalization have not yet been solved. From a practical point of view, Landau gauge has the advantage of being covariant and thus many infrared investigations have been undertaken, however studies are still in progress. A brief review of the results obtained in this gauge, also in correspondence to the confinement mechanism, will be presented at the end of this chapter.

1.1. QCD as non-abelian gauge theory

QCD is a non-abelian gauge theory whose matter constituents, the quarks, are spin 1/2 fermions and obey the Dirac Lagrangian[2]

$$\mathcal{L}_q = \bar{q}_\alpha(x)\left[\mathrm{i}\gamma_\mu D^\mu - m\right]_{\alpha\beta} q_\beta(x), \qquad \bar{q} = q^\dagger\gamma^0 \tag{1.1}$$

where the Dirac γ^μ matrices satisfy the Clifford algebra, $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ and the indices $\alpha, \beta \ldots$ commonly denote the Dirac spinor, flavor and (fundamental) color. The quark

[1] The noncovariant Feynman integrals in Coulomb gauge will be examined in Chapter 3.

[2] Initially we use Minkowski metric defined in the Appendix A.

fields $\bar{q}, q$ transform in the fundamental representation of the gauge group $SU(N_c)$[3], with $N_c = 3$ realized in QCD. The covariant derivative (in the fundamental representation) is given by

$$D_\mu = \partial_\mu - \mathrm{i}gA_\mu, \tag{1.2}$$

where g is the coupling constant of the theory and $A_\mu(x) = A^a_\mu(x)T^a$. The non-abelian gauge field $A^a_\mu(x)$ transforms according to the adjoint representation of the gauge group. Given an infinitesimal transform $U(x) = 1 - \mathrm{i}\theta^a(x)T^a$ the variation of the gauge and quark fields is

$$\delta A^a_\mu(x) = -\frac{1}{g}\hat{D}^{ac}_\mu(x)\theta^c(x) \tag{1.3}$$

$$\delta q_\alpha(x) = -\mathrm{i}T^a\theta^a(x)q_\alpha(x) \tag{1.4}$$

where the covariant derivative in the adjoint representation reads

$$\hat{D}^{ac}_\mu = \delta^{ac}\partial_\mu + gf^{abc}A^b_\mu. \tag{1.5}$$

Defining the field strength tensor

$$F_{\mu\nu} = T^a(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu) \tag{1.6}$$

we can construct a kinetic term for the non-abelian gauge field A_μ

$$\mathcal{L}_{YM} = -\frac{1}{2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}, \tag{1.7}$$

such that the total Lagrange density $\mathcal{L}_{QCD} = \mathcal{L}_q + \mathcal{L}_{YM}$ is invariant under local gauge transformations.

Let us now consider the functional integral

$$Z = \int \mathcal{D}[A\bar{q}q]\exp\{\mathrm{i}\mathcal{S}_{QCD}\}, \tag{1.8}$$

where $\mathcal{D}[A\bar{q}q]$ denotes the functional integration measure for the Yang–Mills and quark fields and the QCD action is given by

$$\mathcal{S}_{QCD} = \int d^4x \left\{ \bar{q}_\alpha(x)\left[\mathrm{i}\gamma_\mu D^\mu - m\right]_{\alpha\beta} q_\beta(x) - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} \right\}. \tag{1.9}$$

The difficulty with the functional integral Eq. (1.8) is that the measure $\mathcal{D}[A\bar{q}q]$ runs over infinitely many gauge equivalent configurations (field configurations that are connected by gauge transformations), whereas the the action $\mathcal{S}_{QCD}$ is gauge invariant. Hence it introduces for every gauge orbit[4] a divergent factor (the volume of the gauge group). The way to handle this problem is to use a method introduced by Faddeev and Popov [14]. The idea is to single out one representative from each orbit by imposing a gauge fixing condition to the functional integral Eq. (1.8). In this thesis we shall be considering only Coulomb gauge, but also other choices such as Landau gauge are possible. As a consequence, a new set of Grassmann fields, known as ghosts, are introduced. The gauge fixing procedure, the ghost fields and the problems associated with them will be discussed in the next section.

[3] Some relevant formulas for the group $SU(N_c)$ are collected in the Appendix A.

[4] A gauge orbit contains all configurations connected by gauge transformations and will be explicitly defined in the next section.

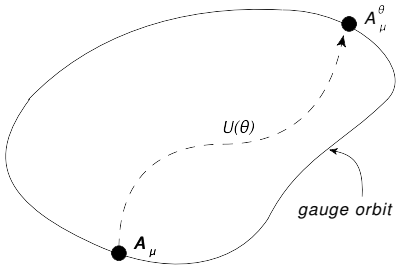

Figure 1.1.: Depiction of a gauge orbit containing A_μ and the gauge transformed field A^θ_μ

1.2. Gauge fixing and ghosts

We start by defining the *gauge orbit* for some configuration A_μ as the set of all gauge equivalent configurations, i.e. each point A^U_μ on the gauge orbit is obtained by acting upon A_μ with the gauge transformation U (see also Fig. 1.1):

$$[A^{orbit}_\mu] =: \{A^U_\mu = UA_\mu U^\dagger - \frac{\mathrm{i}}{g}(\partial_\mu U)U^\dagger \mid U \in SU(N_c)\}. \tag{1.10}$$

Then the integral over all the gauge fields $\int \mathcal{D}A$ can be written as an integral over a full set of gauge-inequivalent configurations $\int \mathcal{D}A^{ref}$, where A^{ref} is a reference gauge field representative for the orbit (i.e., integral *over* all possible gauge orbits), and an integral *around* each gauge orbit $\int \mathcal{D}O(A^{ref})$

$$\int \mathcal{D}A = \int \mathcal{D}A^{ref} \int \mathcal{D}O(A^{ref}). \tag{1.11}$$

Since one integrates over infinitely many equivalent configurations related by a gauge transformation, the integral around the gauge orbit $\int \mathcal{D}O(A^{ref})$ is infinite and must be eliminated. The strategy is to extract only one representative gauge-field configuration out of each orbit by imposing a gauge fixing condition $\chi[A] = 0$. Each gauge field configuration on the orbit $O(A^{ref})$ is a gauge transformation of A^{ref}, i.e. one can (up to a volume factor) rewrite the integral $\int \mathcal{D}O(A^{ref})$ as an integral over the gauge group $\int \mathcal{D}U$. The gauge fixing condition is implemented by inserting the identity

$$1 = \Delta[A] \int \mathcal{D}U \delta[\chi(A^U)] \tag{1.12}$$

into the functional integral Eq. (1.8). Ideally, the gauge fixing condition $\chi(A) = 0$ should be satisfied by only one A_μ of each gauge orbit. [5] The Faddeev-Popov determinant $\Delta[A] = \Delta[A^U]$ accounts for the functional determinant arising from the argument of the delta function. Using the invariance of the action under gauge transformations, one can rewrite the functional integral of the theory as

$$Z = \int \mathcal{D}U Z_{gf} \tag{1.13}$$

where the gauge fixed amplitude is given by

$$Z_{gf} = \int \mathcal{D}[A\bar{q}q]\Delta[A]\delta[\chi(A)] \exp\{\mathrm{i}\mathcal{S}_{QCD}\}, \tag{1.14}$$

[5] But, as will shortly be discussed, this is in practice impossible due to topological restrictions.

and the divergent measure $\int \mathcal{D}U$ has been factorized as an overall constant which can be absorbed in the normalization. In the following, we will use the notation Z (instead of Z_{gf}) for the gauge fixed functional integral, since no confusion can arise.

In Coulomb gauge, the (noncovariant) gauge fixing condition is given by:

$$\chi[A] = \vec{\nabla}\cdot\vec{A}^a = 0 \tag{1.15}$$

and hence the temporal and spatial components of the gauge field must be treated differently. The gauge fixing condition can be implemented by rewriting the delta function with the help of a Lagrange multiplier λ^a

$$\delta[\chi(A)] \to \int \mathcal{D}\lambda \exp\left[-\mathrm{i}\int d^4x\ \lambda^a\vec{\nabla}\cdot\vec{A}^a\right], \tag{1.16}$$

and the Faddeev-Popov determinant

$$\Delta[A] = \mathrm{Det}\ [\vec{\nabla}\cdot\vec{D}^{ab}] \tag{1.17}$$

can be written as a functional integral over two new Grassmann valued fields c and $\bar{c}$, the so-called "Faddeev-Popov ghosts"[6]:

$$\Delta[A] \to \int \mathcal{D}[\bar{c}c] \exp\left[-\mathrm{i}\int d^4x\ \overline{c}^a\vec{\nabla}\cdot\vec{D}^{ab}c^b\right]. \tag{1.18}$$

In the above, we have introduced the spatial component of the covariant derivative Eq. (1.5) (in the adjoint representation)

$$\vec{D}^{ac} = \delta^{ac}\vec{\nabla} - gf^{abc}\vec{A}^b. \tag{1.19}$$

Due to their unusual spin-statistics (they obey Fermi-Dirac statistics, and at the same time are scalar), the ghost fields are only allowed to appear in closed loops in Feynman diagrams and never as initial or final states in a physical process. Putting all these together, we can write for the Coulomb gauge functional integral:

$$Z = \int \mathcal{D}[A\bar{q}q\bar{c}c\lambda] \exp\{\mathrm{i}\mathcal{S}_{QCD} + \mathrm{i}\mathcal{S}_{FP}\}, \tag{1.20}$$

with

$$\mathcal{S}_{QCD} = \int d^4x \left\{\bar{q}_\alpha\left[\mathrm{i}\gamma^0 D_0 + \mathrm{i}\vec{\gamma}\cdot\vec{D} - m\right]_{\alpha\beta} q_\beta - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}\right\}, \tag{1.21}$$

$$\mathcal{S}_{FP} = \int d^4x \left[-\lambda^a\vec{\nabla}\cdot\vec{A}^a - \overline{c}^a\vec{\nabla}\cdot\vec{D}^{ab}c^b\right]. \tag{1.22}$$

In the QCD action, Eq. (1.21), we have separated the covariant derivative into temporal and spatial components (implicitly in the fundamental color representation), given by:

$$D^0 = \partial^0 - \mathrm{i}gT^c\sigma^c, \quad \vec{D} = \vec{\nabla} + \mathrm{i}gT^c\vec{A}^c, \tag{1.23}$$

where the temporal component of the gauge field A^{0a} has been renamed to σ^a. $\mathcal{S}_{FP}$ collects the terms originating from the gauge fixing condition Eq. (1.16) and the Faddev-Popov determinant Eq. (1.18).

[6] A pedagogic introduction on this topic can be found in [4].

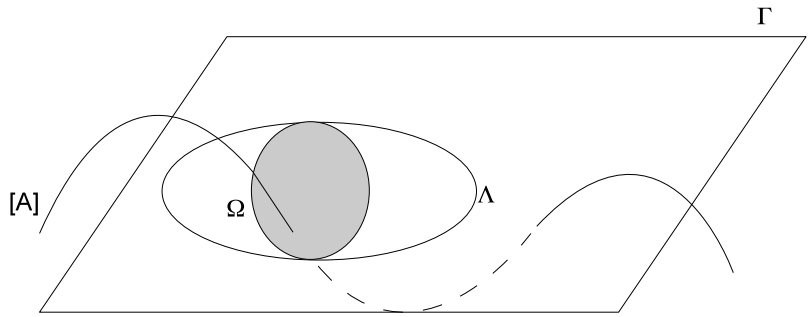

Figure 1.2.: Illustration of the hyperplane Γ in gauge field configuration space obtained by gauge fixing. Furthermore, the first Gribov region Ω and the fundamental modular region Λ are shown. A gauge orbit $[A]$ intersects the hyperplane Γ several times thus generating Gribov copies. The fundamental modular region, by definition, is intersected only once.

The gauge fixing procedure described above is not yet complete, in the sense that the simple Faddeev-Popov trick is not sufficient to extract a single gauge configuration from each gauge orbit. The reason is the presence of the so-called *Gribov copies* [13], i.e. configurations A_μ connected by a gauge transformation that produce multiple intersection points of a gauge orbit with the hyperplane Γ generated by the gauge fixing condition (see also Fig. 1.2). In order to avoid the Gribov copies, it is necessary to restrict the hyperplane Γ to the so-called *Gribov region* Ω. This is obtained by minimizing the L^2-norm of the vector potential

$$F_A[U] = \int d^3x \ \mathrm{tr}\left[A_i^U(x)\right]^2 \tag{1.24}$$

along the gauge orbit [15]. In this region, any *local* minimum of this norm implements the gauge fixing condition (in this case, Coulomb gauge) and the Faddeev-Popov operator is restricted to positive eigenvalues:

$$\Omega \equiv \left\{ \vec{A} : \vec{\nabla}\cdot\vec{A} = 0; -\vec{\nabla}\cdot\vec{D} \geq 0 \right\}. \tag{1.25}$$

Importantly, the Gribov region Ω contains the trivial configuration $g\vec{A} = 0$ (i.e., it contains all configurations relevant for perturbation theory) and therefore the ultraviolet regime is not influenced by this restriction. Moreover, the Faddeev-Popov determinant vanishes on the boundary $\partial\Omega$ of the first *Gribov horizon* [16].

In general, Ω is still not free of Gribov copies[7], thus in principle one has to restrict further the gauge field configurations to the *fundamental modular region* Λ – the region of *global* minima of the norm defined above:

$$\Lambda \equiv \left\{ \vec{A} : F_A[\mathbb{1}] \leq F_A[U] \quad \forall U \right\}. \tag{1.26}$$

This gauge condition is also known as "minimal Coulomb gauge" [12]. In practise, these configurations are extremely complicated to identify[8]. However, in the continuum Zwanziger showed by means of stochastic quantization that Gribov copies inside the Gribov

[7] A proof of the existence of Gribov copies inside Ω was given in [17].
[8] For recent lattice studies see, for example, [18–21].

region do not affect the Green's functions of the theory [16]. Precisely, the dominant configurations lie on the common boundary of Ω and Λ and hence, in practise, restriction to the Gribov region is sufficient. This restriction might eventually generate nontrivial boundary terms which could influence the derivation of the field equations of motion (and implicitly the Dyson–Schwinger equations), but since by definition the Faddeev-Popov operator vanishes on the boundary of Ω, these boundary terms are identically zero.

1.3. Approaches to Coulomb gauge QCD

1.3.1. First order functional formalism

As already mentioned, the gauge of choice in this work is Coulomb gauge. Although this gauge does not have so many practical advantages as Landau gauge for example, there are several reasons motivating our choice: in Coulomb gauge there is a natural picture of confinement, Gauss law is naturally built in (such that in principle gauge invariance is fully accounted for) and the total color charge is conserved and vanishing [12, 22]. However, except for the original work of Khriplovich [23], only recently Dyson–Schwinger studies in Coulomb gauge have been undertaken. The technical barrier stems from the so-called energy divergence problem – the unregulated divergences generated by the ghost loops [24–26]. The way to circumvent this problem is to use the *first order functional formalism*[9]. Within this formalism these divergences cancel exactly and moreover, the system reduces automatically to "would-be-physical" degrees of freedom [12]. All these aspects will be discussed in detail in the course of this section.

Motivation and main idea

The main advantage of working in Coulomb gauge stems from the fact that in this gauge the Gribov-Zwanziger mechanism of confinement[10] becomes particularly transparent. In this picture, the long range confining force is provided by the instantaneous Coulomb interaction, which appears to be enhanced for small three-momenta $\vec{q}^2 \to 0$, whereas the transverse (colored) gluon is suppressed, reflecting the absence of the colored states in the spectrum. In covariant gauges such as Landau gauge, a quantity that leads to the confinement potential has not been identified so far.

The conversion to the first order (or phase-space) formalism is motivated by the fact that within this formalism the famous Coulomb gauge energy divergence [24–26] can be avoided. Energy divergence means that the functional integral Eq. (1.20) gives an ill-defined integration, stemming from the energy-independent ghost loops[11] which are integrated over both 3-momentum *and* energy. As an example, consider the following one-loop integral [30]:

$$\int dk_0 \int d^3\vec{k}[(\vec{k}-\vec{p})^2\vec{k}^2]^{-1}. \tag{1.27}$$

[9] However, since we are mainly concerned with the quark sector of QCD, in the course of our investigations we will come across few points where restriction to second order formalism [27, 28] will be sufficient.

[10] The Gribov-Zwanziger confinement scenario will be presented in more detail in Section 1.4.

[11] The energy independence of the ghost propagator follows from the fact that the Faddeev-Popov operator involves only spatial derivatives and spatial components of the gauge fields (see Refs. [27, 29] for a complete derivation and discussion).

This divergence appears in any number of dimensions and cannot be regularized by usual dimensional regularization, although when taken as a full set in the Dyson–Schwinger equations, all such loops cancel. To handle this type of divergences, Leibbrandt introduced a modified form of the dimensional regularization, the so-called split dimensional regularization [24], in which two complex parameters ω and σ are introduced, $d^3\vec{q} \to d^{2\omega}\vec{q}$ and $dq_0 \to d^{2\sigma}q_0$ with the limits $\omega \to 3/2$ and $\sigma \to 1/2$ to be taken after all the integrations have been completed. An alternative approach is the so-called negative integration method (NDIM) [31, 32], where a "Feynman-like" integral is solved, i.e. a loop integral in negative D-dimensional space with propagators raised to positive powers in the numerator. With these two methods the Coulomb gauge integrals were studied up to one and two-loop level perturbatively and results for the divergent part for several of them have been achieved. These divergences do in principle cancel order by order in perturbation theory (tested up to two-loops [33]), but this cancellation is difficult to isolate.

Furthermore, within the first order formalism we are able to cancel the unphysical ghost fields, i.e. the Faddeev-Popov term. This means that we reduce the functional integral Eq. (1.20) to "physical" degrees of freedom, the transverse gluon and transverse $\vec{\pi}$ fields, which in classical mechanics would be the configuration variables and their momentum conjugates (see below).

We keep the term "physical" into quotations marks because it is realized that the true physical objects are the color singlet states, their observables being the mass spectrum and the decay widths.

The presentation of the first order formalism, together with the reduction to the "physical" degrees of freedom, follows [29] and is discussed in some detail. We start by expressing the field strength tensor Eq. (1.6) in terms of the chromo-electric and -magnetic fields (recall that the temporal component A^{0a} has been renamed to σ^a)

$$\vec{E}^a = -\partial^0 \vec{A}^a - \vec{\nabla}\sigma^a + g f^{abc} \vec{A}^b \sigma^c, \quad B_i^a = \epsilon_{ijk}\left[\nabla_j A_k^a - \frac{1}{2} g f^{abc} A_j^b A_k^c\right], \tag{1.28}$$

such that the Yang–Mills action can be split into chromoelectric and -magnetic terms

$$\mathcal{S}_{YM} = \int d^4x \left[\frac{1}{2}\vec{E}^a \cdot \vec{E}^a - \frac{1}{2}\vec{B}^a \cdot \vec{B}^a\right]. \tag{1.29}$$

Next, we consider the chromoelectric term in the action. We linearize this term and hence convert to the first order formalism by introducing an auxiliary field $\vec{\pi}$ via the identity [12]

$$\exp\left\{\mathrm{i}\int d^4x \frac{1}{2}\vec{E}^a \cdot \vec{E}^a\right\} = \int \mathcal{D}\vec{\pi} \exp\left\{\mathrm{i}\int d^4x \left[-\frac{1}{2}\vec{\pi}^a \cdot \vec{\pi}^a - \vec{\pi}^a \cdot \vec{E}^a\right]\right\}. \tag{1.30}$$

Classically, the $\vec{\pi}$ field is interpreted as the momentum conjugate to $\vec{A}$. The $\vec{\pi}$ field is then split into transverse and longitudinal components using the identity

$$\begin{aligned} \text{const} &= \int \mathcal{D}\phi\delta\left(\vec{\nabla}\cdot\vec{\pi} + \nabla^2\phi\right) \\ &= \int \mathcal{D}\{\phi,\tau\} \exp\left\{-\mathrm{i}\int d^4x \tau^a \left(\vec{\nabla}\cdot\vec{\pi}^a + \nabla^2\phi^a\right)\right\}. \end{aligned} \tag{1.31}$$

After making the change of variables $\vec{\pi} \to \vec{\pi} - \vec{\nabla}\phi$ and collecting together all the parts that contain $\vec{\pi}$, we can write our full functional integral as (Φ denotes the collection of all fields):

$$Z = \int \mathcal{D}\Phi \exp\left\{\mathrm{i}\mathcal{S}_q + \mathrm{i}\mathcal{S}_{YM} + \mathrm{i}\mathcal{S}_{FP}\right\} = \int \mathcal{D}\Phi \exp\left\{\mathrm{i}\mathcal{S}_q + \mathrm{i}\mathcal{S}_\pi + \mathrm{i}\mathcal{S}_B + \mathrm{i}\mathcal{S}_{FP}\right\} \tag{1.32}$$

with

$$\begin{aligned}
\mathcal{S}_q &= \int d^4x\ \bar{q}_\alpha \left[\mathrm{i}\gamma^0 D_0 + \mathrm{i}\vec{\gamma}\cdot\vec{D} - m\right]_{\alpha\beta} q_\beta, \\
\mathcal{S}_\pi &= \int d^4x \left[-\tau^a \vec{\nabla}\cdot\vec{\pi}^a - \frac{1}{2}(\vec{\pi}^a - \vec{\nabla}\phi^a)\cdot(\vec{\pi}^a - \vec{\nabla}\phi^a) + (\vec{\pi}^a - \vec{\nabla}\phi^a)\cdot\left(\partial^0\vec{A}^a + \vec{D}^{ab}\sigma^b\right)\right], \\
\mathcal{S}_B &= \int d^4x \left[-\frac{1}{2}\vec{B}^a\cdot\vec{B}^a\right], \\
\mathcal{S}_{FP} &= \int d^4x \left[-\lambda^a\vec{\nabla}\cdot\vec{A}^a - \overline{c}^a\vec{\nabla}\cdot\vec{D}^{ab}c^b\right].
\end{aligned} \tag{1.33}$$

Having derived the functional integral and the full QCD action in the first order formalism, we are now in the position to show that the ghost loops indeed cancel and the system reduces to the "physical" degrees of freedom. In the next section, we will demonstrate that the above QCD action simplifies to an expression where only the transverse $\vec{A}$ and $\vec{\pi}$ fields appear.

Reduction to "physical" degrees of freedom

Given the functional integral, Eq. (1.32), and Yang–Mills part of the action (the quark field is not considered in this discussion), Eq. (1.33), we start by rewriting the Lagrange multiplier terms as δ-function constraints, which automatically eliminates the $\vec{\nabla}\cdot\vec{A}$ and $\vec{\nabla}\cdot\vec{\pi}$ terms in the action. In addition, we also rewrite the ghost terms as the original Faddeev-Popov determinant. This has clearly the drawback that the local formulation and the BRST invariance of the theory[12] are no longer manifest. The functional integral now reads

$$Z = \int \mathcal{D}\Phi \mathrm{Det}\left[-\vec{\nabla}\cdot\vec{D}\delta^4(x-y)\right]\delta\left(\vec{\nabla}\cdot\vec{A}\right)\delta\left(\vec{\nabla}\cdot\vec{\pi}\right)\exp\left\{\mathrm{i}\mathcal{S}\right\} \tag{1.34}$$

with

$$\mathcal{S} = \int d^4x \left[-\frac{1}{2}\vec{B}^a\cdot\vec{B}^a - \frac{1}{2}\vec{\pi}^a\cdot\vec{\pi}^a + \frac{1}{2}\phi^a\nabla^2\phi^a + \vec{\pi}^a\cdot\partial^0\vec{A}^a + \sigma^a\left(\vec{\nabla}\cdot\vec{D}^{ab}\phi^b + g\hat{\rho}^a\right)\right], \tag{1.35}$$

where we have defined the effective color-charge of the gluons $\hat{\rho}^a = f^{ade}\vec{A}^d\cdot\vec{\pi}^e$. Next, we use the fact that the action Eq. (1.33) has become linear in σ (after introducing the field $\vec{\pi}$), and write the integral over σ as a δ-function constraint. This enforces the chromodynamical equivalent of Gauss' law giving

$$Z = \int \mathcal{D}\Phi \mathrm{Det}\left[-\vec{\nabla}\cdot\vec{D}\delta^4(x-y)\right]\delta\left(\vec{\nabla}\cdot\vec{A}\right)\delta\left(\vec{\nabla}\cdot\vec{\pi}\right)\delta\left(-\vec{\nabla}\cdot\vec{D}^{ab}\phi^b - g\hat{\rho}^a\right)\exp\left\{\mathrm{i}\mathcal{S}\right\} \tag{1.36}$$

[12] The standard BRST and *time-dependent* Gauss–BRST transformations used in this work will be introduced in Chapter 4.

with

$$\mathcal{S} = \int d^4x \left[-\frac{1}{2}\vec{B}^a \cdot \vec{B}^a - \frac{1}{2}\vec{\pi}^a \cdot \vec{\pi}^a + \frac{1}{2}\phi^a \nabla^2 \phi^a + \vec{\pi}^a \cdot \partial^0 \vec{A}^a \right]. \tag{1.37}$$

The implementation of the Gauss' law is very important because this essentially ensures the gauge invariance of the system. Defining the inverse Faddeev-Popov operator M:

$$\left[-\vec{\nabla} \cdot \vec{D}^{ab} \right] M^{bc} = \delta^{ac}, \tag{1.38}$$

we can factorize the Gauß law δ-function constraint as

$$\delta\left(-\vec{\nabla} \cdot \vec{D}^{ab} \phi^b - g\hat{\rho}^a \right) = \mathrm{Det}\left[-\vec{\nabla} \cdot \vec{D} \delta^4(x-y) \right]^{-1} \delta\left(\phi^a - M^{ab} g\hat{\rho}^b \right). \tag{1.39}$$

The inverse functional determinant cancels the original Faddeev-Popov determinant from Eq. (1.36), leaving us with

$$Z = \int \mathcal{D}\Phi \delta\left(\vec{\nabla} \cdot \vec{A} \right) \delta\left(\vec{\nabla} \cdot \vec{\pi} \right) \delta\left(\phi^a - M^{ab} g\hat{\rho}^b \right) \exp\{\mathrm{i}\mathcal{S}\}. \tag{1.40}$$

We now use the δ-function constraint to eliminate the ϕ-field. Since the inverse Faddeev-Popov operator M is Hermitian, we can reorder the operators in the action to give us

$$Z = \int \mathcal{D}\Phi \delta\left(\vec{\nabla} \cdot \vec{A} \right) \delta\left(\vec{\nabla} \cdot \vec{\pi} \right) \exp\{\mathrm{i}\mathcal{S}\} \tag{1.41}$$

with

$$\mathcal{S} = \int d^4x \left[-\frac{1}{2}\vec{B}^a \cdot \vec{B}^a - \frac{1}{2}\vec{\pi}^a \cdot \vec{\pi}^a - \frac{1}{2} g\hat{\rho}^b M^{ba} (-\nabla^2) M^{ac} g\hat{\rho}^c + \vec{\pi}^a \cdot \partial^0 \vec{A}^a \right]. \tag{1.42}$$

As promised, the action Eq. (1.42) contains only transverse $\vec{A}$ and $\vec{\pi}$ fields. All other fields, especially the unphysical ghosts, have been formally eliminated. However, the appearance of the functional δ-functions and the inverse Faddeev-Popov operator M have led to a non-local formalism. It is not known how to do practical calculations within this formulation, but the non-local nature of the above result certainly serves as a guide to the local formulation.

Before we close this section, few more remarks are in order. Quite generally, one can argue that first order formalism in Coulomb gauge is better suited to describe physical phenomena then other gauges such as Landau gauge. Indeed, the natural decomposition of degrees of freedom, both physical and unphysical, inherent to the first order formalism, automatically leads to the cancellations of the unphysical components. The subtlety is then to identify how these cancellations arise, and also it is very important to ensure that approximation schemes employed respect such cancellations. For example, the unphysical ghost loop of the gluon polarization should be cancelled in the Dyson–Schwinger equations. Given that the ghost propagator is energy independent, the temporal component of the gluon propagator must itself have a part that is independent of energy [34] in order to cancel this divergence. Later on in Chapter 5 devoted to heavy quarks, this observation will be used to show that only color singlet quark-antiquark bound states are physically allowed, regardless of the specific form of the energy independent part of the temporal gluon propagator.

On the other hand, one unpleasant feature of this approach is the large number of fields, but as it turns out this does not have serious implications for the Dyson–Schwinger equations [29, 35]. More important though, the issue of renormalisability remains unclear, in the sense that a complete proof of full multiplicative renormalization is still missing and one does not have a Ward identity in the usual sense [12]. A brief overview of the attempts to renormalize Coulomb gauge will be given at the end of this chapter.

1.3.2. Alternative methods

Currently, the most popular continuum formalism to QCD in Coulomb gauge is the Hamiltonian formalism [36–42].[13] In this approach, one starts by imposing Weyl gauge $A^a_0(x) = 0$, and subsequently, the Coulomb gauge is fixed with the help of the Faddeev-Popov method, whereas the Gauss law is imposed as constraint. The Yang–Mills Schrödinger equation is then solved using the variational principle for the vacuum state, with a Gaussian ansatz for the wave functional. Then the vacuum energy is minimized and this leads to a coupled system of non-linear Dyson–Schwinger equations for the gluon energy, the ghost and the Coulomb form factor and for the curvature in configuration space [36]. These have been solved analytically in the infrared and numerically in the whole momentum regime [40]. Similar to Landau gauge (see [43] for a recent review), it has been found that in Coulomb gauge there are two different infrared powerlaw exponents for the gluon and the ghost propagator [41, 42]. The favored solution is the most singular – with the ghost propagator dressing function diverging as $1/|\vec{k}|$ – which generates a linearly rising heavy quark potential at large distances [41].[14] Also the gluon energy is divergent in the infrared, reflecting the absence of the gluons in the physical spectrum at low energies, which is again a signal of confinement. Recently, the static potential between infinitely heavy color sources has been also studied with a method based on the Dyson equation for the Wilson loop. In [44], the authors considered the temporal Wilson loop with the instantaneous part of the gluon propagator and the spatial Wilson loop with the static gluon propagator, and solved the corresponding Dyson equation in Coulomb gauge. In both cases a linearly rising potential has been found.

A second approach to Coulomb gauge is the lattice QCD (see, for example, [45] for a review of the results obtained on the lattice). The first lattice calculations have concentrated on the infrared behavior of the ghost and gluon propagators. In particular, in Ref. [34] it has been shown that the static transverse gluon propagator is suppressed in the IR limit, while the time-time component is enhanced. Moreover, in the infinite-volume limit, it has been found that the transverse gluon is well described by the Gribov's formula [13] but unfortunately this study was not conclusive in the ultraviolet [34, 46]. Recently, in [47] the residual temporal gauge has been fixed and the renormalization of the full gluon propagator has been studied. It has been found that the static propagator is renormalizable only in the limit of continuous time, i.e. the lattice Hamiltonian formulation, and the resulting static propagator satisfies the Gribov's formula at all momenta. For the ghost propagator, lattice results have been reported in [48, 49], and more recently in [50]. An infrared divergence stronger than $1/\vec{k}^2$ has been found, in agreement with the horizon condition necessary in the Zwanziger confinement criterion. Very recently, lattice studies of

[13] In Ref. [39], the results obtained within the Hamiltonian approach to Yang–Mills theory are reviewed.
[14] In the next section, the connection to Landau gauge as well as the infrared ghost dominance will be discussed in more detail.

the Coulomb gauge quark propagator have been also undertaken and preliminary results have been presented [50]. The ambiguities due to the Gribov copies have been analyzed and it was found that their influence on the quark propagator is small. Moreover, the residual gauge has been fixed, and it appears that the impact of the residual gauge fixing is only reflected in the time-dependence of the propagator.

1.4. Connection to Landau gauge and aspects of confinement

Due to its covariance, Landau gauge $\partial_\mu A^a_\mu = 0$ has been for a long time the preferred gauge for non-perturbative Dyson–Schwinger studies. It preserves Lorenz invariance and it has a distinct property which makes it very attractive for practical approximations, namely that the the ghost-gluon vertex remains bare in the infrared[15].

The first non-perturbative calculations in Landau gauge date back to the late seventies, when the infrared gluon propagator was first studied by Mandelstam and Bar-Gadda [52, 53]. In this calculation, the ghost loop and the four-gluon vertex, which do not contribute at first level in perturbation theory, have been neglected and only the (non-abelian) triple gluon vertex has been considered. It was found that the gluon propagator is infrared divergent and moreover, assuming a single gluon exchange, a linearly rising potential between heavy quarks has been derived. Today this picture is known as *infrared slavery*. In Landau gauge this picture has been rather misleading, in the sense that only very late it has been shown that in this gauge it is not the gluons, but the ghosts that drive the infrared properties of the theory[16]. More precisely, the gluon propagator vanishes at zero momentum, whereas the ghosts provide for a long range correlation. The gluon and ghost propagators are characterized by the so-called infrared exponents, which are related by a scaling relation, hence the name *scaling solution*. We briefly mention that there exists a second solution, the *decoupling solution*, which possesses quite different characteristics: the ghost propagator is not infrared enhanced but remains bare in the infrared and the gluon propagator becomes finite instead of going to zero [56, 57]. The connection between the two solutions (scaling and decoupling), different only in the deep infrared, still remains to be understood.

Let us now analyze the consequences of restricting the configuration space to a compact region (Gribov or fundamental modular region), for the infrared properties of the theory. As discussed in the beginning of this chapter (Sec. 1.2), the ultraviolet regime is not affected by this restriction (when the coupling becomes small, all relevant configurations lie in the vicinity of $gA = 0$), whereas the infrared can be in principle governed by any domain within the Gribov region. Zwanziger argued that only the behavior of the gauge field on the Gribov horizon (i.e., the boundary of the Gribov region) is important for the infrared properties [58]. Pictorially, the situation is similar to a compact sphere of radius r in high N dimensions, where the probability distribution is concentrated on the boundary due to the volume measure $r^{N-1}dr$. In Coulomb gauge, the Coulomb potential

[15] Or at least, from a semiperturbative analysis of the Dyson–Schwinger equation for the ghost-gluon vertex (with nonperturbative ghost and gluon propagators and a bare ghost-gluon vertex), it follows that deviations from the tree-level vertex are very small [51].

[16] We refer to the original works [54, 55], and for a review, see Ref. [43].

between two external color charges[17] has been derived [41], and it is related to

$$<M(-\nabla^2)M>^{ab}(\vec{x},\vec{y}), \tag{1.43}$$

where M is the *inverse* Faddeev-Popov operator, defined in Eq. (1.38). Since on the Gribov horizon the Faddeev-Popov operator develops a zero eigenvalue, the Coulomb energy Eq. (1.43) becomes very large in the vicinity of the Gribov horizon, leading directly to an asymptotically linear potential. This mechanism is known as the *Gribov-Zwanziger scenario of confinement*. One can also inspect the relation between confinement and the restriction to the Gribov region by examining the infrared behavior of the ghost propagator. This propagator is given by to the expectation value of the inverse Faddeev-Popov operator and therefore is strongly divergent, due to the vanishing eigenvalues of the Faddeev-Popov operator on the Gribov horizon – this is known as the *horizon condition* [13]. Hence, the ghost propagator becomes infrared enhanced from the effects of the Gribov horizon.

As outlined above, in Landau gauge the infrared ghost dominance has been established, but in Coulomb gauge, the Gribov-Zwanziger scenario has a somewhat different realization, depending on the specific formalism. As discussed in the previous section, in the Hamiltonian formalism the ghost propagator is infrared enhanced, whereas in the functional formalism, apart from the ghost and the transversal spatial gluon propagator, there is a third propagator, the temporal gluon propagator (in the canonical formalism, this would correspond to the non-abelian color Coulomb potential, Eq. (1.43)). Assuming that this propagator is largely independent of energy and diverges like $1/\vec{k}^4$ in the infrared (as indicated by the lattice results [49]) one directly obtains a linearly rising potential for color singlet quark-antiquark states, leading back to the old infrared slavery picture[18]. We also mention that very recently, a relation between the ghost and temporal gluon propagators has been found [60]. Based on the Gribov-Zwanziger scenario, this result, together with the Slavnov–Taylor identities presented in Ref. [27], provide an important element towards connecting the infrared slavery with the ghost dominance picture.

1.5. Issue of renormalizability

As already mentioned, the renormalizability of Coulomb gauge in the continuum has not been proven yet. In the following, we briefly review the efforts made in this direction. Among the various attempts, the most sophisticated approach has been pursued in Ref. [61]. There, the authors define the so-called "interpolating gauge"

$$-a\partial_0 A_0 + \nabla\cdot\vec{A} = 0 \tag{1.44}$$

and recover Coulomb gauge in the limit $a \to 0$. A linear shift in the field variables is performed in order to exhibit a symmetry (called r-symmetry) between the Fermi and Bose unphysical degrees of freedom. Individual closed Fermi-ghost loops and closed unphysical Bose loops diverge like $1/\sqrt{a}$, but they cancel in pairs at every order in perturbation theory by virtue of the r-symmetry. Thus in the Coulomb gauge limit the correlation

[17] This potential is renormalization group invariant and is an upper bound for the gauge invariant potential from the Wilson loop – a statement known as "no confinement without Coulomb confinement" [59].

[18] We postpone the detailed presentation of this mechanism to Chapter 5, concerning the heavy quarks.

functions are finite, and this remains true also for the renormalized correlation functions. However, there have been also identified one-loop graphs that vanish like $\sqrt{a}$, which do not exist in the formal Coulomb gauge (i.e. for $a = 0$). These graphs can not be neglected since they give a finite contribution at two-loop order, when inserted into the graphs that diverge like $1/\sqrt{a}$. One possibility is that these graphs are merely gauge artifacts and decouple from the expectation values of all gauge-invariant quantities such as Wilson loop, but up to know this has not been explicitly shown.

Renormalization of Coulomb gauge QCD has been studied also within the Lagrangian, second order formalism. In Ref. [62], a proof of algebraic renormalizability of the theory has been given with the help of the Zinn-Justin equation. Through diagrammatic analysis the authors have shown that in the strict Coulomb gauge $g^2 D_{00}$ (D_{00} is the time-time component of the gluon propagator) is invariant under renormalization, in accordance with a similar result obtained by Zwanziger [12]. In a covariant gauge, no component of the gluon field has this property.

Chapter 2.

Dyson–Schwinger equations

Dyson–Schwinger equations are the equations of motions in quantum field theory (analogous to the classical Euler-Lagrange equations) and they relate the various Green's functions of the theory. They are very powerful tools to treat nonperturbative phenomena, such as confinement and chiral symmetry breaking, whereas in the weak coupling regime the perturbative series is recovered. The most convenient way to derive Dyson–Schwinger equations (which will also be employed in this work) is to use the functional method, i.e. to derive these equations directly from the invariance of the generating functional under the variation of the field [10, 11]. An alternative method is the Dyson resummation [9], which reorganizes perturbative corrections into subdiagrams.

Dyson–Schwinger equations built an infinite tower of coupled non-linear integral equations, providing a complete description of the theory. Thus, in order to solve the theory it would be in principle necessary to solve the whole set of equations. However, this is in practice impossible and hence the main question is how to truncate the system, i.e. how to find a way to reduce these equations to a smaller subset of simpler equations which can be solved. The only systematic truncation relies on perturbation theory, otherwise one is forced to make an ansatz for the unknown higher order Green's functions. Importantly, the ansatz must respect the symmetry properties of the theory, i.e. it must obey the Ward–Takahashi identities in QED or the Slavnov–Taylor identities in QCD.

In this Chapter we present the formal derivation of the Dyson–Schwinger equations of the two- and three-point quark Green's functions (i.e., the quark gap equation and the quark-gluon vertex functions). In Chapter 3 we will then present results in the perturbative limit, and in the second part of this thesis we will derive the four-point Green's functions and analyze them in the limit of the heavy quark mass.

2.1. Field equations of motion

The full generating functional of the theory is constructed from the functional integral, Eq. (1.32), by adding the corresponding source terms. Explicitly, we have (recall that $\mathcal{D}\Phi$ denotes the integration over all fields):

$$Z[J] = \int \mathcal{D}\Phi \exp\left\{\mathrm{i}\mathcal{S}_q + \mathrm{i}\mathcal{S}_{YM} + \mathrm{i}\mathcal{S}_{FP} + \mathrm{i}\mathcal{S}_s\right\} \tag{2.1}$$

with the action Eq. (1.33) and the sources defined by

$$\mathcal{S}_s = \int d^4x \left[\rho^a\sigma^a + \vec{J}^a\cdot\vec{A}^a + +\kappa^a\phi^a + \vec{K}^a\cdot\vec{\pi}^a + \overline{c}^a\eta^a + \overline{\eta}^a c^a + \xi^a\lambda^a + \overline{q}_\alpha\chi_\alpha + \overline{\chi}_\alpha q_\alpha\right]. \tag{2.2}$$

In the derivation of the quark field equation of motion (from which the Dyson–Schwinger equations will be derived) the gauge-fixing term in the action, S_{FP}, and the terms arising from the conversion to the first order formalism, discussed in the previous Chapter, are unimportant because the quarks are not connected by a primitive vertex to any of the corresponding fields, including the ghosts (i.e., there is no direct coupling term in the quark Lagrange density). What is however important later on is that these extra fields will formally enter the discussion of the Legendre transform (through partial functional derivatives) which, in principle, gives additional terms but which will turn to be vanishing at one-loop order perturbatively.

Also, it is important to note that the generating functional, Eq. (2.1), is restricted to the Gribov region. This restriction might generate complications due to the presence of the Gribov copies inside the Gribov region. However, as discussed in Section 1.3.1, the Gribov copies do not influence the derivation of the Dyson–Schwinger equations in the continuum. Moreover, the boundary terms that may in principle appear are identically zero due to the fact that the Faddeev-Popov operator vanishes on the boundary of the Gribov region.

The quark equation of motion follows from the generating functional Eq. (2.1) and from the observation that the integral of a total derivative vanishes:

$$\int \mathcal{D}\Phi \frac{\delta}{\delta \mathrm{i}\bar{q}_{x\gamma}} \exp\left\{ \mathrm{i}\mathcal{S}_{YM} + \mathrm{i}\int d^4x \left[\bar{q}_{x\alpha} \left(\mathrm{i}\gamma^0 D_0 + \mathrm{i}\vec{\gamma}\cdot\vec{D} - m \right)_{\alpha\beta} q_{x\beta} + \bar{\chi}_{x\alpha} q_{x\alpha} + \bar{q}_{x\alpha}\chi_{x\alpha} \right] + \ldots \right\} = 0. \tag{2.3}$$

In the above, we have inserted the explicit expression for the quark contribution to the QCD action. Also, we have written the quark sources explicitly and denoted the rest with dots. Using the expression for the components of the covariant derivative, Eq. (1.23), it follows that

$$\int \mathcal{D}\Phi \left\{ \left[\mathrm{i}\gamma^0\partial_{0x} + \mathrm{i}\vec{\gamma}\cdot\vec{\nabla}_x + gT^c\gamma^0\sigma^c_x - gT^c\vec{\gamma}\cdot\vec{A}^c_x - m \right]_{\alpha\beta} q_{x\beta} + \chi_{x\alpha} \right\} \exp\{\mathrm{i}\mathcal{S}\} = 0, \tag{2.4}$$

where $\mathcal{S}$ is the full action plus source terms. This expression can be rewritten in terms of derivatives of the generating functional Z:

$$\left[\mathrm{i}\gamma^0\partial_{0x} + \mathrm{i}\vec{\gamma}\cdot\vec{\nabla}_x - m \right]_{\alpha\beta} \frac{\delta Z}{\delta \mathrm{i}\bar{\chi}_{x\beta}} + \left[gT^c\gamma^0 \right]_{\alpha\beta} \frac{\delta^2 Z}{\delta \mathrm{i}\rho^c_x \delta \mathrm{i}\bar{\chi}_{x\beta}} - \left[gT^c\gamma^k \right]_{\alpha\beta} \frac{\delta^2 Z}{\delta \mathrm{i}J^c_{kx} \delta \mathrm{i}\bar{\chi}_{x\beta}} + \chi_{x\alpha} Z = 0 \tag{2.5}$$

Th equation Eq. (2.4) is the starting point for the derivation of the quark Dyson–Schwinger equations. Before we proceed to explicitly derive them, let us first introduce some notations and briefly review the various Green's functions of the theory.

In general, the vacuum expectation values of the time-ordered products of field operators – the *full n-point Green's functions* of the theory (both connected and disconnected) – are obtained by functional differentiation of the generating functional with respect to the sources:

$$G_n(x_1, \ldots, x_n) = \frac{\delta^n Z[J]}{\delta \mathrm{i}J(x_1) \ldots \delta \mathrm{i}J(x_n)}. \tag{2.6}$$

However, in practice we work with connected and one-particle irreducible n-point Green's functions. In order to eliminate the disconnected vacuum to vacuum diagrams,

we use the generating functional of the connected Green's functions W, defined as

$$Z[J] = e^{W[J]}, \tag{2.7}$$

such that the *connected Green's functions* are given by

$$W_n(x_1, \ldots, x_n) = \frac{\delta^n W[J]}{\delta \mathrm{i}J(x_1) \ldots \delta \mathrm{i}J(x_n)}. \tag{2.8}$$

We now introduce a bracket notation for the functional derivatives of W with respect to the sources, such that for a generic source J_α

$$<\mathrm{i}J_\alpha> = \frac{\delta W}{\delta \mathrm{i}J_\alpha}. \tag{2.9}$$

Explicitly, we have:

$$\frac{\delta Z[J]}{\delta \mathrm{i}\bar{\chi}_{x\alpha}} = Z[J] <\mathrm{i}\bar{\chi}_{ax}>, \tag{2.10}$$

$$\frac{\delta^2 Z[J]}{\delta \mathrm{i}\rho^a_x \delta \mathrm{i}\overline{\chi}_{x\alpha}} = Z[J] \left[<\mathrm{i}\rho^a_x \mathrm{i}\overline{\chi}_{x\alpha}> + <\mathrm{i}\rho^a_x><\mathrm{i}\overline{\chi}_{x\alpha}>\right]. \tag{2.11}$$

Using the above equations, we convert Eq. (2.5) into derivatives of $W[J]$ and obtain :

$$\left[\mathrm{i}\gamma^0\partial_{0x} + \mathrm{i}\vec{\gamma}\cdot\vec{\nabla}_x - m\right]_{\alpha\beta} <\mathrm{i}\bar{\chi}_{x\beta}> + gT^c \{ \gamma^0 \left[<\mathrm{i}\rho^c_x><\mathrm{i}\bar{\chi}_{x\alpha}> + <\mathrm{i}\rho^c_x \mathrm{i}\bar{\chi}_{x\alpha}>\right] - \gamma^k \left[<\mathrm{i}J^c_{kx}><\mathrm{i}\bar{\chi}_{x\alpha}> + <\mathrm{i}J^c_{kx}\mathrm{i}\bar{\chi}_{x\alpha}>\right] \} + \chi_{x\alpha} = 0. \tag{2.12}$$

We define the generic classical field (since no confusion can arise, we use the same notation for the quantum fields which are integrated over and for the resulting classical fields) to be:

$$\Phi_\alpha = \frac{1}{Z}\int \mathcal{D}\Phi\Phi_\alpha \exp\{\mathrm{i}\mathcal{S}\} = \frac{1}{Z}\frac{\delta Z}{\delta \mathrm{i}J_\alpha}. \tag{2.13}$$

Explicitly, the classical quark and antiquark fields are given by:

$$\begin{aligned} q_\alpha(x) &= \frac{1}{Z}\int \mathcal{D}\Phi q_\alpha(x)\exp\{\mathrm{i}\mathcal{S}\} &&= \tfrac{1}{Z}\tfrac{\delta Z}{\delta \mathrm{i}\overline{\chi}_\alpha(x)} &&= <\mathrm{i}\overline{\chi}_\alpha(x)> \\ \overline{q}_\alpha(x) &= \frac{1}{Z}\int \mathcal{D}\Phi \overline{q}_\alpha(x)\exp\{\mathrm{i}\mathcal{S}\} &&= -\tfrac{1}{Z}\tfrac{\delta Z}{\delta \mathrm{i}\chi_\alpha(x)} &&= -<\mathrm{i}\chi_\alpha(x)>. \end{aligned} \tag{2.14}$$

Furthermore, we define the effective action (function of the classical fields) via the Legendre transform of $W[J]$ with respect to the fields:

$$\Gamma[\phi, \overline{q}, q] = W[J, \overline{\chi}, \chi] - \mathrm{i}J_\alpha\phi_\alpha - \mathrm{i}\overline{\chi}_\alpha q_\alpha - \mathrm{i}\overline{q}_\alpha\chi_\alpha. \tag{2.15}$$

In the above, we have explicitly separated the quark and Yang–Mills components, such that J_α and ϕ_α denote generic gluonic (Yang–Mills) sources and classical fields, respectively, and we also use the common convention that summation over all discrete indices and integration over continuous arguments is implicit. The generating functional $\Gamma[\Phi]$ then yields the *n-point proper* or *one-particle irreducible (1PI) Green's functions*, which are those Green's functions that are still connected after one internal line is cut:

$$\Gamma_n(x_1, \ldots, x_n) = \frac{\delta^n \Gamma[\Phi]}{\delta \mathrm{i}\Phi(x_1) \ldots \delta \mathrm{i}\Phi(x_n)}. \tag{2.16}$$

In the following, we introduce a bracket to denote derivatives of Γ with respect to fields – although the notation is similar to the derivatives of W with respect to the sources, no confusion can arise since we never mix derivatives with respect to sources and fields. This gives:

$$<\mathrm{i}J_\alpha>=\frac{\delta W}{\delta \mathrm{i}J_\alpha}=\Phi_\alpha \quad \text{and} \quad <\mathrm{i}\Phi_\alpha>=\frac{\delta \Gamma}{\delta \mathrm{i}\Phi_\alpha}=-J_\alpha. \tag{2.17}$$

Note that care must be taken to observe the correct minus signs associated with the quark (Grassmann) fields and sources. Explicitly, this reads:

$$\begin{aligned} q_\alpha(x) =&<\mathrm{i}\overline{\chi}_\alpha(x)>, \qquad \chi_\alpha(x) = -<\mathrm{i}\overline{q}_\alpha(x)>, \\ \overline{q}_\alpha(x) =&-<\mathrm{i}\chi_\alpha(x)>, \qquad \overline{\chi}_\alpha(x) =<iq_\alpha(x)> . \end{aligned} \tag{2.18}$$

Having defined the proper (1PI) Green's function, we can now rewrite the equation of motion, Eq. (2.12), (and from which the Dyson–Schwinger equations will be derived) in terms of proper functions as:

$$\begin{aligned} <\mathrm{i}\bar{q}_{x\alpha}> &= -\mathrm{i}\left[\mathrm{i}\gamma^0\partial_{0x}+\mathrm{i}\vec{\gamma}\cdot\vec{\nabla}_x-m\right]_{\alpha\beta}\mathrm{i}q_{x\beta} \\ &+ gT^c\gamma^0\left[\sigma^c_x q_{x\alpha}+<\mathrm{i}\rho^c_x\mathrm{i}\bar{\chi}_{x\alpha}>\right]-gT^c\gamma^k\left[A^c_{kx}q_{x\alpha}+<\mathrm{i}J^c_{kx}\mathrm{i}\bar{\chi}_{x\alpha}>\right]. \end{aligned} \tag{2.19}$$

In a similar fashion, one can derive the gluon field equations of motion. They are given by[1] (the trace is over Dirac and fundamental color indices):

$$\begin{aligned} <\mathrm{i}A^a_{ix}> &= -\Gamma^{(0)a}_{\bar{q}qAi\alpha\beta}(\mathrm{i}\bar{q}_{x\alpha})(\mathrm{i}q_{x\beta})-g\mathrm{Tr}\left\{[T^a\gamma^i]_{\alpha\beta}<\mathrm{i}\bar{\chi}_{x\beta}\mathrm{i}\chi_{x\alpha}>\right\} \\ &- \int d^4y\,d^4z\,\Gamma^{(0)cab}_{\sigma AAij}(z,x,y)\left[<\mathrm{i}J^b_{jy}\mathrm{i}\rho^c_z>-\mathrm{i}A^b_{jy}\mathrm{i}\sigma^c_z\right] \\ &- \int d^4y\,d^4z\,\frac{1}{2!}\Gamma^{(0)cab}_{\sigma A\sigma i}(z,x,y)\left[<\mathrm{i}\rho^b_y\mathrm{i}\rho^c_z>-\mathrm{i}\sigma^b_y\mathrm{i}\sigma^c_z\right] \\ &- \int d^4y\,d^4z\,\frac{1}{2!}\Gamma^{(0)abc}_{3Aijk}(x,y,z)\left[<\mathrm{i}J^b_{jy}\mathrm{i}J^c_{kz}>-\mathrm{i}A^b_{jy}\mathrm{i}A^c_{kz}\right]+\dots\ , \end{aligned} \tag{2.20}$$

$$\begin{aligned} <\mathrm{i}\sigma^a_x> &= -\Gamma^{(0)a}_{\bar{q}q\sigma\alpha\beta}(\mathrm{i}\bar{q}_{x\alpha})(\mathrm{i}q_{x\beta})+g\mathrm{Tr}\left\{[T^a\gamma^0]_{\alpha\beta}<\mathrm{i}\bar{\chi}_{x\beta}\mathrm{i}\chi_{x\alpha}>\right\} \\ &- \int d^4y\,d^4z\,\frac{1}{2!}\Gamma^{(0)abc}_{\sigma AAjk}(x,y,z)\left[<\mathrm{i}J^b_{jy}\mathrm{i}J^c_{kz}>-\mathrm{i}A^b_{jy}\mathrm{i}A^c_{kz}\right] \\ &- \int d^4y\,d^4z\,\Gamma^{(0)abc}_{\sigma A\sigma j}(x,y,z)\left[<\mathrm{i}J^b_{jy}\mathrm{i}\rho^c_z>-\mathrm{i}A^b_{jy}\mathrm{i}\sigma^c_z\right]+\dots \end{aligned} \tag{2.21}$$

where $\Gamma^{(0)a}_{\bar{q}qAi\alpha\beta}$, $\Gamma^{(0)a}_{\bar{q}q\sigma\alpha\beta}$, and $\Gamma^{(0)cab}_{\sigma AAij}$, $\Gamma^{(0)cab}_{\sigma A\sigma i}$, $\Gamma^{(0)abc}_{3Aijk}$ are the tree-level quark-gluon and triple-gluon vertices, respectively. The first two terms in the above gluon equations of motion, containing the quark-gluon interactions, are needed for the derivation of the quark contributions to the gluon proper two-point functions. The rest of the terms represent the Yang–Mills self-interaction, and are required for the derivation of the Dyson–Schwinger equations for the quark-gluon vertex functions (the dots represent the terms which are not important for the quark sector of the theory and have been left aside). We also mention

[1] The complete derivation, carried on in the context of the Yang–Mills studies, has been presented in Ref. [29] and will not be repeated here.

that in principle one can derive another two equations of motion, for the $\vec{\pi}$ and ϕ fields, but since the quarks do not directly couple to any of these fields, the quark field will not give a contribution to the corresponding proper two-point functions (at least at one loop-perturbative order).

At this stage, it is useful to introduce multiple functional derivatives with respect to quark fields and sources, which will be later on used to derive the Dyson–Schwinger equations. Consider the following partial differentiations, both with respect to sources and fields:

$$\frac{\delta}{\delta \mathrm{i}\Phi_\beta} <X(J)> \;=\; -\mathrm{i}S[\gamma] <\mathrm{i}\Phi_\beta \mathrm{i}\Phi_\gamma><\mathrm{i}J_\gamma X(J)> \tag{2.22a}$$

$$\frac{\delta}{\delta \mathrm{i}J_\beta} <Y(\Phi)> \;=\; \mathrm{i}S[\gamma] <\mathrm{i}J_\beta \mathrm{i}J_\gamma><\mathrm{i}\Phi_\gamma Y(\Phi)> \tag{2.22b}$$

where $S[\gamma] = \pm 1$ accounts for the fact that the quark fields are Grassmann-valued, i.e. a minus sign appears when the index γ refers to the following combinations

$$<\ldots \mathrm{i}q_\gamma><\mathrm{i}\bar{\chi}_\gamma \ldots>, \;\; <\ldots \mathrm{i}\chi_\gamma><\mathrm{i}\bar{q}_\gamma \ldots> \,. \tag{2.23}$$

For $X(J) = \mathrm{i}J_\alpha$, we have that

$$\pm\, \mathrm{i}\frac{\delta}{\delta \mathrm{i}\Phi_\beta} <\mathrm{i}J_\alpha> = \pm S[\gamma] <\mathrm{i}\Phi_\beta \mathrm{i}\Phi_\gamma><\mathrm{i}J_\gamma \mathrm{i}J_\alpha> = \delta_{\alpha\beta} \tag{2.24}$$

(the overall sign is negative for $\Phi_\alpha \equiv \bar{q}_\alpha$). Taking the functional derivative of this with respect to the source $\mathrm{i}J_\delta$ and using the relation Eq. (2.22b), we find

$$\begin{aligned} &\frac{\delta}{\delta \mathrm{i}J_\delta} S[\gamma] <\mathrm{i}\Phi_\beta \mathrm{i}\Phi_\gamma><\mathrm{i}J_\gamma \mathrm{i}J_\alpha> \\ &\quad = \mathrm{i}S[\gamma,\kappa] <\mathrm{i}J_\delta \mathrm{i}J_\kappa><\mathrm{i}\Phi_\kappa \Phi_\beta \mathrm{i}\Phi_\gamma><\mathrm{i}J_\gamma \mathrm{i}J_\alpha> + \eta_{\delta\beta} S[\gamma] <\mathrm{i}\Phi_\beta \mathrm{i}\Phi_\gamma><\mathrm{i}J_\gamma \mathrm{i}J_\delta \mathrm{i}J_\alpha> = 0, \end{aligned} \tag{2.25}$$

where the factor $\eta_{\delta\beta} = -1$ if the fields δ, β anticommute. For $X(J) = \mathrm{i}J_\delta \mathrm{i}J_\alpha$, Eq. (2.22a) becomes

$$\frac{\delta}{\delta \mathrm{i}\Phi_\beta} <\mathrm{i}J_\delta \mathrm{i}J_\alpha> = -\mathrm{i}S[\gamma] <\mathrm{i}\Phi_\beta \mathrm{i}\Phi_\gamma><\mathrm{i}J_\gamma \mathrm{i}J_\delta \mathrm{i}J_\alpha> \tag{2.26}$$

or, with the help of Eq. (2.25)

$$\frac{\delta}{\delta \mathrm{i}\Phi_\beta} <\mathrm{i}J_\delta \mathrm{i}J_\alpha> = -\eta_{\delta\beta} S[\gamma,\kappa] <\mathrm{i}J_\delta \mathrm{i}J_\kappa><\mathrm{i}\Phi_\kappa \Phi_\beta \mathrm{i}\Phi_\gamma><\mathrm{i}J_\gamma \mathrm{i}J_\alpha> \,. \tag{2.27}$$

With this notation multiple functional derivatives of arbitrary order can be efficiently constructed. In particular, this will be used in the next Section for the derivation of the quark gap equation. Also, later on in Chapter 6 the above formula will be employed in the derivation of the 4-point quark Green's functions.

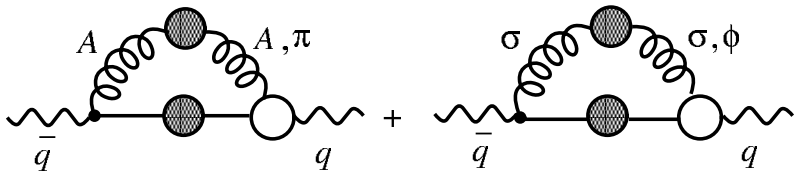

Figure 2.1.: Full nonperturbative diagram for the quark self-energy. Filled circles denote dressed propagators and empty circles denote dressed vertices. Springs denote connected (propagator) functions, solid lines denote quark propagators and wavy lines denote the external legs of the proper functions.

2.2. Quark gap equation

We start the derivation of the gap equation by taking the functional derivative of the quark field equation of motion, Eq. (2.19), with respect to the quark field $\mathrm{i}q_w$, and omit those terms which will eventually vanish when the sources are set to zero. We arrive at:

$$\begin{aligned}<\mathrm{i}\bar{q}_{x\alpha}\mathrm{i}q_{w\beta}> &= \Gamma^{(0)}_{\bar{q}q\alpha\beta}(x)\delta(x-w) \\ &- \int d^4y d^4z\, \delta(x-y)\delta(x-z)\left[\Gamma^{(0)a}_{\bar{q}q\sigma\alpha\gamma}\frac{\delta}{\delta \mathrm{i}q_{w\beta}}<\mathrm{i}\rho^a_y\mathrm{i}\bar{\chi}_{z\gamma}> + \Gamma^{(0)a}_{\bar{q}qAj\alpha\gamma}\frac{\delta}{\delta \mathrm{i}q_{w\beta}}<\mathrm{i}J^a_{jy}\mathrm{i}\bar{\chi}_{z\gamma}>\right].\end{aligned} \tag{2.28}$$

where the (configuration space) tree-level quark proper two-point function $\Gamma^{(0)}_{\bar{q}q}(x)$ and quark-gluon vertices $\Gamma^{(0)a}_{\bar{q}q\sigma}$, $\Gamma^{(0)a}_{\bar{q}qAj}$ are obtained from the quark equation of motion Eq. (2.19). In this thesis we will consider the gap equation both at one-loop perturbative order and in the heavy quark limit. Consequently, the explicit form of the corresponding tree-level quantities will be given separately in Chapter 3, which deals with perturbation theory, and in Chapter 5, where the heavy quark limit is investigated.

We now use the formula Eq. (2.27) to calculate the functional derivatives appearing in the bracket. As already mentioned, simply because of the presence of the $\vec{\pi}$ and ϕ fields arising in the first order formalism, we must allow for the additional terms to be generated. For completeness, we keep all these terms for the moment, bearing in mind that they vanish when we consider the one-loop order in perturbation theory:

$$\begin{aligned}\frac{\delta}{\delta \mathrm{i}q_{w\beta}}<\mathrm{i}\rho^a_y\mathrm{i}\bar{\chi}_{z\gamma}> &= -\int d^4v d^4u\, <\mathrm{i}\bar{\chi}_{z\gamma}\mathrm{i}\chi_{v\delta}><\mathrm{i}\rho^a_y\mathrm{i}\rho^b_u><\mathrm{i}\bar{q}_{v\delta}\mathrm{i}q_{w\beta}\mathrm{i}\sigma^b_u> \\ &\quad -\int d^4v d^4u\, <\mathrm{i}\bar{\chi}_{z\gamma}\mathrm{i}\chi_{v\delta}><\mathrm{i}\rho^a_y\mathrm{i}\kappa^b_u><\mathrm{i}\bar{q}_{v\delta}\mathrm{i}q_{w\beta}\mathrm{i}\phi^b_u>, \qquad (2.29) \\ \frac{\delta}{\delta \mathrm{i}q_{w\beta}}<\mathrm{i}J^a_{jy}\mathrm{i}\bar{\chi}_{z\gamma}> &= -\int d^4v d^4u\, <\mathrm{i}\bar{\chi}_{z\gamma}\mathrm{i}\chi_{v\delta}><\mathrm{i}J^a_{yj}\mathrm{i}J^b_{uk}><\mathrm{i}\bar{q}_{v\delta}\mathrm{i}q_{w\beta}\mathrm{i}A^b_{uk}> \\ &\quad -\int d^4v d^4u\, <\mathrm{i}\bar{\chi}_{z\gamma}\mathrm{i}\chi_{v\delta}><\mathrm{i}J^a_{yj}\mathrm{i}K^b_{uk}><\mathrm{i}\bar{q}_{v\delta}\mathrm{i}q_{w\beta}\mathrm{i}\pi^b_{uk}>\,. \qquad (2.30)\end{aligned}$$

At this point, it is useful to introduce our conventions and notations for the Fourier transform. For a general two-point function (connected or proper) which obeys transla-

tional invariance we have:

$$<\mathrm{i}J_\alpha(y)\mathrm{i}J_\beta(x)> \;=\; \int \mathrm{d}\,k\, W_{\alpha\beta}(k)e^{-\mathrm{i}k\cdot(y-x)} \tag{2.31}$$

$$<\mathrm{i}\Phi_\alpha(y)\mathrm{i}\Phi_\beta(x)> \;=\; \int \mathrm{d}\,k\, \Gamma_{\alpha\beta}(k)e^{-\mathrm{i}k\cdot(y-x)}, \tag{2.32}$$

where $\mathrm{d}\,k = d^4k/(2\pi)^4$. The propagator (connected 2-point function) $W_{\alpha\beta}(y,x)$ and proper (1PI) two-point function $\Gamma_{\alpha\beta}(y,x)$ are related via the Legendre transform. Whereas in covariant gauges this is simply an inversion, in Coulomb gauge this may not always be the case. The relation between the connected and proper two-point functions follows from Eq. (2.24). For the quark propagator, we find (in momentum space) the standard relation

$$W_{\alpha\gamma}(k)\Gamma_{\gamma\beta}(k) = \delta_{\alpha\beta}. \tag{2.33}$$

Returning to the gap equation, we insert the expressions Eq. (2.29), Eq. (2.30) into Eq. (2.28), Fourier transform into momentum space, and we obtain the quark Dyson–Schwinger (or gap) equation:

$$\begin{aligned}
\Gamma_{\bar{q}q\alpha\beta}(k) &= \Gamma^{(0)}_{\bar{q}q\alpha\beta}(k) \\
&+ \int \mathrm{d}\,\omega\, \Gamma^{(0)a}_{\bar{q}q\sigma\alpha\gamma}(k,-\omega,\omega-k) W_{\bar{q}q\gamma\delta}(\omega) \Gamma^{b}_{\bar{q}q\sigma\delta\beta}(\omega,-k,k-\omega) W^{ab}_{\sigma\sigma}(k-\omega) \\
&+ \int \mathrm{d}\,\omega\, \Gamma^{(0)a}_{\bar{q}q\sigma\alpha\gamma}(k,-\omega,\omega-k) W_{\bar{q}q\gamma\delta}(\omega) \Gamma^{b}_{\bar{q}q\phi\delta\beta}(\omega,-k,k-\omega) W^{ab}_{\sigma\phi}(k-\omega) \\
&+ \int \mathrm{d}\,\omega\, \Gamma^{(0)a}_{\bar{q}qAi\alpha\gamma}(k,-\omega,\omega-k) W_{\bar{q}q\gamma\delta}(\omega) \Gamma^{b}_{\bar{q}qAj\delta\beta}(\omega,-k,k-\omega) W^{ab}_{AAij}(k-\omega) \\
&+ \int \mathrm{d}\,\omega\, \Gamma^{(0)a}_{\bar{q}qAi\alpha\gamma}(k,-\omega,\omega-k) W_{\bar{q}q\gamma\delta}(\omega) \Gamma^{b}_{\bar{q}q\pi j\delta\beta}(\omega,-k,k-\omega) W^{ab}_{A\pi ij}(k-\omega).
\end{aligned} \tag{2.34}$$

The self-energy corrections are presented diagrammatically in Fig. 2.1. We see that the $\vec{\pi}$ and ϕ fields do make a contribution thanks to the existence of the mixed propagators $W_{A\pi ij}$ and $W_{\sigma\phi}$ in the first order formalism. But, as emphasized, these contributions will drop out at one-loop order because of the absence of corresponding tree-level vertices, i.e., there exist no direct interaction terms in the action between the quark fields and the auxiliary fields of the first order formalism. However, for future studies one has to bear in mind that additional contributions may arise.

In the second part of this thesis we will also consider the gap equation in the heavy quark limit. In this case we shall work in the standard, second order formalism, where the auxiliary fields do not appear from the first place.

2.3. Quark contributions to the gluon propagators

In order to understand the analytic structure of the gluon propagators, it is necessary to explore the quark contributions to the gluonic proper two-point functions.[2] In contrast

[2] Apart from the quark contribution, the gluon propagators contain ghost loops, as well as a collection of terms generated by the tree-level 4-gluon interactions (which give rise to tadpole and explicit 2-loop contributions to the gluon Dyson–Schwinger equations). All these will not be considered in this work (a complete derivation of the Yang–Mills terms appearing in the Coulomb gauge gluon propagators can be found in Ref. [29]).

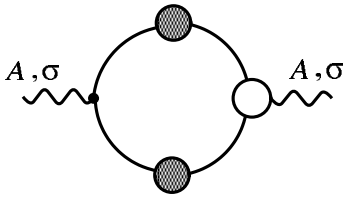

Figure 2.2.: One-loop diagram for the quark contributions to the gluon proper two-point functions. Filled circles denote dressed propagators and empty circles denote dressed vertices. Solid lines denote quark propagators and wavy lines denote the external legs of the proper functions.

to covariant gauges, the various gluonic degrees of freedom (temporal and spatial) are being separated into three proper two-point functions, $\Gamma_{\sigma\sigma}$, Γ_{AA} and $\Gamma_{\sigma A}$. The Dyson–Schwinger equation for the third function, $\Gamma_{\sigma A}$, can be derived, since the quark-gluon vertices entering the equation are defined, even though the function itself does not have a tree level component in the first order formalism.

Starting with the σ equation of motion Eq. (2.21) and following the same procedure as for the gap equation we derive the quark contribution to the proper two-point function $\Gamma_{\sigma\sigma}$ (indicated by the index (q)) in configuration space:

$$<\mathrm{i}\sigma^a_x \mathrm{i}\sigma^b_w>_{(q)} = -\mathrm{Tr}\int d^4y d^4z d^4u d^4v \Gamma^{(0)a}_{\bar{q}q\sigma\alpha\gamma}(z,y,x) <\mathrm{i}\bar{\chi}_{y\gamma}\mathrm{i}\chi_{u\beta}><\mathrm{i}\bar{q}_{u\beta}\mathrm{i}q_{v\delta}\mathrm{i}\sigma^b_w><\mathrm{i}\bar{\chi}_{v\delta}\mathrm{i}\chi_{z\alpha}>. \tag{2.35}$$

In the above, the trace over Dirac and fundamental color indices is taken. Performing the Fourier transform, we get in momentum space:

$$\Gamma^{ab}_{\sigma\sigma(q)}(k) = -\mathrm{Tr}\int d\omega\, \Gamma^{(0)a}_{\bar{q}q\sigma\alpha\gamma}(\omega-k,-\omega,k) W_{\bar{q}q\gamma\beta}(\omega)\Gamma^{b}_{\bar{q}q\sigma\beta\delta}(\omega,k-\omega,-k) W_{\bar{q}q\delta\alpha}(\omega-k). \tag{2.36}$$

Similarly we obtain:

$$\Gamma^{ab}_{\sigma Ai(q)}(k) = -\mathrm{Tr}\int d\omega\, \Gamma^{(0)a}_{\bar{q}q\sigma\alpha\gamma}(\omega-k,-\omega,k) W_{\bar{q}q\gamma\beta}(\omega)\Gamma^{b}_{\bar{q}qAi\beta\delta}(\omega,k-\omega,-k) W_{\bar{q}q\delta\alpha}(\omega-k), \tag{2.37}$$

(it is easy to check that $\Gamma_{\sigma Ai(q)} = \Gamma_{A\sigma i(q)}$) and

$$\Gamma^{ab}_{AAij(q)}(k) = -\mathrm{Tr}\int d\omega\, \Gamma^{(0)a}_{\bar{q}qAi\alpha\gamma}(\omega-k,-\omega,k) W_{\bar{q}q\gamma\beta}(\omega)\Gamma^{b}_{\bar{q}qAj\beta\delta}(\omega,k-\omega,-k) W_{\bar{q}q\delta\alpha}(\omega-k). \tag{2.38}$$

In Chapter 3 we will consider these loop contributions (shown collectively in Fig. 2.2) at one loop perturbative level, and compare the results with the calculations performed in covariant gauges. We can already anticipate that since at one loop perturbative level the quark loop cannot be different from its covariant analog (the difference is that the spatial and temporal degrees of freedom are separated into the corresponding proper two-point functions), the one loop results should equal the covariant gauge calculations. Later on we will see that this is indeed the case.

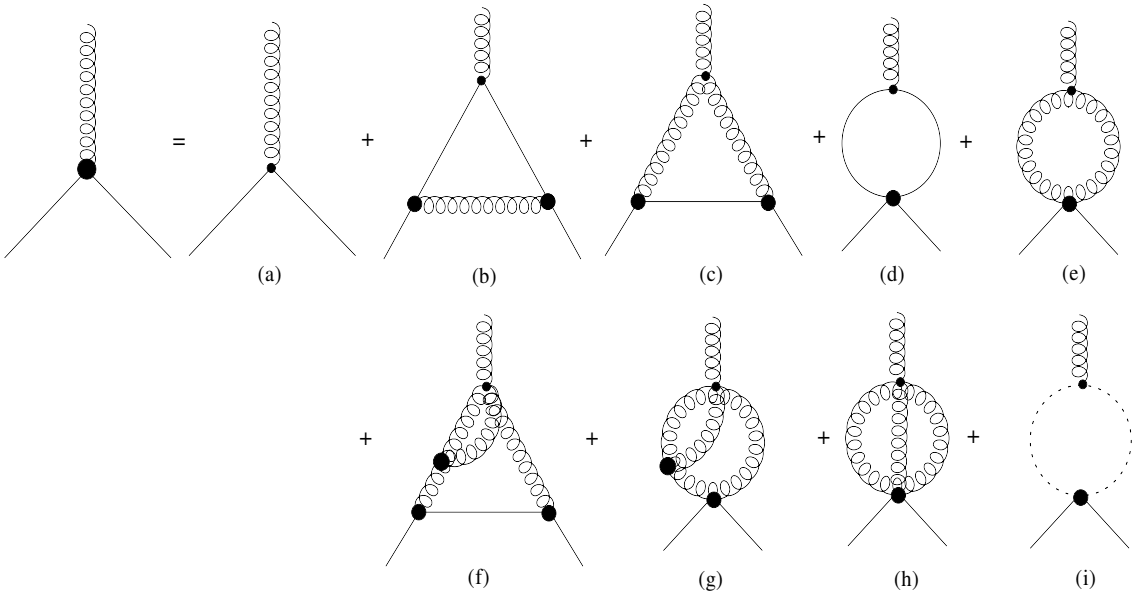

Figure 2.3.: Dyson–Schwinger equation for the (*spatial* or *temporal*) quark-gluon vertex, written in terms of proper (1PI) Green's functions. Blobs denote dressed vertices and all internal propagators are fully dressed. Internal propagators denoted by springs may be either spatial ($\vec{A}$) or temporal (σ) propagators, dashed lines represent the ghost propagator and solid lines represent the quark propagator. Symmetry factors and signs have been omitted.

2.4. Quark-gluon vertex

The quark-gluon vertex, as an important component that relates the Yang–Mills and the quark sector of QCD, has been intensively studied in the last years, mostly in covariant gauges. At perturbative level, it has been analyzed in Ref. [63], in arbitrary (covariant) gauge and dimension. Also, in Landau gauge, nonperturbative studies have been carried on (see, for example, [64] and references therein).

In Coulomb gauge, due to the intrinsic noncovariance, there are *two* quark-gluon vertices, spatial and temporal.[3] Just as for the quark contributions to the gluonic two-point functions, the Dyson–Schwinger equations for the spatial and temporal quark-gluon vertices are obtained by taking the functional derivative of the gluon field equations of motion Eq. (2.20), Eq. (2.21) with respect to $\mathrm{i}\bar{q}_{v\delta}, \mathrm{i}q_{w\gamma}$ and setting sources to zero.

Defining the Fourier transform for the vertex functions (all momenta are incoming):

$$\Gamma(x,y,z) = \int d\,k_1\; d\,k_2\; d\,k_3\,(2\pi)^4\delta(k_1+k_2+k_3)e^{-\mathrm{i}k_1\cdot x-\mathrm{i}k_2\cdot y-\mathrm{i}k_3\cdot z}\Gamma(k_1,k_2,k_3), \quad (2.39)$$

we get for the spatial quark gluon vertex, in terms of proper Green's functions:

[3] In fact, within the first order formalism one can derive the Dyson–Schwinger equations for two more vertex functions, corresponding to the interaction of the quarks with the additional fields π and ϕ. However, since there are no direct interaction terms in the Lagrangian, these terms do not give a contribution at one loop order in perturbation theory.

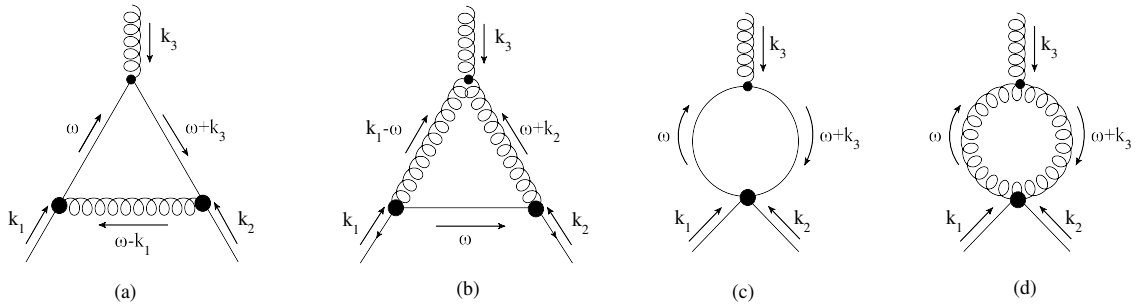

Figure 2.4.: Momentum rooting for the quark-gluon vertex functions. Diagrams (a) and (c) represent the QED-like graphs, and diagrams (b) and (d) are the non-abelian graphs.

$$
\begin{aligned}
&\Gamma^{d}_{\bar{q}qAi\alpha\beta}(k_1,k_2,k_3) = -gT^{d}_{\alpha\beta}\gamma^{i} \\
&\quad - \int \mathrm{d}\omega \,\Big\{\Gamma^{c}_{\bar{q}qAk\alpha\gamma}(\omega-k_1,k_1,-\omega)W_{\bar{q}q\gamma\delta}(\omega)\Gamma^{(0)d}_{\bar{q}qAi\delta\lambda}(k_3,\omega,-k_3-\omega)W_{\bar{q}q\lambda\eta}(k_3+\omega) \\
&\quad \times\Gamma^{b}_{\bar{q}qAj\eta\beta}(k_3+\omega,k_2,k_1-\omega)W^{bc}_{AAjk}(k_1-\omega) \\
&\quad +\Gamma^{c}_{\bar{q}q\sigma\alpha\gamma}(\omega-k_1,k_1,-\omega)W_{\bar{q}q\gamma\delta}(\omega)\Gamma^{(0)d}_{\bar{q}qAi\delta\lambda}(k_3,\omega,-k_3-\omega)W_{\bar{q}q\lambda\eta}(k_3+\omega) \\
&\quad \times\Gamma^{b}_{\bar{q}q\sigma\eta\beta}(k_3+\omega,k_2,k_1-\omega)W^{bc}_{\sigma\sigma}(k_1-\omega) \\
&\quad +\Gamma^{a}_{\bar{q}qAk\alpha\gamma}(\omega-k_1,k_1,-\omega)W_{\bar{q}q\gamma\delta}(\omega)\Gamma^{e}_{\bar{q}q\sigma\delta\beta}(\omega,k_2,-k_2-\omega)W^{ec}_{\sigma\sigma}(k_2+\omega) \\
&\quad \times G^{(0)dbc}_{\sigma AAij}(k_2+\omega,k_3,k_1-\omega)W^{ba}_{AAjk}(\omega-k_1) \\
&\quad +\Gamma^{a}_{\bar{q}q\sigma\alpha\gamma}(\omega-k_1,k_1,-\omega)W_{\bar{q}q\gamma\delta}(\omega)\Gamma^{e}_{\bar{q}qAk\delta\beta}(\omega,k_2,-k_2-\omega)W^{ec}_{AAjk}(k_2+\omega) \\
&\quad \times\Gamma^{(0)dbc}_{\sigma AAij}(k_2+\omega,k_3,k_1-\omega)W^{ba}_{\sigma\sigma}(\omega-k_1) \\
&\quad +\Gamma^{a}_{\bar{q}qAl\alpha\gamma}(\omega-k_1,k_1,-\omega)W_{\bar{q}q\gamma\delta}(\omega)\Gamma^{e}_{\bar{q}qAm\delta\beta}(\omega,k_2,-k_2-\omega)W^{ec}_{AAmk}(k_2+\omega) \\
&\quad \times\Gamma^{(0)dbc}_{3Aijk}(k_2+\omega,k_3,k_1-\omega)W^{ba}_{AAjl}(\omega-k_1) \\
&\quad +\Gamma^{a}_{\bar{q}q\sigma\alpha\gamma}(\omega-k_1,k_1,-\omega)W_{\bar{q}q\gamma\delta}(\omega)\Gamma^{e}_{\bar{q}q\sigma\delta\beta}(\omega,k_2,-k_2-\omega)W^{ec}_{\sigma\sigma}(k_2+\omega) \\
&\quad \times\Gamma^{(0)dbc}_{\sigma A\sigma i}(k_2+\omega,k_3,k_1-\omega)W^{ba}_{\sigma\sigma}(\omega-k_1) \\
&\quad +\Gamma^{ce}_{\bar{q}q\sigma\sigma\alpha\beta}(\omega+k_3,k_2,k_1,\omega)W^{ac}_{\sigma\sigma}(\omega)\Gamma^{(0)abd}_{\sigma A\sigma i}(\omega,k_3,-\omega-k_3)W^{be}_{\sigma\sigma}(\omega+k_3) \\
&\quad +\Gamma^{ce}_{\bar{q}q\bar{q}q\alpha\beta\delta\gamma}(\omega+k_3,k_1,k_2,-\omega)W^{ac}_{\bar{q}q\delta\gamma}(\omega)\Gamma^{(0)d}_{\bar{q}qAi\lambda\eta}(\omega,k_3,-\omega-k_3)W^{be}_{\bar{q}q\eta\gamma}(\omega+k_3)\Big\} \\
&\quad +\ldots .
\end{aligned}
\tag{2.40}
$$

Similarly, the temporal quark gluon vertex is given by:

$$
\begin{aligned}
&\Gamma^{d}_{\bar{q}q\sigma\alpha\beta}(k_1,k_2,k_3) = gT^{d}_{\alpha\beta}\gamma^{0} \\
&\quad - \int \mathrm{d}\omega \,\Big\{\Gamma^{c}_{\bar{q}qAk\alpha\gamma}(\omega-k_1,k_1,-\omega)W_{\bar{q}q\gamma\delta}(\omega)\Gamma^{(0)d}_{\bar{q}q\sigma\delta\lambda}(k_3,\omega,-k_3-\omega)W_{\bar{q}q\lambda\eta}(k_3+\omega) \\
&\quad \times\Gamma^{b}_{\bar{q}qAj\eta\beta}(k_3+\omega,k_2,k_1-\omega)W^{bc}_{AAjk}(k_1-\omega) \\
&\quad +\Gamma^{c}_{\bar{q}q\sigma\alpha\gamma}(\omega-k_1,k_1,-\omega)W_{\bar{q}q\gamma\delta}(\omega)\Gamma^{(0)d}_{\bar{q}q\sigma\delta\lambda}(k_3,\omega,-k_3-\omega)W_{\bar{q}q\lambda\eta}(k_3+\omega) \\
&\quad \times\Gamma^{b}_{\bar{q}q\sigma\eta\beta}(k_3+\omega,k_2,k_1-\omega)W^{bc}_{\sigma\sigma}(k_1-\omega) \\
&\quad +\Gamma^{a}_{\bar{q}qAk\alpha\gamma}(\omega-k_1,k_1,-\omega)W_{\bar{q}q\gamma\delta}(\omega)\Gamma^{e}_{\bar{q}q\sigma\delta\beta}(\omega,k_2,-k_2-\omega)W^{ec}_{\sigma\sigma}(k_2+\omega) \\
&\quad \times\Gamma^{(0)dbc}_{\sigma A\sigma j}(k_2+\omega,k_3,k_1-\omega)W^{ba}_{AAjk}(\omega-k_1)
\end{aligned}
$$

$$
\begin{aligned}
&+\Gamma^{a}_{\bar{q}q\sigma\alpha\gamma}(\omega-k_1,k_1,-\omega)W_{\bar{q}q\gamma\delta}(\omega)\Gamma^{e}_{\bar{q}qAk\delta\beta}(\omega,k_2,-k_2-\omega)W^{ec}_{AAjk}(k_2+\omega)\\
&\Gamma^{(0)dbc}_{\sigma A\sigma j}(k_2+\omega,k_3,k_1-\omega)W^{ba}_{\sigma\sigma}(\omega-k_1)\\
&+\Gamma^{a}_{\bar{q}qAl\alpha\gamma}(\omega-k_1,k_1,-\omega)W_{\bar{q}q\gamma\delta}(\omega)\Gamma^{e}_{\bar{q}qAm\delta\beta}(\omega,k_2,-k_2-\omega)W^{ec}_{AAmk}(k_2+\omega)\\
&\times\Gamma^{(0)dbc}_{\sigma AAjk}(k_2+\omega,k_3,k_1-\omega)W^{ba}_{AAjl}(\omega-k_1)\\
&+\Gamma^{ce}_{\bar{q}q\bar{q}q\alpha\beta\delta\gamma}(\omega+k_3,k_1,k_2,-\omega)W^{ac}_{\bar{q}q\delta\lambda}(\omega)\Gamma^{(0)d}_{\bar{q}q\sigma\lambda\eta}(\omega,k_3,-\omega-k_3)W^{be}_{\bar{q}q\eta\gamma}(\omega+k_3)\Big\}+\ldots.
\end{aligned}
\tag{2.41}
$$

In the expressions Eq. (2.40), Eq. (2.41), the dots represent the interaction of the quarks with the π and ϕ fields, which are not regarded here. Further, the (two-loop) Yang–Mills diagrams arising from the 4-gluon vertex are not considered in this work (see also footnote at the beginning of Section 2.3), but for completeness we have included these two-loop contributions in the graphical representation from Fig. 2.3. The momentum rooting for the vertex functions is shown diagrammatically in Fig. 2.4.

In both equations, the external gluon leg is connected to a bare vertex in the loop diagrams. Alternatively, one can start with the quark equation of motion and take functional derivatives with respect to the gluon fields. Then in the corresponding Dyson–Schwinger equations the external quark legs are attached to the bare internal vertex. In the full theory, both equations should give the same quark gluon vertex, however in a truncated theory one of these equations is eventually easier to solve.[4] In this work we will only use the first "version" of the quark-gluon vertex, based on Eqs. (2.40, 2.41).

In the next chapter, we will explicitly evaluate the divergences of the vertex functions at one-loop perturbative level. Moreover, the corresponding one-loop expressions, combined with the so-called quark-ghost kernels, will be then used to show that the Slavnov–Taylor identity for the quark-gluon vertex is satisfied at leading order in perturbation theory, without explicitly evaluating the integrals.

[4] See also Appendix B of Ref. [64] for an extended discussion of this topic.

Chapter 3.

One-loop perturbative results

This chapter is concerned with the limit when the coupling g between quarks and gluons is small, i.e. the whole formalism can be expanded in powers of g, giving rise to the perturbative expansion.

We will first derive the Feynman rules for the quark sector of the theory (basically, these are the lowest perturbative order Green's functions), and we will establish the general form of the two-point functions. Then we will consider the proper two- and three-point Green's functions derived in the preceding chapter at one-loop order in perturbation theory, and evaluate the corresponding two-point dressing functions. Coming across the problems originating from the energy-divergence and the noncovariance inherent to Coulomb gauge, we will derive the required noncovariant massive integrals, using a technique based on differential equations and integration by parts. We then consider the renormalization of the quark mass and propagator and we verify that the corresponding renormalization factors agree with the results obtained in covariant gauges. We also evaluate the first coefficient of the perturbative β-function, and again we find that our results agree with the covariant gauge calculations. Moreover, we will consider the divergent parts of the temporal and spatial quark-gluon vertices, and shortly discuss the implications for the Slavnov–Taylor identity for the quark-gluon vertices presented in Chapter 4.

3.1. Feynman rules

In this section, we derive the basic Feynman rules and collect all the tree-level quantities required for our one-loop calculations. In addition to tree-level propagators and proper vertices we also derive the proper two-point functions of the theory.

With the field equation of motion written in the forms Eq. (2.4), Eq. (2.12), we can now derive the Feynman rules for the quark components of the theory. We first derive the quark propagator. From the quark equation of motion in terms of connected functions, Eq. (2.12), ignoring interaction terms and functionally differentiating we get the tree-level propagator in configuration space

$$0 = \Big[\mathrm{i}\gamma^0\partial_{0x} + \mathrm{i}\vec{\gamma}\cdot\vec{\nabla}_x - m\Big]_{\alpha\beta} <\mathrm{i}\chi_\gamma(z)\mathrm{i}\overline{\chi}_\beta(x)>^{(0)} -\mathrm{i}\delta_{\gamma\alpha}\delta(z-x). \tag{3.1}$$

For the quark propagator, the Fourier transform Eq. (2.31) explicitly reads (recall that translational invariance is assumed):

$$<\mathrm{i}\overline{\chi}_\beta(z)\mathrm{i}\chi_\alpha(x)>= W_{\overline{q}q\beta\alpha}(z-x) = \int d\,k\, e^{-\mathrm{i}k\cdot(z-x)} W_{\overline{q}q\beta\alpha}(k) \tag{3.2}$$

such that in momentum space, we get

$$0 = \int d\,k\, e^{-\mathrm{i}k\cdot(z-y)} \left\{ W^{(0)}_{\bar{q}q\beta\alpha}(k) \left[\gamma^0 k_0 - \gamma^i k_i + m\right]_{\alpha\gamma} + \mathrm{i}\delta_{\beta\gamma} \right\}. \tag{3.3}$$

The solution is:

$$W^{(0)}_{\bar{q}q\alpha\beta}(k) = (-\mathrm{i}) \frac{\left[\gamma^0 k_0 - \gamma^i k_i + m\right]}{k_0^2 - \vec{k}^2 - m^2} \delta_{\alpha\beta}. \tag{3.4}$$

In a similar fashion, starting with the quark equation of motion in terms of proper functions, Eq. (2.19) the tree-level quark proper two-point function is derived. We obtain:

$$\Gamma^{(0)}_{\bar{q}q\alpha\beta}(x) = \mathrm{i} \left[\mathrm{i}\gamma^0 \partial_{0x} + \mathrm{i}\vec{\gamma}\cdot\vec{\nabla}_x - m \right]_{\alpha\beta} \tag{3.5}$$

or, after Fourier transforming to momentum space:

$$\Gamma^{(0)}_{\bar{q}q\alpha\beta}(k) = \mathrm{i} \left[\gamma^0 k_0 - \gamma^i k_i - m\right] \delta_{\alpha\beta}. \tag{3.6}$$

Due to the noncovariance, we have written out explicitly the components of $\not{k}$, but later on (where appropriate), we will use the usual notation $\not{k} = \gamma^0 k_0 - \gamma^i k_i$.

The (spatial and temporal) tree-level gluon propagators needed in this work have been derived in [29] and are given by:

$$W^{(0)ab}_{AAij}(k) = \delta^{ab} \frac{\mathrm{i}t_{ij}(\vec{k})}{k_0^2 - \vec{k}^2}, \quad W^{(0)ab}_{\sigma\sigma}(k) = \delta^{ab} \frac{\mathrm{i}}{\vec{k}^2} \tag{3.7}$$

where

$$t_{ij}(\vec{k}) = \delta_{ij} - k_i k_j / \vec{k}^2 \tag{3.8}$$

is the transverse spatial projector. It is understood that the denominator factors involving both temporal and spatial components implicitly carry the Feynman prescription, i.e.,

$$\frac{1}{(k_0^2 - \vec{k}^2)} \to \frac{1}{(k_0^2 - \vec{k}^2 + \mathrm{i}0_+)}, \tag{3.9}$$

such that the analytic continuation to the Euclidean space ($k_0 \to \mathrm{i}k_4$) can be performed. This will be explicitly verified at one-loop order in perturbation theory.

There are two tree-level quark-gluon vertices, spatial and temporal, again obtained by taking the functional derivatives with respect to quark, antiquark and gluon field:

$$\Gamma^{(0)a}_{\bar{q}q\sigma\alpha\beta} = \left[gT^a\gamma^0\right]_{\alpha\beta}, \tag{3.10}$$

$$\Gamma^{(0)a}_{\bar{q}qAj\alpha\beta} = -\left[gT^a\gamma^j\right]_{\alpha\beta}. \tag{3.11}$$

Later on, in the evaluation of the vertex functions, we will also need the tree-level gluonic vertices derived in [28] (all momenta are defined as incoming and momentum conservation is assumed):

$$\begin{aligned}
\Gamma^{(0)abc}_{\sigma AAjk}(p_a,p_b,p_c) &= \mathrm{i}gf^{abc}\delta_{jk}(p_b^0 - p_c^0), \\
\Gamma^{(0)abc}_{\sigma A\sigma j}(p_a,p_b,p_c) &= -\mathrm{i}gf^{abc}(p_a - p_c)_j, \\
\Gamma^{(0)abc}_{3Aijk}(p_a,p_b,p_c) &= -\mathrm{i}gf^{abc}\left[\delta_{ij}(p_a - p_b)_k + \delta_{jk}(p_b - p_c)_i + \delta_{ki}(p_c - p_a)_j\right], \\
\Gamma^{(0)abc}_{\bar{c}cAi}(p_{\bar{c}},p_c,p_A) &= -\mathrm{i}gf^{abc}p_{\bar{c}i}.
\end{aligned} \tag{3.12}$$

3.2. Two-point functions

In this section, we introduce the general decompositions of the two-point functions derived in the preceding chapter, and derive the one-loop perturbative expressions for the associated dressing functions. In order to evaluate the noncovariant massive integrals arising in the one-loop calculations, we employ a technique based on differential equations and integration by parts. We then give the perturbative results for the two-point functions.

3.2.1. General decomposition

In order to investigate the Dyson–Schwinger equations for the quark gap equation derived in the previous chapter, in addition to the tree-level forms given in the previous section, we will also require the general decompositions for the quark propagator and proper two-point function, and the relationship between them. Because we work in a noncovariant setting, the usual arguments must be modified to separately account for the temporal and spatial components. Starting with the quark propagator, we observe that (recall that $\mathcal{D}\Phi$ denotes the integration over all fields)

$$W_{\overline{q}q\alpha\beta}(k^0,\vec{k}) \sim \delta_{\alpha\beta}\int \mathcal{D}\Phi\, \overline{q}q\, \exp\{\mathrm{i}\mathcal{S}\} \tag{3.13}$$

such that under both time-reversal and parity transforms, the propagator will remain unchanged – the bilinear combination $\overline{q}q$ is scalar. Since the propagator depends on both k^0 and $\vec{k}$, it has thus *four* components, in distinction to the covariant case where there are only two (a dressing function multiplying $\not{k}$ and a mass term). Hence we can write

$$W_{\bar{q}q\alpha\beta}(k) = \delta_{\alpha\beta}\frac{(-\mathrm{i})}{k_0^2-\vec{k}^2-m^2}\left\{k_0\gamma^0 F_t(k) - k_i\gamma^i F_s(k) + M(k) + k_0k_i\gamma^0\gamma^i F_d(k)\right\} \tag{3.14}$$

where all dressing functions are functions of both k_0^2 and $\vec{k}^2$. At tree-level, one can trivially identify $F_t = F_s = 1$, $F_d = 0$ and $M = m$. The last term with F_d has no covariant counterpart. The possible appearance of this term will be discussed below.

For the proper two-point function, the same arguments apply and we write

$$\Gamma_{\bar{q}q\alpha\beta}(k) = \mathrm{i}\delta_{\alpha\beta}\left\{k_0\gamma^0 A_t(k) - k_i\gamma^i A_s(k) - B_m(k) + k_0k_i\gamma^0\gamma^i A_d(k)\right\} \tag{3.15}$$

and we will refer to A_t, A_s and B_m as the temporal, spatial and massive components, respectively. Again the last component A_d has no covariant counterpart. The relationship between the connected and proper two-point functions is supplied via the Legendre transform (introduced in Chapter 2) and we have

$$\Gamma_{\bar{q}q}(k)W_{\bar{q}q}(k) = 1. \tag{3.16}$$

Let us now discuss the possible appearance of the genuinely noncovariant term corresponding to the dressing function A_d (or equivalently F_d). At one-loop order in perturbation theory, these terms are vanishing. This can be deduced from the Dirac structure of the self-energy loop, stemming from the quark propagator and tree-level vertices. Namely, the tree-level quark propagator does not contain a term with $k_0k_i\gamma^0\gamma^i$ and from the tree-level vertices we only have either two γ^0 or two γ^i matrices together (the gluon propagator

is either purely temporal or spatial). This implies that there is no one-loop contribution that has the overall structure $\gamma^0\gamma^i$ and in turn this means that $A_d = 0$ at one-loop order perturbatively. Hence, the components A_d and F_d will only appear (if at all) at two-loop order or beyond. In in fact, very recently lattice calculations have shown that $F_d = A_d = 0$ [50].

Using the definitions Eq. (3.14), Eq. (3.15), together with the relation Eq. (3.16), we then get the following set of relations for the remaining dressing functions:

$$F_t = \frac{(k_0^2 - \vec{k}^2 - m^2)A_t}{k_0^2 A_t^2 - \vec{k}^2 A_s^2 - B_m^2}, \quad F_s = \frac{(k_0^2 - \vec{k}^2 - m^2)A_s}{k_0^2 A_t^2 - \vec{k}^2 A_s^2 - B_m^2}, \quad M = \frac{(k_0^2 - \vec{k}^2 - m^2)B_m}{k_0^2 A_t^2 - \vec{k}^2 A_s^2 - B_m^2}. \tag{3.17}$$

We again emphasize that these relations only hold up to one-loop perturbatively — in possible future studies it must be recognized that the fourth Dirac structure, $\gamma^0\gamma^i$, may enter in a nontrivial fashion. In this case, the set of relations Eq. (3.17) should be replaced with the following set (which includes the functions A_d and F_d):

$$\begin{aligned} F_t &= \frac{(k_0^2 - \vec{k}^2 - m^2)A_t}{k_0^2 A_t^2 - \vec{k}^2 A_s^2 - B_m^2 + k_0^2\vec{k}^2 A_d^2}, \\ F_s &= \frac{(k_0^2 - \vec{k}^2 - m^2)A_s}{k_0^2 A_t^2 - \vec{k}^2 A_s^2 - B_m^2 + k_0^2\vec{k}^2 A_d^2}, \\ M &= \frac{(k_0^2 - \vec{k}^2 - m^2)B_m}{k_0^2 A_t^2 - \vec{k}^2 A_s^2 - B_m^2 + k_0^2\vec{k}^2 A_d^2}, \\ F_d &= \frac{(k_0^2 - \vec{k}^2 - m^2)A_d}{k_0^2 A_t^2 - \vec{k}^2 A_s^2 - B_m^2 + k_0^2\vec{k}^2 A_d^2}. \end{aligned} \tag{3.18}$$

3.2.2. One-loop perturbative expansions

Let us now consider the one-loop perturbative expansions of the quark gap equation and the quark contributions to the gluon two-point functions. Although so far the formalism has been presented in 4-dimensional Minkowski space, in order to evaluate the resulting loop integrals we have to convert to Euclidean space. This means that we make the analytic continuation $k_0 \to \mathrm{i}k_4$, where k_4 denotes the temporal component of the Euclidean 4-momentum, such that $k^2 = k_4^2 + \vec{k}^2$. Additionally, to regularize the integrals, dimensional regularization is employed[1] with the Euclidean space integration measure

$$d\,\omega = \frac{d\omega_4 d^d\vec{\omega}}{(2\pi)^{d+1}} \tag{3.19}$$

where $d = 3 - 2\varepsilon$ is the spatial dimension. To preserve the dimension of the action we must assign a dimension to the coupling through the replacement

$$g^2 \to g^2\mu^\varepsilon, \tag{3.20}$$

where μ is the square of a non-vanishing mass scale (which may be later associated with a renormalization scale).

[1] An alternative method proposed by Leibbrandt is the so-called split dimensional regularization. This has been discussed in the introductory chapter of this thesis.

The perturbative expansion of a generic two-point dressing function is written as:

$$\Gamma = \Gamma^{(0)} + g^2\Gamma^{(1)} \tag{3.21}$$

where the factor μ^ε is included in $\Gamma^{(1)}$ such that the new coupling and $\Gamma^{(1)}$ are dimensionless.

Let us first consider the (full nonperturbative) gap equation, Eq. (2.34) (depicted in Fig. 2.1). We first insert the various tree-level vertices and propagators given by Eqs. (3.4), (3.7) (3.10), and (3.11). Then we insert the general decomposition of the proper two-point functions, Eq. (3.15), occurring on the left-hand side of the gap equation, take the Dirac projection, solve the color and tensor algebra and lastly perform the Wick rotation. The one-loop temporal, spatial and massive components of the quark gap equation in Euclidean space read (the Casimir invariant $C_F = (N_c^2-1)/2N_c$ is listed in the Appendix A):

$$A_t(k) = 1 - g^2\mu^\varepsilon C_F \frac{1}{k_4^2}\int d\omega \left\{ \frac{k_4\omega_4}{(\omega^2+m^2)(\vec{k}-\vec{\omega})^2} - \frac{k_4\omega_4(\mathrm{d}-1)}{(\omega^2+m^2)(k-\omega)^2}\right\}, \tag{3.22}$$

$$A_s(k) = 1 - g^2\mu^\varepsilon C_F \frac{1}{\vec{k}^2}\int d\omega \left\{ -\frac{2[\vec{k}\cdot(\vec{k}-\vec{\omega})][\vec{\omega}\cdot(\vec{k}-\vec{\omega})]}{(\omega^2+m^2)(k-\omega)^2(\vec{k}-\vec{\omega})^2} - \frac{\vec{k}\cdot\vec{\omega}}{(\omega^2+m^2)(\vec{k}-\vec{\omega})^2} + \frac{\vec{k}\cdot\vec{\omega}(3-\mathrm{d})}{(\omega^2+m^2)(k-\omega)^2}\right\}, \tag{3.23}$$

$$B_m(k) = m + mg^2\mu^\varepsilon C_F \int d\omega \left\{ \frac{1}{(\omega^2+m^2)(\vec{k}-\vec{\omega})^2} + \frac{(\mathrm{d}-1)}{(\omega^2+m^2)(k-\omega)^2}\right\}. \tag{3.24}$$

As mentioned earlier, the possible contribution corresponding to the genuinely noncovariant dressing function A_d does not appear at one-loop. To evaluate the integrals occurring in Eq. (3.23), it is helpful to use the identity:

$$\vec{k}\cdot\vec{\omega} = \frac{1}{2}\left[k^2+\omega^2-(k-\omega)^2\right] - k_4\omega_4, \tag{3.25}$$

which enables us to rewrite A_s as a combination of more straightforward integrals:

$$A_s(k) = 1 - g^2\mu^\varepsilon C_F \frac{1}{\vec{k}^2}\int d\omega \left\{ -\frac{1}{2}\frac{(k^2+m^2)^2}{\omega^2[(k-\omega)^2+m^2]\vec{\omega}^2} + \frac{2(k^2+m^2)k_4\omega_4}{\omega^2[(k-\omega)^2+m^2]\vec{\omega}^2} + \frac{2\varepsilon\vec{k}^2+2k_4^2}{[(k-\omega)^2+m^2]\omega^2} + \frac{2\vec{k}\cdot\vec{\omega}(1-\varepsilon)}{[(k-\omega)^2+m^2]\omega^2} - \frac{1}{2}\frac{m^2+3k^2}{[(k-\omega)^2+m^2]\vec{\omega}^2}\right\}. \tag{3.26}$$

Let us now examine the quark contributions to the various proper two-point gluon dressing functions given by Eqs. (2.36-2.38) (presented in Fig. 2.2). Again, we insert the tree-level factors given by Eqs. (3.4),(3.7), (3.10) and (3.11), solve the color and tensor algebra and perform a Wick rotation. The one-loop integral expressions are (recall that

we have N_f flavors of identical quarks):

$$\vec{k}^2\Gamma^{(1)}_{\sigma\sigma(q)}(k) = \mu^\varepsilon N_f 2 \int d\omega \, \frac{\vec{\omega}^2 - \omega_4^2 - \vec{\omega}\cdot\vec{k} + \omega_4 k_4 + m^2}{(\omega^2+m^2)[(k-\omega)^2+m^2]}, \tag{3.27}$$

$$k_i k_4 \Gamma^{(1)}_{\sigma A(q)}(k) = \mu^\varepsilon N_f 4 \int d\omega \, \frac{\omega_i\omega_4 - k_i\omega_4}{(\omega^2+m^2)[(k-\omega)^2+m^2]}, \tag{3.28}$$

$$\begin{aligned}&\vec{k}^2 t_{ij}(\vec{k})\Gamma^{(1)}_{AA(q)}(k) + k_i k_j \bar{\Gamma}^{(1)}_{AA(q)}(k) \\ &\quad = 2N_f\mu^\varepsilon \int d\omega \, \frac{2\omega_i\omega_j - 2\omega_i k_j + \delta_{ij}(\omega_4 k_4 + \vec{\omega}\cdot\vec{k} - \omega_4^2 - \vec{\omega}^2 - m^2)}{(\omega^2+m^2)[(k-\omega)^2+m^2]},\end{aligned} \tag{3.29}$$

where $\Gamma^{(1)}_{AA(q)}$ and $\bar{\Gamma}^{(1)}_{AA(q)}$ are the transversal and longitudinal components of the proper two-point function $\Gamma^{(1)ab}_{AAij(q)}$ (given by Eq. (2.38)), respectively[2].

Having derived the one-loop perturbative expressions for the propagator dressing functions and the quark contributions to the gluonic two-point functions, we can now proceed to evaluate the corresponding loop integrals. There are two categories of integrals arising: those which can be solved using standard techniques (such as Schwinger parametrization or Mellin representation — the details of these techniques are presented in the Appendices B, C), and those which require a more complex approach. In the next section, we will concentrate on this later variety. Since this part is rather technical, the reader might skip this and go directly to Section 3.2.4, where the physical results are presented.

3.2.3. Noncovariant massive loop integrals

The noncovariant massive loop integrals appearing in the one-loop expansions from the previous section are studied by using a technique based on differential equations and integration by parts developed previously in Ref. [30]. We will consider the two integrals:

$$A_m(k_4^2,\vec{k}^2) = \int \frac{d\omega}{\omega^2[(k-\omega)^2+m^2]\vec{\omega}^2}, \tag{3.30}$$

$$A^4_m(k_4^2,\vec{k}^2) = \int \frac{d\omega\ \omega_4}{\omega^2[(k-\omega)^2+m^2]\vec{\omega}^2}. \tag{3.31}$$

3.2.3.1. Derivation of the differential equations

Let us first write Eqs. (3.30) and (3.31) in the general form ($n = 0, 1$)

$$I^n(k_4^2,\vec{k}^2) = \int \frac{d\omega\ \omega_4^n}{\omega^2[(k-\omega)^2+m^2]\vec{\omega}^2}. \tag{3.32}$$

In this derivation k_4^2 and $\vec{k}^2$ are treated as variables whereas the mass, m, is treated as a parameter. The two first derivatives are:

$$k_4\frac{\partial I^n}{\partial k_4} = \int \frac{d\omega\ \omega_4^n}{\omega^2[(k-\omega)^2+m^2]\vec{\omega}^2}\left\{-2\frac{k_4(k_4-\omega_4)}{(k-\omega)^2+m^2}\right\}, \tag{3.33}$$

$$k_k\frac{\partial I^n}{\partial k_k} = \int \frac{d\omega\ \omega_4^n}{\omega^2[(k-\omega)^2+m^2]\vec{\omega}^2}\left\{-2\frac{\vec{k}\cdot(\vec{k}-\vec{\omega})}{(k-\omega)^2+m^2}\right\}. \tag{3.34}$$

[2] The decomposition of the spatial two-point function Γ_{AAij} is explained in detail in Ref. [29] .

There are also two integration by parts identities:

$$\begin{aligned} 0 &= \int đ\omega\, \frac{\partial}{\partial\omega_4} \frac{\omega_4^{n+1}}{\omega^2[(k-\omega)^2+m^2]\vec{\omega}^2} \\ &= \int \frac{đ\omega\ \omega_4^n}{\omega^2[(k-\omega)^2+m^2]\vec{\omega}^2} \left\{ n+1-2\frac{\omega_4^2}{\omega^2} - 2\frac{\omega_4(\omega_4-k_4)}{(k-\omega)^2+m^2} \right\}, \qquad (3.35) \\ 0 &= \int đ\omega\, \frac{\partial}{\partial\omega_i} \frac{\omega_i\omega_4^n}{\omega^2[(k-\omega)^2+m^2]\vec{\omega}^2} \\ &= \int \frac{đ\omega\ \omega_4^n}{\omega^2[(k-\omega)^2+m^2]\vec{\omega}^2} \left\{ d-2-2\frac{\vec{\omega}^2}{\omega^2} - 2\frac{\vec{\omega}\cdot(\vec{\omega}-\vec{k})}{(k-\omega)^2+m^2} \right\}. \qquad (3.36) \end{aligned}$$

Adding these two expressions gives

$$0 = \int \frac{đ\omega\ \omega_4^n}{\omega^2[(k-\omega)^2+m^2]\vec{\omega}^2} \left\{ d+n-3-2\frac{\omega\cdot(\omega-k)}{(k-\omega)^2+m^2} \right\}. \qquad (3.37)$$

Expanding the numerator factor, we can rewrite Eq. (3.37) as

$$0 = \int \frac{đ\omega\ \omega_4^n}{\omega^2[(k-\omega)^2+m^2]\vec{\omega}^2} \left\{ d+n-4+\frac{k^2-\omega^2+m^2}{(k-\omega)^2+m^2} \right\}. \qquad (3.38)$$

Combining this with Eq. (3.35) then yields:

$$\begin{aligned} &\int \frac{đ\omega\ \omega_4^n}{\omega^2[(k-\omega)^2+m^2]\vec{\omega}^2} \left\{ -2\frac{k_4(k_4-\omega_4)}{(k-\omega)^2+m^2} \right\} = \\ &\int \frac{đ\omega\ \omega_4^n}{\omega^2[(k-\omega)^2+m^2]\vec{\omega}^2} \left[\frac{2k_4^2}{k^2+m^2}(n+d-4)-n+1 \right] + \frac{\vec{k}^2+m^2}{k^2+m^2} \int \frac{đ\omega\,\omega_4^n}{[(k-\omega)^2+m^2]^2\vec{\omega}^2} \\ &-2\int \frac{đ\omega\,\omega_4^n}{\omega^4[(k-\omega)^2+m^2]} - 2\int \frac{đ\omega\,\omega_4^n}{\omega^2[(k-\omega)^2+m^2]^2}. \qquad (3.39) \end{aligned}$$

This leads to the temporal differential equations for A_m and A_m^4:

$$\begin{aligned} k_4\frac{\partial A_m}{\partial k_4} &= \left[1+2\frac{(d-4)k_4^2}{k^2+m^2}\right] A_m + 2\frac{\vec{k}^2+m^2}{k^2+m^2}\int \frac{đ\omega}{[(k-\omega)^2+m^2]^2\vec{\omega}^2} \\ &- 2\int \frac{đ\omega}{\omega^4[(k-\omega)^2+m^2]} - 2\int \frac{đ\omega}{\omega^2[(k-\omega)^2+m^2]^2}, \qquad (3.40) \\ k_4\frac{\partial A_m^4}{\partial k_4} &= 2\frac{(d-3)k_4^2}{k^2+m^2}A_m^4 + 2\frac{\vec{k}^2+m^2}{k^2+m^2}\int \frac{đ\omega\ \omega_4}{[(k-\omega)^2+m^2]^2\vec{\omega}^2} \\ &- 2\int \frac{đ\omega\ \omega_4}{\omega^4[(k-\omega)^2+m^2]} - 2\int \frac{đ\omega\ \omega_4}{\omega^2[(k-\omega)^2+m^2]^2}. \qquad (3.41) \end{aligned}$$

In the same manner, we derive the differential equations involving the spatial components:

$$\begin{aligned} k_i\frac{\partial A_m}{\partial k_i} &= \left[2-d+2\frac{(d-4)\vec{k}^2}{k^2+m^2}\right]A_m - \frac{2\vec{k}^2}{k^2+m^2}\int\frac{d\,\omega}{[(k-\omega)^2+m^2]^2\vec{\omega}^2} \\ &+ 2\int\frac{d\,\omega}{\omega^4[(k-\omega)^2+m^2]} + 2\int\frac{d\,\omega}{\omega^2[(k-\omega)^2+m^2]^2}, \end{aligned} \tag{3.42}$$

$$\begin{aligned} k_i\frac{\partial A^4_m}{\partial k_i} &= \left[2-d+2\frac{(d-3)\vec{k}^2}{k^2+m^2}\right]A^4_m - \frac{2\vec{k}^2}{k^2+m^2}\int\frac{d\,\omega\;\omega_4}{[(k-\omega)^2+m^2]^2\vec{\omega}^2} \\ &+ 2\int\frac{d\,\omega\;\omega_4}{\omega^4[(k-\omega)^2+m^2]} + 2\int\frac{d\,\omega\;\omega_4}{\omega^2[(k-\omega)^2+m^2]^2}. \end{aligned} \tag{3.43}$$

It is in fact possible to write down a mass differential equation, but in the light of the method presented here this would not bring any new information. However, this third differential equation will turn to be useful in checking our solutions – the detailed derivation will be presented in the Appendix C.

At this point, it is instructive to show how the differential equations for the massless integrals considered in Ref. [30] are regained. In the massless limit, there are potential ambiguities arising in the integrals appearing in Eqs. (3.40 -3.43), because in part, the limits $m \to 0$ and $\varepsilon \to 0$ do not interchange. Let us start by considering the following integral given by Eq. (B.16) (similar arguments apply to all the integrals appearing in the differential equations):

$$I = \int\frac{d\,\omega}{\vec{\omega}^2[(k-\omega)^2+m^2]^2} = \frac{[m^2]^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}}\frac{\Gamma(\frac{1}{2}-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(3/2-\varepsilon)}{}_2F_1\left(1,1+\varepsilon;3/2-\varepsilon;-\frac{\vec{k}^2}{m^2}\right). \tag{3.44}$$

It is useful here to invert the argument of the hypergeometric with the help of the formula (see, for instance, Ref. [65]):

$$\begin{aligned} {}_2F_1(a,b;c;t) &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-t)^{-a}{}_2F_1\left(a,1-c+a;1-b+a;\frac{1}{t}\right) \\ &+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-t)^{-b}{}_2F_1\left(b,1-c+b;1-a+b;\frac{1}{t}\right). \end{aligned} \tag{3.45}$$

Then we have:

$$\begin{aligned} I &= \frac{1}{\vec{k}^2}\frac{[m^2]^{-\varepsilon}\Gamma(\varepsilon)}{(4\pi)^{2-\varepsilon}}{}_2F_1\left(1,\frac{1}{2}+\varepsilon;1-\varepsilon;-\frac{m^2}{\vec{k}^2}\right) \\ &+\frac{[\vec{k}^2]^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}}\frac{\Gamma\left(\frac{1}{2}-\varepsilon\right)\Gamma(-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma\left(\frac{1}{2}-2\varepsilon\right)}{}_2F_1\left(1+\varepsilon,\frac{1}{2}+2\varepsilon;1+\varepsilon;-\frac{m^2}{\vec{k}^2}\right). \end{aligned} \tag{3.46}$$

In the expression above, the problem of the non-interchangeable limits is seen explicitly in the first term. However, when all the integrals occurring in the various differential equations are put together, such terms explicitly cancel and only the second term of Eq. (3.46) (which leads to the correct massless limit) contributes.

Returning to the differential equations, we evaluate the standard integrals in terms of

ε (see Appendix B) and with the notation $x = k_4^2$, $y = \vec{k}^2$ we obtain for A_m:

$$\begin{aligned}2x\frac{\partial A_m}{\partial x} &= \left[1 - 2(1+2\varepsilon)\frac{x}{x+y+m^2}\right] A_m \\ &+2\frac{[m^2]^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}}\left\{\frac{y+m^2}{x+y+m^2}X\,{}_2F_1\left(1,1+\varepsilon;3/2-\varepsilon;-\frac{y}{m^2}\right)\right. \\ &\left.-\frac{\Gamma(-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(2-\varepsilon)}\,{}_2F_1\left(2,1+\varepsilon;2-\varepsilon;-\frac{x+y}{m^2}\right) - Y\,{}_2F_1\left(1,1+\varepsilon;2-\varepsilon;-\frac{x+y}{m^2}\right)\right\},\end{aligned} \tag{3.47}$$

$$\begin{aligned}2y\frac{\partial A_m}{\partial y} &= \left[-1 + 2\varepsilon - 2(1+2\varepsilon)\frac{y}{x+y+m^2}\right] A_m \\ &-2\frac{[m^2]^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}}\left\{\frac{y}{x+y+m^2}X\,{}_2F_1\left(1,1+\varepsilon;3/2-\varepsilon;-\frac{y}{m^2}\right)\right. \\ &\left.-\frac{\Gamma(-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(2-\varepsilon)}\,{}_2F_1\left(2,1+\varepsilon;2-\varepsilon;-\frac{x+y}{m^2}\right) - Y\,{}_2F_1\left(1,1+\varepsilon;2-\varepsilon;-\frac{x+y}{m^2}\right)\right\}\end{aligned} \tag{3.48}$$

and for the integral $A^4 = k_4\overline{A}_m$:

$$\begin{aligned}2x\frac{\partial \overline{A}_m}{\partial x} &= \left[-1 - 4\varepsilon\frac{x}{x+y+m^2}\right] \overline{A}_m \\ &+2\frac{[m^2]^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}}\left\{\frac{y+m^2}{x+y+m^2}X\,{}_2F_1\left(1,1+\varepsilon;3/2-\varepsilon;-\frac{y}{m^2}\right)\right. \\ &\left.-\frac{Y}{2-\varepsilon}\left[{}_2F_1\left(2,1+\varepsilon;3-\varepsilon;-\frac{x+y}{m^2}\right) + (1-\varepsilon)\,{}_2F_1\left(1,1+\varepsilon;3-\varepsilon;-\frac{x+y}{m^2}\right)\right]\right\},\end{aligned} \tag{3.49}$$

$$\begin{aligned}2y\frac{\partial \overline{A}_m}{\partial y} &= \left[-1 + 2\varepsilon - 4\varepsilon\frac{y}{x+y+m^2}\right] \overline{A}_m \\ &-2\frac{[m^2]^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}}\left\{\frac{y}{x+y+m^2}X\,{}_2F_1\left(1,1+\varepsilon;3/2-\varepsilon;-\frac{y}{m^2}\right)\right. \\ &\left.-\frac{Y}{2-\varepsilon}\left[{}_2F_1\left(2,1+\varepsilon;3-\varepsilon;-\frac{x+y}{m^2}\right) + (1-\varepsilon)\,{}_2F_1\left(1,1+\varepsilon;3-\varepsilon;-\frac{x+y}{m^2}\right)\right]\right\},\end{aligned} \tag{3.50}$$

where

$$X = \frac{\Gamma(1/2-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(3/2-\varepsilon)}, \quad Y = \frac{\Gamma(1-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(2-\varepsilon)}. \tag{3.51}$$

3.2.3.2. Solving the differential equations

Let us first consider the integral A_m. By the same method as in Ref. [30], we make the following ansatz:

$$A_m(x,y) = F_{Am}(x,y)G_{Am}(x,y) \tag{3.52}$$

such that

$$2x\frac{\partial F_{Am}}{\partial x} = \left[1-(2+4\varepsilon)\frac{x}{x+y+m^2}\right]F_{Am}, \tag{3.53}$$

$$2y\frac{\partial F_{Am}}{\partial y} = \left[-1+2\varepsilon-(2+4\varepsilon)\frac{y}{x+y+m^2}\right]F_{Am}, \tag{3.54}$$

$$\begin{aligned} F_{Am}2x\frac{\partial G_{Am}}{\partial x} &= 2\frac{[m^2]^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}}\left\{\frac{y+m^2}{x+y+m^2}X_2F_1\left(1,1+\varepsilon;3/2-\varepsilon;-\frac{y}{m^2}\right)\right. \\ &\quad -\frac{\Gamma(-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(2-\varepsilon)}{}_2F_1\left(2,1+\varepsilon;2-\varepsilon;-\frac{x+y}{m^2}\right). \\ &\quad \left.+Y_2F_1\left(1,1+\varepsilon;2-\varepsilon;-\frac{x+y}{m^2}\right)\right\}, \end{aligned} \tag{3.55}$$

$$\begin{aligned} F_{Am}2y\frac{\partial G_{Am}}{\partial y} &= 2\frac{[m^2]^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}}\left\{-\frac{y}{x+y+m^2}X_2F_1\left(1,1+\varepsilon;3/2-\varepsilon;-\frac{y}{m^2}\right)\right. \\ &\quad +\frac{\Gamma(-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(2-\varepsilon)}{}_2F_1\left(2,1+\varepsilon;2-\varepsilon;-\frac{x+y}{m^2}\right) \\ &\quad \left.+Y_2F_1\left(1,1+\varepsilon;2-\varepsilon;-\frac{x+y}{m^2}\right)\right\}. \end{aligned} \tag{3.56}$$

By inspection, it is simple to determine the solution for the two homogeneous equations, Eq. (3.53) and Eq. (3.54):

$$F_{Am}(x,y) = x^{1/2}y^{-1/2+\varepsilon}(x+y+m^2)^{-1-2\varepsilon}. \tag{3.57}$$

Since the mass m is treated as a parameter, the (dimensionful) solution, Eq. (3.57), may have an integration constant proportional to $[m^2]^{-1-\varepsilon}$. However, returning to the original equations, Eq. (3.53) and Eq. (3.54), we see that the only consistent solution is the one for which this constant vanishes.

Let us now make the following ansatz for the function G_{Am}, which will be verified below ($z = x/y$):

$$G_{Am}(x,y) = G^0_{Am}(x,y) + \tilde{G}_{Am}(z). \tag{3.58}$$

The component $G^0_{Am}(x,y)$ can be found by adding the differential equations Eq. (3.55) and Eq. (3.56), which lead to:

$$x\frac{\partial G^0_{Am}}{\partial x} + y\frac{\partial G^0_{Am}}{\partial y} = \frac{1}{(4\pi)^{2-\varepsilon}}\left\{\frac{m^2}{\sqrt{x(y+m^2)}}\ln\left(\frac{\sqrt{1+\frac{m^2}{y}}+1}{\sqrt{1+\frac{m^2}{y}}-1}\right)+\mathcal{O}(\varepsilon)\right\} \tag{3.59}$$

Because the function G^0_{Am} is multiplied by the function F_{Am} (which does not have an ε pole), the term of order $\mathcal{O}(\varepsilon)$ will not contribute. The solution of this equation is:

$$G^0_{Am}(x,y) = -\frac{2}{(4\pi)^{2-\varepsilon}}\left\{\frac{1}{\sqrt{z}}\left[\sqrt{1+\frac{m^2}{y}}\ln\left(\frac{\sqrt{1+\frac{m^2}{y}}+1}{\sqrt{1+\frac{m^2}{y}}-1}\right)-\ln y\right]+\mathcal{O}(\varepsilon)\right\}+\mathcal{C}_1. \tag{3.60}$$

Before we proceed to determine $\tilde{G}_{Am}$, we justify the ansatz for G_{Am}, given by Eq. (3.58). First we observe that:

$$2xF_{Am}\frac{\partial G_{Am}}{\partial x} = 2zF_{Am}\frac{\partial \tilde{G}_{Am}}{\partial z} + 2xF_{Am}\frac{\partial G^0_{Am}}{\partial x}. \tag{3.61}$$

Then we subtract the above equation, Eq. (3.61), from Eq. (3.55). This gives:

$$\begin{aligned} z\frac{\partial \tilde{G}_{Am}}{\partial z} = -x\frac{\partial G^0_{Am}}{\partial x} + \frac{1}{F_{Am}}\frac{[m^2]^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}}\Bigg\{ & \frac{y+m^2}{x+y+m^2} X_2F_1\left(1,1+\varepsilon;3/2-\varepsilon;-\frac{y}{m^2}\right) \\ & + Y_2F_1\left(1,1+\varepsilon;2-\varepsilon;-\frac{x+y}{m^2}\right) - \frac{\Gamma(-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(2-\varepsilon)}{}_2F_1\left(2,1+\varepsilon;2-\varepsilon;-\frac{x+y}{m^2}\right)\Bigg\}. \end{aligned} \tag{3.62}$$

Evaluation to the first order in ε is straightforward and we see that the right hand side of the above expression is only a function of the variable z. This allows us to write down a differential equation for $\tilde{G}_{Am}(z)$ in the form:

$$z\frac{\partial \tilde{G}_{Am}}{\partial z} = \frac{1}{(4\pi)^{2-\varepsilon}}\frac{1}{\sqrt{z}}\left\{\frac{1}{\varepsilon} - \gamma + \ln m^2 + \mathcal{O}(\varepsilon)\right\}, \tag{3.63}$$

from which we get immediately

$$\tilde{G}_{Am}(z) = -\frac{2}{(4\pi)^{2-\varepsilon}}\frac{1}{\sqrt{z}}\left\{\frac{1}{\varepsilon} - \gamma + \ln m^2 + \mathcal{O}(\varepsilon)\right\} + \mathcal{C}_2. \tag{3.64}$$

Returning to the original differential equations (3.53 - 3.56) with the function $G(x,y) = G^0_{Am}(x,y) + \tilde{G}_{Am}(z)$, we see that the only consistent solution is the one for which the overall constant $\mathcal{C}_1 + \mathcal{C}_2$ vanishes.

We may now put together the solutions Eqs. (3.57), (3.60) and (3.64) and write for the function A_m:

$$\begin{aligned} A_m(x,y) \;=\; & \frac{(x+y+m^2)^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}}\Bigg\{ -\frac{2}{\varepsilon} + 2\gamma + 2\ln\left(\frac{x+y+m^2}{m^2}\right) \\ & -2\sqrt{1+\frac{m^2}{y}}\ln\left(\frac{\sqrt{1+\frac{m^2}{y}}+1}{\sqrt{1+\frac{m^2}{y}}-1}\right) + \mathcal{O}(\varepsilon)\Bigg\}. \end{aligned} \tag{3.65}$$

We see that for $m^2 = 0$ we regain the result from Ref. [30] and that the singularities are located at $x+y+m^2 = 0$ (with $m^2, y \geq 0$). Two more useful checks arise from the study of the power expansion around $x = 0$ and the mass differential equation (this will be explained in detail in Appendix C).

We now proceed in the same fashion to determine the function $\overline{A}_m(x,y) = F_{\overline{A}m}(x,y)G_{\overline{A}m}(x,y)$.

The resulting partial differential equations are in this case:

$$2x\frac{\partial F_{\overline{A}m}}{\partial x} = \left[-1-4\varepsilon\frac{x}{x+y+m^2}\right]F_{\overline{A}m}, \tag{3.66}$$

$$2y\frac{\partial F_{\overline{A}m}}{\partial y} = \left[-1+2\varepsilon-4\varepsilon\frac{y}{x+y+m^2}\right]F_{\overline{A}m}, \tag{3.67}$$

$$\begin{aligned} F_{\overline{A}m}2x\frac{\partial G_{\overline{A}m}}{\partial x} &= 2\frac{[m^2]^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}}\left\{\frac{y+m^2}{x+y+m^2}X\,{}_2F_1\left(1,1+\varepsilon;3/2-\varepsilon;-\frac{y}{m^2}\right)\right. \\ &\quad -\frac{Y}{2-\varepsilon}\left[{}_2F_1\left(2,1+\varepsilon;3-\varepsilon;-\frac{x+y}{m^2}\right)\right. \\ &\quad \left.\left.+(1-\varepsilon)\,{}_2F_1\left(1,1+\varepsilon;3-\varepsilon;-\frac{x+y}{m^2}\right)\right]\right\}, \end{aligned} \tag{3.68}$$

$$\begin{aligned} F_{\overline{A}m}2y\frac{\partial G_{\overline{A}m}}{\partial y} &= 2\frac{[m^2]^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}}\left\{-\frac{y}{x+y+m^2}X\,{}_2F_1\left(1,1+\varepsilon;3/2-\varepsilon;-\frac{y}{m^2}\right)\right. \\ &\quad +\frac{Y}{2-\varepsilon}\left[{}_2F_1\left(2,1+\varepsilon;3-\varepsilon;-\frac{x+y}{m^2}\right)\right. \\ &\quad \left.\left.+(1-\varepsilon)\,{}_2F_1\left(1,1+\varepsilon;3-\varepsilon;-\frac{x+y}{m^2}\right)\right]\right\} \end{aligned} \tag{3.69}$$

with X, Y defined previously. The solution to the first pair is

$$F_{\overline{A}m}(x,y) = x^{-1/2}y^{-1/2+\varepsilon}(x+y+m^2)^{-2\varepsilon}. \tag{3.70}$$

For brevity, in the above expression and also in the derivation of the function $G_{\overline{A}m} = G^0_{\overline{A}m} + \tilde{G}_{\overline{A}m}$ (the analogue of G_{Am}) we omit the constants of integration – they vanish as in the case of the functions F_{Am} and G_{Am}.

As before, for $G_{\overline{A}m}(x,y)$ we make the ansatz:

$$G_{\overline{A}m}(x,y) = G^0_{\overline{A}m}(x,y) + \tilde{G}_{\overline{A}m}(z). \tag{3.71}$$

In the limit $\varepsilon \to 0$, the component $G^0_{\overline{A}m}(x,y)$ is determined from the differential equation:

$$x\frac{\partial G^0_{\overline{A}m}}{\partial x} + y\frac{\partial G^0_{\overline{A}m}}{\partial y} = \frac{1}{(4\pi)^{2-\varepsilon}}\left\{\frac{\sqrt{x}}{x+y+m^2}\frac{m^2}{\sqrt{y+m^2}}\ln\left(\frac{\sqrt{1+\frac{m^2}{y}}+1}{\sqrt{1+\frac{m^2}{y}}-1}\right)+\mathcal{O}(\varepsilon)\right\}.$$

The solution of this equation is:

$$\begin{aligned} G^0_{\overline{A}m}(x,y) = \frac{1}{(4\pi)^{2-\varepsilon}}\left\{\mathrm{i}\ln\left(\frac{\sqrt{1+\frac{m^2}{y}}-\mathrm{i}\sqrt{z}}{\sqrt{1+\frac{m^2}{y}}+\mathrm{i}\sqrt{z}}\right)\ln\left(\frac{\mathrm{i}\sqrt{z}+1}{\mathrm{i}\sqrt{z}-1}\right)\right. \\ \left.-\mathrm{iLi}_2\left(\frac{1-\mathrm{i}\sqrt{z}}{1+\mathrm{i}\sqrt{z}}\cdot\frac{\sqrt{1+\frac{m^2}{y}}-\mathrm{i}\sqrt{z}}{\sqrt{1+\frac{m^2}{y}}+\mathrm{i}\sqrt{z}}\right)+\mathrm{iLi}_2\left(\frac{1+\mathrm{i}\sqrt{z}}{1-\mathrm{i}\sqrt{z}}\cdot\frac{\sqrt{1+\frac{m^2}{y}}-\mathrm{i}\sqrt{z}}{\sqrt{1+\frac{m^2}{y}}+\mathrm{i}\sqrt{z}}\right)+\mathcal{O}(\varepsilon)\right\}, \end{aligned} \tag{3.72}$$

where $\mathrm{Li}_2(z)$ is the dilogarithmic function [66]:

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\ln(1-t)}{t} dt. \tag{3.73}$$

As before, we check that the ansatz for $G_{\overline{A}m}(x,y)$ given in Eq. (3.71) is correct and derive the differential equation for the function $\tilde{G}_{\overline{A}m}$, in the limit $\varepsilon \to 0$:

$$z\frac{\partial \tilde{G}_{\overline{A}m}}{\partial z} = \frac{1}{(4\pi)^{2-\varepsilon}} \left\{ \frac{\sqrt{z}}{z+1} \left[\ln(1+z) - \ln z - 2\ln 2 \right] + \mathcal{O}(\varepsilon) \right\}. \tag{3.74}$$

The result we leave for the moment in the form:

$$\begin{aligned} \tilde{G}_{\overline{A}m}(z) &= \frac{1}{(4\pi)^{2-\varepsilon}} \{ -4\ln 2 \arctan(\sqrt{z}) + \int_0^z \frac{dt}{\sqrt{t}(1+t)} \ln(1+t) \\ &- \int_0^z \frac{dt}{\sqrt{t}(1+t)} \ln t + \mathcal{O}(\varepsilon) \}. \end{aligned} \tag{3.75}$$

With the solutions, Eqs. (3.70), (3.72) and (3.75), after some further manipulation we can write down the following simplified expression for the integral A_m^4:

$$\begin{aligned} A_m^4(x,y) = k_4 \frac{(x+y+m^2)^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}} \frac{(1+z+\frac{m^2}{y})}{\sqrt{z}} \Bigg\{ &- \int_0^z \frac{dt}{\sqrt{t}(1+t)} \ln\left(1+t+\frac{m^2}{y}\right) \\ &+ 2\ln\left(\frac{\sqrt{1+\frac{m^2}{y}}+1}{\sqrt{1+\frac{m^2}{y}}-1}\right) \arctan\left(\frac{\sqrt{z}}{\sqrt{\frac{m^2}{y}+1}}\right) + 2\ln\left(\frac{m^2}{y}\right) \arctan(\sqrt{z}) + \mathcal{O}(\varepsilon) \Bigg\}, \end{aligned} \tag{3.76}$$

with the integral

$$\begin{aligned} \int_0^z \frac{dt}{\sqrt{t}(1+t)} \ln\left(1+t+\frac{m^2}{y}\right) &= \pi \ln 2 - \mathrm{i} \ln\left(\frac{1-\mathrm{i}\sqrt{z}}{1+\mathrm{i}\sqrt{z}}\right) 2\ln 2 \\ &+ \mathrm{i} \ln\left(\frac{1-\mathrm{i}\sqrt{z}}{1+\mathrm{i}\sqrt{z}} \frac{\sqrt{1+\frac{m^2}{y}}-\mathrm{i}\sqrt{z}}{\sqrt{1+\frac{m^2}{y}}+\mathrm{i}\sqrt{z}}\right) \ln\left(\sqrt{1+\frac{m^2}{y}}+1\right) \\ &+ \mathrm{i} \ln\left(\frac{1-\mathrm{i}\sqrt{z}}{1+\mathrm{i}\sqrt{z}} \frac{\sqrt{1+\frac{m^2}{y}}+\mathrm{i}\sqrt{z}}{\sqrt{1+\frac{m^2}{y}}-\mathrm{i}\sqrt{z}}\right) \ln\left(\sqrt{1+\frac{m^2}{y}}-1\right) \\ &- \mathrm{i} \ln(\sqrt{z}-\mathrm{i}) \left[\ln 2 + \ln(1+z+\frac{m^2}{y}) - \ln(1-\mathrm{i}\sqrt{z}) - \frac{1}{2}\ln(\sqrt{z}-\mathrm{i}) \right] \\ &- \mathrm{i}\mathrm{Li}_2\left(\frac{1}{2}-\frac{\mathrm{i}}{2}\sqrt{z}\right) + \mathrm{i}\mathrm{Li}_2\left(\frac{1}{2}+\frac{\mathrm{i}}{2}\sqrt{z}\right) - \mathrm{i}\mathrm{Li}_2\left(\frac{\mathrm{i}+\sqrt{z}}{-\mathrm{i}+\sqrt{z}}\right) + \mathrm{i}\mathrm{Li}_2\left(\frac{-\mathrm{i}+\sqrt{z}}{\mathrm{i}+\sqrt{z}}\right) \\ &+ \mathrm{i} \ln(\sqrt{z}+\mathrm{i}) \left[\ln 2 + \ln(1+z+\frac{m^2}{y}) - \ln(1+\mathrm{i}\sqrt{z}) - \frac{1}{2}\ln(\sqrt{z}+\mathrm{i}) \right] \\ &+ \mathrm{i}\mathrm{Li}_2\left(\frac{\mathrm{i}\sqrt{1+\frac{m^2}{y}}+\sqrt{z}}{-\mathrm{i}+\sqrt{z}}\right) - \mathrm{i}\mathrm{Li}_2\left(\frac{-\mathrm{i}\sqrt{1+\frac{m^2}{y}}+\sqrt{z}}{\mathrm{i}+\sqrt{z}}\right) \\ &+ \mathrm{i}\mathrm{Li}_2\left(\frac{-\mathrm{i}\sqrt{1+\frac{m^2}{y}}+\sqrt{z}}{-\mathrm{i}+\sqrt{z}}\right) + \mathrm{i}\mathrm{Li}_2\left(\frac{\mathrm{i}\sqrt{1+\frac{m^2}{y}}+\sqrt{z}}{\mathrm{i}+\sqrt{z}}\right). \end{aligned} \tag{3.77}$$

We see that for $m^2 = 0$ we get the correct limit for the function A_m^4. We also mention that the singularities are located at $x + y + m^2 = 0$ and the apparent singularities at $z = -1$ (i.e., $x + y = 0$) in the expression Eq. (3.76) are canceling out. This can be easily seen by making a series expansion of Eq. (3.76) around $z = -1$:

$$\begin{aligned} A_m^4 \quad \overset{z\to -1}{=} \quad & \frac{1}{\sqrt{z}}\left[\frac{y}{m^2} - \frac{z+1}{2}\left(\frac{y}{m^2}\right)^2 + \mathcal{O}\left((z+1)^3\right)\right] \\ & - \frac{\sqrt{1+\frac{m^2}{y}}}{\sqrt{z}(z+1+\frac{m^2}{y})} \ln\left(\frac{\sqrt{1+\frac{m^2}{y}}+1}{\sqrt{1+\frac{m^2}{y}}-1}\right) + \mathcal{O}(\varepsilon). \end{aligned} \tag{3.78}$$

Again, the result Eq. (3.76) has been checked by performing an expansion around $x = 0$ and by studying the mass differential equation (see Appendix C).

3.2.4. Results in the limit $\varepsilon \to 0$

Having derived the noncovariant massive loop integrals in the previous section, and collecting the results for the standard massive integrals from Appendix B, we are now in the position to write down the perturbative results for the dressing functions under consideration. For the temporal, spatial and massive components of the quark gap equation, Eqs. (3.22), (3.24) and (3.26), we find in the limit $\varepsilon \to 0$:

$$\begin{aligned} A_t(k) = 1 + \frac{C_F g^2}{(4\pi)^{2-\varepsilon}}\Bigg\{ & \frac{1}{\varepsilon} - \gamma - \ln\frac{m^2}{\mu} \\ & + 1 - \frac{m^2}{k^2} + \left(\frac{m^4}{k^4} - 1\right)\ln\left(1 + \frac{k^2}{m^2}\right) + \mathcal{O}(\varepsilon)\Bigg\}, \end{aligned} \tag{3.79}$$

$$\begin{aligned} A_s(k) = 1 + \frac{C_F g^2}{(4\pi)^{2-\varepsilon}}\Bigg\{ & \frac{1}{\varepsilon} - \gamma - \ln\frac{m^2}{\mu} \\ & + 1 + 8\frac{k^2}{\vec{k}^2} + 4\frac{m^2}{\vec{k}^2} - \frac{m^2}{k^2} + \left(1 + \frac{m^2}{k^2}\right)\left(4\frac{k^2}{\vec{k}^2} - 1 + \frac{m^2}{k^2}\right)\ln\left(1 + \frac{k^2}{m^2}\right) \\ & - \left(4\frac{k^2}{\vec{k}^2} + 2\frac{m^2}{\vec{k}^2}\right)\sqrt{1 + \frac{m^2}{\vec{k}^2}}\ln\left(\frac{\sqrt{1+\frac{m^2}{\vec{k}^2}}+1}{\sqrt{1+\frac{m^2}{\vec{k}^2}}-1}\right) - 2\frac{k_4^2}{\vec{k}^4}(k^2+m^2) f_m\left(k_4^2, \vec{k}^2\right) + \mathcal{O}(\varepsilon)\Bigg\}, \end{aligned} \tag{3.80}$$

$$\begin{aligned} B_m(k) = m + m\frac{C_F g^2}{(4\pi)^{2-\varepsilon}}\Bigg\{ & \frac{4}{\varepsilon} - 4\gamma - 4\ln\frac{m^2}{\mu} \\ & + 10 - 2\sqrt{1 + \frac{m^2}{\vec{k}^2}}\ln\left(\frac{\sqrt{1+\frac{m^2}{\vec{k}^2}}+1}{\sqrt{1+\frac{m^2}{\vec{k}^2}}-1}\right) - 2\left(1 + \frac{m^2}{k^2}\right)\ln\left(1 + \frac{k^2}{m^2}\right) + \mathcal{O}(\varepsilon)\Bigg\}, \end{aligned} \tag{3.81}$$

where the function $f_m(x, y)$ is given by ($x = k_4^2, y = \vec{k}^2, z = x/y$):

$$\begin{aligned} f_m(x,y) \quad = \quad & \frac{2}{\sqrt{z}}\ln\left(\frac{m^2}{y}\right)\arctan\left(\sqrt{z}\right) + \frac{2}{\sqrt{z}}\ln\left(\frac{\sqrt{1+\frac{m^2}{y}}+1}{\sqrt{1+\frac{m^2}{y}}-1}\right)\arctan\left(\frac{\sqrt{z}}{\sqrt{\frac{m^2}{y}+1}}\right) \\ & - \int_0^1 \frac{dt}{\sqrt{t}(1+zt)}\ln\left(1 + zt + \frac{m^2}{y}\right). \end{aligned} \tag{3.82}$$

The last integral has been rewritten using the identity:

$$\frac{1}{\sqrt{z}}\int_0^z \frac{dt}{\sqrt{t}(1+t)}\ln\left(1+t+\frac{m^2}{y}\right)=\int_0^1 \frac{dt}{\sqrt{t}(1+zt)}\ln\left(1+zt+\frac{m^2}{y}\right). \tag{3.83}$$

As a useful check, we can set $m = 0$ and show that the results for the temporal and spatial components are in agreement with the calculation performed independently using the one-loop massless integrals derived in Ref. [30].

Another important point is related to the singularity structure of the above dressing functions. As has been shown in the previous section, in the noncovariant integrals the singularities appear at $x + y + m^2 = 0$. Further, it is easy to see that the standard integrals have the same singularity structure. As promised, since the singularities in both the Euclidean and spacelike Minkowski regions are absent, we find that the validity of the Wick rotation is justified. In addition, the results for A_t,A_s and B_m can be compared (allowing for the color factors) and agree with those of Quantum Electrodynamics [67, 68].

Having calculated the dressing functions for the quark proper two-point Green's function, we are now able to discuss the structure of the quark propagator. In Eq. (3.17) we first analyze the denominator factor. Let us denote (in Euclidean space):

$$D(k) = k_4^2 A_t^2(k) + \vec{k}^2 A_s^2(k) + B_m^2(k). \tag{3.84}$$

Inserting the expressions from Eqs. (3.79), (3.81) and (3.81) into the above equation, we have:

$$\begin{aligned} D(k) \;=\; & k^2+m^2\left\{1+6\frac{g^2C_F}{(4\pi)^{2-\varepsilon}}\left[\frac{1}{\varepsilon}-\gamma-\ln\frac{m^2}{\mu}+\frac{4}{3}\right]\right\} \\ & +(k^2+m^2)\frac{2C_Fg^2}{(4\pi)^{2-\varepsilon}}\left\{\frac{1}{\varepsilon}-\gamma-\ln\frac{m^2}{\mu}+9\right. \\ & \left.+\left(3-\frac{m^2}{k^2}\right)\ln\left(1+\frac{k^2}{m^2}\right)-4\sqrt{1+\frac{m^2}{\vec{k}^2}}\ln\left(\frac{\sqrt{1+\frac{m^2}{\vec{k}^2}}+1}{\sqrt{1+\frac{m^2}{\vec{k}^2}}-1}\right)-2\frac{k_4^2}{\vec{k}^2}f_m(k_4^2,\vec{k}^2)\right\}. \end{aligned} \tag{3.85}$$

Defining the renormalized mass, m_R, via:

$$m^2 = Z_m^2 m_R^2 \quad \text{with} \quad Z_m^2 = 1-6\frac{g^2C_F}{(4\pi)^{2-\varepsilon}}\left\{\frac{1}{\varepsilon}-\gamma-\ln\frac{m^2}{\mu}+\frac{4}{3}\right\}, \tag{3.86}$$

we see that the expression for $D(k)$, Eq. (3.85), then contains explicitly the overall factor $k^2 + m_R^2$. This means that the simple pole mass of the quark emerges, just as it does in covariant gauges. For the remaining part, the singularity structure is such that non-analytic structures do not appear for spacelike or Euclidean momenta. Moreover, we see that the renormalization factor, Z_m, which defines the physical perturbative pole mass and hence should be a gauge invariant quantity, agrees with the result obtained in covariant gauges [3].

Because of the Dirac structure, it is more convenient to write the quark propagator in Minkowski space, i.e. to analytically continue $k_4^2 \to -k_0^2$. We have shown that the

analytic continuation of the functions A_t, A_s, B_m (and implicitly, of the function $D(k)$) back into the Minkowski space is allowed and this enables us simply to write:

$$W_{\bar{q}q\alpha\beta}(k) = \mathrm{i}\delta_{\alpha\beta}\left\{\gamma^0 k_0 A_t(k) - \gamma^i k_i A_s(k) + B_m(k)\right\} D^{-1}(k). \tag{3.87}$$

Inserting the denominator factor, Eq. (3.85), in the limit $\varepsilon \to 0$ and replacing the mass with its renormalized counterpart, the above expression gives:

$$W_{\bar{q}q\alpha\beta}(k) = -\delta_{\alpha\beta}\frac{i}{k_0^2 - \vec{k}^2 - m_R^2}\left\{(\not{k} + m_R)\left[1 - C_F\frac{g^2}{(4\pi)^{2-\varepsilon}}\left(\frac{1}{\varepsilon} - \gamma\right)\right] + \text{ finite terms}\right\}. \tag{3.88}$$

We can thus write down for the quark propagator:

$$W_{\bar{q}q\alpha\beta}(k) = (-\mathrm{i})\delta_{\alpha\beta}\frac{\not{k} + m_R}{k_0^2 - \vec{k}^2 - m_R^2}Z_2 + \text{ finite terms} \tag{3.89}$$

and identify the renormalization constant (omitting the prescription dependent constants)

$$Z_2 = 1 - \frac{g^2 C_F}{(4\pi)^{2-\varepsilon}}\left(\frac{1}{\varepsilon} - \gamma\right). \tag{3.90}$$

Turning to the quark loop contributions to the gluon two-point proper functions, in evaluating the integral structure of Eqs. (3.27), (3.28) and (3.29) we observe the following relations (in Euclidean space):

$$\Gamma^{(1)}_{\sigma\sigma(q)}(k) = \Gamma^{(1)}_{\sigma A(q)}(k) = -\frac{\vec{k}^2}{k^2}\Gamma^{(1)}_{AA(q)}(k) = -\frac{\vec{k}^2}{k_4^2}\bar{\Gamma}^{(1)}_{AA,q}(k) = I(k_4^2, \vec{k}^2), \tag{3.91}$$

where the integral $I(k_4^2, \vec{k}^2)$ reads (using the results of Appendix B), as $\varepsilon \to 0$:

$$\begin{aligned} I(k_4^2, \vec{k}^2) \;=\; & \frac{N_f}{(4\pi)^{2-\varepsilon}}\left\{-\frac{2}{3}\left[\frac{1}{\varepsilon} - \gamma - \ln\frac{k^2}{\mu}\right] - \frac{10}{9} + \frac{2}{3}\left(4\frac{m^2}{k^2} + \ln\frac{m^2}{k^2}\right)\right. \\ & \left. + \frac{2}{3}\sqrt{1 + \frac{4m^2}{k^2}}\left(1 - 2\frac{m^2}{k^2}\right)\ln\left(\frac{\sqrt{1 + \frac{4m^2}{k^2}} + 1}{\sqrt{1 + \frac{4m^2}{k^2}} - 1}\right) + \mathcal{O}(\varepsilon)\right\}. \end{aligned} \tag{3.92}$$

The relations Eq. (3.91) are similar to the Slavnov–Taylor identities for the Yang–Mills part of the theory, derived in [28]. Also, the above integral Eq. (3.92) agrees with the results obtained in covariant gauges (see for instance [3]). This was in fact expected, since at one-loop level the quark loop as a whole is identical with its covariant counterpart — the only difference is that the various degrees of freedom (temporal and spatial) are being separated into the corresponding proper two-point functions, i.e., Γ_{AA}, $\Gamma_{A\sigma}$ and $\Gamma_{\sigma\sigma}$.

Let us now consider the one-loop gluon propagator dressing functions, in connection to the first coefficient of the perturbative β-function. Analogously to the two-point proper functions, these are constructed by writing $D = D^{(0)} + g^2 D^{(1)}$. As mentioned previously, in the first order formalism we have to account for the presence of the additional $\vec{\pi}, \phi$ and ghost fields, and implicitly the corresponding propagators (for example $D_{A\pi}$). We have already seen that at one-loop the quarks only contribute to three of the gluon proper two-point functions ($\Gamma_{AA}, \Gamma_{A\sigma}$ and $\Gamma_{\sigma\sigma}$). But since these gluon two-point functions are related

to the various connected (propagator) two-point functions arising from the first order formalism, there will be (quark loop) contributions to many more of these propagators. The relationship between the connected and proper gluon two-point functions in the first order formalism is derived in Ref. [30] and will not be reproduced here. The full set of quark contributions to these gluonic type propagators is given by:

$$D^{(1)}_{AA(q)}(k) = D^{(1)}_{\sigma\sigma(q)}(k) = \Gamma^{(1)}_{\sigma\sigma(q)}(k) = I(k_4^2, \vec{k}^2), \tag{3.93}$$

$$D^{(1)}_{A\pi(q)}(k) = -\frac{\vec{k}^2}{k_4^2} D^{(1)}_{\pi\pi(q)}(k) = D^{(1)}_{\sigma\phi(q)}(k) = -D^{(1)}_{\phi\phi(q)}(k) = I(k_4^2, \vec{k}^2). \tag{3.94}$$

We are now able to identify the first coefficient of the β-function. As is well known in Landau gauge, a renormalization group invariant running coupling can be defined through the following perturbative combination of gluon and ghost propagator dressing functions [43]:

$$g^2 D_{AA} D_c^2 \sim g^2 \left[1 + \frac{g^2}{16\pi^2} \frac{1}{\varepsilon} \left(\frac{11 N_c}{3} - \frac{2 N_f}{3} \right) \right]. \tag{3.95}$$

At one-loop in perturbation theory, the coefficient of the $1/\varepsilon$ pole above is simply minus the first coefficient of the β-function ($\beta_0 = -11N_c/3 + 2N_f/3$). By inspecting the relations Eq. (3.93), containing the quark contribution to the propagator D_{AA} , and those obtained in Ref. [30] for the Yang–Mills part of the propagator D_{AA} and the propagator D_c, we see that the same result is achieved in Coulomb gauge. Moreover, in Coulomb gauge, a second renormalization group invariant combination of propagators appears and is given by $g^2 D_{\sigma\sigma}$ [12]. Again, combining our results Eq. (3.93) and those obtained in Ref. [30] we see that indeed the coefficient of $1/\varepsilon$ agrees with this.

3.3. Quark-gluon vertices

We now evaluate the divergent components of the quark-gluon vertex functions at one loop perturbative level. Just as for the two-point functions discussed in the preceding section, we write the the perturbative expansion of the vertex function as

$$\Gamma = \Gamma^{(0)} + g^3 \Gamma^{(1)} \tag{3.96}$$

such that the new coupling and $\Gamma^{(1)}$ are dimensionless.

We first consider the temporal part of the quark-gluon vertex, Eq. (2.41). By the same method as for the two-point functions, we insert the appropriate tree-level vertices and propagators from Eq. (3.4), Eq. (3.7), Eq. (3.12), Eq. (3.10) and Eq. (3.11) and solve the color algebra. For the one-loop expansion of the temporal part of the quark-gluon vertex we find, in Minkowski space (the factor $C_F - N_c/2$ arising from the color algebra is derived

in the Appendix A):

$$\Gamma^{(1)d}_{\bar{q}q\sigma\alpha\beta}(k_1,k_2,k_3) = \left(C_F - \frac{N_c}{2}\right) T^d_{\alpha\beta} \int \mathrm{d}\,\omega$$
$$\left\{ -\gamma^k \frac{\left[\gamma^0(k_1-\omega)_0 - \gamma^i(k_1-\omega)_i + m\right]}{(k_1-\omega)^2 - m^2} \gamma^0 \frac{\left[-\gamma^0(k_2+\omega)_0 + \gamma^i(k_2+\omega)_i + m\right]}{(k_2+\omega)^2 - m^2} \gamma^j \frac{t_{jk}(\vec{\omega})}{\omega^2} \right.$$
$$\left. -\gamma^0 \frac{\left[\gamma^0(k_1-\omega)_0 - \gamma^i(k_1-\omega)_i + m\right]}{(k_1-\omega)^2 - m^2} \gamma^0 \frac{\left[-\gamma^0(k_2+\omega)_0 + \gamma^i(k_2+\omega)_i + m\right]}{(k_2+\omega)^2 - m^2} \gamma^0 \frac{1}{\vec{\omega}^2} \right\}$$
$$-\frac{1}{2} N_c T^d \int \mathrm{d}\,\omega \left\{ \gamma^k \frac{\left[\gamma^0(k_1-\omega)_0 - \gamma^i(k_1-\omega)_i + m\right]}{(k_1-\omega)^2 - m^2} \gamma^0 \frac{(2k_3+\omega)_j}{(\vec{\omega}+\vec{k}_3)^3} \frac{t_{jk}(\vec{\omega})}{\omega^2} \right.$$
$$+\gamma^0 \frac{\left[\gamma^0(\omega-k_2)_0 - \gamma^i(\omega-k_2)_i + m\right]}{(k_2-\omega)^2 - m^2} \gamma^k \frac{(-2k_3-\omega)_j}{(\vec{\omega}+\vec{k}_3)^3} \frac{t_{jk}(\vec{\omega})}{\omega^2}$$
$$\left. +\gamma^l \frac{\left[\gamma^0(k_1-\omega)_0 - \gamma^i(k_1-\omega)_i + m\right]}{(k_1-\omega)^2 - m^2} \gamma^k (k_3+2\omega)_0 \frac{t_{jk}(\vec{\omega}+\vec{k}_3)}{(\omega+k_3)^2} \frac{t_{jl}(\vec{\omega})}{\omega^2} \right\} \tag{3.97}$$

Since we are only interested in the divergent part of the above expression, we first make use of the identity Eq. (3.25) to simplify the non-abelian terms containing two noncovariant denominator factors, and then we employ a simple power analysis to separate the convergent and divergent integrals (at leading order in perturbation theory). After eliminating the convergent terms and using the expression Eq. (3.8) for the transversal projector $t_{ij}(\vec{k})$, we are left with the following divergent part of the temporal quark-gluon vertex:

$$\Gamma^{(1)d}_{\bar{q}q\sigma,div}(k_1,k_2,k_3) = \left(-\frac{\mathrm{i}}{2} T^d N_c + \mathrm{i} T^d C_F\right)$$
$$\int \mathrm{d}\,\omega \left\{ \frac{\gamma^k(\gamma^0\omega_0^2 + \gamma^i\gamma^0\gamma^l\omega_i\omega_l)\gamma^j}{\omega^2[(k_1-\omega)^2 - m^2][(k_2+\omega)^2 - m^2]} \left(\delta_{jk} - \frac{\omega_j\omega_k}{\vec{\omega}^2}\right) \right.$$
$$\left. + \frac{\gamma^0(\gamma^0\omega_0^2 + \gamma^i\gamma^0\gamma^l\omega_i\omega_l)\gamma^0}{\vec{\omega}^2[(k_1-\omega)^2 - m^2][(k_2+\omega)^2 - m^2]} \right\}$$
$$-\mathrm{i} T^d N_c \int \mathrm{d}\,\omega \, \frac{\gamma^l\gamma^0\gamma^k\omega_0^2}{[(k_1-\omega)^2 - m^2](\omega+k_3)^2\omega^2} \left[\delta_{kl} - \frac{\omega_k\omega_l}{2\vec{\omega}^2} - \frac{(\omega+k_3)_k\omega_l}{2(\vec{\omega}+\vec{k}_3)^2}\right] \tag{3.98}$$

Collecting the results for the divergent factors of the three-point loop integrals listed in the Appendix D, it is straightforward to calculate the divergent part of the temporal quark gluon vertex, in the limit $\varepsilon \to 0$. It is given by (in Minkowski space):

$$\Gamma^{(1)d}_{\bar{q}q\sigma,div} = T^d \gamma^0 \frac{C_F}{(4\pi)^2} \frac{1}{\varepsilon} + \text{finite terms.} \tag{3.99}$$

Repeating this calculation for the spatial component of the quark-gluon vertex, Eq. (2.40), we obtain:

$$\Gamma^{(1)d}_{\bar{q}qAi,div} = -T^d \gamma^i \left(C_F + \frac{4}{3} N_c\right) \frac{1}{(4\pi)^2} \frac{1}{\varepsilon} + \text{finite terms.} \tag{3.100}$$

The results Eq. (3.99), Eq. (3.100) will then provide a useful check for the Slavnov–Taylor identity for the quark-gluon vertices. We will return to this subject at the end of the next chapter, when we analyze the divergent structure of the Slavnov–Taylor identity.

Chapter 4.

Slavnov–Taylor identity for the quark-gluon vertices

The Slavnov–Taylor identities relate Green's functions with a different number of external legs, analogous to the Ward–Takahashi identities in QED. They play an important role in connection to nonperturbative Dyson–Schwinger studies, since they provide constraints for the higher n-point functions that enter the Dyson–Schwinger equations.

In order to derive these identities, we start with the observation that the BRST invariance is an unbroken symmetry of the gauge fixed theory, at the level of the Lagrangian. The BRST symmetry (and its variation in Coulomb gauge) will be presented in detail in the first Section of this chapter. Next, we will present the formal derivation of the Slavnov–Taylor identities for the quark-gluon vertex functions, based on the BRST invariance of the QCD Green's functions, and in particular, we will examine the so-called quark-ghost scattering kernels. We will then demonstrate that these identities are satisfied at one-loop perturbative level. The proof is based on the translational invariance of the loop integrals and it does not require the explicit evaluation of the loop integrals.

This chapter concerns the Slavnov–Taylor for the quark-gluon vertices, but similar identities can be derived for Green's functions of arbitrary order. Some of the higher order identities will be explored in the second part of this thesis, in the limit of heavy quark mass. Moreover, their implications for the Dyson–Schwinger equations will also be discussed.

4.1. Gauss–BRST symmetry

Regardless of the specific gauge, the gauge fixing term in the QCD action violates the invariance under the local gauge transformation Eq. (A.3). Except for the axial gauge, the invariance of the full (gauge-fixed) action can be recovered by performing a local gauge transformation similar to Eq. (A.3), but with the following ansatz [69, 70]

$$\theta^a(x) = c^a(x)\lambda \tag{4.1}$$

where both c and λ are Grassman-valued quantities and λ is a constant. By demanding that the gluon, quark and ghost fields transform as

$$\begin{aligned} \delta A^a &= \frac{1}{g}\hat{D}^{ac}c^c\delta\lambda, \\ \delta q_\alpha &= -\mathrm{i}\,[T^a]_{\alpha\beta}\,c^a\delta\lambda q_\beta \\ \delta\overline{c}^a &= \frac{1}{g}\lambda^a\delta\lambda \end{aligned}$$

$$\delta c^a \quad = \quad -\frac{1}{2} f^{abc} c^b c^c \delta\lambda \tag{4.2}$$

(c and $\bar{c}$ are independent Grassmann fields, and thus we have independent transformations), we find that the gauge-fixing term remains also invariant. This new symmetry is called BRST (Becchi-Rouet-Stora-Tyutin) and is the basis for deriving the generalized Ward–Takahashi identities (also called Slavnov–Taylor identities).

While the "standard" BRST symmetry is independent of the gauge considered, in Coulomb gauge the time derivatives are absent and hence one can further define a *time-dependent* — so-called Gauss–BRST symmetry [12]. As before, the spacetime-dependent parameter θ^a is factorized into two Grassmann components

$$\theta^a(x) = c^a(x)\delta\lambda_t, \tag{4.3}$$

but in this case $\delta\lambda_t$ (not to be confused with the colored Lagrange multiplier λ^a) is a *time dependent* infinitesimal variation. Within the standard, second order formalism, the fields transform as:

$$\begin{aligned}
&\delta\sigma_a = -\frac{1}{g}\partial_0\theta^a - f^{abc}\sigma^b\theta^c, \quad \delta\vec{A}^a = \frac{1}{g}\vec{\nabla}\theta^a - f^{abc}\vec{A}^b\theta^c, \\
&\delta q_\alpha = -\mathrm{i}\theta^a \left[T^a\right]_{\alpha\beta} q_\beta, \quad \delta\overline{q}_\alpha = \mathrm{i}\theta^a \overline{q}_\beta \left[T^a\right]_{\beta\alpha}, \\
&\delta\overline{c}^a = \frac{1}{g}\lambda^a\delta\lambda_t, \quad \delta c^a = -\frac{1}{2} f^{abc} c^b c^c \delta\lambda_t, \quad \delta\lambda^a = 0.
\end{aligned} \tag{4.4}$$

The Gauss-BRST transform, Eq. (4.3), is the starting point for the derivation of the Slavnov–Taylor identities in Coulomb gauge. These identities, together with the peculiar features introduced by the time dependent transformation, will be discussed in the next section.

4.2. Formal derivation

As discussed in the previous section, the full (gauge fixed) QCD action in the standard, second order formalism, Eq. (1.21), is invariant under a Gauss–BRST transform, Eq. (4.3). The Coulomb gauge Slavnov–Taylor identities arise from regarding the Gauss–BRST transform as a change of integration variables under which the generating functional Eq. (2.1) (together with the source terms Eq. (2.2)) is invariant. Since the Jacobian factor is trivial [29] and only the source term varies, we deduce that

$$\begin{aligned}
0 &= \int \mathcal{D}\Phi \frac{\delta}{\delta\left[\mathrm{i}\delta\lambda_t\right]} \exp\left\{\mathrm{i}\mathcal{S}_{QCD} + \mathrm{i}\mathcal{S}_{FP} + \mathrm{i}\mathcal{S}_s + \mathrm{i}\delta\mathcal{S}_s\right\}\Big|_{\delta\lambda_t=0} \\
&= \int \mathcal{D}\Phi \exp\left\{\mathrm{i}\mathcal{S}_{QCD} + \mathrm{i}\mathcal{S}_{FP} + \mathrm{i}\mathcal{S}_s\right\} \int d^4x \delta(t-x_0) \left\{ -\frac{1}{g}\left(\partial_x^0 \rho_x^a\right) c_x^a + f^{abc}\rho_x^a \sigma_x^b c_x^c \right. \\
&\left. - \frac{1}{g} J_{ix}^a \nabla_{ix} c_x^a + f^{abc} J_{ix}^a A_{ix}^b c_x^c - \mathrm{i}\overline{\chi}_{\alpha x} c_x^a T_{\alpha\beta}^a q_{\beta x} - \mathrm{i} c_x^a \overline{q}_{\beta x} T_{\beta\alpha}^a \chi_{\alpha x} + \frac{1}{g}\lambda_x^a \eta_x^a + \frac{1}{2} f^{abc} \overline{\eta}_x^a c_x^b c_x^c \right\}.
\end{aligned} \tag{4.5}$$

Notice the $\delta(t-x_0)$ constraint, which arises because the time-dependent variation $\delta\lambda_t$ and is characteristic to the Gauss-BRST transform. It leads eventually to a non-trivial energy injection in the Slavnov–Taylor identities which is not present in the covariant gauge case.

The above identity is best expressed in terms of proper functions. Using the definitions Eq. (2.17) for the derivatives of W with respect to the sources and of Γ with respect to the classical fields, we arrive at the identity

$$
\begin{aligned}
0 = & \int d^4x \delta(t-x_0) \\
& \times \Bigg\{ \frac{1}{g} \left(\partial_x^0 <\mathrm{i}\sigma_x^a> \right) c_x^a - f^{abc} <\mathrm{i}\sigma_x^a> \left[<\mathrm{i}\rho_x^b \mathrm{i}\overline{\eta}_x^c> + \sigma_x^b c_x^c \right] - \frac{1}{g} \left[\frac{\nabla_{ix}}{(-\nabla_x^2)} <\mathrm{i}A_{ix}^a> \right] <\mathrm{i}\overline{c}_x^a> \\
& - f^{abc} <\mathrm{i}A_{ix}^a> t_{ij}(\vec{x}) \left[<\mathrm{i}J_{jx}^b \mathrm{i}\overline{\eta}_x^c> + A_{jx}^b c_x^c \right] - \frac{1}{g} \lambda_x^a <\overline{c}_x^a> + \frac{1}{2} f^{abc} <\mathrm{i}c_x^a> \left[<\mathrm{i}\overline{\eta}_x^b \mathrm{i}\overline{\eta}_x^c> + c_x^b c_x^c \right] \\
& + \mathrm{i}T_{\alpha\beta}^a <\mathrm{i}q_{\alpha x}> \left[<\mathrm{i}\overline{\chi}_{\beta x} \mathrm{i}\overline{\eta}_x^a> - c_x^a q_{\beta x} \right] + \mathrm{i}T_{\beta\alpha}^a \left[<\mathrm{i}\chi_{\beta x} \mathrm{i}\overline{\eta}_x^a> + c_x^a \overline{q}_{\beta x} \right] <\mathrm{i}\overline{q}_{\alpha x}> \Bigg\}. \qquad (4.6)
\end{aligned}
$$

Note that functional derivatives involving the Lagrange multiplier result merely in a trivial identity such that the classical field λ_x^a can be set to zero [27]. Further, to derive the quark Slavnov–Taylor identities, one functional derivative with respect to $\mathrm{i}c_z^d$ is needed and then the ghost fields/sources can be set to zero. Implementing this then gives

$$
\begin{aligned}
0 = & \int d^4x \delta(t-x_0) \\
& \times \Bigg\{ -\frac{\mathrm{i}}{g} \left(\partial_x^0 <\mathrm{i}\sigma_x^d> \right) \delta(z-x) - f^{abc} <\mathrm{i}\sigma_x^a> \left[\frac{\delta}{\delta \mathrm{i}c_z^d} <\mathrm{i}\rho_x^b \mathrm{i}\overline{\eta}_x^c> - \mathrm{i}\sigma_x^b \delta^{dc} \delta(z-x) \right] \\
& + \frac{1}{g} \left[\frac{\nabla_{ix}}{(-\nabla_x^2)} <\mathrm{i}A_{ix}^a> \right] <\mathrm{i}\overline{c}_x^a \mathrm{i}c_z^d> - f^{abc} <\mathrm{i}A_{ix}^a> t_{ij}(\vec{x}) \left[\frac{\delta}{\delta \mathrm{i}c_z^d} <\mathrm{i}J_{jx}^b \mathrm{i}\overline{\eta}_x^c> - \mathrm{i}A_{jx}^b \delta^{dc} \delta(z-x) \right] \\
& - \mathrm{i}T_{\alpha\beta}^a <\mathrm{i}q_{\alpha x}> \left[\frac{\delta}{\delta \mathrm{i}c_z^d} <\mathrm{i}\overline{\chi}_{\beta x} \mathrm{i}\overline{\eta}_x^a> + \delta^{da} \delta(z-x) \mathrm{i}q_{\beta x} \right] \\
& + \mathrm{i}T_{\beta\alpha}^a \left[\frac{\delta}{\delta \mathrm{i}c_z^d} <\mathrm{i}\chi_{\beta x} \mathrm{i}\overline{\eta}_x^a> - \delta^{da} \delta(z-x) \mathrm{i}\overline{q}_{\beta x} \right] <\mathrm{i}\overline{q}_{\alpha x}> \Bigg\}. \qquad (4.7)
\end{aligned}
$$

Two further functional derivatives with respect to $\mathrm{i}q_{\gamma\omega}$ and $\mathrm{i}\overline{q}_{\delta v}$ are taken and all remaining fields/sources set to zero. Now we use the following relation, derived in Ref. [27]

$$
\frac{\delta}{\delta \mathrm{i}c_z^d} <\mathrm{i}\rho_x^b \mathrm{i}\overline{\eta}_x^c> \Big|_{J=0} = t_{ij}(\vec{x}) \frac{\delta}{\delta \mathrm{i}c_z^d} <\mathrm{i}J_{jx}^b \mathrm{i}\overline{\eta}_x^c> \Big|_{J=0} = 0, \qquad (4.8)
$$

and obtain the Slavnov–Taylor identity for the quark-gluon vertices in configuration space:

$$
\begin{aligned}
0 = & \int d^4x \delta(t-x_0) \\
& \times \Bigg\{ -\frac{\mathrm{i}}{g} \left(\partial_x^0 <\mathrm{i}\overline{q}_{\delta v} \mathrm{i}q_{\gamma\omega} \mathrm{i}\sigma_x^d> \right) \delta(z-x) + \frac{1}{g} \left[\frac{\nabla_{ix}}{(-\nabla_x^2)} <\mathrm{i}\overline{q}_{\delta v} \mathrm{i}q_{\gamma\omega} \mathrm{i}A_{ix}^a> \right] <\mathrm{i}\overline{c}_x^a \mathrm{i}c_z^d> \\
& + \mathrm{i}T_{\alpha\beta}^a <\mathrm{i}\overline{q}_{\delta v} \mathrm{i}q_{\alpha x}> \left[\frac{\delta^2}{\delta \mathrm{i}q_{\gamma\omega} \delta \mathrm{i}c_z^d} <\mathrm{i}\overline{\chi}_{\beta x} \mathrm{i}\overline{\eta}_x^a> \Big|_{J=0} + \delta^{da} \delta(z-x) \delta_{\gamma\beta} \delta(\omega - x) \right] \\
& + \mathrm{i}T_{\beta\alpha}^a \left[\frac{\delta^2}{\delta \mathrm{i}\overline{q}_{\delta v} \delta \mathrm{i}c_z^d} <\mathrm{i}\chi_{\beta x} \mathrm{i}\overline{\eta}_x^a> \Big|_{J=0} - \delta^{da} \delta(z-x) \delta_{\delta\beta} \delta(v-x) \right] <\mathrm{i}\overline{q}_{\alpha x} \mathrm{i}q_{\gamma\omega}> \Bigg\}. \qquad (4.9)
\end{aligned}
$$

We now introduce the following definitions:

$$
\begin{aligned}
\tilde{\Gamma}^{d}_{\overline{q};\overline{c}cq\alpha\gamma}(x,z,\omega) &\equiv \mathrm{i}gT^{a}_{\alpha\beta}\,\frac{\delta^{2}}{\delta \mathrm{i}q_{\gamma\omega}\delta \mathrm{i}c^{d}_{z}}\,<\mathrm{i}\overline{\chi}_{\beta x}\mathrm{i}\overline{\eta}^{a}_{x}>\Big|_{J=0} \\
&= \mathrm{i}gT^{a}_{\alpha\beta}\Big\{-<\mathrm{i}\overline{\chi}_{\beta x}\mathrm{i}\chi_{\varepsilon}><\mathrm{i}\overline{q}_{\varepsilon}\mathrm{i}q_{\gamma\omega}\mathrm{i}\Phi_{\lambda}><J_{\lambda}\mathrm{i}J_{\kappa}><\overline{\eta}^{a}_{x}\mathrm{i}\eta_{\tau}><\mathrm{i}\overline{c}_{\tau}\mathrm{i}c^{d}_{z}\mathrm{i}\Phi_{\kappa}> \\
&\quad + <\mathrm{i}\overline{\chi}_{\beta x}\mathrm{i}\chi_{\kappa}><\mathrm{i}\overline{q}_{\kappa}\mathrm{i}q_{\gamma\omega}\mathrm{i}\overline{c}_{\tau}\mathrm{i}c^{d}_{z}><\mathrm{i}\overline{\eta}^{a}_{x}\mathrm{i}\eta_{\tau}>\Big\}\,, \\
\tilde{\Gamma}^{d}_{q;\overline{c}c\overline{q}\delta\alpha}(x,z,v) &\equiv \frac{\delta^{2}}{\delta \mathrm{i}\overline{q}_{\delta v}\delta \mathrm{i}c^{d}_{z}}\,<\mathrm{i}\chi_{\beta x}\mathrm{i}\overline{\eta}^{a}_{x}>\Big|_{J=0}\mathrm{i}gT^{a}_{\beta\alpha} \\
&= \Big\{<\overline{\eta}^{a}_{x}\mathrm{i}\eta_{\tau}><\mathrm{i}\overline{c}_{\tau}\mathrm{i}c^{d}_{z}\mathrm{i}\Phi_{\kappa}><\mathrm{i}J_{\kappa}\mathrm{i}J_{\lambda}><\mathrm{i}\overline{q}_{\delta v}\mathrm{i}q_{\varepsilon}\mathrm{i}\Phi_{\lambda}><\mathrm{i}\overline{\chi}_{\varepsilon}\mathrm{i}\chi_{\beta x}> \\
&\quad - <\mathrm{i}\overline{\eta}^{a}_{x}\mathrm{i}\eta_{\tau}><\mathrm{i}\overline{c}_{\tau}\mathrm{i}c^{d}_{z}\mathrm{i}\overline{q}_{\delta v}\mathrm{i}q_{\kappa}><\mathrm{i}\overline{\chi}_{\kappa}\mathrm{i}\chi_{\beta x}>\Big\}\,\mathrm{i}gT^{a}_{\beta\alpha}. \qquad (4.10)
\end{aligned}
$$

In the above, the sources/fields J and Φ refer to either $\vec{A}$ or σ, and the internal indices refer to all attributes of the object in question (summed or integrated over). A few more comments concerning the above definitions and notations are in order. Recall that in the "standard" Dyson–Schwinger equations, the loop expressions contain a propagator ($<\mathrm{i}\overline{\chi}_{\alpha z}\mathrm{i}\chi_{\beta x}>$ or any other propagator) and an associated tee-level vertex, stemming directly from the interaction terms in the Lagrangian. However, as seen in Eq. (4.10), in the case of the (nonabelian) Slavnov–Taylor identities one has a different structure which arises from the Gauss-BRST transform (or generally from the BRST transform): namely, one has to functionally differentiate objects such as $<\mathrm{i}\overline{\chi}_{\beta x}\mathrm{i}\overline{\eta}^{a}_{x}>$, but in this case the vertex arising from a direct interaction is missing. The resulting expressions still have a (partial) meaning as loop integrals, even though the tree-level vertex is absent. These types of would-be loop expressions, denoted with a tilde, are called *quark-ghost scattering kernels* and are common to the Slavnov–Taylor identities of nonabelian theories.[1]

With these definitions, we return to the configuration space Slavnov–Taylor identity Eq. (4.9), and rewritte it as

$$
\begin{aligned}
0 = &\int d^{4}x\,\delta(t-x_{0}) \\
&\times\Big\{-\Big(\mathrm{i}\partial^{0}_{x}\,<\mathrm{i}\overline{q}_{\delta v}\mathrm{i}q_{\gamma\omega}\mathrm{i}\sigma^{d}_{x}>\Big)\,\delta(z-x)+\Big[\frac{\nabla_{ix}}{(-\nabla^{2}_{x})}\,<\mathrm{i}\overline{q}_{\delta v}\mathrm{i}q_{\gamma\omega}\mathrm{i}A^{a}_{ix}>\Big]\,<\mathrm{i}\overline{c}^{a}_{x}\mathrm{i}c^{d}_{z}> \\
&+<\mathrm{i}\overline{q}_{\delta v}\mathrm{i}q_{\alpha x}>\Big[\tilde{\Gamma}^{d}_{\overline{q};\overline{c}cq\alpha\gamma}(x,z,\omega)+\mathrm{i}gT^{d}_{\alpha\gamma}\delta(z-x)\delta(\omega-x)\Big] \\
&+\Big[\tilde{\Gamma}^{d}_{q;\overline{c}c\overline{q}\delta\alpha}(x,z,v)-\mathrm{i}gT^{d}_{\delta\alpha}\delta(z-x)\delta(v-x)\Big]\,<\mathrm{i}\overline{q}_{\alpha x}\mathrm{i}q_{\gamma\omega}>\Big\}\,. \qquad (4.11)
\end{aligned}
$$

Using the formula Eq. (2.39) for the Fourier transform of the vertex functions, one can write the Slavnov–Taylor identity in momentum space (momentum conservation is

[1] In Coulomb gauge Yang–Mills theory these objects have been studied in detail in [27], and in linear covariant gauges, some well-known examples are the ghost-gluon and quark-ghost scattering-like kernels, appearing in the identities for the three-gluon and quark-gluon vertices (a nice presentation of this subject can be found in Ref. [71]).

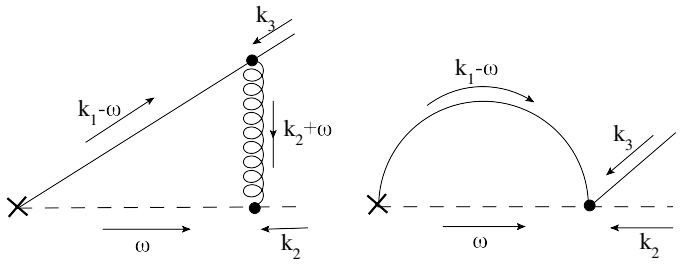

Figure 4.1.: Quark-ghost scattering-like kernels. Solid lines denote quarks, dashed lines denote ghosts and spring lines denote gluons. All internal propagators are dressed. The cross indicates the absence of a tree-level vertex (see text for details).

assumed)

$$\begin{aligned} k_3^0 \Gamma^d_{\overline{q}q\sigma\alpha\beta}(k_1,k_2,k_3) &= \mathrm{i}\frac{k_{3i}}{\vec{k}_3^2}\Gamma^a_{\overline{q}qA\alpha\beta i}(k_1,k_2,k_3)\Gamma^{ad}_{\overline{c}c}(-k_3) \\ &\quad +\Gamma_{\overline{q}q\alpha\delta}(k_1)\left[\tilde{\Gamma}^d_{\overline{q};\overline{c}cq}(k_1+q_0,k_3-q_0;k_2)+\mathrm{i}gT^d\right]_{\delta\beta} \\ &\quad +\left[\tilde{\Gamma}^d_{q;\overline{c}c\overline{q}}(k_2+q_0,k_3-q_0;k_1)-\mathrm{i}gT^d\right]_{\alpha\delta}\Gamma_{\overline{q}q\delta\beta}(-k_2) \quad (4.12) \end{aligned}$$

In the above, $\Gamma_{\overline{c}c}$ is the proper ghost two-point function and q_0 is the (arbitrary) energy injection scale that arises from the time-dependence of the Gauss-BRST transform. This expression is very similar to the one derived in Ref. [27] for the Yang–Mills case.

We now write the quark-ghost kernels Eq. (4.10) in momentum space. After performing the Fourier transform, we arrive at the following expressions (represented diagrammatically in Fig. 4.1):

$$\begin{aligned} \tilde{\Gamma}^d_{\overline{q};\overline{c}cq}(p_1,p_2,p_3) &= \mathrm{i}gT^a\int d\omega\, W^{ab}_{\overline{c}c}(\omega)W_{\overline{q}q}(p_1-\omega)\Big[\Gamma^{bd}_{\overline{c}cq\overline{q}}(\omega,p_2,p_1-\omega,p_3) \\ &\quad -\Gamma^{bdc}_{\overline{c}c\kappa}(\omega,p_2,-p_2-\omega)W^{ce}_{\kappa\lambda}(p_2+\omega)\Gamma^e_{\overline{q}q\lambda}(p_1-\omega,p_3,p_2+\omega)\Big], \\ \tilde{\Gamma}^d_{q;\overline{c}c\overline{q}}(p_1,p_2,p_3) &= \int d\omega\Big[\Gamma^{bdc}_{\overline{c}c\kappa}(\omega,p_2,-p_2-\omega)W^{ce}_{\kappa\lambda}(p_2+\omega)\Gamma^e_{\overline{q}q\lambda}(p_3,p_1-\omega,p_2+\omega) \\ &\quad -\Gamma^{bd}_{\overline{c}c\overline{q}q}(\omega,p_2,p_3,p_1-\omega)\Big]W^{ab}_{\overline{c}c}(\omega)W_{\overline{q}q}(\omega-p_1)\mathrm{i}gT^a, \qquad (4.13) \end{aligned}$$

where the indices κ, λ, refer to the gluonic field types σ or $\vec{A}$ (with the associated spatial index).

Just as in the Yang–Mills case, the above Slavnov–Taylor identity, Eq. (4.12) is used as a constraint, i.e. it helps to find meaningful truncations for the vertex functions of the theory, which are then used to solve the Dyson–Schwinger equations. In particular, in the limit of the heavy quark mass, this equation will be used to write the temporal quark-gluon vertex $\Gamma_{\overline{q}q\sigma}$ in terms of purely spatial, ghost or quark propagators and proper functions. This will then be used to obtain an exact solution of the quark gap equation and of the Bethe–Salpeter equation for quark-antiquark bound states.

Using the Feynman rules presented in chapter 3, it is easy to show that the identity Eq. (4.12) is trivially satisfied at tree-level. Also, it can be verified that this identity is

fulfilled at one-loop perturbatively. In the next section, we will present this derivation in detail.

4.3. Perturbative analysis

Let us now briefly show that the Slavnov-Taylor identity is fulfilled at one-loop perturbative level. The proof is based on the translational invariance of the loop-integrals, without explicitly evaluating them. At one-loop perturbative level, Eq. (4.12) reads:

$$\begin{aligned}
&k_3^0\Gamma^{(1)d}_{\bar{q}q\sigma}(k_1,k_2,k_3)+k_3^i\Gamma^{(1)d}_{\bar{q}qAi}(k_1,k_2,k_3)+\mathrm{i}gT^a\gamma^i\frac{k_{3i}}{\vec{k}_3^2}\Gamma^{(1)ad}_{\bar{c}c}(-\vec{k}_3)\\
&\quad=\mathrm{i}g\left[\Gamma^{(1)}_{\bar{q}q}(k_1)T^d-T^d\Gamma^{(1)}_{\bar{q}q}(-k_2)\right]\\
&\quad+\Gamma^{(0)}_{\bar{q}q}(k_1)\tilde{\Gamma}^d_{\bar{q};\bar{c}cq}(k_1+q_0,k_3-q_0,k_2)+\tilde{\Gamma}^d_{q;\bar{c}c\bar{q}}(k_2+q_0,k_3-q_0,k_1)\Gamma^{(0)}_{\bar{q}q}(-k_2).
\end{aligned}\tag{4.14}$$

In the above, we have inserted the tree-level spatial quark-gluon vertex, Eq. (3.11), and the tree-level ghost two-point function from Ref. [28]

$$\Gamma^{ad(0)}_{\bar{c}c}(k)=\delta^{ad}\mathrm{i}\vec{k}^2.\tag{4.15}$$

Using the expressions Eq. (2.40), Eq. (2.41) for the quark-gluon vertices, we obtain for the (nonperturbative) combination of quark-gluon vertex functions, on the left-hand side of Eq. (4.14):

$$\begin{aligned}
&k_3^0\Gamma^{(1)d}_{\bar{q}q\sigma}(k_1,k_2,k_3)+k_3^i\Gamma^{(1)d}_{\bar{q}qAi}(k_1,k_2,k_3)\\
&\quad=-\int d\omega\Big\{\times\Gamma^c_{\bar{q}qAk}(-k_1+\omega,k_1,-\omega)W_{\bar{q}q}(\omega)\\
&\quad\Big[k_3^0\Gamma^{(0)d}_{\bar{q}q\sigma}(k_3,\omega,-k_3-\omega)+k_{3i}\Gamma^{(0)d}_{\bar{q}qAi}(k_3,\omega,-k_3-\omega)\Big]\\
&\quad\times W_{\bar{q}q}(k_3+\omega)\Gamma^b_{\bar{q}qAj}(k_3+\omega,k_2,k_1-\omega)W^{bc}_{AAjk}(k_1-\omega)\\
&\quad-\Gamma^c_{\bar{q}q\sigma}(-k_1+\omega,k_1,-\omega)W_{\bar{q}q}(\omega)\Big[k_3^0\Gamma^{(0)d}_{\bar{q}q\sigma}(k_3,\omega,-k_3-\omega)+k_{3i}\Gamma^{(0)d}_{\bar{q}qAi}(k_3,\omega,-k_3-\omega)\Big]\\
&\quad\times W_{\bar{q}q}(k_3+\omega)\Gamma^b_{\bar{q}q\sigma}(k_3+\omega,k_2,k_1-\omega)W^{bc}_{\sigma\sigma}(k_1-\omega)\\
&\quad-\Gamma^a_{\bar{q}qAk}(-k_1+\omega,k_1,-\omega)W_{\bar{q}q}(\omega)\Gamma^e_{\bar{q}q\sigma}(\omega,k_2,-k_2-\omega)W^{ec}_{\sigma\sigma}(k_2+\omega)W^{ba}_{AAjk}(-k_1+\omega)\\
&\quad\times\Big[k_3^0\Gamma^{(0)dbc}_{\sigma A\sigma j}(k_2+\omega,k_3,k_1-\omega)+k_{3i}\Gamma^{(0)dbc}_{\sigma AAij}(k_2+\omega,k_3,k_1-\omega)\Big]\\
&\quad-\Gamma^a_{\bar{q}q\sigma}(-k_1+\omega,k_1,-\omega)W_{\bar{q}q}(\omega)\Gamma^e_{\bar{q}qAk}(\omega,k_2,-k_2-\omega)W^{ec}_{AAjk}(k_2+\omega)W^{ba}_{\sigma\sigma}(-k_1+\omega)\\
&\quad\times\Big[k_3^0\Gamma^{(0)dbc}_{\sigma A\sigma j}(k_2+\omega,k_3,k_1-\omega)+k_{3i}\Gamma^{(0)dbc}_{\sigma AAij}(k_2+\omega,k_3,k_1-\omega)\Big]\\
&\quad-\Gamma^a_{\bar{q}qAl}(-k_1+\omega,k_1,-\omega)W_{\bar{q}q}(\omega)\Gamma^e_{\bar{q}qAm}(\omega,k_2,-k_2-\omega)W^{ec}_{AAmk}(k_2+\omega)W^{ba}_{AAjl}(-k_1+\omega)\\
&\quad\times\Big[k_3^0\Gamma^{(0)dbc}_{\sigma AAjk}(k_2+\omega,k_3,k_1-\omega)+k_{3i}\Gamma^{(0)dbc}_{3Aijk}(k_2+\omega,k_3,k_1-\omega)\Big]\\
&\quad-\Gamma^a_{\bar{q}q\sigma}(-k_1+\omega,k_1,-\omega)W_{\bar{q}q}(\omega)\Gamma^e_{\bar{q}q\sigma}(\omega,k_2,-k_2-\omega)W^{ec}_{\sigma\sigma}(k_2+\omega)W^{ba}_{\sigma\sigma}(-k_1+\omega)\\
&\quad\times k_{3i}\Gamma^{(0)dbc}_{A\sigma\sigma i}(k_2+\omega,k_3,k_1-\omega)\Big\}
\end{aligned}\tag{4.16}$$

The lengthy expression above can be simplified if we make use of the tree-level Yang-Mills Slavnov-Taylor identities derived in Ref. [27].[2] For completeness we list them all here (together with the tree-level quark Slavnov-Taylor identity, which is trivial):

$$
\begin{aligned}
&k_3^0\Gamma^{fed(0)}_{\sigma A\sigma k}(k_1,k_2,k_3)+k_{3i}\Gamma^{fed(0)}_{\sigma AAki}=-gf^{ade}\Gamma^{fa(0)}_{\sigma Ak}(k_1)-gf^{adf}\Gamma^{ea(0)}_{\sigma Ak}(k_2)\\
&k_3^0\Gamma^{abc(0)}_{AA\sigma jk}(k_1,k_2,k_3)+k_{3i}\Gamma^{abc}_{3Aijk(0)}=-gf^{dcb}\Gamma^{ad(0)}_{AAjk}(k_2)-gf^{dea}\Gamma^{bd(0)}_{AAjk}(k_1)\\
&k_{3i}\Gamma^{abc(0)}_{\sigma\sigma Ai}(k_1,k_2,k_3)=-gf^{dcb}\Gamma^{ad(0)}_{\sigma\sigma}(k_2)-gf^{dca}\Gamma^{bd(0)}_{\sigma\sigma}(k_1)\\
&k_3^0\Gamma^{(0)d}_{\bar{q}q\sigma}(k_1,k_2,k_3)+k_{3i}\Gamma^{(0)d}_{\bar{q}qAi}(k_1,k_2,k_3)=\Gamma^{(0)}_{\bar{q}q}(k_1)\mathrm{i}gT^d-\mathrm{i}gT^d\Gamma^{(0)}_{\bar{q}q}(-k_2)
\end{aligned}
\tag{4.17}
$$

Further, in order to derive the one-loop expression, in the full nonperturbative equation Eq. (4.16) we insert the tree-level vertices and propagators given by Eq. (3.4), Eq. (3.7), Eq. (3.10), Eq. (3.11). Putting all these together, we arrive at the following simplified expression:

$$
\begin{aligned}
&k_3^0\Gamma^{(1)d}_{\bar{q}q\sigma}(k_1,k_2,k_3)+k_3^i\Gamma^{(1)d}_{\bar{q}qAi}(k_1,k_2,k_3)\\
&\quad=-\mathrm{i}g^3T^aT^aT^d\\
&\quad\int \mathrm{d}\omega\,\Big\{\gamma^kW_{\bar{q}q}(k_3+\omega)\gamma^jW_{AAjk}(k_1-\omega)-\gamma^kW_{\bar{q}q}(\omega)\gamma^jW_{AAjk}(k_1-\omega)\\
&\quad+\gamma^0W_{\bar{q}q}(k_3+\omega)\gamma^0W_{\sigma\sigma}(k_1-\omega)-\gamma^0W_{\bar{q}q}(\omega)\gamma^0W_{\sigma\sigma}(k_1-\omega)\Big\}\\
&\quad=\frac{\mathrm{i}}{2}T^dN_cg^3\int \mathrm{d}\omega\,\frac{k_{3j}t_{ij}(\vec{\omega})}{\omega^2(\vec{k}_3-\vec{\omega})^2}\\
&\quad\times\Big\{\Big[\gamma^0(k_3-\omega)_0-\gamma^k(k_3-\omega)_k\Big]\frac{\big[\gamma^0(\omega+k_2)_0-\gamma^k(\omega+k_2)_k-m\big]}{(\omega+k_2)^2-m^2}\gamma^i\\
&\quad+\gamma^i\frac{\big[\gamma^0(\omega+k_1)_0-\gamma^k(\omega+k_1)_k+m\big]}{(\omega+k_1)^2-m^2}\Big[\gamma^0(k_3-\omega)_0-\gamma^k(k_3-\omega)_k\Big]\Big\}
\end{aligned}
\tag{4.18}
$$

We first compare the first integral of Eq. (4.18) with the combination of one-loop quark two-point functions (without ghost factors) on the r.h.s. of Eq. (4.14). Using the expression Eq. (2.34) for the gap equation and recalling that the $\vec{\pi}$ and ϕ fields do not give a contribution at one-loop order in perturbation theory, we deduce that these terms are equal and hence they cancel in the Slavnov–Taylor identity. For the rest of the terms (the combination of quark- ghost kernels on the r.h.s. and the remaining ghost term on the l.h.s. of Eq. (4.14)) we get:

$$
\begin{aligned}
&\Gamma^{(0)}_{\bar{q}q}(k_1)\tilde{\Gamma}^d_{\bar{q};\bar{c}cq}(k_1+q_0,k_3-q_0,k_2)+\tilde{\Gamma}^d_{q;\bar{c}c\bar{q}}(k_2+q_0,k_3-q_0,k_1)\Gamma^{(0)}_{\bar{q}q}(-k_2)\\
&\quad-\mathrm{i}gT^a\gamma^i\frac{k_{3i}}{\vec{k}_3^2}\Gamma^{(1)ad}_c(-\vec{k}_3)=\frac{\mathrm{i}}{2}T^dN_cg^3\int \mathrm{d}\omega\,\frac{k_{3j}t_{ij}(\vec{\omega})}{\omega^2(\vec{k}_3-\vec{\omega})^2}\\
&\quad\Big\{-(\gamma^0k_{10}-\gamma^kk_{1k}-m)\frac{\big[\gamma^0(\omega+k_2)_0-\gamma^k(\omega+k_2)_k-m\big]}{(\omega+k_2)^2-m^2}\gamma^i\\
&\quad+\gamma^i\frac{\big[\gamma^0(\omega+k_1)_0-\gamma^k(\omega+k_1)_k+m\big]}{(\omega+k_1)^2-m^2}(-\gamma^0k_{20}+\gamma^kk_{2k}-m)-2k_{3i}\frac{\gamma^kk_{3k}}{\vec{k}_3^2}\Big\}
\end{aligned}
\tag{4.19}
$$

[2] In principle, the Yang–Mills identities also contain ghost-gluon scattering-like kernels, but at tree-level these factors do not give a contribution.

After rearranging the terms, we find that the second integral in the formula Eq. (4.18) equals the above combination of quark-ghost kernels and the ghost term. Collecting all the results, we deduce that the l.h.s. of Eq. (4.14) equals the r.h.s., and hence the Slavnov–Taylor identity for the quark-gluon vertex is satisfied at one-loop perturbative level.

As a useful check, we can also show that the divergent parts of the quantities entering the formula Eq. (4.14) combine such that the Slavnov–Taylor identity is again satisfied. For this we use the results derived in Chapter 3, i.e. the divergent part of the quark gluon vertices Eq. (3.99), Eq. (3.100), and the divergent part of the quark gap equation, obtained by inverting the result Eq. (3.89) for the quark propagator

$$\Gamma_{\bar{q}q}(k) = \mathrm{i}\frac{C_F}{(4\pi)^2}\frac{1}{\varepsilon}(\gamma^0 k_0 - \gamma^i k_i - 4m) + \text{finite terms}, \tag{4.20}$$

together with the ghost two-point function (previously calculated in [30])

$$\Gamma_c^{ab}(k) = -\mathrm{i}\delta^{ab}\vec{k}^2\frac{4}{3}\frac{N_c}{(4\pi)^2}\frac{1}{\varepsilon} + \text{finite terms}. \tag{4.21}$$

Further, writing out the explicit form of the quark-ghost kernels Eq. (4.13) and making a simple power analysis, we notice that these factors do not contain divergences.[3] Collecting the divergent factors Eq. (3.99), Eq. (3.100), Eq. (4.20) and Eq. (4.21), it is straightforward to verify that that the Slavnov–Taylor identity is again satisfied.

[3] The convergence of the quark-ghost kernels has been also shown in Ref. [72], in the framework of split dimensional regularization.

Chapter 5.

Heavy quarks

So far, we have investigated the Dyson–Schwinger equations (along with the Slavnov–Taylor identities) for two- and three-point Green's functions at one-loop perturbative level, where the coupling between quarks and gluons is small. In the second part of this thesis, we are concerned with the nonperturbative (i.e., strong coupling) region of the theory, with the aim to describe bound states of quarks, and interpret them in relation to the linearly rising potential which confines them.

Traditionally, most of the nonperturbative studies are performed in covariant gauges. They concentrate on the light quark sector, where one is concerned with dynamical chiral symmetry breaking (a review of this subject can be found in Ref. [73], see also Ref. [74] for the phenomenological implications for meson and baryon states). So far, the Landau gauge studies of the heavy quark sector are unfortunately not conclusive, since the heavy quark propagator contains spurious poles which prevent a solution of the bound state Bethe–Salpeter equation [75]. The difficulty arises from the fact that nonperturbative Dyson–Schwinger equations and heavy quark limit are not automatically compatible, i.e., both the heavy quark mass and the ultraviolet-cutoff scale of the loop integrals (supposing that a ultraviolet-cutoff regularization is employed) tend to be the largest scale in the problem.

An alternative approach is to reformulate the heavy quark sector of QCD as an effective theory – the Heavy Quark Effective Theory [HQET][1]. In this approach, the main simplification stems from the fact that the *effective* quark Lagrange function is rewritten as an expansion in powers of $1/m$, where m is the heavy quark mass. Naturally, when the quarks are very heavy one can keep only the first few terms in the mass expansion. The effective Lagrange function also enjoys two additional symmetries, which turn to be very useful for practical applications [77]: the heavy flavor symmetry[2] and the so called spin symmetry, which arises because in the heavy quark limit the spin degrees of freedom do not couple to the gluon field. Moreover, the apparent incompatibility of the heavy quark mass and the ultraviolet cutoff has been recognized. To handle this problem, the so-called matching procedure has been developed [81]: QCD operators are written as series in $1/m$ via HQET operators and the coefficients in these series are determined by matching on-shell matrix elements in both theories. These calculations have been performed up to three loops in perturbation theory [84–86].

In the present chapter, the aim is to explore the heavy quark sector of the theory by combining nonperturbative Dyson–Schwinger equations with the heavy quark mass expansion [87]. We will employ the full nonperturbative QCD functional formalism, in-

[1] In [76–80] we collected the original papers; extended reviews can be found, among others, in Refs. [81, 82] and in the textbook [83].

[2] However, as we consider identical quarks, this is not of particular interest for the present work.

volving the complete quark fields, rather than HQET expressions that refer to the heavy quark degrees of freedom. The effect will be that, due to the complexity of the equations, we will be obliged to restrict ourselves to leading order in the mass expansion, and this means that we will not be able to describe real quarks (which are not infinitely heavy). However, this does not alter our goal to study the confining potential, since all quarks should be confined, regardless of their mass.

5.1. Heavy quark mass expansion

Let us start by writing out the explicit quark contribution to the full QCD generating functional, Eq. (2.1),

$$\begin{aligned} Z[\overline{\chi},\chi] &= \int \mathcal{D}\Phi \exp\left\{ \mathrm{i} \int d^4x \overline{q}_\alpha(x) \left[\mathrm{i}\gamma^0 D_0 + \mathrm{i}\vec{\gamma}\cdot\vec{D} - m \right]_{\alpha\beta} q_\beta(x) \right\} \\ &\quad \times \exp\left\{ \mathrm{i} \int d^4x \left[\overline{\chi}_\alpha(x) q_\alpha(x) + \overline{q}_\alpha(x)\chi_\alpha(x) \right] + \mathrm{i}\mathcal{S}_{YM} \right\}, \end{aligned} \tag{5.1}$$

with the temporal and spatial components of the covariant derivative (in the fundamental color representation) given by Eq. (1.23). The Yang–Mills contribution to the generating functional is in the standard, second order formalism [27, 28], where the auxiliary fields $\vec{\pi}$ and ϕ do not appear.

Now consider the following decomposition of the quark and antiquark fields:

$$\begin{aligned} q_\alpha(x) &= e^{-\mathrm{i}mx_0} \left[h(x) + H(x) \right]_\alpha , \\ \overline{q}_\alpha(x) &= e^{\mathrm{i}mx_0} \left[\overline{h}(x) + \overline{H}(x) \right]_\alpha , \end{aligned} \tag{5.2}$$

such that

$$\begin{aligned} h_\alpha(x) &= e^{\mathrm{i}mx_0} \left[P_+ q(x) \right]_\alpha , \quad H_\alpha(x) = e^{\mathrm{i}mx_0} \left[P_- q(x) \right]_\alpha \\ \overline{h}_\alpha(x) &= e^{-\mathrm{i}mx_0} \left[\overline{q}(x) P_+ \right]_\alpha , \quad \overline{H}_\alpha(x) = e^{-\mathrm{i}mx_0} \left[\overline{q}(x) P_- \right]_\alpha . \end{aligned} \tag{5.3}$$

In the above, the (spinor) projection operators are given by[3]

$$P_\pm = \frac{1}{2}(\mathbb{1} \pm \gamma^0), \quad P_+ + P_- = \mathbb{1}, \quad P_+ P_- = 0, \quad P_\pm^2 = P_\pm . \tag{5.4}$$

This decomposition is a particular case of the heavy quark transform underlying HQET [81]. There, the starting point is the observation that a heavy quark within a hadron is almost on-shell and moves with the hadron velocity v. Its 4-momentum can be written

$$p^\mu = m v^\mu + k^\mu \tag{5.5}$$

where $|k| \ll m|v|$ and $v^2 = 1$ (such that when $|k| = 0$, $p^2 = m^2$). One then uses the general projectors $P_\pm = (\mathbb{1} \pm \not{v})/2$ and the exponential terms are generalized to $e^{\pm \mathrm{i}mv\cdot x}$. The case used here corresponds to the rest frame of the quark, $v^\mu = (1, \vec{0})$. Intuitively, this corresponds to a shift in the position of the "zero energy" level, such that the energy of the

[3] Recall that in the free Dirac theory the operators $P_\pm$ project on the positive and negative energy eigenstates (they select the upper and lower components of the quark spinor).

state m (the heavy quark at rest) is the new zero level. In other words, instead of the true energy $p_0 = m+k_0$ of the heavy quark, the residual energy k_0 is used. However, within the context of the generating functional Eq. (5.1), the decomposition Eq. (5.2) can be initially regarded simply as and arbitrary decomposition, which will later on prove to be useful in Coulomb gauge. In fact, this choice will result in an important simplification: the spatial components of the Yang–Mills Green's functions are absent at leading order in the mass expansion and they apear only in next-to-leading order. The projection operators satisfy the following further relations

$$P_+\gamma^0 P_+ = P_+P_+, \quad P_+\gamma^0 P_- = 0, \quad P_+\gamma^i P_+ = 0 \tag{5.6}$$

such that the following relations hold for the components of the quark field:

$$\overline{h}\gamma^0 h = \overline{h}h, \;\; \overline{H}\gamma^0 H = -\overline{H}H, \;\; \overline{h}\gamma^0 H = \overline{H}\gamma^0 h = \overline{h}\gamma^i h = \overline{H}\gamma^i H = 0. \tag{5.7}$$

Inserting the decomposition of the quark fields given by Eq. (5.2) into the generating functional Eq. (5.1) and using these relationships one obtains

$$\begin{aligned} Z[\overline{\chi},\chi] = \int \mathcal{D}\Phi \exp\Big\{ \mathrm{i}\int d^4x \Big[& \overline{h}_\alpha(x)\left[\mathrm{i}D_0\right]_{\alpha\beta} h_\beta(x) \\ & + \overline{h}_\alpha(x)\left[\mathrm{i}\vec{\gamma}\cdot\vec{D}\right]_{\alpha\beta} H_\beta(x) + \overline{H}_\alpha(x)\left[\mathrm{i}\vec{\gamma}\cdot\vec{D}\right]_{\alpha\beta} h_\beta(x) + \overline{H}_\alpha(x)\left[-2m-\mathrm{i}D_0\right]_{\alpha\beta} H_\beta(x)\Big]\Big\} \\ \times \exp\Big\{ \mathrm{i}\int d^4x \left[e^{-\mathrm{i}mx_0}\overline{\chi}_\alpha(x)\left[h(x)+H(x)\right]_\alpha + e^{\mathrm{i}mx_0}\left[\overline{h}(x)+\overline{H}(x)\right]_\alpha \chi_\alpha(x)\right] + \mathrm{i}\mathcal{S}_{YM}\Big\}. \end{aligned} \tag{5.8}$$

At this point, it is illuminating to discuss the difference between the mass expansion (as used here) and HQET. As already emphasized in the introduction of this chapter, in this approach we work with the full quark fields, i.e. we retain the source terms for the quark fields $q, \bar{q}$ in order to derive the full gap, Bethe–Salpeter and Faddeev equations from $Z[\overline{\chi},\chi]$ (see below). Hence, the source term expression in the above is slightly modified by the appearance of the exponential factors. Since the Jacobian of the transformation is field independent (and thus trivial), our generating functional has not been altered, but merely rewritten in terms of different integration variables. This is in contrast to HQET where the quark sources are replaced with sources for the projected fields h and H. In principle, differentiation with respect to the sources of the generating functional Eq. (5.8) would then lead to the Green's functions of the projections h, H of the spinor q. In order to derive a Green's function with an external heavy quark, the components H (multiplied with twice the quark mass) are integrated out (as below) *and* in addition their sources are set to zero [82].

Noticing that for the h-field ('large') components, the quark mass parameter m does not appear directly, we integrate out the H-fields and get the following expression

$$\begin{aligned} Z[\overline{\chi},\chi] \;=\; & \int \mathcal{D}\Phi \mathrm{Det}\left[\mathrm{i}D_0+2m\right] \exp\Big\{ \mathrm{i}\int d^4x \Big[\overline{h}_\alpha(x)\left[\mathrm{i}D_0\right]_{\alpha\beta} h_\beta(x) \\ & + \Big[\overline{h}(x)\mathrm{i}\vec{\gamma}\cdot\vec{D} + e^{-\mathrm{i}mx_0}\overline{\chi}(x)\Big]_\alpha \left[\mathrm{i}D_0+2m\right]^{-1}_{\alpha\beta}\Big[\mathrm{i}\vec{\gamma}\cdot\vec{D}h(x) + e^{\mathrm{i}mx_0}\chi(x)\Big]_\beta\Big]\Big\} \\ & \times \exp\Big\{ \mathrm{i}\int d^4x \left[e^{-\mathrm{i}mx_0}\overline{\chi}_\alpha(x)h_\alpha(x) + e^{\mathrm{i}mx_0}\overline{h}_\alpha(x)\chi_\alpha(x)\right] + \mathrm{i}\mathcal{S}_{YM}\Big\}. \end{aligned} \tag{5.9}$$

Obviously, since we have integrated out a nontrivial component of the original quark field, our expression is nonlocal and this is where the heavy mass expansion is necessary. The determinant arising from the fermion functional integration depends on the temporal component of the gauge field and it can be written in the following way:

$$\begin{aligned}\mathrm{Det}\left[\mathrm{i}D_0+2m\right] &= \exp\ln\left[\mathrm{i}D_0+2m\right] = N\exp\mathrm{Tr}\ln\left[1+gT^a\sigma^a\frac{1}{\mathrm{i}\partial_0+2m}\right]\\ &\sim N+\mathcal{O}\left(1/m^2\right)\end{aligned} \tag{5.10}$$

where N is a (noninteracting) constant. If we expand the logarithm in power series we see that the first term, of $\mathcal{O}(1/m)$, vanishes due to the fact that the generators T^a are traceless matrices, and the term at $O(1/m^2)$ is proportional to $Tr(T^aT^b) = N_f$ (but, since we restrict to leading order in the mass expansion, this term will not appear in our calculations).

Leaving the $e^{\pm\mathrm{i}mx_0}$ factors as they are, we can thus write

$$\begin{aligned}Z[\overline{\chi},\chi] = &\int\mathcal{D}\Phi\exp\left\{\mathrm{i}\int d^4x\,[\,\overline{h}_\alpha(x)\left[\mathrm{i}D_0\right]_{\alpha\beta}h_\beta(x)\right.\\ &\left.+\frac{1}{2m}\left[\overline{h}(x)\mathrm{i}\vec{\gamma}\cdot\vec{D}+e^{-\mathrm{i}mx_0}\overline{\chi}(x)\right]_\alpha\left[\mathrm{i}\vec{\gamma}\cdot\vec{D}h(x)+e^{\mathrm{i}mx_0}\chi(x)\,]_\alpha\right]\right\}\\ &\times\exp\left\{\mathrm{i}\int d^4x\left[e^{-\mathrm{i}mx_0}\overline{\chi}_\alpha(x)h_\alpha(x)+e^{\mathrm{i}mx_0}\overline{h}_\alpha(x)\chi_\alpha(x)\right]+\mathrm{i}\mathcal{S}_{YM}\right\}+\mathcal{O}\left(1/m^2\right).\end{aligned} \tag{5.11}$$

Our generating functional is now local in the fields and arranged in an expansion in the parameter $1/m$ (this will be referred to as the mass expansion although strictly speaking, it is an expansion in the *inverse* mass). However, locality does not mean that the above expression can be directly applied. Let us consider the classical (full) quark field in the presence of sources:

$$\begin{aligned}&\frac{1}{Z}\int\mathcal{D}\Phi q_\alpha(x)\exp\left\{\mathrm{i}\mathcal{S}\right\} = \frac{1}{Z}\frac{\delta Z}{\delta\mathrm{i}\overline{\chi}_\alpha(x)}\\ &= \frac{1}{Z}\int\mathcal{D}\Phi\left\{e^{-\mathrm{i}mx_0}h_\alpha(x)+\frac{e^{-\mathrm{i}mx_0}}{2m}\left[\mathrm{i}\vec{\gamma}\cdot\vec{D}h(x)+e^{\mathrm{i}mx_0}\chi(x)\right]_\alpha\right\}\exp\left\{\mathrm{i}\mathcal{S}\right\}+\mathcal{O}\left(1/m^2\right).\end{aligned} \tag{5.12}$$

One sees immediately that even at $\mathcal{O}(1/m)$, the classical quark field has components that involve interaction type terms ($\vec{D}h\sim\vec{A}h$) brought about by the truncation of the nonlocality. Of course, this is nothing more than the statement that the h-field is nontrivially (and dynamically) related to the full q-field. It also means that if we want to use the nonperturbative gap and Bethe–Salpeter equations (i.e., those equations derived from the action for the full quark fields q and their sources χ and which are the equations of QCD as opposed to HQET) then we cannot expect that the mass expansion can realistically be extended far beyond the leading order in order to do practical calculations. As stated previously though, the aim is to investigate the connection between the Yang–Mills sector and the physical world made of quarks; the string tension that represents our goal is not dependent on the quark mass (both light and heavy quarks are confined in the same way, as far as we know). Therefore, we restrict our attention to the leading order in the mass

expansion as follows (and writing D_0 explicitly):

$$\begin{aligned} Z[\overline{\chi},\chi] &= \int \mathcal{D}\Phi \exp\left\{ \mathrm{i}\int d^4x \overline{h}_\alpha(x) \left[\mathrm{i}\partial_{0x} + gT^a\sigma^a(x)\right]_{\alpha\beta} h_\beta(x)\right\} \\ &\quad \times \exp\left\{ \mathrm{i}\int d^4x \left[e^{-\mathrm{i}mx_0}\overline{\chi}_\alpha(x)h_\alpha(x) + e^{\mathrm{i}mx_0}\overline{h}_\alpha(x)\chi_\alpha(x)\right] + \mathrm{i}\mathcal{S}_{YM}\right\} + \mathcal{O}\left(1/m\right). \end{aligned} \tag{5.13}$$

The standard machinery of functional methods is now employed. Just as for the usual QCD gap equation derived in Chapter 2, we start with the observation that up to boundary terms (which are assumed to vanish) the integral of a total derivative vanishes, and obtain for the quark field equation of motion:

$$\begin{aligned} 0 &= \int \mathcal{D}\Phi \frac{\delta}{\delta \mathrm{i}\overline{h}_\alpha(x)} \exp\left\{\mathrm{i}\mathcal{S}\right\} \\ &= \int \mathcal{D}\Phi \left\{ \left[\mathrm{i}\partial_{0x} + gT^a\sigma^a(x)\right]_{\alpha\beta} h_\beta(x) + e^{\mathrm{i}mx_0}\chi_\alpha(x)\right\} \exp\left\{\mathrm{i}\mathcal{S}\right\} + \mathcal{O}\left(1/m\right). \end{aligned} \tag{5.14}$$

The field equation of motion for the antiquark gives equivalent results and can be neglected. From the generating functional of connected Green's functions $W[\overline{\chi},\chi]$, Eq. (2.7), and using the bracket notation, Eq. (2.9), for the derivatives of W with respect to sources, we obtain the classical quark fields, Eq. (2.14).

Using the equation Eq. (2.11) and noticing that from Eq. (5.13), the following relations hold:

$$\begin{aligned} \frac{\delta Z[\overline{\chi},\chi]}{\delta \mathrm{i}\overline{\chi}_\alpha(x)} &= \int \mathcal{D}\Phi e^{-\mathrm{i}mx_0} h_\alpha(x) \exp\left\{\mathrm{i}\mathcal{S}\right\} + \mathcal{O}\left(1/m\right), \\ \frac{\delta Z[\overline{\chi},\chi]}{\delta \mathrm{i}\chi_\alpha(x)} &= -\int \mathcal{D}\Phi e^{\mathrm{i}mx_0} \overline{h}_\alpha(x) \exp\left\{\mathrm{i}\mathcal{S}\right\} + \mathcal{O}\left(1/m\right) \end{aligned} \tag{5.15}$$

(recalling the earlier discussion of neglecting the $\mathcal{O}\left(1/m\right)$ terms), the field equation of motion, Eq. (5.14), can be written in terms of derivatives of W:

$$\begin{aligned} 0 &= \left[\mathrm{i}\partial_{0x}\right]_{\alpha\beta} e^{\mathrm{i}mx_0} <\mathrm{i}\overline{\chi}_\beta(x)> + \left[gT^a\right]_{\alpha\beta} e^{\mathrm{i}mx_0} \left[<\mathrm{i}\rho^a(x)\mathrm{i}\overline{\chi}_\beta(x)> + <\mathrm{i}\rho^a(x)><\mathrm{i}\overline{\chi}_\beta(x)>\right] \\ &\quad + e^{\mathrm{i}mx_0}\chi_\alpha(x) + \mathcal{O}\left(1/m\right). \end{aligned} \tag{5.16}$$

Factoring out the exponential terms gives then

$$\begin{aligned} 0 &= \left[\mathrm{i}\partial_{0x} - m\right]_{\alpha\beta} <\mathrm{i}\overline{\chi}_\beta(x)> + \left[gT^a\right]_{\alpha\beta} \left[<\mathrm{i}\rho^a(x)\mathrm{i}\overline{\chi}_\beta(x)> + <\mathrm{i}\rho^a(x)><\mathrm{i}\overline{\chi}_\beta(x)>\right] \\ &\quad + \chi_\alpha(x) + \mathcal{O}\left(1/m\right). \end{aligned} \tag{5.17}$$

To continue, we employ the Legendre transform Eq. (2.15) in order to construct the effective action for the full quark fields (with the notations and conventions introduced in Chapter 2). The field equation of motion can then be rewritten in terms of derivatives of the effective action (which are the proper Green's functions when sources are set to zero):

$$<\mathrm{i}\overline{q}_\alpha(x)> = \left[\mathrm{i}\partial_{0x} - m\right]_{\alpha\beta} q_\beta(x) + \left[gT^a\right]_{\alpha\beta} \left[<\mathrm{i}\rho^a(x)\mathrm{i}\overline{\chi}_\beta(x)> + \sigma^a(x)q_\beta(x)\right] + \mathcal{O}\left(1/m\right). \tag{5.18}$$

With the field equation of motion written in the above forms, we can now derive the Feynman rules for the quark components of the theory (the Yang–Mills parts are already known [28]). Following the derivation presented in Chapter 3, we functionally differentiate Eq. (5.17), ignoring the interaction terms, and we get the tree-level propagator in configuration space

$$0 = [\mathrm{i}\partial_{0x} - m]_{\alpha\beta} <\mathrm{i}\chi_\gamma(z)\mathrm{i}\overline{\chi}_\beta(x)>^{(0)} -\mathrm{i}\delta_{\gamma\alpha}\delta(z-x) + \mathcal{O}\left(1/m\right). \tag{5.19}$$

Naively, one would write the solutions as

$$W^{(0)}_{\overline{q}q\beta\alpha}(k) = \frac{-\mathrm{i}\delta_{\beta\alpha}}{[k_0 - m]} + \mathcal{O}\left(1/m\right), \quad W^{(0)}_{q\overline{q}\gamma\beta}(k) = \frac{-\mathrm{i}\delta_{\gamma\beta}}{[k_0 + m]} + \mathcal{O}\left(1/m\right). \tag{5.20}$$

These propagators will turn to have a couple of striking features, which will be extensively discussed after deriving the nonperturbative solution for the gap equation. At this stage, we only emphasize that the quark and antiquark propagators must be treated separately, and this will turn to have important consequences for the bound state equations. Due to the mass expansion, we only have a single pole in the complex k_0-plane, as opposed to the conventional quark propagator, which possesses a pair of simple poles. Hence, in order to define the Fourier transform, one must first define the Feynman prescription for handling the poles in the energy integral. For the quark propagator, we write

$$W^{(0)}_{\overline{q}q\beta\alpha}(k) = \frac{-\mathrm{i}\delta_{\beta\alpha}}{[k_0 - m + \mathrm{i}\varepsilon]} + \mathcal{O}\left(1/m\right). \tag{5.21}$$

Let us state for the moment that for the antiquark propagator, the following Feynman prescription is assigned (this will be explained in the context of the Bethe–Salpeter equation for bound states, see below):

$$W^{(0)}_{q\overline{q}\gamma\beta}(k) = \frac{-\mathrm{i}\delta_{\gamma\beta}}{[k_0 + m + \mathrm{i}\varepsilon]} + \mathcal{O}\left(1/m\right). \tag{5.22}$$

For the proper two-point function, we use functional derivatives of Eq. (5.18) and we get directly in momentum space

$$\begin{aligned} \Gamma^{(0)}_{\overline{q}q\alpha\beta}(k) &= \mathrm{i}\left[k_0 - m\right]\delta_{\alpha\beta} + \mathcal{O}\left(1/m\right), \\ \Gamma^{(0)}_{q\overline{q}\alpha\beta}(k) &= \mathrm{i}\left[k_0 + m\right]\delta_{\alpha\beta} + \mathcal{O}\left(1/m\right), \end{aligned} \tag{5.23}$$

Since the two-point function requires no Feynman prescription and is diagonal in the outer product of the fundamental color, flavor and spinor spaces, we have the following relation between the quark and antiquark two-point functions:

$$\Gamma^{(0)}_{\overline{q}q\alpha\beta}(k) = -\Gamma^{(0)}_{q\overline{q}\alpha\beta}(-k) \tag{5.24}$$

In addition, $W_{\overline{q}q}\Gamma_{\overline{q}q} = \mathbb{1}$ as usual. For the three-point functions we obtain

$$\begin{aligned} \Gamma^{(0)a}_{\overline{q}q\sigma\alpha\beta}(k_1,k_2,k_3) &= \left[gT^a\right]_{\alpha\beta} + \mathcal{O}\left(1/m\right), \\ \Gamma^{(0)a}_{q\overline{q}\sigma\alpha\beta}(k_1,k_2,k_3) &= -\left[gT^a\right]_{\beta\alpha} + \mathcal{O}\left(1/m\right). \end{aligned} \tag{5.25}$$

Notice the ordering of the indices for the $\Gamma_{q\bar{q}\sigma}$ vertex. Importantly, the tree-level spatial quark-gluon vertex does not appear at leading order in the mass expansion:

$$\Gamma^{(0)a}_{\bar{q}qA\alpha\beta i} = \Gamma^{(0)a}_{q\bar{q}A\alpha\beta i} \sim \mathcal{O}\left(1/m\right). \tag{5.26}$$

Let us again stress that, because of our insistence of using the full quark sources, all the nonperturbative equations involving the quarks (the Dyson–Schwinger equations for two-point and three-point functions, the Slavnov–Taylor identities, the Bethe–Salpeter equation and the Faddeev equation) will not alter their form at leading order in the mass expansion, the only alterations being at the level of the tree-level factors.

5.2. Truncation scheme

In order to solve the nonperturbative system we must further specify our truncation scheme. In the context of the heavy mass expansion, we propose to consider only the dressed two-point functions of the Yang–Mills sector (i.e., the nonperturbative gluon propagators) and to set all the pure Yang–Mills vertices and higher n-point functions occurring in the quark equations to zero.

Since the tree-level spatial quark-gluon vertex is suppressed by the mass expansion, in our approximation any diagram containing this vertex will not contribute at one-loop order in perturbation theory. Further, the fully temporal gluon Green's functions $\Gamma_{\sigma\sigma\sigma}, \Gamma_{\sigma\sigma\sigma\sigma}, \ldots$ are zero at tree-level [28], and this means that the leading order perturbative corrections containing this vertices again vanish. This implies that that the number of loop diagrams arising because of the Yang–Mills vertices is heavily restricted and they first contribute to the next to leading order in perturbation theory. Physically, the most important point that will emerge is that when we set the Yang–Mills vertices to zero, we exclude the non-Abelian part of the charge screening mechanism of the quark color charge and any potential glueball states. On the other hand, the charge screening mechanism and glueball contributions of the gluon field (i.e., the color string) is implicitly encoded in the nonperturbative form of the temporal gluon propagator. We will come back to the physical implications of this truncation after deriving the confining potential between a quark and an antiquark from the Bethe–Salpeter equation.

With the truncation scheme as outlined above, the Yang–Mills sector reduces to the inclusion of a single object: the temporal gluon propagator which is written as [28]

$$W^{ab}_{\sigma\sigma}(k) = \delta^{ab}\frac{\mathrm{i}}{\vec{k}^2}D_{\sigma\sigma}(\vec{k}^2). \tag{5.27}$$

There are three important features to this propagator. Firstly, there are indications on the lattice that the dressing function $D_{\sigma\sigma}$ is largely independent of energy [49], justifying the energy independence of the above form. As we already emphasized in the first chapter, within the first order formalism one can justify that the temporal gluon propagator must have some part that is constant in the energy in order to cancel closed ghost loops and resolve the Coulomb gauge energy divergences[4]. Second, the lattice analysis indicates that the dressing function $D_{\sigma\sigma}$ is infrared divergent and is likely to behave as $1/\vec{k}^2$ for vanishing $\vec{k}^2$. Since we are interested mainly in the relationship between $D_{\sigma\sigma}$ (as the input

[4] See also the more formal considerations of Ref. [34].

of the Yang–Mills sector) and the string tension, we will not need the specific form until towards the end. Third, the product $g^2 D_{\sigma\sigma}$ is a renormalization group invariant quantity and thus a good candidate for being related to the physical string tension [12, 27].

5.3. Gap equation

5.3.1. Nonperturbative treatment

Let us begin by considering the Dyson–Schwinger equation for the quark two-point proper function. In full, second order formalism, QCD it is given by[5]:

$$\begin{aligned}\Gamma_{\overline{q}q\alpha\delta}(k) &= \Gamma^{(0)}_{\overline{q}q\alpha\delta}(k) \\ &\quad + \int d\omega \, \Big\{ \Gamma^{(0)a}_{\overline{q}q\sigma\alpha\beta}(k,-\omega,\omega-k) W_{\overline{q}q\beta\gamma}(\omega) \Gamma^{b}_{\overline{q}q\sigma\gamma\delta}(\omega,-k,k-\omega) W^{ab}_{\sigma\sigma}(k-\omega) \\ &\quad + \Gamma^{(0)a}_{\overline{q}qA\alpha\beta i}(k,-\omega,\omega-k) W_{\overline{q}q\beta\gamma}(\omega) \Gamma^{b}_{\overline{q}qA\gamma\delta j}(\omega,-k,k-\omega) W^{ab}_{AAij}(k-\omega) \Big\}\end{aligned} \tag{5.28}$$

(the spatial gluon propagator W_{AA} will be unimportant here because of the suppression Eq. (5.26) of the spatial quark-gluon vertex). The gap equation is supplemented by the Slavnov–Taylor identity, Eq. (4.12), derived in Chapter 4 .

In order to use the Slavnov–Taylor identity as input for solving the gap equation, we first apply our truncation scheme in the context of the heavy mass expansion at leading order. Starting with the dressed spatial quark-gluon vertex, consider the terms that contribute to the Dyson–Schwinger equation shown schematically in Fig. 2.3. According to the truncation scheme, we set all Yang–Mills vertices to zero, meaning that diagrams (c) and (e-i) are excluded. This then leaves us with the tree-level term (a) and the quark contributions (b), (d). However, all of these involve at least one tree-level spatial quark-gluon vertex, which is not present at leading order in the mass expansion. Thus, we obtain the nonperturbative result that

$$\Gamma^{a}_{\overline{q}qA\alpha\beta i}(k_1,k_2,k_3) \sim \mathcal{O}\left(1/m\right). \tag{5.29}$$

Similarly, the ghost-quark kernels of the Slavnov–Taylor identity, given their definition, Eq. (4.13), involve Yang–Mills vertices and can be neglected. Thus, in our truncation scheme and at leading order in the mass expansion, the Slavnov–Taylor identity reduces to

$$k_3^0 \Gamma^{d}_{\overline{q}q\sigma\alpha\beta}(k_1,k_2,k_3) = \Gamma_{\overline{q}q\alpha\delta}(k_1) \left[\mathrm{i}gT^d\right]_{\delta\beta} - \left[\mathrm{i}gT^d\right]_{\alpha\delta} \Gamma_{\overline{q}q\delta\beta}(-k_2) + \mathcal{O}\left(1/m\right). \tag{5.30}$$

Clearly, the Slavnov–Taylor identity under truncations results in an Abelian type Ward identity. Moreover, since the temporal quark-gluon vertex is simply multiplied by the temporal gluon energy (an essential feature of Coulomb gauge Slavnov–Taylor identities, as opposed to covariant gauges) and the quark proper two-point function is color diagonal, we can immediately write the solution:

$$\Gamma^{d}_{\overline{q}q\sigma\alpha\beta}(k_1,k_2,k_3) = \frac{\mathrm{i}g}{k_3^0} \Big\{ T^d \left[\Gamma_{\overline{q}q}(k_1) - \Gamma_{\overline{q}q}(-k_2)\right] \Big\}_{\alpha\beta} + \mathcal{O}\left(1/m\right). \tag{5.31}$$

[5] This can be directly inferred from the expression Eq. (2.34), in the first order formalism, by setting the auxiliary fields $\vec{\pi}$, ϕ to zero.

The above solution is trivially satisfied at tree-level. The apparent singularity arising for vanishing gluon energy ($k_3^0 = 0$, but $\vec{k}_3 \neq 0$) must somehow be canceled by the difference of proper quark two-point functions. Since the spatial momentum configuration is arbitrary, this leads to the requirement that $\Gamma_{\overline{q}q}(k) \to \Gamma_{\overline{q}q}(k_0) + \mathcal{O}(1/m)$. Later on we shall see that our solution does indeed satisfy this condition. The demand that the nonperturbative vertex solution to the Coulomb gauge Slavnov–Taylor identity be free of kinematic divergences (here, simply the $1/k_3^0$ factor) is a variation of the familiar covariant gauge situation considered in Ref. [88].

Inserting the results, Eq. (5.31) and Eq. (5.29), for the vertices, using the Feynman rules given by Eq. (5.23), Eq. (5.25), and the temporal gluon propagator given by Eq. (5.27) and resolving the color structure, the nonperturbative gap equation, Eq. (5.28), under truncation and at leading order in the mass expansion thus reads

$$\begin{aligned}\Gamma_{\overline{q}q\alpha\delta}(k_0) &= \mathrm{i}\,[k_0 - m]\,\delta_{\alpha\delta} - g^2 C_F \int \frac{d\,\omega\, D_{\sigma\sigma}(\vec{k}-\vec{\omega})}{(k_0-\omega_0)(\vec{k}-\vec{\omega})^2} W_{\overline{q}q\alpha\beta}(\omega_0)\,[\Gamma_{\overline{q}q}(\omega_0) - \Gamma_{\overline{q}q}(k_0)]_{\beta\delta} \\ &\quad + \mathcal{O}(1/m)\,. \end{aligned} \tag{5.32}$$

There exists one particularly simple solution to this equation, given by

$$\Gamma_{\overline{q}q\alpha\beta}(k) = \mathrm{i}\delta_{\alpha\beta}\,[k_0 - m - \mathcal{I}_r] + \mathcal{O}(1/m) \tag{5.33}$$

$$W_{\overline{q}q\alpha\beta}(k_0) = \frac{-\mathrm{i}\delta_{\alpha\beta}}{[k_0 - m - \mathcal{I}_r + \mathrm{i}\varepsilon]} + \mathcal{O}(1/m)\,, \tag{5.34}$$

with the constant $[\,d\,\vec{\omega} = d^3\vec{\omega}/(2\pi)^3]$

$$\mathcal{I}_r = \frac{1}{2} g^2 C_F \int_r \frac{d\,\vec{\omega}\, D_{\sigma\sigma}(\vec{\omega})}{\vec{\omega}^2} + \mathcal{O}(1/m)\,. \tag{5.35}$$

A short comment regarding the ordering of the limits in the spatial and temporal integrals and the potential divergences in the constant $\mathcal{I}_r$ is in order. Here, we state that the temporal integral is performed first under the condition that the spatial integral is somehow regularized, i.e. finite (the implicit regularization is signaled by the subscript "r")[6]. Extracting the expression for the constant $\mathcal{I}_r$ from the gap equation under truncation, Eq. (5.32), we insert the solution given by Eq. (5.33), Eq. (5.34) (assuming that the spatial integral is regularized), and obtain

$$\begin{aligned}\mathcal{I}_r &= g^2 C_F \int_r \frac{d\,\vec{\omega}\, D_{\sigma\sigma}(\vec{k}-\vec{\omega})}{(\vec{k}-\vec{\omega})^2} \frac{\mathrm{i}}{2\pi} \lim_{R\to\infty} \int_{-R}^{R} \frac{d\omega_0}{[\omega_0 - m - \mathcal{I}_r + \mathrm{i}\varepsilon]} + \mathcal{O}(1/m) \\ &= \frac{1}{2} g^2 C_F \int_r \frac{d\,\vec{\omega}\, D_{\sigma\sigma}(\vec{k}-\vec{\omega})}{(\vec{k}-\vec{\omega})^2} + \mathcal{O}(1/m)\,. \end{aligned} \tag{5.36}$$

Clearly, the effect of performing the temporal integral first is that the regularized constant $\mathcal{I}_r$ in the denominator factor can be shifted away and hence becomes irrelevant. Further, by shifting the integration variable in the spatial integral we find that this integral involves no external scale, and thus we arrive at our above result, Eq. (5.35).

[6] This regularization will be assumed throughout the rest of this work.

Inserting the solution Eq. (5.33) into the Slavnov–Taylor identity, we also have that for the vertex

$$\Gamma^{d}_{\overline{q}q\sigma\alpha\beta}(k_1,k_2,k_3) = \left[gT^d\right]_{\alpha\beta} + \mathcal{O}\left(1/m\right), \tag{5.37}$$

and this means that the dressed temporal quark-gluon vertex is trivial and the gap equation reduces to the rainbow truncation.

Let us now briefly discuss the physical interpretation of these results. Firstly, it might be the case that there exist other solutions to the truncated gap equation. However, as will be shown in the next section, the above solution can also be derived from a semi-perturbative type of expansion. One has to bear in mind that in principle the fully nonperturbative solution might not be the same as the resummed perturbative solution.

Secondly, as already outlined when defining the Feynman prescription, the quark propagator has a single pole, so cannot represent physical propagation (which requires a covariant double pole) and this arises obviously from the truncation of the mass expansion. From Eq. (5.34) it then follows that the closed quark loops (virtual quark-antiquark pairs) vanish due to the energy integration, which implies that the theory is quenched in the heavy mass limit (see also Ref. [81] for an alternate discussion on this topic):

$$\int \frac{dk_0}{\left[k_0 - m - \mathcal{I}_r + \mathrm{i}\varepsilon\right]\left[k_0 + p_0 - m - \mathcal{I}_r + \mathrm{i}\varepsilon\right]} = 0. \tag{5.38}$$

The above result can be interpreted as being a consequence of the breaking of time reversal symmetry of the full Dirac equation (quark and antiquark moving either forwards or backwards in time according to causality), as a result of the mass expansion. Time reversal symmetry breaking means that there is only the quark (or only the antiquark) moving forward in time, corresponding to the above Feynman prescription, and, in this sense, the corresponding antiquark (or quark) is not present. On the other hand, the closed quark loop involves a quark going backwards in time (similarly for an antiquark loop), which is prohibited according to our argumentation, so that such closed quark loop integrals vanish at leading order. Clearly, the breaking of the time reversal symmetry is also reflected in the presence of only a single pole in the heavy quark propagator.

Thirdly, the quark propagator Eq. (5.34) is diagonal in the outer product of the fundamental color, flavor and spinor spaces as a consequence of the mass expansion – physically this corresponds to the decoupling of the spin from the heavy quark system. In fact, $W^{(0)}_{\overline{q}q}$ is identical to the heavy quark tree-level propagator [81] up to the appearance of the mass term, and this is due to the fact that in HQET one uses the sources for the large h-fields directly, while we retain the sources of the full quark fields. Also, note that the kinetic term of the (tree-level) propagator would read $-\vec{k}^2/2m$ in the denominator factor and hence appears at higher order in the mass expansion. Such terms are obviously important to the UV properties of the loop integrals but do not play any role in the infrared limit considered here.

Finally, the fact that the solution involves potentially divergent constants is not a comfortable situation but does not necessarily contradict the physics, since the position of the pole has no physical meaning (the quark can never be on-shell). The poles in the quark propagator are situated at infinity (thanks to $\mathcal{I}_r$ as the regularization is removed) meaning that either one requires infinite energy to create a quark from the vacuum or, if a a hadronic system is considered, only the relative energy is important. Indeed, it was shown some time ago [89] that the divergence of the absolute energy has no physical

meaning and only the relative energy (derived from the Bethe–Salpeter equation, see below) must be considered. Further, the divergences appearing in the quark propagator have no interpretation with regards to renormalization, at least within the context of the mass expansion to leading order. The mass parameter cannot be renormalized simply because one cannot construct an appropriate counterterm in the action. Also, the quark field renormalization is trivial at leading order, as one sees from the explicit form of the temporal quark-gluon vertex, Eq. (5.37).

Having discussed the quark propagator, let us now discuss the antiquark propagator. Recall that at tree-level, we used a different Feynman prescription for the two denominator factors and this gives rise to some rather interesting physical consequences. As previously discussed, the heavy mass expansion employed here breaks the charge conjugation symmetry relating particle and antiparticle, so we cannot expect that the two propagators are necessarily equivalent. Starting with the gap equation for full QCD, Eq. (5.28), we reverse the ordering of the quark and antiquark functional derivatives that form the quark Green's functions (still within the context of the full quark fields and sources) and rearrange the ordering to get the gap equation for the antiquark propagator:

$$\begin{aligned}-\Gamma_{q\overline{q}\delta\alpha}(-k) &= -\Gamma^{(0)}_{q\overline{q}\delta\alpha}(-k) \\ &\quad - \int đ\omega \, \Big\{ \Gamma^{b}_{q\overline{q}\sigma\delta\gamma}(-k,\omega,k-\omega) W_{q\overline{q}\gamma\beta}(-\omega) \Gamma^{(0)a}_{q\overline{q}\sigma\beta\alpha}(-\omega,k,\omega-k) W^{ab}_{\sigma\sigma}(k-\omega) \\ &\quad + \Gamma^{b}_{q\overline{q}Aj\gamma j}(-k,\omega,k-\omega) W_{q\overline{q}\gamma\beta}(-\omega) \Gamma^{(0)a}_{q\overline{q}A\beta\alpha i}(-\omega,k,\omega-k) W^{ab}_{Aij}(k-\omega) \, \Big\} . \end{aligned} \tag{5.39}$$

Applying our truncation scheme reduces the above to

$$\begin{aligned}\Gamma_{q\overline{q}\delta\alpha}(-k) &= \Gamma^{(0)}_{q\overline{q}\delta\alpha}(-k) \\ &\quad + \int đ\omega \, \Gamma^{b}_{q\overline{q}\sigma\delta\gamma}(-k,\omega,k-\omega) W_{q\overline{q}\gamma\beta}(-\omega) \Gamma^{(0)a}_{q\overline{q}\sigma\beta\alpha}(-\omega,k,\omega-k) W^{ab}_{\sigma\sigma}(k-\omega) + \mathcal{O}\left(1/m\right) . \end{aligned} \tag{5.40}$$

In similar fashion, we have the Slavnov–Taylor identity for the antiquark-gluon vertex:

$$-k_3^0 \Gamma^{d}_{q\overline{q}\sigma\beta\alpha}(k_2,k_1,k_3) = +\Gamma_{q\overline{q}\beta\delta}(k_2) \left[igT^d \right]^T_{\delta\alpha} - \left[igT^d \right]^T_{\beta\delta} \Gamma_{q\overline{q}\delta\alpha}(-k_1) + \mathcal{O}\left(1/m\right) . \tag{5.41}$$

Before we give the solution to Eq. (5.40), let us remark that in order to construct a quark-antiquark pair in the Bethe–Salpeter equation (which has a physical interpretation of a bound state equation), one is not considering a virtual quark-antiquark pair but rather a system composed of two separate unphysical particles.[7] Since the quark and antiquark lines of the Bethe–Salpeter equation are never connected by a primitive vertex (unlike the closed quark loop discussed above), we are allowed to assign the following Feynman prescription for the antiquark propagator:

$$W_{q\overline{q}\alpha\beta}(k) = \frac{-\mathrm{i}\delta_{\alpha\beta}}{[k_0 + m - \mathcal{I}_r + \mathrm{i}\varepsilon]} + \mathcal{O}\left(1/m\right) , \tag{5.42}$$

$$\Gamma_{q\overline{q}\alpha\beta}(k) = \mathrm{i}\delta_{\alpha\beta} \left[k_0 + m - \mathcal{I}_r \right] + \mathcal{O}\left(1/m\right) \tag{5.43}$$

[7] It is known that in Coulomb gauge, Gauss' law forbids the creation of a colored state in isolation (the total color charge is conserved and vanishing [90]) and so, the Feynman prescription for the quark (or the antiquark) propagator has no physical meaning in isolation.

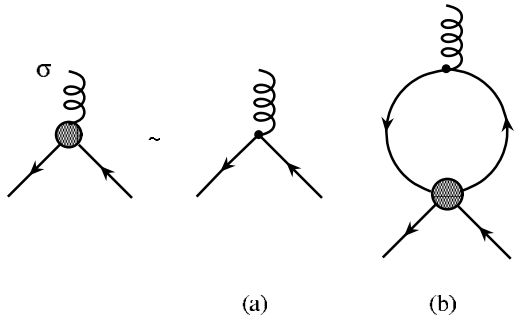

Figure 5.1.: Diagrams that contribute under our truncation scheme to the Dyson–Schwinger equation for the *temporal* quark-gluon vertex (without prefactors or signs). Internal propagators are fully dressed and blobs represent dressed proper vertex and (reducible) kernels. Internal propagators represented by solid lines represent the quark propagator.

with the corresponding solution for the vertex

$$\Gamma^{d}_{q\overline{q}\sigma\alpha\beta}(k_1,k_2,k_3) = -\left[gT^d\right]_{\beta\alpha} + \mathcal{O}\left(1/m\right). \tag{5.44}$$

In the above, notice that the sign of the loop correction has changed and this will result (in the context of the bound state studies for mesons and diquarks) in a physical interpretation for the quark-antiquark Bethe–Salpeter equation as a whole.

5.3.2. Semiperturbative treatment

After solving the gap (and anti-gap) equation, we have seen that the solutions for the proper two-point function leads to a (nonperturbatively) bare temporal quark-gluon (and antiquark-gluon) vertex. In this section, we will reconsider this vertex within the context of a semiperturbative analysis. This will then introduce a technical feature crucial for considering the Bethe–Salpeter equation nonperturbatively.

Under our truncation scheme, the nonperturbative Dyson–Schwinger equation for the temporal quark-gluon vertex involves the diagrams shown in Fig. 5.1. Semiperturbative expansion means in this case that in the loop expansion all internal propagators are taken to be dressed, but all internal vertices are tree-level. Clearly, the goal is to show that all the loop corrections (contained in diagram (b) of Fig. 5.1) vanish. For this, it suffices to consider two types of diagram, given in Figs. 5.2 and 5.3.

The easiest to start with is Fig. 5.2, where a single ladder exchange correction to the temporal quark-gluon vertex is considered. This diagram (neglecting the overall color and prefactors) gives rise to the following scalar integral:

$$\int_r \frac{d\vec{\omega}\, D_{\sigma\sigma}(\vec{k}-\vec{\omega})}{(\vec{k}-\vec{\omega})^2} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega_0}{\left[\omega_0 - m - \mathcal{I}_r + \mathrm{i}\varepsilon\right]\left[\omega_0 + q_0 - m - \mathcal{I}_r + \mathrm{i}\varepsilon\right]}. \tag{5.45}$$

Now we apply the following identity (for finite, real a, b; the case $a = b$ is trivial):

$$\int_{-\infty}^{\infty} \frac{dz}{\left[z-a+\mathrm{i}\varepsilon\right]\left[z-b+\mathrm{i}\varepsilon\right]} = \frac{1}{(a-b)} \int_{-\infty}^{\infty} dz \left\{ \frac{1}{\left[z-a+\mathrm{i}\varepsilon\right]} - \frac{1}{\left[z-b+\mathrm{i}\varepsilon\right]} \right\} = 0. \tag{5.46}$$

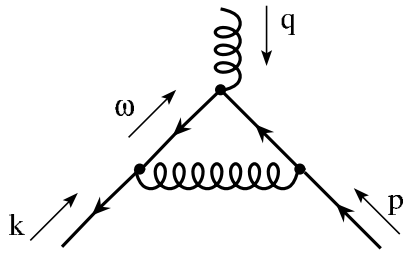

Figure 5.2.: Ladder type loop correction to the temporal quark-gluon vertex. Internal propagators are fully dressed: solid lines represent the quark propagator and springs denote the temporal gluon propagator.

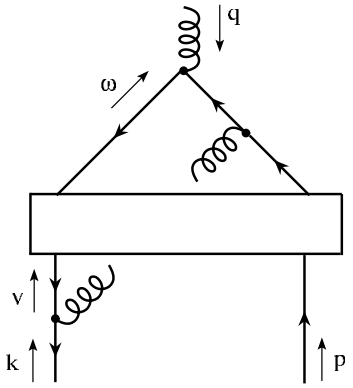

Figure 5.3.: Generic crossed box type loop correction to the temporal quark-gluon vertex. Internal propagators are fully dressed: solid lines represent the quark propagator and springs denote the temporal gluon propagator. The box represents any combination of interactions allowed under our truncation.

Thus we see that where the spatial integral is regularized, the temporal integral vanishes. It is simple to see that the planar one-loop diagrams with two or more external temporal gluon legs (which under the truncation scheme considered here connect only to the internal quark line) and one internal temporal gluon will also vanish.

Now let us consider a generic crossed box (nonplanar) type of diagram, illustrated in Fig. 5.3. Considering only the temporal double integral components of the explicit internal quark propagators, we have the following form

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{d\omega_0\, dv_0}{[v_0 - a_1 + \mathrm{i}\varepsilon]\,[\omega_0 - a_2 + \mathrm{i}\varepsilon]\,[\omega_0 + q_0 - a_3 + \mathrm{i}\varepsilon]\,[\omega_0 - v_0 - p_0 - a_4 + \mathrm{i}\varepsilon]} \\
&= \int_{-\infty}^{\infty} \frac{d\omega_0}{[\omega_0 - a_2 + \mathrm{i}\varepsilon]\,[\omega_0 + q_0 - a_3 + \mathrm{i}\varepsilon]\,[\omega_0 - p_0 - a_1 - a_4 + 2\mathrm{i}\varepsilon]} \\
&\times \int_{-\infty}^{\infty} dv_0 \left\{ \frac{1}{[v_0 - a_1 + \mathrm{i}\varepsilon]} - \frac{1}{[v_0 - \omega_0 + p_0 + a_4 - \mathrm{i}\varepsilon]} \right\} \\
&= -2\pi\mathrm{i} \int_{-\infty}^{\infty} \frac{d\omega_0}{[\omega_0 - a_2 + \mathrm{i}\varepsilon]\,[\omega_0 + q_0 - a_3 + \mathrm{i}\varepsilon]\,[\omega_0 - p_0 - a_1 - a_4 + 2\mathrm{i}\varepsilon]} \\
&= 0
\end{aligned} \tag{5.47}$$

where in the last line, we have used a variation of the identity Eq. (5.46). Thus we have

the result that the generic crossed box type of diagram shown in Fig. 5.3 also vanishes.

With the result that both the single ladder type exchange diagram and the generic crossed box diagrams considered so far vanish, it is easy to see that any vertex dressing diagram will vanish (including all subdiagrams such as internal vertex corrections and so on), since all associated diagrams are merely variations or combinations of these two under our truncation scheme. We stress that this result is a consequence of the fact that the energy and Feynman prescription of the denominator factors follow the quark line through the diagram so that eventually the identity, Eq. (5.46), can be used. It is also precisely the reason why all closed quark loops vanish, according to the relation Eq. (5.38).

Thus, the semiperturbative expansion confirms our previous result that the temporal quark-gluon vertex remains bare to all orders. Clearly, the result also applies to the antiquark-gluon vertex. With the corresponding simple forms for the self-energy integrals of the gap and anti-gap equations (which as we recall, reduce to the rainbow truncation), the results for the quark propagator functions are also confirmed. Notice though that whilst the all orders semiperturbative result must match the nonperturbative result, the converse is not necessarily true. It remains the case that there may exist further solutions, but these must be purely nonperturbative in character if they exist.

To summarize, in this section we have introduced a set of valuable identities for the energy integrals, which confirm the nonperturbative case previously studied. This is very useful for further investigations, since we are allowed to apply them with confidence to study the bound state equations in the next section.

5.4. Bound state equations

Bound states of quarks appear as free-particle poles in their respective n-point Green's functions. For example, the quark-antiquark, and three-quark Green's functions exhibit meson and baryon poles, respectively. In this section, we will consider the corresponding bound state equations in the limit of the heavy quark mass – named Bethe–Salpeter equation for mesons and Faddeev equation for baryons – and interpret them in connection with the linearly rising potential which confines the quarks. As has been emphasized, due to the fact that we include the full quark sources and fields in our generating functional, we are allowed to use the full functional (nonperturbative) equations as a starting point and then subsequently apply our mass expansion and truncation scheme. In this context, we will also seek for a relationship between Yang–Mills sector of the theory and nonperturbative external physical scale (i.e., the string tension).

5.4.1. Bethe–Salpeter equation for mesons and diquarks

Let us now consider the homogeneous Bethe–Salpeter equation for quark-antiquark bound states. In full QCD this equation reads

$$\Gamma_{\alpha\beta}(p;P) = -\int d\,k\, K_{\alpha\beta;\delta\gamma}(p,k;P)\,[W_{\overline{q}q}(k_+)\Gamma(k;P)W_{\overline{q}q}(k_-)]_{\gamma\delta} \tag{5.48}$$

In the above, the minus sign arises from the definitions of the Legendre transform and Green's functions in Coulomb gauge. This will be shown in Chapter 6, where the 4-point Green's function (which includes the homogeneous Bethe–Salpeter equation) will be explicitly derived. The momenta of the quarks are given by $k_+ = k+\xi P$, $k_- = k+(\xi-1)P$

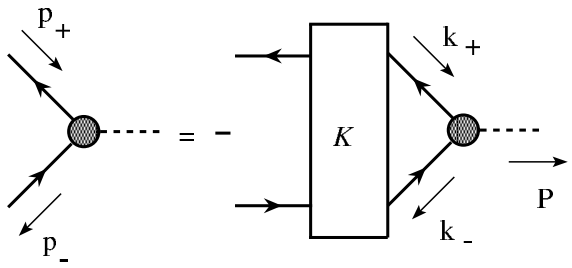

Figure 5.4.: Homogeneous Bethe–Salpeter equation for quark-antiquark bound states. Internal propagators are fully dressed and solid lines represent the quark propagator. The box represents the Bethe–Salpeter kernel K and filled blobs represent the Bethe–Salpeter vertex function Γ with the (external) bound state leg given by a dashed line. See text for details.

(similarly for $p_\pm$), P is the pole 4-momentum of the bound state (assuming that a solution exists), $\xi = [0, 1]$ is the so-called momentum sharing fraction that dictates how much of the total meson momentum is carried by each quark constituent. K represents the Bethe–Salpeter kernel and Γ is the Bethe–Salpeter vertex function for the particular bound state that one is considering and whose indices explicitly denote only its quark content. Later on, we will explicitly investigate what color, flavor and spin structure the solutions may have. We also mention that physically, the results should be independent of ξ and this has been numerically observed in phenomenological studies [91]. The Bethe–Salpeter equation is shown pictorially in Fig. 5.4.

To solve the Bethe–Salpeter equation, we firstly address the problem of constructing the kernel K. For technical reasons the most widely studied system is based on the ladder kernel which is either constructed via the interchange of a single gluon (for example [92]) or as a phenomenological potential (see for example Ref. [89]). However, there has been much recent attention focused on the construction of more sophisticated kernels. One key element of the construction is the axialvector Ward–Takahashi identity [AXWTI], which relates the gap equation to the Bethe–Salpeter kernel and which ensures that chiral symmetry and its spontaneous breaking are consistently implemented (e.g., Refs. [89, 93, 94]). Here, we shall show that the ladder Bethe–Salpeter kernel is *exact* at leading order in the heavy mass expansion and under our truncation scheme.

To construct the kernel K, we follow the semiperturbative analysis of the previous section. As before, at leading order in the mass expansion and under our truncation scheme we only have the temporal quark-gluon and antiquark-gluon vertices, both of which have been shown to be given by their tree-level forms. Consider now the generic semiperturbative crossed box contribution, which contains all possible nontrivial contributions allowed within our truncation scheme. This diagram is depicted in Fig. 5.5. Such a diagram has at least the following terms in the temporal integral (as before, we assume that the spatial integral is regularized and finite so that we are able to firstly perform the temporal integral without complication)

$$\int \frac{d\omega_0}{\left[\omega_0 + p_+^0 - m - \mathcal{I}_r + \mathrm{i}\varepsilon\right] \ldots \left[\omega_0 - k_-^0 + m - \mathcal{I}_r + \mathrm{i}\varepsilon\right]}. \tag{5.49}$$

The first factor corresponds to the explicit quark (upper) propagator, the last factor to the explicit antiquark (lower) propagator. The dots represent the multiple internal

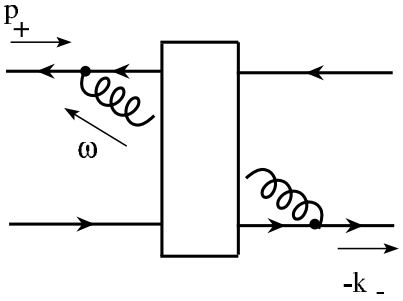

Figure 5.5.: Generic crossed box type of diagram that contributes to the Bethe–Salpeter kernel. Internal propagators are fully dressed, whereas vertices are tree-level. The upper (lower) solid line denotes the quark (antiquark) propagator; springs denote the temporal gluon propagator and the box represents any combination of nontrivial interactions allowed under our truncation scheme. See text for details.

propagator factors which carry the same dependence on the integration energy ω_0 *and* have the same relative sign for the Feynman prescription term, i.e., $\omega_0 + \mathrm{i}\varepsilon$, regardless of whether they originate from internal quark or antiquark propagators. Therefore, this type of integral can always be reduced to the difference of integrals over a simple pole and with the same sign for carrying out the analytic integration, just as in Eq. (5.46). Thus, all generic crossed box diagrams in the Bethe–Salpeter kernel are zero and one is left with simply the ladder contribution to the kernel. In fact, this result can also be inferred from the AXWTI: this identity connects the self-energy term of the gap equation and the Bethe–Salpeter kernel and since it has been shown explicitly that the self-energy integral reduces to the rainbow truncation, the corresponding Bethe–Salpeter kernel is simply given by ladder exchange.

Having analyzed the kernel K, let us now inspect the quark and antiquark pieces of the Bethe–Salpeter equation. Recalling that the antiquark propagator must be treated as distinct from the quark propagator, in Fig. 5.4 we must explicitly specify which line is assigned to the quark propagator and which one, to the antiquark propagator. For the quark-antiquark system under consideration (later on we will analyze the quark-quark, or diquark system), the Bethe–Salpeter equation more properly reads

$$\Gamma_{\alpha\beta}(p;P) = -\int \mathrm{d}\,k\, K_{\alpha\beta;\delta\gamma}(p,k;P)\left[W_{\overline{q}q}(k_+)\Gamma(k;P)(-)W^T_{q\overline{q}}(-k_-)\right]_{\gamma\delta} \tag{5.50}$$

where we have explicitly identified the antiquark propagator contribution (it corresponds to the lower line of Fig. 5.4) by reordering the functional derivatives, i.e. $W_{q\overline{q}}(k_-) = -W^T_{q\overline{q}}(-k_-)$. The above equation is still valid in full QCD. Further, we explicitly replace the tree-level form for the kernel, and we arrive at the following expression Bethe–Salpeter equation for the quark-antiquark system, at leading order in the mass expansion and within our truncation scheme:

$$\begin{aligned}\Gamma_{\alpha\beta}(p;P) \;=\; & -\int \mathrm{d}\,k\,\left[\Gamma^a_{\overline{q}q\sigma}(p_+,-k_+,k-p)W_{\overline{q}q}(k_+)\right]_{\alpha\lambda} \\ & \times\left[(-)W^T_{q\overline{q}}(-k_-)(-)\Gamma^b_{q\overline{q}\sigma}(-p_-,k_-,p-k)\right]_{\beta\kappa} W^{ab}_{\sigma\sigma}(p-k)\Gamma_{\lambda\kappa}(k;P) + \mathcal{O}\,(1/m).\end{aligned} \tag{5.51}$$

Notice the antiquark contribution within the kernel which absorbs the explicit minus sign from Eq. (5.50). Also, recall that we are implicitly considering only the flavor non-singlet case. The above equation can be trivially rewritten as:

$$\begin{aligned}\Gamma_{\alpha\beta}(p;P) &= -\int d\,k\,\Gamma^{a}_{\overline{q}q\sigma\alpha\gamma}(p_+,-k_+,k-p)W^{ab}_{\sigma\sigma}(p-k)\Gamma^{bT}_{q\overline{q}\sigma\beta\delta}(-p_-,k_-,p-k)\\ &\quad\times W_{\overline{q}q\gamma\lambda}(k_+)W^{T}_{q\overline{q}\delta\kappa}(-k_-)\Gamma_{\lambda\kappa}(k;P)+\mathcal{O}\left(1/m\right). \end{aligned} \tag{5.52}$$

Inserting the nonperturbative results for the propagators and vertices so far, Eqs. (5.34), (5.43), (5.37), (5.44) and taking the form, Eq. (5.27), for the temporal gluon propagator, we obtain the equation

$$\begin{aligned}\Gamma_{\alpha\beta}(p;P) &= g^2\int_r \frac{d\,\vec{k}\,D_{\sigma\sigma}(\vec{p}-\vec{k})}{(\vec{p}-\vec{k})^2}\\ &\quad\times\frac{\mathrm{i}}{2\pi}\int_{-\infty}^{\infty}\frac{dk_0}{\left[k^0_+-m-\mathcal{C}_r+\mathrm{i}\varepsilon\right]\left[k^0_--m+\mathcal{I}_r-\mathrm{i}\varepsilon\right]}\left[T^a\Gamma(k;P)T^a\right]_{\alpha\beta}+\mathcal{O}\left(1/m\right). \end{aligned} \tag{5.53}$$

We see immediately that the flavor and spin structure of the meson decouples from the problem — this is well a known property of the heavy mass expansion. The color structure will be discussed shortly. Since the external energy p_0 does not enter the right-hand side of the above equation, we further have that the Bethe–Salpeter vertex $\Gamma_{\alpha\beta}(p;P)$ must be independent of the quark energy (and only implicitly dependent on the bound state energy P_0). Thus we can write

$$\begin{aligned}\Gamma_{\alpha\beta}(\vec{p};P) &= g^2\int_r \frac{d\,\vec{k}\,D_{\sigma\sigma}(\vec{p}-\vec{k})}{(\vec{p}-\vec{k})^2}\left[T^a\Gamma(\vec{k};P)T^a\right]_{\alpha\beta}\\ &\quad\times\frac{\mathrm{i}}{2\pi}\int_{-\infty}^{\infty}\frac{dk_0}{\left[k^0_+-m-\mathcal{I}_r+\mathrm{i}\varepsilon\right]\left[k^0_--m+\mathcal{I}_r-\mathrm{i}\varepsilon\right]}+\mathcal{O}\left(1/m\right)\\ &= -g^2\int_r \frac{d\,\vec{k}\,D_{\sigma\sigma}(\vec{p}-\vec{k})}{(\vec{p}-\vec{k})^2}\frac{\left[T^a\Gamma(\vec{k};P)T^a\right]_{\alpha\beta}}{\left[P_0-2\mathcal{I}_r+2\mathrm{i}\varepsilon\right]}+\mathcal{O}\left(1/m\right). \end{aligned} \tag{5.54}$$

Thus, at leading order in the mass expansion, inserting the expression Eq. (5.35) for $\mathcal{I}_r$, it is now clear that

$$\begin{aligned}\left[P_0-g^2C_F\int_r\frac{d\,\vec{\omega}\,D_{\sigma\sigma}(\vec{\omega})}{\vec{\omega}^2}\right]\Gamma_{\alpha\beta}(\vec{p};P) &= -g^2\int_r\frac{d\,\vec{k}\,D_{\sigma\sigma}(\vec{p}-\vec{k})}{(\vec{p}-\vec{k})^2}\left[T^a\Gamma(\vec{k};P)T^a\right]_{\alpha\beta}\\ &\quad+\mathcal{O}\left(1/m\right). \end{aligned} \tag{5.55}$$

We notice that the explicit quark mass contributions of the self-energy expressions cancel. This is a feature of the quark-antiquark Bethe–Salpeter equation — it does not make any reference to the origins of its constituents (this explains, for example, why the pion can be a massless bound state of massive constituents). Physically, one can visualize that the quark and antiquark are moving with equal and opposite velocities such that the center of mass is stationary. This is related explicitly to the choice of Feynman prescription for the constituent quark and antiquark. Were the Feynman prescription for the antiquark chosen to coincide with that of the quark, the right-hand side of Eq. (5.54) would simply vanish and there would be certainly no physical quark-antiquark state. The Feynman

prescription for the antiquark corresponds precisely to a particle moving with the opposite velocity. Also, at leading order in the mass expansion, the momentum sharing parameter, ξ, has dropped out, and therefore the results are independent of ξ. Shifting the integration momenta, we can write

$$P_0\Gamma_{\alpha\beta}(\vec{p};P) = g^2\int_r \frac{d\,\vec{\omega}\,D_{\sigma\sigma}(\vec{\omega})}{\vec{\omega}^2}\left\{C_F\Gamma_{\alpha\beta}(\vec{p};P) - [T^a\Gamma(\vec{p}-\vec{\omega};P)T^a]_{\alpha\beta}\right\} + \mathcal{O}\left(1/m\right). \tag{5.56}$$

To see the physical meaning of this equation, we rewrite the Bethe–Salpeter vertex function as a Fourier transform:

$$\Gamma_{\alpha\beta}(\vec{p};P) = \int d\vec{y}e^{-\mathrm{i}\vec{p}\cdot\vec{y}}\Gamma_{\alpha\beta}(\vec{y}) \tag{5.57}$$

(in the homogeneous Bethe–Salpeter equation, the total momentum P denotes the solution and is not a variable). We also assign the following color structure

$$[T^a\Gamma(\vec{y})T^a]_{\alpha\beta} = C_M\Gamma_{\alpha\beta}(\vec{y}) \tag{5.58}$$

where C_M is yet to be identified. Then, the Bethe–Salpeter equation reduces to

$$\begin{aligned}\int d\vec{y}e^{-\mathrm{i}\vec{p}\cdot\vec{y}}P_0\Gamma_{\alpha\beta}(\vec{y}) \quad &= \quad \int d\vec{y}e^{-\mathrm{i}\vec{p}\cdot\vec{y}}g^2\int_r \frac{d\,\vec{\omega}\,D_{\sigma\sigma}(\vec{\omega})}{\vec{\omega}^2}\left\{C_F\Gamma_{\alpha\beta}(\vec{y}) - e^{\mathrm{i}\vec{\omega}\cdot\vec{y}}C_M\Gamma_{\alpha\beta}(\vec{y})\right\} \\ &\quad +\mathcal{O}\left(1/m\right)\end{aligned} \tag{5.59}$$

with the simple solution

$$P_0 = g^2\int_r \frac{d\,\vec{\omega}\,D_{\sigma\sigma}(\vec{\omega})}{\vec{\omega}^2}\left\{C_F - e^{\mathrm{i}\vec{\omega}\cdot\vec{y}}C_M\right\} + \mathcal{O}\left(1/m\right). \tag{5.60}$$

As already mentioned, because the total color charge of the system is conserved and vanishing [90], neither the quark or antiquark can exist as an independent asymptotic physical state. Thus, the $\bar{q}q$ system is either confined, such that the bound state energy P_0 increases linearly as the separation between the quark and antiquark increases, or the system cannot be physically created, such that the energy P_0 is infinite when the hypothetical regularization is removed. Whether the system is confining or disallowed can only depend on the color structure, since the temporal gluon propagator dressing function would be common to both situations.

An infrared confining solution is characterized in configuration space by the solution

$$P_0 = \sigma|\vec{y}| \text{ for large } |\vec{y}|, \tag{5.61}$$

where $\sigma \sim 1GeV/fm$ is called the string tension, such that as the separation between the quark and antiquark increases, the energy of the system should increase linearly without bound and infinite energy input is required to fully separate them[8] (at least in the absence of unquenching). The Fourier transform integral needed to obtain the above form of the solution is

$$\int \frac{d\,\vec{\omega}}{\vec{\omega}^4}\left[1 - e^{\mathrm{i}\vec{\omega}\cdot\vec{y}}\right] = \frac{|\vec{y}|}{8\pi}. \tag{5.62}$$

[8]The small $|\vec{y}|$ (and large $|\vec{\omega}|$) properties are of no concern here.

This implies that the temporal gluon propagator dressing function diverges like $1/\vec{\omega}^2$ and in addition the condition

$$C_F = C_M \tag{5.63}$$

must be satisfied. Moreover, with the these conditions, the spatial integral in Eq. (5.60) becomes automatically convergent and hence the energy of the system is well-defined, as the regularization is removed (since we are interested in the low $|\vec{\omega}|$ regime, it becomes clear that the regularization here would be infrared in character). Using the Fierz identity for the generators, Eq. (A.12), we get the condition

$$C_F\Gamma_{\alpha\gamma}(\vec{y}) \equiv C_M\Gamma_{\alpha\gamma}(\vec{y}) = \frac{1}{2}\delta_{\alpha\gamma}\Gamma_{\beta\beta}(\vec{y}) - \frac{1}{2N_c}\Gamma_{\alpha\gamma}(\vec{y}), \tag{5.64}$$

or with the definition Eq. (A.11),

$$\Gamma_{\alpha\gamma}(\vec{y}) = \delta_{\alpha\gamma}\Gamma(\vec{y}). \tag{5.65}$$

In other words, the quark-antiquark Bethe–Salpeter equation can only have a finite solution for *color singlet* states where the divergent constant integral coming from the unphysical quark self-energy cancels; otherwise the energy of the system is divergent.

Assuming that in the infrared (as is indicated by the lattice data [49] or by the above argument about the non-existence of asymptotic quark states), $D_{\sigma\sigma} = X/\vec{\omega}^2$ where X is some combination of constants (and further knowing that g^2X is a renormalization group invariant [12, 27]), then

$$P_0 \equiv \sigma|\vec{y}| = \frac{g^2C_FX}{8\pi}|\vec{y}| + \mathcal{O}\left(1/m\right). \tag{5.66}$$

The above result shows that there exists a direct connection between the string tension and the nonperturbative Yang–Mills sector of QCD (i.e., the temporal gluon propagator) at least under the truncation scheme considered here.

Let us now shortly discuss what are the possible consequences of including the pure Yang–Mills vertices in our approach. Recall that the confining potential follows from the rainbow-ladder Bethe–Salpeter kernel (i.e., dressed temporal gluon propagator and tree-level quark-gluon vertices). This suggests that the dressing function $D_{\sigma\sigma}$ does implicitly contain all nonperturbative effects associated with the dynamical dressing of the color charge (including, for example, potential glueball states), whereas the quark-gluon vertices correspond to the naked quark color charge. Pictorially, one can visualize this as a dressed color string confining two naked color sources. Consequently, we anticipate that the effect of including the non-Abelian corrections to the formalism presented here would not result in the removal of the linearly rising bound state energy (this would correspond to the cancellation of the ladder exchange). Instead, one expects a lowering of the string tension σ by shifting the pole position by some finite amount. Physically, this corresponds to the screening of the quark color charge — this statement has been phrased by Zwanziger as "no confinement without Coulomb confinement".

Let us now turn to the diquark Bethe–Salpeter equation. From a technical point of view, the difference between this and the previously considered quark-antiquark system is rather simple, but as we shall see it leads to a completely different physical result. Since the quark and the antiquark propagators share the same Feynman prescription relative

to their energy, the result that the crossed box contributions to the Bethe–Salpeter kernel vanish extends to the diquark case. This means that we can immediately write down the Bethe–Salpeter equation for diquarks, at leading order in the mass expansion and within our truncation scheme:

$$\begin{aligned}\Gamma_{\alpha\beta}(p;P) &= -\int \mathrm{d}\,k\,\Gamma^{a}_{\overline{q}q\sigma\alpha\gamma}(p_+,-k_+,k-p)W^{ab}_{\sigma\sigma}(p-k)\Gamma^{b}_{\overline{q}q\sigma\beta\delta}(-p_-,k_-,p-k)\\ &\quad\times W_{\overline{q}q\gamma\lambda}(k_+)W_{\overline{q}q\delta\kappa}(-k_-)\Gamma_{\lambda\kappa}(k;P)+\mathcal{O}\left(1/m\right). \end{aligned} \tag{5.67}$$

Again, the indices of the Bethe–Salpeter vertex function correspond to the quark content of the diquark and since the flavor and spin content decouple from the system, we shall only be interested in the color content of the diquark. Expanding this out as before, we get the analogous result

$$\begin{aligned}\left[P_0-2m-g^2C_F\int_r\frac{\mathrm{d}\,\vec{\omega}\,D_{\sigma\sigma}(\vec{\omega})}{\vec{\omega}^2}\right]\Gamma_{\alpha\beta}(\vec{p};P) &= g^2\int_r\frac{\mathrm{d}\,\vec{\omega}\,D_{\sigma\sigma}(\vec{\omega})}{\vec{\omega}^2}\left[T^a\right]_{\alpha\lambda}\left[T^a\right]_{\beta\kappa}\Gamma_{\lambda\kappa}(\vec{p}-\vec{\omega};P)\\ &\quad+\mathcal{O}\left(1/m\right). \end{aligned} \tag{5.68}$$

Fourier transforming as previously, and writing

$$\left[T^a\right]_{\alpha\lambda}\left[T^a\right]_{\beta\kappa}\Gamma_{\lambda\kappa}(\vec{y})=C_D\Gamma_{\alpha\beta}(\vec{y}) \tag{5.69}$$

gives the solution

$$P_0=2m+g^2\int_r\frac{\mathrm{d}\,\vec{\omega}\,D_{\sigma\sigma}(\vec{\omega})}{\vec{\omega}^2}\left\{C_F+e^{\mathrm{i}\vec{\omega}\cdot\vec{y}}C_D\right\}+\mathcal{O}\left(1/m\right). \tag{5.70}$$

The dependence of the solution on the quark mass simply indicates that in contrast to the quark-antiquark system, there are now two co-moving quarks. For the anti-diquark system the solution is identical to the above, but with *minus* twice the mass – their velocities are simply reversed. The diquark is antisymmetric under interchange of the two quark legs and this means that the color structure must be antisymmetric. Similar to the quark-antiquark system, the system can only have a confining (finite) energy solution, i.e., if $C_D=-C_F$, or no finite solution at all. Using the Fierz identity Eq. (A.12), the color condition then reads

$$-C_F\Gamma_{\alpha\beta}(\vec{y})\equiv C_D\Gamma_{\alpha\beta}(\vec{y})=\frac{1}{2}\Gamma_{\beta\alpha}(\vec{y})-\frac{1}{2N_c}\Gamma_{\alpha\beta}(\vec{y}). \tag{5.71}$$

Demanding the diquark color antisymmetry and with the definition Eq. (A.11) this becomes

$$N_c^2-N_c-2=0 \quad\Rightarrow\quad N_c=-1,2. \tag{5.72}$$

This means that in $SU(N_c=2)$ there exists a confined, antisymmetric bound state of two quarks – the $SU(2)$ baryon – and otherwise no physical states are allowed. In the next section, we will consider the $SU(3)$ baryon, and explicitly demonstrate how the confinement potential is obtained in this case.

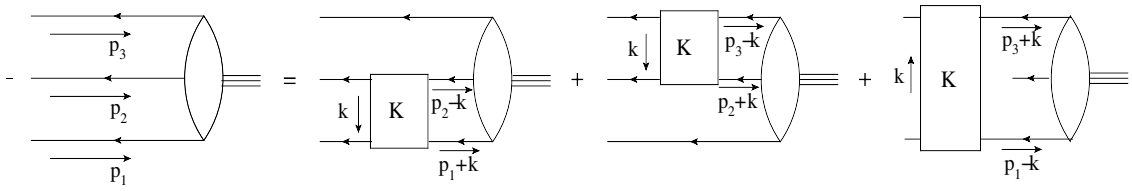

Figure 5.6.: Faddeev equation for three quark bound states. Solid lines represent the quark propagator, the box represents the diquark kernel K and the ellipse represents the Faddeev vertex function with the bound state leg depicted by a triple-line. See text for details.

5.4.2. Faddeev equation for baryons

Baryons appear as three-quark bound-states in the Faddeev equation, which generally can be written as:

$$\Gamma = K^{(3)}\Gamma \tag{5.73}$$

where Γ is the quark-baryon vertex and $K^{(3)}$ is the three-body kernel, containing an irreducible term $K^{(3)}_{ir}$ and the sum of permuted diquark kernels:

$$K^{(3)} = K^{(3)}_{ir} + \sum_{i=1}^{3} K^{(2)}_{i} \tag{5.74}$$

The Faddeev equation [95] and its subsequent developments [96, 97] (for an extended review see [98] and the textbook [99]) provide a general formulation of the relativistic three-body problem. It is a bound state equation (the direct analogue of the homogeneous two-body Bethe–Salpeter equation) and it has been efficiently applied in QCD to study baryon states, via the Green's functions of the theory. Typically, these studies are performed in Landau gauge and, due to the complexity of the equations, they have been mainly restricted to rainbow-ladder truncation, where the kernel is reduced to the single exchange of a dressed gluon. Within this approximation and by employing phenomenological ansätze for the Yang–Mills part of the theory, the nucleon and Δ properties have been studied [100–103]. Other simplifications include the three-body spectator formalism [98], a Salpeter-type equation with instantaneous interaction [104] or the diquark correlations [105].

In the following, we investigate the Faddeev equation for three-quark bound states, by employing only the permuted two quark kernels $K^{(2)}_i$ (which coincide with kernel appearing in the Bethe–Salpeter equation for diquark states) and neglecting the three-quark irreducible diagrams, i.e., genuine three-body forces [106]. This approximation is also motivated by the fact that in the quark-diquark model the binding energy is assumed to be mainly provided by the two-quark correlations [107]. In this truncation, the Faddeev equation reads (see also Fig. 5.6):

$$\begin{aligned}
\Gamma_{\alpha\beta\gamma}(p_1,p_2,p_3;P) = \\
&- \int d\,k \,\Big\{ K_{\beta\alpha;\alpha'\beta'}(k) W_{\bar{q}q\alpha'\alpha''}(p_1+k) W_{\bar{q}q\beta'\beta''}(p_2-k) \Gamma_{\alpha''\beta''\gamma}(p_1+k,p_2-k,p_3;P) \\
&+ K_{\gamma\beta;\beta'\gamma'}(k) W_{\bar{q}q\beta'\beta''}(p_2+k) W_{\bar{q}q\gamma'\gamma''}(p_3-k) \Gamma_{\alpha\beta''\gamma''}(p_1,p_2+k,p_3-k;P) \\
&+ K_{\alpha\gamma;\gamma'\alpha'}(k) W_{\bar{q}q\gamma'\gamma''}(p_3+k) W_{\bar{q}q\alpha'\alpha''}(p_1-k) \Gamma_{\alpha\beta''\gamma''}(p_1-k,p_2,p_3+k;P) \Big\}
\end{aligned} \tag{5.75}$$

where p_1, p_2, p_3 are the momenta of the quarks, $P = p_1 + p_2 + p_3$ is the pole 4-momentum of the bound baryon state and Γ is the so-called quark-baryon Faddeev vertex for the particular bound state under consideration and whose indices denote explicitly only its quark content. Due to the fact that in the heavy mass limit the spin degrees of freedom decouple from the system, at leading order in the mass expansion the Faddeev baryon amplitude $\Gamma_{\alpha\beta\gamma}$ becomes a Dirac scalar, similar to the heavy quark propagator Eq. (5.34). The explicit momentum dependence of the kernels K is abbreviated for notational convenience. As in the homogeneous Bethe–Salpeter equation, the integral equation depends only parametrically on the total four momentum P.

As discussed in the case of the Bethe–Salpeter equation, the kernel K reduces to the ladder approximation (constructed via gluon exchange) and it reads

$$K_{\beta\alpha;\alpha'\beta'}(k) = \Gamma^a_{\bar{q}q\sigma\alpha\alpha'} W^{ab}_{\sigma\sigma}(\vec{k}) \Gamma^b_{\bar{q}q\sigma\beta\beta'} = g^2 T^a_{\alpha\alpha'} W^{ab}_{\sigma\sigma}(\vec{k}) T^b_{\beta\beta'} \tag{5.76}$$

with the temporal quark-gluon vertex and the temporal gluon propagator given by Eq. (5.27) and Eq. (5.23), respectively. Similar to the Bethe–Salpeter equation for meson bound states, the energy independence of this propagator will turn to be crucial in the derivation of the confining potential.

Let us now investigate the energy dependence of the equation Eq. (5.75). As shown in the previous section, the Bethe–Salpeter kernel was energy independent, and thus it was straightforward to show that the Bethe–Salpeter vertex itself did not contain an energy dependent part. This observation was then used to calculate the confining potential from the Bethe–Salpeter equation, via a simple analytical integration over the relative energy variable. Unfortunately this approach cannot be extended to baryon states: despite the instantaneous kernel, a relative energy dependence still remains and thus one cannot assume an energy-independent Faddeev vertex. Therefore, in order to proceed, we make the following separable ansatz for the Faddeev vertex:

$$\Gamma_{\alpha\beta\gamma}(p_1, p_2, p_3; P) = \Psi_{\alpha\beta\gamma} \Gamma_t(p_1^0, p_2^0, p_3^0; P) \Gamma_s(\vec{p}_1, \vec{p}_2, \vec{p}_3; P) \tag{5.77}$$

where we have introduced a purely antisymmetric (in the quark legs) color factor Ψ (the possible baryon color index is omitted) and the symmetric (Dirac scalar) temporal and spatial components Γ_t and Γ_s, respectively.

Inserting the explicit form of the kernel Eq. (5.76) and the quark-baryon vertex ansatz Eq. (5.77), the Faddeev equation Eq. (5.75) can be explicitly written as (for simplicity we drop the label P in the arguments of the vertex functions):

$$\begin{aligned}\Gamma_{\alpha\beta\gamma}(p_1, p_2, p_3) = -g^2 T^a_{\alpha\tau} T^a_{\beta\kappa} \Psi_{\tau\kappa\gamma} \int d\,k\, W_{\sigma\sigma}(\vec{k}) \\ \times W_{\bar{q}q}(p_1 + k) W_{\bar{q}q}(p_2 - k) \Gamma_t(p_1^0 + k_0, p_2^0 - k_0, p_3^0) \Gamma_s(\vec{p}_1 + \vec{k}, \vec{p}_2 - \vec{k}, \vec{p}_3) + \text{c.p.},\end{aligned} \tag{5.78}$$

where the explicit color structure has been extracted ($W^{ab}_{\sigma\sigma} = \delta^{ab} W_{\sigma\sigma}, W_{\bar{q}q\alpha\beta} = \delta_{\alpha\beta} W_{\bar{q}q}$).

Analogous to the Bethe–Salpeter equation, we use the Fierz identity for the generators, Eq. (A.12), to write the color structure as

$$T^a_{\alpha\alpha'} T^a_{\beta\beta'} \Psi_{\alpha'\beta'\gamma} = -C_B \Psi_{\alpha\beta\gamma} \tag{5.79}$$

with $C_B = (N_c + 1)/2N_c$, where N_c is the number of colors, yet to be identified (i.e., the baryon is not assumed to be a color singlet).

In the next step we perform the Fourier transform for the spatial part of the equation, recalling that the heavy quark propagator is only a function of energy. We define the coordinate space vertex function via its Fourier transform

$$\Gamma_s(\vec{p_1},\vec{p_2},\vec{p_3}) = \int d\vec{x}_1 d\vec{x}_2 d\vec{x}_3 e^{-\mathrm{i}\vec{p}_1\cdot\vec{x}_1-\mathrm{i}\vec{p}_2\cdot\vec{x}_2-\mathrm{i}\vec{p}_3\cdot\vec{x}_3}\Gamma_s(\vec{x}_1,\vec{x}_2,\vec{x}_3) \tag{5.80}$$

such that

$$\int d\,\vec{k}\,W_{\sigma\sigma}(\vec{k})\Gamma_s(\vec{p}_1+\vec{k},\vec{p}_2-\vec{k},\vec{p}_3) = \int d\,\vec{x}_1\; d\,\vec{x}_2\; d\,\vec{x}_3\, e^{-\mathrm{i}\vec{p}_1\cdot\vec{x}_1-\mathrm{i}\vec{p}_2\cdot\vec{x}_2-\mathrm{i}\vec{p}_3\cdot\vec{x}_3}W_{\sigma\sigma}(\vec{x}_2-\vec{x}_1)\Gamma_s(\vec{x}_1,\vec{x}_2,\vec{x}_3). \tag{5.81}$$

Clearly, the component Γ_s trivially simplifies (as before, we have separated the temporal and spatial integrals, under the assumption the spatial integral is regularized and finite) and the equation Eq. (5.78) reduces to [$d\,k_0 = dk_0/(2\pi)$]:

$$\Gamma_t(p_1^0,p_2^0,p_3^0) = g^2 C_B W_{\sigma\sigma}(\vec{x}_2-\vec{x}_1)\int d\,k_0\, W_{\bar{q}q}(p_1^0+k_0)W_{\bar{q}q}(p_2^0-k_0)\Gamma_t(p_1^0+k_0,p_2^0-k_0,p_3^0) + \text{ c.p..} \tag{5.82}$$

At this point we make a further simplification, motivated by the symmetry of the three-quark system: we restrict to a particular geometry, namely to equal quark separations, i.e. $|\vec{r}| = |\vec{x}_2-\vec{x}_1| = |\vec{x}_3-\vec{x}_2| = |\vec{x}_1-\vec{x}_3|$. By inserting the explicit form of the quark propagators, Eq. (5.34), we have

$$\Gamma_t(p_1^0,p_2^0,p_3^0) = -g^2 C_B W_{\sigma\sigma}(|\vec{r}|)\int d\,k_0\,\frac{\Gamma_t(p_1^0+k_0,p_2^0-k_0,p_3^0)}{\left[p_1^0+k_0-m-\mathcal{I}_r+\mathrm{i}\varepsilon\right]\left[p_2^0-k_0-m-\mathcal{I}_r+\mathrm{i}\varepsilon\right]} + \text{c.p..} \tag{5.83}$$

With the assumption that the vertex Γ_t is symmetric under permutation of quark legs, an ansatz that satisfies this equation is:

$$\Gamma_t(p_1^0,p_2^0,p_3^0) = \sum_{i=1,2,3}\frac{1}{2P_0-3(p_i^0+m+\mathcal{I}_r)+\mathrm{i}\varepsilon}. \tag{5.84}$$

Since the explicit derivation is rather technical, we only give here the solution and defer the details to the Appendix E.

Notice that in the expression Eq. (5.84) there are simple poles (in the energy) present. These poles however do not occur for finite energies and cannot be physical. As discussed, this is also the case for the quark propagator. Intuitively, when a single heavy quark is pulled apart from the system, the qqq state becomes equivalent (i.e., it has the same color quantum numbers) to the $\bar{q}q$ system in the sense that the remaining two quarks form a diquark which for $N_c = 3$ would be a color antitriplet configuration, and hence the physical interpretation of the vertex Eq. (5.84) can be directly related to the heavy quark propagator Eq. (5.34): the presence of the single pole in Eq. (5.84) simply means that this cannot have the meaning of physical propagation (this would require a covariant double pole). Moreover, the divergent constant $\mathcal{I}_r$ appearing in the absolute energy does

not contradict the physics – the only relevant quantity is the relative energy of the three quark system.

With this ansatz at hand, we return to the formula Eq. (5.83), insert the definitions Eq. (5.27) and Eq. (5.35) for $W_{\sigma\sigma}(\vec{x})$ and $\mathcal{I}_r$, and arrive at the following solution for the bound state energy P_0, in the case of equal quark separations:

$$P_0 = 3m + \frac{3}{2}g^2 \int đ\vec{\omega}\, \frac{D_{\sigma\sigma}(\vec{\omega})}{\vec{\omega}^2} \left[C_F - 2C_B e^{i\vec{\omega}\cdot\vec{r}}\right]. \tag{5.85}$$

The following reasoning is similar to our discussion from the case of the bound states of a meson or diquark system. Since the quarks can not be prepared as isolated states, the only possibilities for the qqq state are either that the system is confined (i.e., the bound state energy P_0 increases with the separation), or the system is physically not allowed (i.e., the energy P_0 is infinite). From the formula Eq. (5.85) and knowing that $D_{\sigma\sigma}(\vec{\omega})$ is infrared enhanced, it is clear that in order to have an infrared confining solution (corresponding to a convergent three-momentum integral), the condition

$$C_B = \frac{C_F}{2} \tag{5.86}$$

must be satisfied. This is fulfilled for $N_c = 3$ colors, implying that $\Psi_{\alpha\beta\gamma} = \varepsilon_{\alpha\beta\gamma}$ and that the baryon is a color singlet (confined) bound state of three quarks; otherwise, for $N_c \neq 3$ the energy of the the system is infinite for any separation $|\vec{r}|$.

As in the preceding section, with the assumption that in the infrared $D_{\sigma\sigma}(\vec{\omega}) = X/\vec{\omega}^2$ (as indicated by the lattice data [47–49, 108, 109] and by the variational calculations in the continuum [41]), where X is some combination of constants, it is straightforward to perform the integration on the right hand side of Eq. (5.85), with the result that for large separation $|\vec{r}|$:

$$P_0 = 3m + \frac{3}{2}\frac{g^2 C_F X}{8\pi}|\vec{r}|. \tag{5.87}$$

This mimics the previous findings for $\bar{q}q$ and qq systems, namely that there exists a direct connection between the string tension and the nonperturbative Yang–Mills Green's functions (at least under truncation). In this case, the standard term "string tension" refers to the coefficient of the three-body linear confinement term $\sigma_{3q}|\vec{r}|$. Also, comparing with the result of the previous section, we find that the string tension corresponding to the qqq system is 3/2 times that of the $\bar{q}q$. To our knowledge, no direct comparison between the string tensions of the two systems has been made and hence this relation would be interesting to investigate on the lattice. The appearance of three times the quark mass stems from the presence of the mass term in the heavy quark propagator Eq. (5.34) which enters the Faddeev equation.

Chapter 6.

Higher order Green's functions

We have seen that in general, the underlying equation for the description of meson bound states is the two-body homogeneous Bethe–Salpeter equation. In the rainbow-ladder approximation, this equation has been successfully used to describe the properties of light mesons (see, for example [91, 92] and for a recent review [43]), where the driving mechanism is the chiral symmetry breaking. Beyond this approximation, models with dressed vertex contributions [93, 94, 110–113] and unquenching effects [114–116] have been considered, and more sophisticated numerical methods to solve both the homogeneous and the inhomogeneous Bethe–Salpeter equation have been recently developed [117]. In spite of this success, an exact derivation of the meson or diquark bound state energies via Green's functions techniques has not been yet reported. The difficulty stems from the fact that the (irreducible) interaction kernel contains higher order vertex functions which in general can not be calculated exactly.

In this chapter, we continue our investigations of the quark-antiquark and diquark states by using Green's functions techniques. As before, our study is based on the heavy mass expansion underlining HQET and with the truncation of the Yang–Mills sector to include only dressed two-point functions. By means of functional methods, we explicitly derive the (fully amputated) quark 4-point Green's functions and give an exact, analytical solution. This will enable us to verify that bound states are related to the occurrence of the poles in the Green's functions and hence we will be able to provide a direct connection between the homogeneous Bethe–Salpeter equation considered in the previous chapter and the singularities of the Green's function, at least within the scheme considered here.

6.1. 4-point Green's functions for quark-antiquark systems

In this section we derive the Dyson–Schwinger equations for the one-particle irreducible and for the fully amputated 4-point quark-antiquark Green's functions (from which later on the bound state energy for the heavy quark system will be extracted).

Let us start by deriving the Dyson–Schwinger equation for the one-particle irreducible (1PI) 4-point function. As an illustration of the functional differentiation techniques, we present the explicit derivation of the first term in this function and notice that the rest of the terms follow from an identical calculation. Just as for the gap equation, we first take the functional derivative of the quark Eq. (2.19) with respect to $\mathrm{i}q_{\gamma z}$, and obtain the result Eq. (2.29); further, we functionally differentiate with respect to $\mathrm{i}\bar{q}_{\tau w}$ and, using the

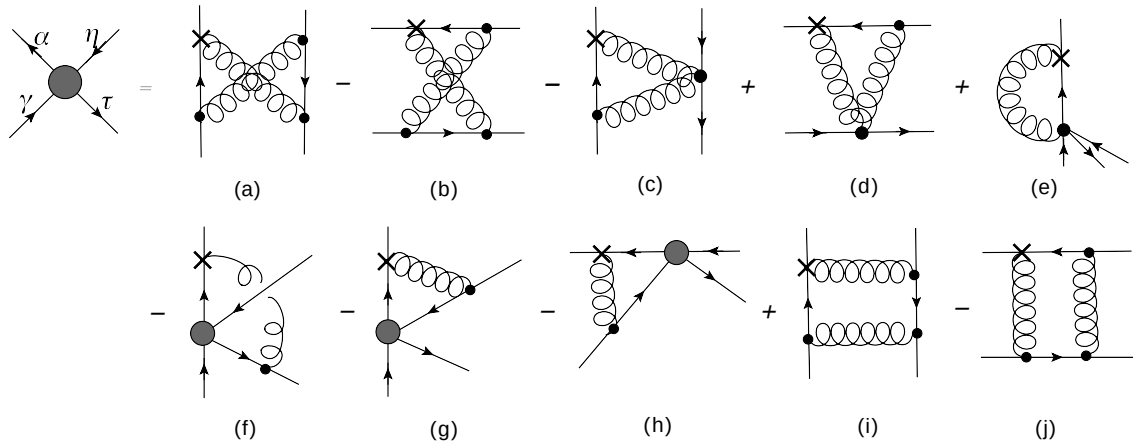

Figure 6.1.: Diagrammatic representation of the one particle irreducible 4-point quark-antiquark Green's function. Blobs represent dressed proper (1PI) 4-point vertex, solid lines represent the quark propagator, springs denote either spatial ($\vec{A}$) or temporal (σ) gluon propagator and cross denotes the tree level quark-gluon vertex. Internal propagators are fully dressed.

product rule, we obtain:

$$\begin{aligned}\frac{\delta^2}{\delta \mathrm{i}\bar{q}_{\tau w}\delta \mathrm{i}q_{\gamma z}} <\mathrm{i}\rho^a_y \mathrm{i}\bar{\chi}_{\beta x}> &= -S[\lambda,\kappa]\left\{\left[\frac{\delta}{\delta \mathrm{i}\bar{q}_{\tau w}} <\mathrm{i}\rho^a_y \mathrm{i}J_\lambda>\right] <\mathrm{i}\Phi_\lambda \mathrm{i}q_{\gamma z}\mathrm{i}\Phi_\kappa><\mathrm{i}J_\kappa \mathrm{i}\bar{\chi}_{\beta x}>\right.\\ &+ <\mathrm{i}\rho^a_y \mathrm{i}J_\lambda><\mathrm{i}\Phi_\lambda \mathrm{i}\bar{q}_{\tau w}\mathrm{i}q_{\gamma z}\mathrm{i}\Phi_\kappa><\mathrm{i}J_\kappa \mathrm{i}\bar{\chi}_{\beta x}>\\ &\left.+ <\mathrm{i}\rho^a_y \mathrm{i}J_\lambda><\mathrm{i}\Phi_\lambda \mathrm{i}q_{\gamma z}\mathrm{i}\Phi_\kappa>\left[\frac{\delta}{\delta \mathrm{i}\bar{q}_{\tau w}} <\mathrm{i}J_\kappa \mathrm{i}\bar{\chi}_{\beta x}>\right]\right\} \qquad (6.1)\end{aligned}$$

As explained above, we only retain the first term in the product (and denote the rest with dots). Again, we make use of the formula Eq. (2.27) and obtain:

$$\begin{aligned}\frac{\delta^2}{\delta \mathrm{i}\bar{q}_{\tau w}\delta \mathrm{i}q_{\gamma z}} <\mathrm{i}\rho^a_y \mathrm{i}\bar{\chi}_{\beta x}> &= S[\lambda,\kappa,\mu,\nu]\\ &\times <\mathrm{i}\rho^a_y \mathrm{i}J_\mu><\mathrm{i}\Phi_\mu \mathrm{i}\bar{q}_{\tau w}\mathrm{i}\Phi_\nu><\mathrm{i}J^a_\nu \mathrm{i}J_\lambda><\mathrm{i}\Phi_\lambda \mathrm{i}q_{\gamma z}\mathrm{i}\Phi_\kappa><\mathrm{i}J_\kappa \mathrm{i}\bar{\chi}_{\beta x}> + \ldots . \qquad (6.2)\end{aligned}$$

A last functional derivative with respect to the quark field $q_{\eta t}$ gives

$$\begin{aligned}&\frac{\delta^3}{\delta \mathrm{i}\bar{q}_{\tau w}\delta \mathrm{i}q_{\gamma z}\delta \mathrm{i}q_{\eta t}} <\mathrm{i}\rho^a_y \mathrm{i}\bar{\chi}_{\beta x}> = S[\lambda,\kappa,\mu,\nu]\\ &\times\left\{\left[\frac{\delta}{\delta \mathrm{i}q_{\eta t}} <\mathrm{i}\rho^a_y \mathrm{i}J_\mu>\right] <\mathrm{i}\Phi_\mu \mathrm{i}\bar{q}_{\tau w}\mathrm{i}\Phi_\nu><\mathrm{i}J^a_\nu \mathrm{i}J_\lambda><\mathrm{i}\Phi_\lambda \mathrm{i}q_{\gamma z}\mathrm{i}\Phi_\kappa><\mathrm{i}J_\kappa \mathrm{i}\bar{\chi}_{\beta x}>\right.\\ &+ <\mathrm{i}\rho^a_y \mathrm{i}J_\mu><\mathrm{i}\Phi_\mu \mathrm{i}q_{\eta t}\mathrm{i}\bar{q}_{\tau w}\mathrm{i}\Phi_\nu><\mathrm{i}J^a_\nu \mathrm{i}J_\lambda><\mathrm{i}\Phi_\lambda \mathrm{i}q_{\gamma z}\mathrm{i}\Phi_\kappa><\mathrm{i}J_\kappa \mathrm{i}\bar{\chi}_{\beta x}>\\ &+ <\mathrm{i}\rho^a_y \mathrm{i}J_\mu><\mathrm{i}\Phi_\mu \mathrm{i}\bar{q}_{\tau w}\mathrm{i}\Phi_\nu>\left[\frac{\delta}{\delta \mathrm{i}q_{\eta t}} <\mathrm{i}J^a_\nu \mathrm{i}J_\lambda>\right] <\mathrm{i}\Phi_\lambda \mathrm{i}q_{\gamma z}\mathrm{i}\Phi_\kappa><\mathrm{i}J_\kappa \mathrm{i}\bar{\chi}_{\beta x}>\\ &+ <\mathrm{i}\rho^a_y \mathrm{i}J_\mu><\mathrm{i}\Phi_\mu \mathrm{i}\bar{q}_{\tau w}\mathrm{i}\Phi_\nu><\mathrm{i}J^a_\nu \mathrm{i}J_\lambda><\mathrm{i}\Phi_\lambda \mathrm{i}q_{\eta t}\mathrm{i}q_{\gamma z}\mathrm{i}\Phi_\kappa><\mathrm{i}J_\kappa \mathrm{i}\bar{\chi}_{\beta x}>\\ &\left.+ <\mathrm{i}\rho^a_y \mathrm{i}J_\mu><\mathrm{i}\Phi_\mu \mathrm{i}\bar{q}_{\tau w}\mathrm{i}\Phi_\nu><\mathrm{i}J^a_\nu \mathrm{i}J_\lambda><\mathrm{i}\Phi_\lambda \mathrm{i}q_{\gamma z}\mathrm{i}\Phi_\kappa>\left[\frac{\delta}{\delta \mathrm{i}q_{\eta t}} <\mathrm{i}J_\kappa \mathrm{i}\bar{\chi}_{\beta x}>\right]\right\}+\ldots . \qquad (6.3)\end{aligned}$$

Again, we take only the first term from the above sum and as before we use the formula Eq. (2.27). We obtain:

$$\frac{\delta^3}{\delta \mathrm{i}\bar{q}_{\tau w}\delta \mathrm{i}q_{\gamma z}\delta \mathrm{i}q_{\eta t}} <\mathrm{i}\rho^a_y \mathrm{i}\bar{\chi}_{\beta x}> \;=\; -S[\lambda,\kappa,\mu,\nu,\varepsilon,\delta]\; <\mathrm{i}\rho^a_y \mathrm{i}J_\varepsilon><\mathrm{i}\Phi_\varepsilon \mathrm{i}\bar{q}_{\eta t}\mathrm{i}\Phi_\delta><\mathrm{i}J^a_\delta \mathrm{i}J_\mu> \times <\mathrm{i}\Phi_\mu \mathrm{i}\bar{q}_{\tau w}\mathrm{i}\Phi_\nu><\mathrm{i}J^a_\nu \mathrm{i}J_\lambda><\mathrm{i}\Phi_\lambda \mathrm{i}q_{\gamma z}\mathrm{i}\Phi_\kappa><\mathrm{i}J_\kappa \mathrm{i}\bar{\chi}_{\beta x}> + \ldots . \quad (6.4)$$

Identifying the various fields and sources, we arrive at the the first term in the expression below for the 4-point Green's function (all the other terms are derived by a similar calculation and we omit the terms which will vanish when the sources are set to zero):

$$\begin{aligned}
<\mathrm{i}\overline{q}_\alpha \mathrm{i}q_\gamma \mathrm{i}\overline{q}_\tau \mathrm{i}q_\eta> = [g\gamma^0 T^a]_{\alpha\beta} \int dy\, \delta(x-y) \\
\times \Big\{ \Big[<\mathrm{i}\overline{\chi}_\beta \mathrm{i}\chi_\kappa><\mathrm{i}\overline{q}_\kappa \mathrm{i}q_\gamma \mathrm{i}\sigma^c_\lambda><\mathrm{i}\rho^c_\lambda \mathrm{i}\rho^d_\nu> \Big] \Big[<\mathrm{i}\overline{q}_\tau \mathrm{i}q_\mu \mathrm{i}\sigma^d_\nu><\mathrm{i}\overline{\chi}_\mu \mathrm{i}\chi_\delta><\mathrm{i}\overline{q}_\delta \mathrm{i}q_\eta \mathrm{i}\sigma^b_\varepsilon><\mathrm{i}\rho^b_\varepsilon \mathrm{i}\rho^a_y> \Big] \\
- \Big[<\mathrm{i}\overline{\chi}_\beta \mathrm{i}\chi_\kappa><\mathrm{i}\overline{q}_\kappa \mathrm{i}q_\gamma \mathrm{i}\sigma^c_\lambda><\mathrm{i}\rho^c_\lambda \mathrm{i}\rho^d_\nu><\overline{q}_\tau \mathrm{i}q_\eta \mathrm{i}\sigma^d_\nu \mathrm{i}\sigma^b_\mu><\mathrm{i}\rho^b_\mu \mathrm{i}\rho^a_y> \Big] \\
+ \Big[<\mathrm{i}\overline{\chi}_\beta \mathrm{i}\chi_\kappa><\mathrm{i}\overline{q}_\kappa \mathrm{i}q_\gamma \mathrm{i}\sigma^c_\lambda><\mathrm{i}\rho^c_\lambda \mathrm{i}\rho^d_\delta> \Big] \Big[<\mathrm{i}\overline{q}_\tau \mathrm{i}q_\nu \mathrm{i}\sigma^b_\mu><\mathrm{i}\overline{\chi}_\nu \mathrm{i}\chi_\varepsilon><\mathrm{i}\overline{q}_\varepsilon \mathrm{i}q_\eta \mathrm{i}\sigma^d_\delta><\mathrm{i}\rho^b_\mu \mathrm{i}\rho^a_y> \Big] \\
- \Big[<\mathrm{i}\overline{\chi}_\beta \mathrm{i}\chi_\kappa><\mathrm{i}\overline{q}_\kappa \mathrm{i}q_\gamma \mathrm{i}\overline{q}_\lambda \mathrm{i}q_\eta><\mathrm{i}\overline{q}_\tau \mathrm{i}q_\nu \mathrm{i}\sigma^b_\mu><\mathrm{i}\overline{\chi}_\nu \mathrm{i}\chi_\lambda><\mathrm{i}\rho^b_\mu \mathrm{i}\rho^a_y> \Big] \\
- \Big[<\mathrm{i}\overline{\chi}_\beta \mathrm{i}\chi_\delta><\mathrm{i}\overline{q}_\delta \mathrm{i}q_\eta \mathrm{i}\sigma^c_\varepsilon><\mathrm{i}\rho^c_\varepsilon \mathrm{i}\rho^d_\kappa> \Big] \Big[<\mathrm{i}\overline{q}_\tau \mathrm{i}q_\nu \mathrm{i}\sigma^b_\mu><\mathrm{i}\overline{\chi}_\nu \mathrm{i}\chi_\lambda><\mathrm{i}\overline{q}_\lambda \mathrm{i}q_\gamma \mathrm{i}\sigma^d_\kappa><\mathrm{i}\rho^b_\mu \mathrm{i}\rho^a_y> \Big] \\
- \Big[<\mathrm{i}\overline{\chi}_\beta \mathrm{i}\chi_\kappa><\mathrm{i}\overline{q}_\kappa \mathrm{i}q_\gamma \mathrm{i}\overline{q}_\tau \mathrm{i}q_\lambda><\mathrm{i}\overline{\chi}_\lambda \mathrm{i}\chi_\delta><\mathrm{i}\overline{q}_\delta \mathrm{i}q_\eta \mathrm{i}\sigma^b_\varepsilon><\mathrm{i}\rho^b_\varepsilon \mathrm{i}\rho^a_y> \Big] \\
+ \Big[<\mathrm{i}\overline{\chi}_\beta \mathrm{i}\chi_\kappa><\mathrm{i}\overline{q}_\kappa \mathrm{i}q_\gamma \mathrm{i}\overline{q}_\tau \mathrm{i}q_\eta \sigma^b_\lambda><\mathrm{i}\rho^b_\lambda \mathrm{i}\rho^a_y> \Big] \\
+ \Big[<\mathrm{i}\overline{\chi}_\beta \mathrm{i}\chi_\delta><\mathrm{i}\overline{q}_\delta \mathrm{i}q_\eta \mathrm{i}\sigma^c_\varepsilon><\mathrm{i}\rho^c_\varepsilon \mathrm{i}\rho^d_\kappa><\mathrm{i}\overline{q}_\tau \mathrm{i}q_\gamma \mathrm{i}\sigma^d_\kappa \mathrm{i}\sigma^b_\lambda><\mathrm{i}\rho^b_\lambda \mathrm{i}\rho^a_y> \Big] \\
- \Big[<\mathrm{i}\overline{\chi}_\beta \mathrm{i}\chi_\nu><\mathrm{i}\overline{q}_\nu \mathrm{i}q_\mu \mathrm{i}\overline{q}_\tau \mathrm{i}q_\eta><\mathrm{i}\overline{\chi}_\mu \mathrm{i}\chi_\kappa><\mathrm{i}\overline{q}_\kappa \mathrm{i}q_\gamma \mathrm{i}\sigma^b_\lambda><\mathrm{i}\rho^b_\lambda \mathrm{i}\rho^a_y> \Big] \\
- \Big[<\mathrm{i}\overline{\chi}_\beta \mathrm{i}\chi_\delta><\mathrm{i}\overline{q}_\delta \mathrm{i}q_\eta \mathrm{i}\sigma^c_\varepsilon><\mathrm{i}\rho^c_\varepsilon \mathrm{i}\rho^d_\nu> \Big] \Big[<\mathrm{i}\overline{q}_\tau \mathrm{i}q_\mu \mathrm{i}\sigma^d_\nu><\mathrm{i}\overline{\chi}_\mu \mathrm{i}\chi_\kappa><\mathrm{i}\overline{q}_\kappa \mathrm{i}q_\gamma \mathrm{i}\sigma^d_\lambda><\mathrm{i}\rho^d_\lambda \mathrm{i}\rho^a_y> \Big] \Big\} \\
+ \ldots
\end{aligned} \quad (6.5)$$

where the dots represent the $\vec{A}$ vertex terms which are not considered here and we have already replaced the tree-level temporal quark-gluon vertex with its expression Eq. (3.10). Also, notice the minus sign in the fourth term, which corresponds to the minus sign arising in the Bethe–Salpeter equation considered in Chapter 5. This equation is diagrammatically represented in Fig. 6.1.

The next step is to derive the Dyson–Schwinger equation for the connected 4-point quark-antiquark Green's function, which is related to the 1PI Green's function via Legendre transform. Starting with the identity

$$<\mathrm{i}q_\beta \mathrm{i}\overline{q}_\gamma><\mathrm{i}\chi_\gamma \mathrm{i}\overline{\chi}_\alpha> = \delta_{\beta\alpha}, \quad (6.6)$$

we first take a functional derivative with respect to the source $\mathrm{i}J_\delta$ and, with the help of the formula Eq. (2.22b), we derive the relation:

$$0 = <\mathrm{i}q_\beta \mathrm{i}\overline{q}_\rho><\mathrm{i}\chi_\rho \mathrm{i}\overline{\chi}_\alpha \mathrm{i}J_\delta> + \mathrm{i}S[\kappa]\; <\mathrm{i}J_\delta \mathrm{i}J_\kappa><\mathrm{i}\Phi_\kappa \mathrm{i}q_\beta \mathrm{i}\bar{q}_\gamma><\mathrm{i}\chi_\gamma \mathrm{i}\overline{\chi}_\alpha> . \quad (6.7)$$

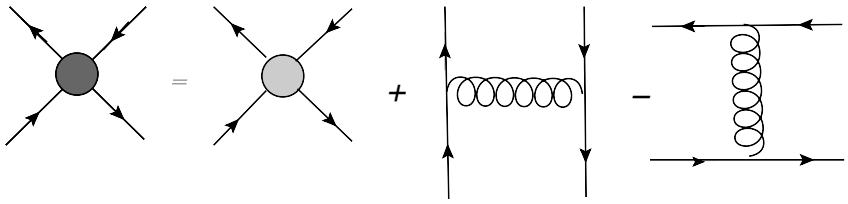

Figure 6.2.: Relation between the one particle irreducible (filled blob) and amputated (dashed blob) 4-point Greens function for the quark-antiquark system. Internal propagators are fully dressed.

Identifying J_δ with a gluon source, we have that

$$<\mathrm{i}\overline{\chi}_\alpha \mathrm{i}\chi_\rho \mathrm{i}J_\delta> = (-\mathrm{i}) <\mathrm{i}\overline{\chi}_\alpha \mathrm{i}\chi_\gamma><\mathrm{i}\overline{q}_\gamma \mathrm{i}q_\beta \mathrm{i}\phi_\kappa><\mathrm{i}\overline{\chi}_\beta \mathrm{i}\chi_\rho><\mathrm{i}J_\kappa \mathrm{i}J_\delta> \,. \tag{6.8}$$

Taking a further functional derivative of Eq. (6.7) with respect to the quark source $\mathrm{i}\overline{\chi}_\lambda$ (in this case, J_δ is identified as quark source), we arrive at the following expression

$$\begin{aligned} <\mathrm{i}\overline{\chi}_\alpha \mathrm{i}\chi_\delta \mathrm{i}\overline{\chi}_\lambda \mathrm{i}\chi_\eta> \;=\; & -\mathrm{i} <\mathrm{i}\overline{\chi}_\alpha \mathrm{i}\chi_\delta \mathrm{i}J_\rho><\mathrm{i}\overline{q}_\varepsilon \mathrm{i}q_\beta \mathrm{i}\phi_\rho><\mathrm{i}\overline{\chi}_\lambda \mathrm{i}\chi_\varepsilon><\mathrm{i}\overline{\chi}_\beta \mathrm{i}\chi_\eta> \\ & +\mathrm{i} <\mathrm{i}\overline{\chi}_\alpha \mathrm{i}\chi_\gamma><\mathrm{i}\overline{q}_\gamma \mathrm{i}q_\beta \mathrm{i}\phi_\kappa><\mathrm{i}\overline{\chi}_\lambda \mathrm{i}\chi_\delta \mathrm{i}J_\kappa><\mathrm{i}\overline{\chi}_\beta \mathrm{i}\chi_\eta> \\ & - <\mathrm{i}\overline{\chi}_\alpha \mathrm{i}\chi_\gamma><\mathrm{i}\overline{q}_\gamma \mathrm{i}q_\kappa \mathrm{i}\overline{q}_\varepsilon \mathrm{i}q_\beta><\mathrm{i}\overline{\chi}_\lambda \mathrm{i}\chi_\varepsilon><\mathrm{i}\overline{\chi}_\kappa \mathrm{i}\chi_\delta><\mathrm{i}\overline{\chi}_\beta \mathrm{i}\chi_\eta> \,. \end{aligned} \tag{6.9}$$

We now replace the the quark-gluon vertices with the expressions Eq. (6.8) and obtain for the 4-point quark-antiquark connected Green's functions, written in terms of 1PI Green's functions:

$$\begin{aligned} & <\mathrm{i}\overline{\chi}_\alpha \mathrm{i}\chi_\delta \mathrm{i}\overline{\chi}_\lambda \mathrm{i}\chi_\eta> = \\ & \quad - \left[<\mathrm{i}\overline{\chi}_\alpha \mathrm{i}\chi_\tau><\mathrm{i}\overline{q}_\tau \mathrm{i}q_\mu \mathrm{i}\phi_\nu><\mathrm{i}\overline{\chi}_\mu \mathrm{i}\chi_\delta>\right] \left[<\mathrm{i}\overline{\chi}_\lambda \mathrm{i}\chi_\varepsilon><\mathrm{i}\overline{q}_\varepsilon \mathrm{i}q_\beta \mathrm{i}\phi_\rho><\mathrm{i}\overline{\chi}_\beta \mathrm{i}\chi_\eta>\right] <\mathrm{i}J_\rho \mathrm{i}J_\nu> \\ & \quad + \left[<\mathrm{i}\overline{\chi}_\alpha \mathrm{i}\chi_\gamma><\mathrm{i}\overline{q}_\gamma \mathrm{i}q_\beta \mathrm{i}\phi_\kappa><\mathrm{i}\overline{\chi}_\beta \mathrm{i}\chi_\eta>\right] \left[<\mathrm{i}\overline{\chi}_\lambda \mathrm{i}\chi_\tau><\mathrm{i}\overline{q}_\tau \mathrm{i}q_\mu \mathrm{i}\phi_\nu><\mathrm{i}\overline{\chi}_\mu \mathrm{i}\chi_\delta>\right] <\mathrm{i}J_\kappa \mathrm{i}J_\nu> \\ & \quad + \left[<\mathrm{i}\overline{\chi}_\alpha \mathrm{i}\chi_\gamma><\mathrm{i}\overline{\chi}_\lambda \mathrm{i}\chi_\varepsilon><\mathrm{i}\overline{q}_\gamma \mathrm{i}q_\kappa \mathrm{i}\overline{q}_\varepsilon \mathrm{i}q_\beta><\mathrm{i}\overline{\chi}_\kappa \mathrm{i}\chi_\delta><\mathrm{i}\overline{\chi}_\beta \mathrm{i}\chi_\eta>\right] . \end{aligned} \tag{6.10}$$

The relation Eq. (6.10) can be further simplified by introducing the fully amputated Green's function, i.e. dividing by the quark propagators (cut the quark legs), as shown in Fig. 6.2. The resulting expression for the 1PI Green's function (as function of the amputated Green's function), is then replaced in Eq. (6.5).

6.2. Solution in the heavy mass limit

In the heavy mass limit and under our truncation, introduced at the beginning of the previous chapter, Eqs. (6.5, 6.10) simplify dramatically, allowing us to derive exact solutions for both the 1PI and the fully amputated 4-point quark Green's functions. In this section, the consecutive steps required to arrive at the simplified form of these equations will be discussed in detail. Furthermore, the physical implications of the corresponding solutions will be discussed.

6.2.1. One Particle Irreducible Green's Function

Let us start with the 1PI Green's function, Eq. (6.5), and the corresponding diagrammatic representation, Fig. 6.1, and apply our truncation scheme in the heavy quark limit, at leading order in the mass expansion.

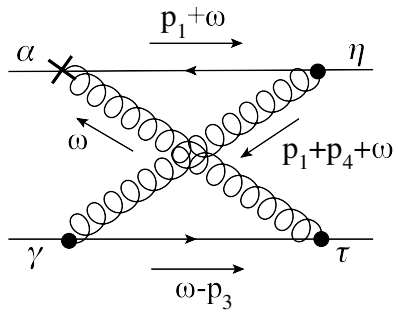

Figure 6.3.: Crossed ladder diagram that contributes to the 1PI 4-point Green's function. The upper line denotes the quark propagator, the lower one, the antiquark propagator and springs denote the temporal gluon propagator.

For the quark-antiquark system, we consider the flavor non-singlet Green's function in the s-channel, recalling that the quark and the antiquark are regarded as two distinct flavors. Hence, the diagrams (a), (c) and (i) are excluded. The diagram (b) (crossed ladder type exchange diagram) explicitly reads (see also Fig. 6.3):

$$\begin{aligned}\int d\omega \, \Big[\Gamma^{(0)a}_{\bar{q}q\sigma\alpha\delta}(p_1,-p_1-\omega,\omega)W_{\bar{q}q\delta\phi}(p_1+\omega)\Gamma^{b}_{\bar{q}q\sigma\phi\eta}(p_1+\omega,p_4,-p_1-p_4-\omega)\Big] \\ \times\Big[\Gamma^{c}_{q\bar{q}\sigma\tau\mu}(p_3,\omega-p_3,-\omega)W_{\bar{q}q\mu\lambda}(p_3-\omega)\Gamma^{d}_{q\bar{q}\sigma\lambda\gamma}(p_3-\omega,p_2,p_1+p_4+\omega)\Big] \\ \times W^{ac}_{\sigma\sigma}(-\vec{\omega})W^{bd}_{\sigma\sigma}(\vec{p}_1+\vec{p}_4+\vec{\omega}) \qquad (6.11)\end{aligned}$$

Isolating the energy integral over the quark propagators and using the fact that $W_{\bar{q}q}(k) = -W^{T}_{q\bar{q}}(-k)$ (before the truncation), we find that the energy integral vanishes, due to the fact that both quark and antiquark propagators have the same Feynman prescription, just as the crossed box contributions from the kernel of the Bethe–Salpeter equation discussed in Section 5.4 of Chapter 5:

$$\begin{aligned}\int d\omega_0 \, W_{\bar{q}q}(p_1+\omega)W_{q\bar{q}}(\omega-p_3) = \\ \frac{1}{-(p_1^0+p_3^0)+2m}\int d\omega_0 \left\{\frac{1}{\omega_0+p_1^0-m-\mathcal{I}_r+\mathrm{i}\varepsilon}-\frac{1}{\omega_0-p_3^0+m-\mathcal{I}_r+\mathrm{i}\varepsilon}\right\} = 0. \qquad (6.12)\end{aligned}$$

In the above, the first factor corresponds to the explicit quark propagator, and the second factor, to the explicit antiquark propagator (upper and lower line in the diagram Fig. 6.3, respectively).

Let us now analyze the diagram (d), containing a quark-2 gluon vertex. This vertex can be obtained as a solution of the corresponding Slavnov–Taylor identity. The derivation is identical to the Slavnov–Taylor identity for the quark-gluon vertex presented in Chapter 4. Starting with Eq. (4.7), we functionally differentiate with respect to the quark, antiquark and gluon fields. In principle, the resulting equation contains a large number of terms, however most of them simplify in the heavy mass limit and under truncation. To be specific, after taking the functional derivatives there are four categories of terms entering the equation. Firstly, the terms multiplied by a spatial quark-gluon vertex do

not contribute, since this vertex is suppressed at leading order in the mass expansion, according to Eq. (5.29) (see also the related discussion). Secondly, the terms containing a 4-point function $\Gamma_{\bar{q}qA\sigma}$ are also of $\mathcal{O}(1/m)$, due to the fact that in the corresponding Dyson–Schwinger equation at least one vertex must be at tree-level, and we are at liberty to chose this to be $\Gamma_{\bar{q}qA}$, which is truncated out by the mass expansion. Thirdly, the ghost kernels arising from the functional derivatives also vanish, since they only interact with the spatial Yang–Mills sector of the theory. Hence, under truncation and in the heavy mass limit, only the terms that involve a temporal quark-gluon vertex will survive. Explicitly, the equation Eq. (4.7), from which the Slavnov–Taylor identity for the quark-2 gluon vertex is derived, reduces to:

$$\begin{aligned} 0 \;=\; & \int d^4x \delta(t-x_0) \\ & \times \left\{ -\frac{\mathrm{i}}{g}\left(\partial_x^0 <\mathrm{i}\sigma_x^d>\right)\delta(z-x) + f^{abd} <\mathrm{i}\sigma_x^a> \mathrm{i}\sigma_x^b \delta(z-x) \right. \\ & \left. -\mathrm{i}T^d_{\alpha\beta} <\mathrm{i}q_{\alpha x}> \mathrm{i}q_{\beta x}\delta(z-x) - \mathrm{i}T^d_{\beta\alpha}\mathrm{i}\overline{q}_{\beta x} <\mathrm{i}\overline{q}_{\alpha x}> \delta(z-x)\right\}. \end{aligned} \tag{6.13}$$

We now functionally differentiate with respect to $\mathrm{i}q_{\varepsilon y}, \mathrm{i}\bar{q}_{\rho t}$ and $\mathrm{i}\sigma_w^e$ and arrive at the following expression:

$$\begin{aligned} 0 = & \int dx_0 \delta(t-x_0) \\ & \times \left\{ -\frac{1}{g}\partial_z^0 <\mathrm{i}\bar{q}_{\rho t}\mathrm{i}q_{\varepsilon y}\mathrm{i}\sigma_x^d\mathrm{i}\sigma_w^e> \delta(z-x) - \mathrm{i}f^{aed} <\mathrm{i}\bar{q}_{\rho t}\mathrm{i}q_{\varepsilon y}\mathrm{i}\sigma_w^a> \delta(z-x)\delta(x-w) \right. \\ & \left. + T^d_{\alpha\varepsilon} <\mathrm{i}\bar{q}_{\rho t}\mathrm{i}q_{\alpha y}\mathrm{i}\sigma_w^e> \delta(z-x)\delta(x-y) - T^d_{\rho\alpha} <\mathrm{i}\bar{q}_{\alpha z}\mathrm{i}q_{\varepsilon y}\mathrm{i}\sigma_w^e> \delta(z-x)\delta(x-t)\right\} \end{aligned} \tag{6.14}$$

Replacing the temporal quark-gluon vertex with the expression Eq. (5.23), and using the identity Eq. (A.4) for the generators, it is straightforward to see that the color structure cancels and hence $\Gamma_{\bar{q}q\sigma\sigma}$ is zero (and correspondingly, the diagram (d) vanishes).

Let us for the moment discard the diagrams (f) and (g), which include the 1PI 4-point quark Green's function, and the diagram (e), containing a 4 quark-gluon vertex. Then we are only left with the diagram (h) and the rainbow-ladder term (j). This simplification enables us to derive a solution for the corresponding (truncated) equation for the 1PI 4-point quark Green's function. With this result at hand, we will then return to the diagrams (f), (g) and (e), and explicitly show that they cancel (and hence our assumption is justified). With these observations, Eq. (6.5) reduces to:

$$\begin{aligned} <\mathrm{i}\overline{q}_\alpha \mathrm{i}q_\gamma \mathrm{i}\overline{q}_\tau \mathrm{i}q_\eta> = & -\left[gT^a\gamma^0\right]_{\alpha\beta}\left\{ <\mathrm{i}\overline{\chi}_\beta\mathrm{i}\chi_\nu><\mathrm{i}\overline{q}_\nu\mathrm{i}q_\mu\mathrm{i}\overline{q}_\tau\mathrm{i}q_\eta><\mathrm{i}\overline{\chi}_\mu\mathrm{i}\chi_\kappa><\mathrm{i}\overline{q}_\kappa\mathrm{i}q_\gamma\mathrm{i}\sigma_\lambda^b><\mathrm{i}\rho_\lambda^b\mathrm{i}\rho^a> \right. \\ & \left. + \left[<\mathrm{i}\overline{\chi}_\beta\mathrm{i}\chi_\delta><\mathrm{i}\overline{q}_\delta\mathrm{i}q_\eta\mathrm{i}\sigma_\varepsilon^c><\mathrm{i}\rho_\varepsilon^c\mathrm{i}\rho_\kappa^d>\right]\left[<\mathrm{i}\overline{q}_\tau\mathrm{i}q_\nu\mathrm{i}\sigma_\mu^b><\mathrm{i}\overline{\chi}_\nu\mathrm{i}\chi_\lambda><\mathrm{i}\overline{q}_\lambda\mathrm{i}q_\gamma\mathrm{i}\sigma_\kappa^d><\mathrm{i}\rho_\mu^b\mathrm{i}\rho^a>\right]\right\}. \end{aligned} \tag{6.15}$$

It is convenient to express the resulting equation in momentum space. We define the momentum space Green's functions via their respective Fourier transform (in order to

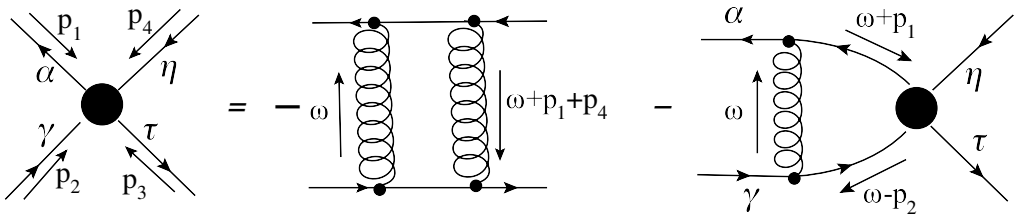

Figure 6.4.: Dyson-Schwinger equation for the 1PI 4-point Green's function in the s-channel. Same conventions as in Fig. 6.1 apply.

avoid proliferation of indices, we introduce the convention that x_1, k_1 correspond to the index α etc.)

$$<\mathrm{i}\overline{q}_\alpha \mathrm{i}q_\gamma \mathrm{i}\overline{q}_\tau \mathrm{i}q_\eta>= \int d\,k_1\; d\,k_2\; d\,k_3\; d\,k_4\, e^{-\mathrm{i}k_1x_1-\mathrm{i}k_2x_2-\mathrm{i}k_3x_3-\mathrm{i}k_4x_4}\Gamma^{(4)}_{\alpha\gamma\tau\eta}(k_1,k_2,k_3,k_4) \tag{6.16}$$

and arrive at the following Dyson–Schwinger equation for the 1PI 4-point quark Green's function in the s-channel (shown diagrammatically in Fig. 6.4):

$$\begin{aligned}
&\Gamma^{(4)}_{\alpha\gamma\tau\eta}(p_1,p_2,p_3,p_4) = \\
&\quad -\int d\,\omega\, \left[\Gamma^{(0)a}_{\bar{q}q\sigma\alpha\delta}(p_1,-p_1-\omega,\omega)W_{\bar{q}q\delta\phi}(p_1+\omega)\Gamma^{b}_{\bar{q}q\sigma\phi\eta}(p_1+\omega,p_4,-p_1-p_4-\omega)\right] \\
&\quad \times\left[\Gamma^{c}_{\bar{q}q\sigma\tau\mu}(p_3,p_2-\omega,p_1+p_4+\omega)W_{\bar{q}q\mu\lambda}(\omega-p_2)\Gamma^{d}_{\bar{q}q\sigma\lambda\gamma}(\omega-p_2,p_2,-\omega)\right] \\
&\quad \times W^{ad}_{\sigma\sigma}(-\vec{\omega})W^{bc}_{\sigma\sigma}(\vec{p}_1+\vec{p}_4+\vec{\omega}) \\
&\quad -\int d\,\omega\,\Gamma^{(0)a}_{\bar{q}q\sigma\alpha\delta}(p_1,-p_1-\omega,\omega)W_{\bar{q}q\delta\phi}(p_1+\omega)W_{\bar{q}q\mu\lambda}(\omega-p_2)\Gamma^{b}_{\bar{q}q\sigma\lambda\gamma}(\omega-p_2,p_2,-\omega) \\
&\quad \times W^{ab}_{\sigma\sigma}(-\vec{\omega})\Gamma^{(4)}_{\phi\mu\tau\eta}(p_1+\omega,p_2-\omega,p_3,p_4).
\end{aligned} \tag{6.17}$$

In order to proceed, we make the following assumption for the function $\Gamma^{(4)}$:

$$\Gamma^{(4)}(p_1,p_2,p_3,p_4) = \Gamma^{(4)}(P_0;\vec{p}_1+\vec{p}_4), \tag{6.18}$$

with $P_0 = p_1^0+p_2^0$. This implies that in the above equation the 4-point function $\Gamma^{(4)}_{\phi\mu\tau\eta}(p_1+\omega,p_2-\omega,p_3,p_4)$ does not depend on the integration variable ω_0, and hence we can separate the energy and three-momentum integrals and perform the energy integration over the quark propagators. Moreover, with this ansatz we are allowed to Fourier transform the resulting spatial integral back to coordinate space, as shall be explained shortly below.

After inserting the expressions Eq. (5.34), Eq. (5.43), for the quark and antiquark propagators and completing the energy integration, Eq. (6.17) simplifies to :

$$\begin{aligned}
&\left[p_1^0+p_2^0-2\mathcal{I}_r+2\mathrm{i}\varepsilon\right]\Gamma^{(4)}_{\alpha\gamma\tau\eta}(P_0;\vec{p}_1+\vec{p}_4) \\
&\quad = \mathrm{i}\frac{g^4}{4}\left[\left(N_c-\frac{2}{N_c}\right)\delta_{\alpha\gamma}\delta_{\tau\eta}+\frac{1}{N_c^2}\delta_{\alpha\eta}\delta_{\gamma\tau}\right]\int d\,\vec{\omega}\,W_{\sigma\sigma}(\vec{\omega})W_{\sigma\sigma}(\vec{k}+\vec{\omega}) \\
&\quad +\mathrm{i}\frac{g^2}{2}\left[\delta_{\alpha\gamma}\delta_{\mu\phi}-\frac{1}{N_c}\delta_{\alpha\phi}\delta_{\mu\gamma}\right]\int d\,\vec{\omega}\,W_{\sigma\sigma}(\vec{\omega})\Gamma^{(4)}_{\phi\mu\tau\eta}(P_0;\vec{p}_1+\vec{p}_4+\vec{\omega}).
\end{aligned} \tag{6.19}$$

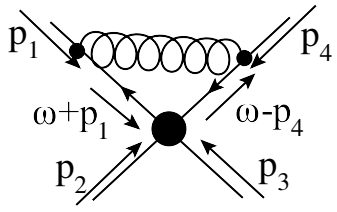

Figure 6.5.: Momentum routing for the diagram (g). See text for details.

Let us now make the following color decomposition for the function $\Gamma^{(4)}$:

$$\Gamma^{(4)}_{\alpha\gamma\tau\eta} = \delta_{\alpha\gamma}\delta_{\tau\eta}\Gamma_1 + \delta_{\alpha\eta}\delta_{\gamma\tau}\Gamma_2. \tag{6.20}$$

where (for a given flavor structure) $\Gamma^{(4)}_1$ and $\Gamma^{(4)}_2$ are Dirac scalar functions.

At this point, it is convenient to Fourier transform back to coordinate space. In general, since Eq. (6.17) might in principle contain momentum-dependent vertex functions, as well as mixing of energy and three-momentum variables, this transformation could not be carried out. However, in our case momentum-dependent vertices are absent and moreover, with the ansatz Eq. (6.18), the energy and tree-momentum integrals have separated such that the spatial integral is performed only over two functions (spatial gluon propagator and the spatial component of the quark 4-point function). Hence the Fourier transform simplifies to the usual convolution product:

$$\int d\vec{\omega}\, W_{\sigma\sigma}(\vec{\omega})\Gamma^{(4)}(\vec{q}+\vec{\omega}) = \int d\vec{x} e^{-\mathrm{i}\vec{q}\cdot\vec{x}} W_{\sigma\sigma}(\vec{x})\Gamma^{(4)}(-\vec{x}), \tag{6.21}$$

Using the Fierz identity for the generators, Eq. (A.12), and sorting out the color factors, it is straightforward to obtain for the components $\Gamma^{(4)}_1$, $\Gamma^{(4)}_2$:

$$\begin{aligned}
\Gamma^{(4)}_1(P_0;-\vec{x}) &= \mathrm{i}\left(\frac{g^2}{2N_c}\right)^2 \frac{W_{\sigma\sigma}(\vec{x})W_{\sigma\sigma}(-\vec{x})}{P_0 - 2\mathcal{I}_r + \mathrm{i}\frac{g^2}{2N_c}W_{\sigma\sigma}(\vec{x}) + 2\mathrm{i}\varepsilon} \\
&\quad \times \frac{N_c\left[(P_0-2\mathcal{I}_r)(N_c^2-2) + \mathrm{i}g^2 C_F W_{\sigma\sigma}(\vec{x})\right]}{P_0 - 2\mathcal{I}_r - \mathrm{i}g^2 C_F W_{\sigma\sigma}(\vec{x}) + 2\mathrm{i}\varepsilon} \\
\Gamma^{(4)}_2(P_0;-\vec{x}) &= \mathrm{i}\left(\frac{g^2}{2N_c}\right)^2 \frac{W_{\sigma\sigma}(\vec{x})W_{\sigma\sigma}(-\vec{x})}{P_0 - 2\mathcal{I}_r + \mathrm{i}\frac{g^2}{2N_c}W_{\sigma\sigma}(\vec{x}) + 2\mathrm{i}\varepsilon},
\end{aligned} \tag{6.22}$$

where $\vec{x}$ is the separation associated with the momentum $\vec{p}_1 + \vec{p}_4$. Inserting the above results into the decomposition Eq. (6.20), we find the final formula for the 1PI quark Green's function:

$$\begin{aligned}
\Gamma^{(4)}_{\alpha\gamma\tau\eta}(P_0;-\vec{x}) &= \mathrm{i}\left(\frac{g^2}{2N_c}\right)^2 \frac{W_{\sigma\sigma}(\vec{x})^2}{P_0 - 2\mathcal{I}_r + \mathrm{i}\frac{g^2}{2N_c}W_{\sigma\sigma}(\vec{x}) + 2\mathrm{i}\varepsilon} \\
&\quad \times \left\{\delta_{\alpha\gamma}\delta_{\tau\eta}\frac{(P_0-2\mathcal{I}_r)N_c(N_c^2-2) + \mathrm{i}g^2 N_c C_F W_{\sigma\sigma}(\vec{x})}{P_0 - 2\mathcal{I}_r - \mathrm{i}g^2 C_F W_{\sigma\sigma}(\vec{x}) + 2\mathrm{i}\varepsilon} + \delta_{\alpha\eta}\delta_{\gamma\tau}\right\}
\end{aligned} \tag{6.23}$$

where we recall that $P_0 = p_1^0 + p_2^0$.

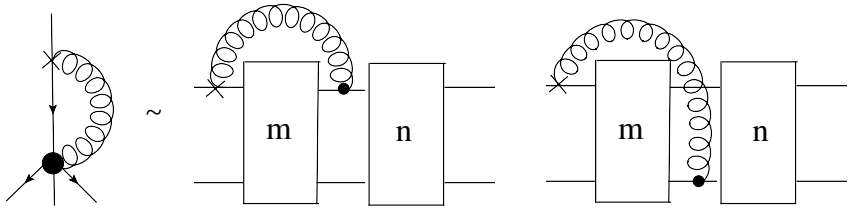

Figure 6.6.: Perturbative expansion of the diagram (e). Boxes comprise m and n gluon legs, respectively, with $m, n \geq 1$. See text for details.

Having derived the solution Eq. (6.23) for the 1PI Green's function, we return to the diagrams (f), (g) and (e) and show that they do not contribute to the final result. To see this, we use our usual trick and consider the energy integral. In the case of the diagram (g), this reads (with the momentum routing from Fig. 6.5):

$$\begin{aligned}
&\int d\omega_0\, W_{\bar{q}q}(p_1+\omega) W_{\bar{q}q}(\omega - p_4) \Gamma_E^{(4)}(p_1+\omega, p_2, p_3, \omega - p_4) \\
&\quad \sim \int d\omega_0\, \frac{1}{\left[p_1^0 + \omega_0 + m - \mathcal{I}_r + \mathrm{i}\varepsilon\right]\left[\omega^0 - p_4^0 + m - \mathcal{I}_r + \mathrm{i}\varepsilon\right]} \Gamma_E^{(4)}(\omega_0 + P_0) \\
&\quad = \int d\omega_0\, \frac{1}{\left[p_1^0 + \omega_0 + m - \mathcal{I}_r + \mathrm{i}\varepsilon\right]\left[\omega^0 - p_4^0 + m - \mathcal{I}_r + \mathrm{i}\varepsilon\right]} \\
&\quad \times \frac{P_0 + \omega_0 + \alpha}{\left[P_0 + \omega_0 + C_1 + 2\mathrm{i}\varepsilon\right]\left[P_0 + \omega_0 + C_2 + 2\mathrm{i}\varepsilon\right]}
\end{aligned} \tag{6.24}$$

where α, C_1 and C_2 are energy independent constants that appear in the expression Eq. (6.23) for the 1PI Green's function. The above formula can be trivially rewritten as

$$\begin{aligned}
&\int d\omega_0\, \frac{1}{\left[p_1^0 + \omega_0 + m - \mathcal{I}_r + \mathrm{i}\varepsilon\right]\left[\omega^0 - p_4^0 + m - \mathcal{I}_r + \mathrm{i}\varepsilon\right]} \\
&\quad \times \left\{ \frac{1}{P_0 + \omega_0 + C_1 + 2\mathrm{i}\varepsilon} + \frac{\alpha - C_1}{\left[P_0 + \omega_0 + C_1 + 2\mathrm{i}\varepsilon\right]\left[P_0 + \omega_0 + C_2 + 2\mathrm{i}\varepsilon\right]} \right\}.
\end{aligned} \tag{6.25}$$

Clearly, both terms in the above sum (having the same Feynman prescription) can be reduced to differences of integrals over a simple pole, and with the same sign for performing the integration in the complex plane and this vanishes (same as for the kernel of the Bethe–Salpeter equation and the diagram (b) from above). An identical calculation for the integral (f) leads us to the fact that this integral is also vanishing.

Finally, we are now in the position to show that the diagram (e), containing the 4 quark-gluon vertex, is also vanishing. The argumentation is based on our previous findings, namely that the diagrams (f) and (g), containing the 1PI quark Green's functions, are zero. We first observe that the diagram (e) can be written as a combination of diagrams of the form shown in Fig. 6.6, where the boxes contain an arbitrary number of gluon legs (ladder resummation).[1] On the other hand, as a result of the Dyson–Schwinger equation for the 1PI quark Green's functions, the diagrams (f) and (g) can also be written as a ladder resummation, which exactly coincide with the two terms in the diagram Fig. 6.6.

[1] To see this, it is enough to analyze the first few terms in perturbation theory of the 4 quark-gluon vertex, which are then included into the diagram (e).

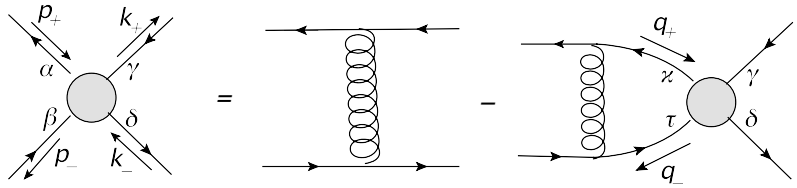

Figure 6.7.: Dyson-Schwinger equation for the fully amputated quark-antiquark 4-point Green's function in the s-channel.

Hence, the perturbative series of diagram (e) has been reorganized such that although the function $\Gamma_{\bar{q}q\bar{q}q\sigma}$ itself does not vanish, this 5-point interaction vertex and and the gluon line on top of it provide the cancellation at every order perturbatively. In turn, this implies that our original assumption is correct and the solution Eq. (6.23) is valid nonperturbatively.

6.2.2. Amputated Green's function

In the following, we return to the Dyson–Schwinger equation for the fully amputated 4-point quark-antiquark Green's function in the s-channel and, with the simplifications outlined in the previous section, we will derive a solution to this equation. We will verify that this is consistent with the 1PI Green's function obtained in the previous section and moreover, we will analyze the position of the poles and compare them with the Bethe–Salpeter equation for physical states.

The Dyson–Schwinger equation for the fully amputated 4-point quark-antiquark Green's function is obtained from the formula Eq. (6.15), by replacing the 1PI Green's function with the expression Eq. (6.10) and cutting the legs. Thus this equation (shown diagrammatically in Fig. 6.7) is equivalent to the Dyson–Schwinger equation Eq. (6.17), derived for 1PI Green's functions. It is given by:

$$\begin{aligned} G^{(4)}_{\alpha\beta;\delta\gamma}(p_+,p_-;k_+,k_-) &= W^{ab}_{\sigma\sigma}(\vec{p}-\vec{k})\left[\Gamma^a_{\bar{q}q\sigma}\right]_{\alpha\gamma}\left[\Gamma^b_{\bar{q}q\sigma}\right]_{\delta\beta} \\ &- \int d\,q\,\left[\Gamma^a_{\bar{q}q\sigma}W_{\bar{q}q}(q_+)\right]_{\alpha\kappa}\left[W^T_{q\bar{q}}(-q_-)\Gamma^{Tb}_{q\bar{q}\sigma}\right]_{\beta\tau}W^{ab}_{\sigma\sigma}(\vec{p}-\vec{q})G^{(4)}_{\kappa\tau;\delta\gamma}(q_+,q_-;k_+,k_-). \end{aligned} \tag{6.26}$$

In the above, the momenta of the quarks are given by $p_+ = p + \xi P$, $p_- = p - (1-\xi)P$ (similarly for k and q), and ξ is the momentum sharing fraction. P indicates the dependence on the total four momentum, which will become important for the investigation of the bound-state contributions to the Green's function. Eq. (6.17) is an inhomogeneous integral equation which – unlike the Bethe–Salpeter equation – contains both a resonant component (later on used to reproduce the bound state confining energy of the $\bar{q}q$ system), *and* a nonresonant term.

The right hand side of equation Eq. (6.17) does not depend on the external energy p_0, implying that the 4-point function $G^{(4)}$ has to be independent on the relative energy q_0, and we further assume that $G^{(4)}$ depends only on the relative momentum $\vec{p}-\vec{k}$. Accordingly, we replace the quark and antiquark propagators with the expressions Eq. (5.34), Eq. (5.43), and as before we perform the energy integration. We arrive at the following

expression:

$$\begin{aligned} G^{(4)}_{\alpha\beta;\delta\gamma}(P_0;\vec{p}-\vec{k}) &= g^2 T^a_{\alpha\gamma} T^b_{\delta\beta} W^{ab}_{\sigma\sigma}(\vec{p}-\vec{k}) \\ &\quad + g^2 T^a_{\alpha\kappa} T^a_{\tau\beta} \frac{\mathrm{i}}{P_0 - 2\mathcal{I}_r + 2\mathrm{i}\varepsilon} \int d\,\vec{q}\, W_{\sigma\sigma}(\vec{p}-\vec{q}) G^{(4)}_{\kappa\tau;\delta\gamma}(P_0;\vec{q}-\vec{k}). \end{aligned} \tag{6.27}$$

In the above, we have also replaced the vertex functions with their tree-level expressions Eq. (5.37), Eq. (5.44). Having integrated out the energy, it is convenient to rewrite the above formula back into coordinate space. Using the definition Eq. (6.16) and the relation Eq. (6.21), the equation Eq. (6.27) simplifies to:

$$G^{(4)}_{\alpha\beta;\delta\gamma}(P_0;\vec{x}) = g^2 T^a_{\alpha\gamma} T^a_{\delta\beta} W_{\sigma\sigma}(\vec{x}) + g^2 T^a_{\alpha\kappa} T^a_{\tau\beta} \frac{\mathrm{i}}{P_0 - 2\mathcal{I}_r + 2\mathrm{i}\varepsilon} W_{\sigma\sigma}(\vec{x}) G^{(4)}_{\kappa\tau;\delta\gamma}(P_0;\vec{x}). \tag{6.28}$$

Again, we decompose the function $G^{(4)}$:

$$G^{(4)}_{\alpha\beta;\delta\gamma} = \delta_{\alpha\beta}\delta_{\gamma\delta} G^{(4)}_1 + \delta_{\alpha\gamma}\delta_{\beta\delta} G^{(4)}_2, \tag{6.29}$$

(for a given flavor structure, $G^{(4)}_1$ and $G^{(4)}_2$ are Dirac scalar functions), use the Fierz identity, Eq. (A.12), to sort out the color factors, and obtain the following results for the components $G^{(4)}_1$, $G^{(4)}_2$:

$$\begin{aligned} G^{(4)}_1(P_0;\vec{x}) &= \left(\frac{g^2}{2}\right) \frac{(P_0 - 2\mathcal{I}_r) W_{\sigma\sigma}(\vec{x})}{P_0 - 2\mathcal{I}_r + \mathrm{i}\frac{g^2}{2N_c} W_{\sigma\sigma}(\vec{x}) + 2\mathrm{i}\varepsilon} \\ &\quad \times \frac{(P_0 - 2\mathcal{I}_r)}{P_0 - 2\mathcal{I}_r - \mathrm{i}\frac{g^2}{2}\left(N_c - \frac{1}{N_c}\right) W_{\sigma\sigma}(\vec{x}) + 2\mathrm{i}\varepsilon} \end{aligned} \tag{6.30}$$

$$G^{(4)}_2(P_0;\vec{x}) = -\left(\frac{g^2}{2N_c}\right) \frac{(P_0 - 2\mathcal{I}_r) W_{\sigma\sigma}(\vec{x})}{P_0 - 2\mathcal{I}_r + \mathrm{i}\frac{g^2}{2N_c} W_{\sigma\sigma}(\vec{x}) + 2\mathrm{i}\varepsilon} \tag{6.31}$$

Replacing these results in the formula Eq. (6.29), we get the final result for the function $G^{(4)}$:

$$\begin{aligned} G^{(4)}(P_0;\vec{x}) &= \frac{g^2}{2} \frac{(P_0 - 2\mathcal{I}_r)\, W_{\sigma\sigma}(\vec{x})}{P_0 - 2\mathcal{I}_r + \mathrm{i}\frac{g^2}{2N_c} W_{\sigma\sigma}(\vec{x}) + 2\mathrm{i}\varepsilon} \\ &\quad \times \left[\delta_{\alpha\beta}\delta_{\gamma\delta} \frac{(P_0 - 2\mathcal{I}_r)}{P_0 - 2\mathcal{I}_r - \mathrm{i}\frac{g^2}{2} W_{\sigma\sigma}(\vec{x}) C_F + 2\mathrm{i}\varepsilon} - \delta_{\alpha\gamma}\delta_{\beta\delta} \frac{1}{N_c} \right] \end{aligned} \tag{6.32}$$

A few comments regarding the structure of the above equation are here in order. Firstly, a direct calculation shows that our result for the amputated 4-point function is related to the result Eq. (6.23) for the 1PI Green's function, via the formula Eq. (6.10) (or alternatively, Fig. 6.2).

Also, notice that despite the truncation, the poles in the 1PI and amputated Green's functions are identical. Moreover, in this approach the physical and nonphysical poles disentangle automatically, as opposite to the "standard" QCD setting, where this separation does not occur. Using the form Eq. (5.27) for the temporal gluon propagator, we

notice that the bound state (infrared confining) energy $P_0 = \sigma|\vec{x}|$ emerges as the pole of the resonant component (first term in the bracket of Eq. (6.32)), for arbitrary number of colors. Hence, this provides an explicit analytical dependence of the 4-point Green's function on the $\bar{q}q$ bound state energy, which results from the Bethe–Salpeter equation presented in Chapter 5. The term multiplying the bracket has a pole for $N_c = 0$ and hence this cannot represent a physical state. This (common) pole in the nonresonant term can be shifted to infinity (as the regularization of I_r is removed) and, as discussed in the previous chapter, is nonphysical just like the poles in the quark propagator or in the baryon vertex. In the case of the 4-point Green's function, this can be simply absorbed in the normalization. Also, the appearance of this spurious pole does not contradict the physical results, since the bound state energy, stemming from the first pole of Eq. (6.32), is the only relevant quantity.

6.3. 4-point Green's functions for diquarks

Let us now consider the diquark 4-point Green's function. This can be easily obtained from the equation Eq. (6.5) for quark-antiquark systems, by interchanging the quark legs and inserting the appropriate minus signs. We obtain (see also the diagrammatic representation from Eq. (6.8)):

$$
\begin{aligned}
<\mathrm{i}\overline{q}_\alpha \mathrm{i}\overline{q}_\tau \mathrm{i}q_\gamma \mathrm{i}q_\eta> = [g\gamma^0 T^a]_{\alpha\beta} \int dy\, \delta(x-y) \\
\times \Big\{ \Big[<\mathrm{i}\overline{\chi}_\beta \mathrm{i}\chi_\kappa><\mathrm{i}\overline{q}_\kappa \mathrm{i}q_\gamma \mathrm{i}\sigma^c_\lambda><\mathrm{i}\rho^c_\lambda \mathrm{i}\rho^d_\nu><\overline{q}_\tau \mathrm{i}q_\eta \mathrm{i}\sigma^d_\nu \mathrm{i}\sigma^b_\mu><\mathrm{i}\rho^b_\mu \mathrm{i}\rho^a_y> \Big] \\
- \Big[<\mathrm{i}\overline{\chi}_\beta \mathrm{i}\chi_\kappa><\mathrm{i}\overline{q}_\kappa \mathrm{i}q_\gamma \mathrm{i}\sigma^c_\lambda><\mathrm{i}\rho^c_\lambda \mathrm{i}\rho^d_\nu> \Big] \Big[<\mathrm{i}\overline{q}_\tau \mathrm{i}q_\mu \mathrm{i}\sigma^d_\nu><\mathrm{i}\overline{\chi}_\mu \mathrm{i}\chi_\delta><\mathrm{i}\overline{q}_\delta \mathrm{i}q_\eta \mathrm{i}\sigma^b_\varepsilon><\mathrm{i}\rho^b_\varepsilon \mathrm{i}\rho^a_y> \Big] \\
- \Big[<\mathrm{i}\overline{\chi}_\beta \mathrm{i}\chi_\kappa><\mathrm{i}\overline{q}_\kappa \mathrm{i}q_\gamma \mathrm{i}\sigma^c_\lambda><\mathrm{i}\rho^c_\lambda \mathrm{i}\rho^d_\delta> \Big] \Big[<\mathrm{i}\overline{q}_\tau \mathrm{i}q_\nu \mathrm{i}\sigma^b_\mu><\mathrm{i}\overline{\chi}_\nu \mathrm{i}\chi_\varepsilon><\mathrm{i}\overline{q}_\varepsilon \mathrm{i}q_\eta \mathrm{i}\sigma^d_\delta><\mathrm{i}\rho^b_\mu \mathrm{i}\rho^a_y> \Big] \\
- \Big[<\mathrm{i}\overline{\chi}_\beta \mathrm{i}\chi_\kappa><\mathrm{i}\overline{q}_\kappa \mathrm{i}\overline{q}_\lambda \mathrm{i}q_\gamma \mathrm{i}q_\eta><\mathrm{i}\overline{q}_\tau \mathrm{i}q_\nu \mathrm{i}\sigma^b_\mu><\mathrm{i}\overline{\chi}_\nu \mathrm{i}\chi_\lambda><\mathrm{i}\rho^b_\mu \mathrm{i}\rho^a_y> \Big] \\
+ \Big[<\mathrm{i}\overline{\chi}_\beta \mathrm{i}\chi_\delta><\mathrm{i}\overline{q}_\delta \mathrm{i}q_\eta \mathrm{i}\sigma^c_\varepsilon><\mathrm{i}\rho^c_\varepsilon \mathrm{i}\rho^d_\kappa> \Big] \Big[<\mathrm{i}\overline{q}_\tau \mathrm{i}q_\nu \mathrm{i}\sigma^b_\mu><\mathrm{i}\overline{\chi}_\nu \mathrm{i}\chi_\lambda><\mathrm{i}\overline{q}_\lambda \mathrm{i}q_\gamma \mathrm{i}\sigma^d_\kappa><\mathrm{i}\rho^b_\mu \mathrm{i}\rho^a_y> \Big] \\
- \Big[<\mathrm{i}\overline{\chi}_\beta \mathrm{i}\chi_\kappa><\mathrm{i}\overline{q}_\kappa \mathrm{i}\overline{q}_\tau \mathrm{i}q_\gamma \mathrm{i}q_\lambda><\mathrm{i}\overline{\chi}_\lambda \mathrm{i}\chi_\delta><\mathrm{i}\overline{q}_\delta \mathrm{i}q_\eta \mathrm{i}\sigma^b_\varepsilon><\mathrm{i}\rho^b_\varepsilon \mathrm{i}\rho^a_y> \Big] \\
+ \Big[<\mathrm{i}\overline{\chi}_\beta \mathrm{i}\chi_\kappa><\mathrm{i}\overline{q}_\kappa \mathrm{i}\overline{q}_\tau \mathrm{i}q_\gamma \mathrm{i}q_\eta \sigma^b_\lambda><\mathrm{i}\rho^b_\lambda \mathrm{i}\rho^a_y> \Big] \\
- \Big[<\mathrm{i}\overline{\chi}_\beta \mathrm{i}\chi_\delta><\mathrm{i}\overline{q}_\delta \mathrm{i}q_\eta \mathrm{i}\sigma^c_\varepsilon><\mathrm{i}\rho^c_\varepsilon \mathrm{i}\rho^d_\kappa><\mathrm{i}\overline{q}_\tau \mathrm{i}q_\gamma \mathrm{i}\sigma^d_\kappa \mathrm{i}\sigma^b_\lambda><\mathrm{i}\rho^b_\lambda \mathrm{i}\rho^a_y> \Big] \\
- \Big[<\mathrm{i}\overline{\chi}_\beta \mathrm{i}\chi_\nu><\mathrm{i}\overline{q}_\nu \mathrm{i}\overline{q}_\tau \mathrm{i}q_\mu \mathrm{i}q_\eta><\mathrm{i}\overline{\chi}_\mu \mathrm{i}\chi_\kappa><\mathrm{i}\overline{q}_\kappa \mathrm{i}q_\gamma \mathrm{i}\sigma^b_\lambda><\mathrm{i}\rho^b_\lambda \mathrm{i}\rho^a_y> \Big] \\
+ \Big[<\mathrm{i}\overline{\chi}_\beta \mathrm{i}\chi_\delta><\mathrm{i}\overline{q}_\delta \mathrm{i}q_\eta \mathrm{i}\sigma^c_\varepsilon><\mathrm{i}\rho^c_\varepsilon \mathrm{i}\rho^d_\nu> \Big] \Big[<\mathrm{i}\overline{q}_\tau \mathrm{i}q_\mu \mathrm{i}\sigma^d_\nu><\mathrm{i}\overline{\chi}_\mu \mathrm{i}\chi_\kappa><\mathrm{i}\overline{q}_\kappa \mathrm{i}q_\gamma \mathrm{i}\sigma^d_\lambda><\mathrm{i}\rho^d_\lambda \mathrm{i}\rho^a_y> \Big] \Big\} \\
+ \ldots . \qquad (6.33)
\end{aligned}
$$

As in the $\bar{q}q$ case, the dots represent the $\vec{A}$ vertex terms which are not considered here and the tree-level temporal quark-gluon vertex has been replaced with its expression Eq. (3.10). Also, notice that the diquark is antisymmetric under the exchange of two quark legs. In this case we explicitly take into account the flavor structure, i.e. we consider equal (heavy) mass quarks but with different flavors.

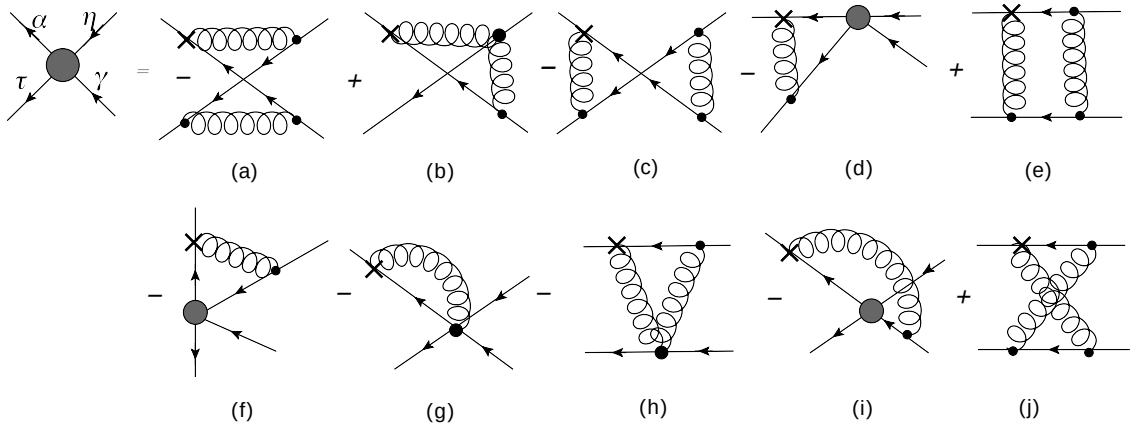

Figure 6.8.: Diagramatic representation of the one particle irreducible 4-point diquark. Green's function. Same conventions as in Fig. 6.1 apply.

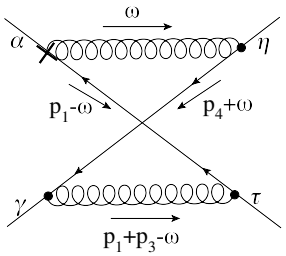

Figure 6.9.: Crossed ladder diagram that contributes to the 1PI 4-point Green's function in the diquark channel.

As before, we analyze the diagrammatic representation from Fig. 6.8 and show that the same type of cancellations occur. Starting with the diagram (a), we notice that this is a crossed ladder type exchange diagram (see also Fig. 6.9):

$$\begin{aligned}\int \bar{d}\,\omega\ &\Big[\Gamma^{(0)a}_{\bar{q}q\sigma\alpha\beta}(p_1,\omega-p_1,-\omega)W_{\bar{q}q\beta\kappa}(p_1-\omega)\Gamma^{d}_{\bar{q}q\sigma\kappa\gamma}(p_1-\omega,p_3,\omega-p_1-p_3)\Big]\\ &\times\Big[\Gamma^{c}_{q\bar{q}\sigma\tau\mu}(p_2,p_4+\omega,p_1+p_3-\omega)W_{\bar{q}q\mu\delta}(-p_4-\omega)\Gamma^{b}_{q\bar{q}\sigma\delta\eta}(-p_4-\omega,p_4,\omega)\Big]\\ &\times W^{ab}_{\sigma\sigma}(\vec{\omega})W^{dc}_{\sigma\sigma}(\vec{p}_1+\vec{p}_3-\vec{\omega}).\end{aligned}\tag{6.34}$$

It has been already shown that the integral over the quark propagators (with the same Feynman prescription) vanishes and thus the diagram (a) is zero. A similar type of integral arises in the diagram (j), and hence this term is also not giving a contribution. Further, the diagrams (b) and (h) are zero, since the corresponding quark-2 gluon vertex vanishes according to the Slavnov–Taylor identity, Eq. (6.14). As in the case of the $\bar{q}q$ system, let us for the moment assume that the integrals (f), (i), containing the diquark 4-point vertex, and (g), containing the 5-point functions $\Gamma_{\bar{q}\bar{q}qq\sigma}$ are also zero, and solve the Dyson–Schwinger equation with the remaining terms. Having derived the solution, we will then return to the diagrams (f),(i) and (g) and show that they vanish.

Putting all these together, the Dyson–Schwinger equation for the diquark 4-point func-

tion with the remaining diagrams, i.e. diagrams (c), (d) and (e) explicitly reads:

$$
\begin{aligned}
&\Gamma_{\bar{q}q\bar{q}q\sigma\alpha\tau\gamma\eta}(p_1,p_2,p_3,p_4) = \\
&\quad -\int d\omega \left[\Gamma^{(0)a}_{\bar{q}q\sigma\alpha\beta}(p_1,-\omega-p_1,\omega)W_{\bar{q}q\beta\kappa}(p_1+\omega)\Gamma^{c}_{\bar{q}q\sigma\kappa\gamma}(p_1+\omega,p_3,-\omega-p_1-p_3)\right] \\
&\quad \times\left[\Gamma^{b}_{q\bar{q}\sigma\tau\nu}(p_2,\omega-p_2,-\omega)W_{\bar{q}q\nu\varepsilon}(p_2-\omega)\Gamma^{d}_{q\bar{q}\sigma\varepsilon\eta}(p_2-\omega,p_4,p_1+p_3+\omega)\right] \\
&\quad \times W^{ab}_{\sigma\sigma}(-\vec{\omega})W^{dc}_{\sigma\sigma}(\vec{p}_1+\vec{p}_3+\vec{\omega}) \\
&\quad +\int d\omega \left[\Gamma^{(0)a}_{\bar{q}q\sigma\alpha\beta}(p_1,-\omega-p_1,\omega)W_{\bar{q}q\beta\delta}(p_1+\omega)\Gamma^{d}_{\bar{q}q\sigma\delta\eta}(p_1+\omega,p_4,-\omega-p_1-p_4)\right] \\
&\quad \times\left[\Gamma^{b}_{q\bar{q}\sigma\tau\nu}(p_2,\omega-p_2,-\omega)W_{\bar{q}q\nu\lambda}(p_2-\omega)\Gamma^{c}_{q\bar{q}\sigma\lambda\gamma}(p_2-\omega,p_3,p_1+p_4+\omega)\right] \\
&\quad \times W^{ab}_{\sigma\sigma}(-\vec{\omega})W^{dc}_{\sigma\sigma}(\vec{p}_1+\vec{p}_4+\vec{\omega}) \\
&\quad -\int d\omega \left[\Gamma^{(0)a}_{\bar{q}q\sigma\alpha\beta}(p_1,-\omega-p_1,\omega)W_{\bar{q}q\beta\kappa}(p_1+\omega)\right] \\
&\quad \times\left[\Gamma^{b}_{\bar{q}q\sigma\tau\nu}(p_2,\omega-p_2,-\omega)W_{\bar{q}q\nu\lambda}(p_2-\omega)\right]\Gamma_{\bar{q}q\bar{q}q\sigma\kappa\lambda\gamma\eta}(p_1+\omega,p_2-\omega,p_3,p_4)W^{ab}_{\sigma\sigma}(\vec{\omega})
\end{aligned}
\tag{6.35}
$$

After sorting out the color factors and applying the usual separation of the energy and three-momentum integrals, we arrive at the following formula:

$$
\begin{aligned}
&\Gamma^{(4)}_{\alpha\tau\gamma\eta}(p_1,p_2,p_3,p_4) = \frac{1}{P^0-2m-2\mathcal{I}_r+2\mathrm{i}\varepsilon} \\
&\quad \Bigg\{-\mathrm{i}\left(\frac{g^2}{2N_c}\right)^2\delta^f_{\alpha\gamma}\delta^f_{\tau\eta}\left[\left(N_c^2+1\right)\delta_{\alpha\gamma}\delta_{\tau\eta}-2N_c\delta_{\alpha\eta}\delta_{\tau\gamma}\right]\int d\vec{\omega}\, W_{\sigma\sigma}(\vec{\omega})W_{\sigma\sigma}(\vec{p}_1+\vec{p}_3+\vec{\omega}) \\
&\quad +\mathrm{i}\left(\frac{g^2}{2N_c}\right)^2\delta^f_{\alpha\eta}\delta^f_{\tau\gamma}\left[\left(N_c^2+1\right)\delta_{\alpha\eta}\delta_{\tau\gamma}-2N_c\delta_{\alpha\gamma}\delta_{\tau\eta}\right]\int d\vec{\omega}\, W_{\sigma\sigma}(\vec{\omega})W_{\sigma\sigma}(\vec{p}_1+\vec{p}_4+\vec{\omega}) \\
&\quad +\frac{g^2}{2N_c}\delta^f_{\alpha\lambda}\delta^f_{\tau\kappa}\left[N_c\delta_{\alpha\kappa}\delta_{\tau\lambda}-\delta_{\alpha\lambda}\delta_{\tau\kappa}\right] \\
&\quad \times\int d\omega_0\left[\frac{1}{\omega_0+p_1^0-m-\mathcal{I}_r+\mathrm{i}\varepsilon}-\frac{1}{\omega_0-p_2^0+m+\mathcal{I}_r-\mathrm{i}\varepsilon}\right] \\
&\quad \times\int d\vec{\omega}\, W_{\sigma\sigma}(\vec{\omega})\Gamma^{(4)}_{\lambda\kappa\gamma\eta}(p_1+\omega,p_2-\omega,p_3,p_4)\Bigg\}
\end{aligned}
\tag{6.36}
$$

Since the diquarks we are not restricted to flavor singlet, we make the following flavor decomposition:

$$
\Gamma^{(4)}_{\alpha\tau\gamma\eta} = \delta^f_{\alpha\gamma}\delta^f_{\tau\eta}\Gamma^{(c)}_{\alpha\tau\gamma\eta}+\delta^f_{\alpha\eta}\delta^f_{\gamma\tau}\Gamma^{(p)}_{\alpha\tau\gamma\eta} \tag{6.37}
$$

(the superscripts c and p stand for crossed and parallel configurations, respectively). As before, in order to carry on the energy integration, and in the view of Fourier transforming to coordinate space the resulting spatial integral, we make the following ansatz for the components $\Gamma^{(c)}$ and $\Gamma^{(p)}$:

$$
\Gamma^{(c)}(p_1+\omega,p_2-\omega,p_3,p_4) = \Gamma^{(c)}(P_0;\vec{p}_1+\vec{p}_3+\vec{\omega}) \tag{6.38}
$$

$$
\Gamma^{(p)}(p_1+\omega,p_2-\omega,p_3,p_4) = \Gamma^{(c)}(P_0;\vec{p}_1+\vec{p}_4+\vec{\omega}), \tag{6.39}
$$

with $P_0 = p_1^0+p_2^0$.

With these notations, Eq. (6.36) decouples to:

$$\Gamma^{(c)}_{\alpha\tau\gamma\eta}(P_0;\vec{p}_1+\vec{p}_3) = (-\mathrm{i})\frac{g^2}{2N_c}\frac{1}{P^0-2m-2\mathcal{I}_r+2\mathrm{i}\varepsilon}$$
$$\times\left\{\frac{g^2}{2N_c}\left[\left(N_c^2+1\right)\delta_{\alpha\gamma}\delta_{\tau\eta}-2N_c\delta_{\alpha\eta}\delta_{\gamma\tau}\right]\int d\vec{\omega}\, W_{\sigma\sigma}(\vec{\omega})W_{\sigma\sigma}(\vec{p}_1+\vec{p}_3+\vec{\omega})\right.$$
$$\left.+\left[N_c\delta_{\alpha\lambda}\delta_{\tau\kappa}-\delta_{\alpha\kappa}\delta_{\tau\lambda}\right]\int d\vec{\omega}\, W_{\sigma\sigma}(\vec{\omega})\Gamma^{(c)}_{\kappa\lambda\gamma\eta}(P_0;\vec{p}_1+\vec{p}_3+\vec{\omega})\right\} \tag{6.40}$$

$$\Gamma^{(p)}_{\alpha\tau\gamma\eta}(P_0;\vec{p}_1+\vec{p}_4) = \mathrm{i}\frac{g^2}{2N_c}\frac{1}{P^0-2m-2\mathcal{I}_r+2\mathrm{i}\varepsilon}$$
$$\times\left\{\frac{g^2}{2N_c}\left[\left(N_c^2+1\right)\delta_{\alpha\eta}\delta_{\tau\gamma}-2N_c\delta_{\alpha\gamma}\delta_{\tau\eta}\right]\int d\vec{\omega}\, W_{\sigma\sigma}(\vec{\omega})W_{\sigma\sigma}(\vec{p}_1+\vec{p}_4+\vec{\omega})\right.$$
$$\left.-\left[N_c\delta_{\alpha\lambda}\delta_{\tau\kappa}-\delta_{\alpha\kappa}\delta_{\tau\lambda}\right]\int d\vec{\omega}\, W_{\sigma\sigma}(\vec{\omega})\Gamma^{(p)}_{\kappa\lambda\gamma\eta}(P_0;\vec{p}_1+\vec{p}_4+\vec{\omega})\right\} \tag{6.41}$$

Further, we make the color decomposition:

$$\Gamma^{(c,p)}_{\alpha\tau\gamma\eta} = \delta_{\alpha\gamma}\delta_{\tau\eta}\Gamma^{(c,p1)}+\delta_{\alpha\eta}\delta_{\tau\gamma}\Gamma^{(c,p2)}, \tag{6.42}$$

where $\Gamma^{(c,p1)}$, $\Gamma^{(c,p2)}$ are Dirac scalar functions. Inserting this into Eqs. (6.40, 6.41), we obtain the following set of equations:

$$\Gamma^{(c1)}(P_0;\vec{p}_1+\vec{p}_3) = (-\mathrm{i})\frac{g^2}{2N_c}\frac{1}{P^0-2m-2\mathcal{I}_r+2\mathrm{i}\varepsilon}$$
$$\times\left\{\frac{g^2}{2N_c}\left(N_c^2+1\right)\int d\vec{\omega}\, W_{\sigma\sigma}(\vec{\omega})W_{\sigma\sigma}(\vec{p}_1+\vec{p}_3+\vec{\omega})\right.$$
$$\left.+\int d\vec{\omega}\, W_{\sigma\sigma}(\vec{\omega})\left[N_c\Gamma^{(c2)}(P_0;\vec{p}_1+\vec{p}_3+\vec{\omega})-\Gamma^{(c1)}(P_0;\vec{p}_1+\vec{p}_3+\vec{\omega})\right]\right\} \tag{6.43}$$

$$\Gamma^{(p1)}(P_0;\vec{p}_1+\vec{p}_4) = (-\mathrm{i})\frac{g^2}{2N_c}\frac{1}{P^0-2m-2\mathcal{I}_r+2\mathrm{i}\varepsilon}$$
$$\times\left\{g^2\int d\vec{\omega}\, W_{\sigma\sigma}(\vec{\omega})W_{\sigma\sigma}(\vec{p}_1+\vec{p}_4+\vec{\omega})\right.$$
$$\left.+\int d\vec{\omega}\, W_{\sigma\sigma}(\vec{\omega})\left[N_c\Gamma^{(p2)}(P_0;\vec{p}_1+\vec{p}_4+\vec{\omega})-\Gamma^{(p1)}(P_0;\vec{p}_1+\vec{p}_4+\vec{\omega})\right]\right\} \tag{6.44}$$

$$\Gamma^{(c2)}(P_0;\vec{p}_1+\vec{p}_3) = \mathrm{i}\frac{g^2}{2N_c}\frac{1}{P^0-2m-2\mathcal{I}_r+2\mathrm{i}\varepsilon}$$
$$\times\left\{g^2\int d\vec{\omega}\, W_{\sigma\sigma}(\vec{\omega})W_{\sigma\sigma}(\vec{p}_1+\vec{p}_3+\vec{\omega})\right.$$
$$\left.-\int d\vec{\omega}\, W_{\sigma\sigma}(\vec{\omega})\left[N_c\Gamma^{(c1)}(P_0;\vec{p}_1+\vec{p}_3+\vec{\omega})-\Gamma^{(c2)}(P_0;\vec{p}_1+\vec{p}_3+\vec{\omega})\right]\right\} \tag{6.45}$$

$$\Gamma^{(p2)}(P_0;\vec{p}_1+\vec{p}_4) = \mathrm{i}\frac{g^2}{2N_c}\frac{1}{P^0-2m-2\mathcal{I}_r+2\mathrm{i}\varepsilon}$$
$$\times\left\{\frac{g^2}{2N_c}\left(N_c^2+1\right)\int d\vec{\omega}\, W_{\sigma\sigma}(\vec{\omega})W_{\sigma\sigma}(\vec{p}_1+\vec{p}_4+\vec{\omega})\right.$$
$$\left.-\int d\vec{\omega}\, W_{\sigma\sigma}(\vec{\omega})\left[N_c\Gamma^{(p1)}(P_0;\vec{p}_1+\vec{p}_4+\vec{\omega})-\Gamma^{(p2)}(P_0;\vec{p}_1+\vec{p}_4+\vec{\omega})\right]\right\} \tag{6.46}$$

In the next step, we transform to coordinate space and rearrange the terms, such that we obtain:

$$\Gamma^{(c1)}(P_0;\vec{y})+\Gamma^{(c2)}(P_0;\vec{y})=(-\mathrm{i})\left(\frac{g^2}{2N_c}\right)^2\frac{(N_c-1)^2\,W_{\sigma\sigma}(\vec{y})^2}{P_0-2m-2\mathcal{I}_r-\mathrm{i}\frac{g^2}{2N_c}(1-N_c)W_{\sigma\sigma}(\vec{y})+2\mathrm{i}\varepsilon} \tag{6.47}$$

$$\Gamma^{(c1)}(P_0;\vec{y})-\Gamma^{(c2)}(P_0;\vec{y})=(-\mathrm{i})\left(\frac{g^2}{2N_c}\right)^2\frac{(N_c+1)^2\,W_{\sigma\sigma}(\vec{y})^2}{P_0-2m-2\mathcal{I}_r-\mathrm{i}\frac{g^2}{2N_c}(1+N_c)W_{\sigma\sigma}(\vec{y})+2\mathrm{i}\varepsilon} \tag{6.48}$$

$$\Gamma^{(p1)}(P_0;\vec{x})+\Gamma^{(p2)}(P_0;\vec{x})=(+\mathrm{i})\left(\frac{g^2}{2N_c}\right)^2\frac{(N_c-1)^2\,W_{\sigma\sigma}(\vec{x})^2}{P_0-2m-2\mathcal{I}_r-\mathrm{i}\frac{g^2}{2N_c}(1-N_c)W_{\sigma\sigma}(\vec{x})+2\mathrm{i}\varepsilon} \tag{6.49}$$

$$\Gamma^{(p1)}(P_0;\vec{x})-\Gamma^{(p2)}(P_0;\vec{x})=(-\mathrm{i})\left(\frac{g^2}{2N_c}\right)^2\frac{(N_c+1)^2\,W_{\sigma\sigma}(\vec{x})^2}{P_0-2m-2\mathcal{I}_r-\mathrm{i}\frac{g^2}{2N_c}(1+N_c)W_{\sigma\sigma}(\vec{x})+2\mathrm{i}\varepsilon} \tag{6.50}$$

In the above, $\vec{x}$ and $\vec{y}$ represent the separations corresponding to the momentum $\vec{p}_1+\vec{p}_4$ and $\vec{p}_1+\vec{p}_3$, respectively. The final expression for the diquark 4-point Green's function reads:

$$\begin{aligned}\Gamma_{\alpha\tau\gamma\eta}(P_0;\vec{x},\vec{y}) &= \delta^f_{\alpha\gamma}\delta^f_{\tau\eta}\left\{\frac{1}{2}\left(\delta_{\alpha\gamma}\delta_{\tau\eta}+\delta_{\alpha\eta}\delta_{\tau\gamma}\right)\left[\Gamma^{(c1)}(P_0;\vec{y})+\Gamma^{(c2)}(P_0;\vec{y})\right]\right.\\ &\left.+\frac{1}{2}\left(\delta_{\alpha\gamma}\delta_{\tau\eta}-\delta_{\alpha\eta}\delta_{\tau\gamma}\right)\left[\Gamma^{(c1)}(P_0;\vec{y})-\Gamma^{(c2)}(P_0;\vec{y})\right]\right\}\\ &+\delta^f_{\alpha\eta}\delta^f_{\tau\gamma}\left\{\frac{1}{2}\left(\delta_{\alpha\gamma}\delta_{\tau\eta}+\delta_{\alpha\eta}\delta_{\tau\gamma}\right)\left[\Gamma^{(p1)}(P_0;\vec{x})+\Gamma^{(p2)}(P_0;\vec{x})\right]\right.\\ &\left.+\frac{1}{2}\left(\delta_{\alpha\gamma}\delta_{\tau\eta}-\delta_{\alpha\eta}\delta_{\tau\gamma}\right)\left[\Gamma^{(p1)}(P_0;\vec{x})-\Gamma^{(p2)}(P_0;\vec{x})\right]\right\}\end{aligned} \tag{6.51}$$

with the components given by Eqs. (6.47 – 6.50).

As in the case of the $\bar{q}q$ systems, with this solution we return to the diagrams (f), (i) and (g). Writing out the explicit form of the energy integrals we notice that their form is identical to the quark-antiquark case, since the ε prescription is similar, regardless of the internal quark (or antiquark) propagator. Thus, these diagrams are also vanishing.

The above equation can be rewritten such that the pole structure becomes manifest. Introducing the notations

$$f_+(\vec{y}) = (-\mathrm{i})\left(\frac{g^2}{2N_c}\right)^2\frac{(N_c-1)^2\,W_{\sigma\sigma}(\vec{y})^2}{P_0-2m-2\mathcal{I}_r-\mathrm{i}\frac{g^2}{2N_c}(1-N_c)W_{\sigma\sigma}(\vec{y})+2\mathrm{i}\varepsilon} \tag{6.52}$$

$$f_-(\vec{y}) = (-\mathrm{i})\left(\frac{g^2}{2N_c}\right)^2\frac{(N_c+1)^2\,W_{\sigma\sigma}(\vec{y})^2}{P_0-2m-2\mathcal{I}_r-\mathrm{i}\frac{g^2}{2N_c}(1+N_c)W_{\sigma\sigma}(\vec{y})+2\mathrm{i}\varepsilon}, \tag{6.53}$$

we have that

$$\begin{aligned}\Gamma_{\alpha\tau\gamma\eta}(P_0;\vec{x},\vec{y}) &= \frac{1}{2}\left\{\left(\delta_{\alpha\gamma}\delta_{\tau\eta}+\delta_{\alpha\eta}\delta_{\tau\gamma}\right)\left[\delta^f_{\alpha\gamma}\delta^f_{\tau\eta}f_+(\vec{y})-\delta^f_{\alpha\eta}\delta^f_{\tau\gamma}f_+(\vec{x})\right]\right.\\ &\left.+\left(\,\delta_{\alpha\gamma}\delta_{\tau\eta}-\delta_{\alpha\eta}\delta_{\tau\gamma}\,\right)\left[\delta^f_{\alpha\gamma}\delta^f_{\tau\eta}f_-(\vec{y})+\delta^f_{\alpha\eta}\delta^f_{\tau\gamma}f_-(\vec{x})\right]\right\}\end{aligned} \tag{6.54}$$

Analyzing the pole structure of the above equation we notice that, as in the case of the $\bar{q}q$ systems, we have two different pole conditions:

$$P_0 - 2m - 2\mathcal{I}_r - \mathrm{i}\frac{g^2}{2N_c}(1 \pm N_c)W_{\sigma\sigma}(\vec{x}) = 0 \tag{6.55}$$

From the first equation (corresponding to the color antisymmetric term in the above formula), we find that $N_c = 2, -1$, and from the second equation, we obtain $N_c = -2, 1$. Hence, the only physical solution, with $N_c = 2$, corresponds to color antisymmetric and flavor symmetric configuration, in agreement with our findings from the previous chapter that bound states exist only for color antisymmetric $SU(2)$ baryons (in that case flavor symmetry was implicit). Notice also that the poles in the case of the diquark system disconnect, as opposite to the meson case, where they multiply.

Chapter 7.

Summary and conclusions

This thesis has been constructed from three major building blocks: the general derivation of the quark Dyson–Schwinger equations in Coulomb gauge first order formalism, the perturbative studies at one-loop order, and the nonperturbative investigations in the limit of the heavy quark mass. We will present our summary and conclusions separately for each part.

7.1. Functional derivation of Dyson–Schwinger equations

Starting with the QCD Lagrangian, we have introduced the gauge fixing technique and the Faddeev-Popov method in Coulomb gauge. Further, we have converted to first order formalism. There are two reasons motivating this choice: firstly, the (unphysical) ghost degrees of freedom can be formally eliminated and secondly, the system can be reduced to the "would-be physical" degrees of freedom. Formal here refers to the fact that the resulting equations are nonlocal and hence very difficult to approach in practical calculations.

Using functional derivation techniques, in Chapter 2 we have explicitly derived the quark field equations of motion. Based on these equations, we have obtained the Feynman rules and the relevant tree-level proper two-point and vertex functions. We have also discussed the general decomposition of the quark (proper and connected) two-point functions. Based on the Lagrange transformation, we have derived exact relations between the corresponding dressing functions. Given that at one-loop order in perturbation theory the fourth Dirac structure $\gamma^0\gamma^i$ of the quark gap equation vanishes, the set of equations relating these functions is significantly reduced. Starting with the quark field equation of motion, we have derived the Dyson–Schwinger equations for the quark proper two-point function, quark contributions to the gluon proper two-point functions and quark-gluon vertex functions. Moreover, in Chapter 6 we have presented the explicit derivation of the four-point Green's function for quark-antiquark and diquark systems.

Since the Dyson–Schwinger equations build an infinite tower of coupled integral equations, they cannot be solved exactly, and hence approximation schemes have to be employed. These have to preserve the symmetries of the theory, which are reflected in the Slavnov–Taylor identities. Thus, starting with the *time-dependent* BRST transform we have explicitly derived the Slavnov–Taylor identities for the quark-gluon vertices and, in the limit of heavy quark mass, for the four-quark-gluon vertex.

7.2. One-loop perturbative results

In the second part of this thesis, a one-loop perturbative analysis of the quark gap equation, quark contribution to the gluonic two-point functions and quark-gluon vertex functions has been undertaken. The various propagator and two-point dressing functions have been evaluated at this order. To this end, the (dimensionally regularized) results for the required noncovariant massive integrals have been obtained, using differential equations and integration by part techniques.

The results for the various two-point functions are rather illuminating. Since the singularities are absent in both Euclidean and spacelike Minkowski regions, the analytic continuation between Euclidian and Minkowski is justified. The second important physical results is the renormalization of the quark mass and propagator. Namely, we have verified that the one-loop renormalized quark mass agrees explicitly with the calculation performed in linear covariant gauges. Also, up to color factors the results for the quark propagator dressing function agree with the corresponding results obtained in Quantum Electrodynamics. Finally, the correct one-loop coefficient of the perturbative β-function has been obtained.

Turning to the quark-gluon vertex functions, we have explicitly evaluated their divergent parts and considered them in conjunction with the corresponding Slavnov–Taylor identity. This identity contains some peculiar objects, the quark-ghost scattering kernels (analogous to ghost-gluon kernels known from Yang–Mills theory). In order to gain some more information about them, it has been helpful to analyze the Slavnov–Taylor identity at one-loop perturbative order. This has been done in two different ways: firstly by using the translation invariance of the loop integrals, without explicitly evaluating them, and secondly by considering only the divergent parts of the integrals entering the identity. In this later investigation, we have employed the results for the divergent parts of the quark-gluon vertex functions previously derived and have found that the equation is satisfied. Also, the quark-ghost kernels appear not to contain divergences, in agreement with the calculations performed using the method of split dimensional regularization.

7.3. Heavy quarks

In the third part of the thesis, we have considered the limit of the heavy quark mass. After performing a heavy quark transform of the quark field (adapted for Coulomb gauge), we have expanded the generating functional of the theory in the mass parameter and retained only the leading order. In this case, the system has simplified dramatically: we have found that only the temporal gluon propagator contributes at leading order, whereas the spatial gluon is suppressed by the mass expansion. Further, we have truncated the Yang–Mills sector to include only the non-perturbative (energy-independent) temporal gluon propagator and neglect all the higher order Yang–Mills Green's functions.

In this setting, we have used the full nonperturbative quark equations of QCD in order to study the confinement properties. The gap equation, supplemented by the Slavnov–Taylor identity, has been solved and we have shown that the rainbow-ladder approximation to the quark (and antiquark) propagators is exact in this case. These equations have been used along with the Bethe–Salpeter equation for mesons and Faddeev equation for baryons, and we have demonstrated that the ladder approximation to the Bethe–Salpeter equation

is also exact.

Using the assumption that the temporal gluon propagator is energy independent, it was then straightforward to find solutions for these equations. In the meson case, from the Bethe–Salpeter equation we have found that only color singlet mesons and $SU(2)$ baryons have finite energy that increases linearly with the distance, i.e. confining solution (and otherwise the system is physically not allowed). Further, we have found that there exist a direct connection between the temporal gluon propagator and the string tension (at least within the approximation scheme considered here).

Turning to the baryon case, we have considered the Faddeev equation for three-quark states in a symmetric configuration. As in the case of the $\bar{q}q$ systems, we have found that the bound state energy between three quarks increases linearly with the separation and we have provided a direct connection between the temporal gluon propagator and the physical string tension. Furthermore, we predict that the string tension for three-quark states is $3/2$ times that of the $\bar{q}q$ system.

At this point, we make a short comment regarding our approximations. Firstly, we have assumed that the temporal gluon propagator is energy independent. As explained in the text, this assumption is supported by the lattice results and moreover, from formal arguments it can be inferred that this propagator must have at least a part that is energy independent, in order to cancel the ghost loops in the Yang–Mills expressions and to solve the energy divergence problem of Coulomb gauge. The second approximation was to neglect the Yang–Mills vertices and, given that the spatial quark-gluon vertices are suppressed by the mass, we have found that the temporal vertex remains nonperturbatively bare. Since the temporal gluon propagator is dressed, we have the situation of a gluon string connecting two naked color charges, and thus the inclusion the Yang–Mills vertices would represent the dressing of these naked charges, i.e. the screening mechanism. Thus, we anticipate that this would only lower the value of the string tension, and would not alter the linear behavior of the confining potential. The fact that the three-gluon vertex is irrelevant for the infrared properties of the theory has been also shown in the Hamiltonian approach [36].

Finally, in the heavy mass limit and within the same truncation scheme, we have also considered the 4-point quark Green's functions, and in particular we have studied the role of singularities of these functions. Firstly, we have found that both for $\bar{q}q$ and diquark systems the poles (physical and unphysical) naturally separate — as is well-known, this would not be the case in the usual QCD calculations. Also, we have found that the bound state energies emerge as a pole of the resonant component, thus providing and explicit connection between the (nonperturbative) 4-point Green's function and the bound state energy resulting from the Bethe–Salpeter equation.

Appendix A.

Conventions and useful formulae for the gauge group $SU(N_c)$

In this work, natural units are used:

$$\hbar = c = 1. \tag{A.1}$$

Initially we work in Minkowski space, with the following metric:

$$g_{\mu\nu} = \mathrm{diag}(1, -\vec{1}). \tag{A.2}$$

Where perturbative integrals are to be explicitly evaluated, we analytically continue to Euclidean space, i.e. $k_0 \to \mathrm{i}k_4$, as explained in the text.

The group elements of $SU(N_c)$ are unitary $N_c \times N_c$ matrices with determinant one and can be written in the form

$$U(x) = \exp\{-\mathrm{i}\theta^a(x)T^a\} \tag{A.3}$$

where $\theta^a(x)$ specify the angle of rotation in color space.

The $N_c^2 - 1$ (Hermitian) generators of $SU(N_c)$ obey the Lie algebra

$$[T^a, T^b] = \mathrm{i}f^{abc}T^c \tag{A.4}$$

where the numbers f^{abc} are the completely antisymmetric structure constants of the group. From the relation

$$\det U = \exp\left\{\mathrm{Tr}\ln U\right\} = \exp\{-\mathrm{i}\theta^a(x)\mathrm{Tr}\ T^a\} \tag{A.5}$$

it follows that the generators are traceless matrices:

$$\mathrm{Tr}\ T^a = 0. \tag{A.6}$$

In addition, we have the normalization condition

$$\mathrm{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab}. \tag{A.7}$$

The sum rules used in the text are:

$$T^aT^a = C_F\mathbb{1} \tag{A.8}$$

$$T^bT^cf^{dbc} = \frac{\mathrm{i}}{2}T^dN_c \tag{A.9}$$

$$T^aT^bT^a = \left(C_F - \frac{N_c}{2}\right)T^a \tag{A.10}$$

where the Casimir invariant is given by

$$C_F = \frac{N_c^2 - 1}{2N_c}. \tag{A.11}$$

We note also the Fierz identity:

$$2\left[T^a\right]_{\alpha\beta}\left[T^a\right]_{\delta\gamma} = \delta_{\alpha\gamma}\delta_{\delta\beta} - \frac{1}{N_c}\delta_{\alpha\beta}\delta_{\delta\gamma}. \tag{A.12}$$

Appendix B.

Standard massive integrals

In this Appendix, we present the derivation of some standard Coulomb gauge massive integrals used in Chapter 3. Consider the integral:

$$J_m(k^2) = \int \frac{d\,\omega}{[\omega^2+m^2]^\mu[(k-\omega)^2+m^2]^\nu}. \tag{B.1}$$

In the case $\mu = \nu = 1$ this gives the scalar integral associated with, for example, the fermion loop in quantum electrodynamics [3].

We present here a method to evaluate such integrals for arbitrary denominator powers (developed originally in Ref. [118]) and generalize to the various additional noncovariant integrals. We start by writing the Taylor expansion of the massive propagator in terms of a hypergeometric function in the following way:

$$\frac{1}{[\omega^2+m^2]^\mu} = \frac{1}{[\omega^2]^\mu}\,{}_1F_0\left(\mu; -\frac{m^2}{\omega^2}\right). \tag{B.2}$$

Now, the idea is to use the Mellin-Barnes representation of the hypergeometric function ${}_1F_0(\mu;z)$:

$${}_1F_0(\mu;z) = \frac{1}{\Gamma(\mu)}\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} ds(-z)^s\Gamma(-s)\Gamma(\mu+s), \tag{B.3}$$

where the contour in the complex plane separates the left poles of the Γ functions from the right poles. A first advantage of this representation is that the "mass term" gets separated from the massless propagator and the remaining integrals can be calculated with the Cauchy residue theorem, as we shall see below. Also, in order to study various momentum regimes, the results can be written as a function of either k^2/m^2, or m^2/k^2. This we do by using the standard formulas of analytic continuation of the hypergeometric function (for an extended discussion, see [118]).

Applying Eq. (B.3) to the massive propagator we can rewrite the integral $J_m(k^2)$ as:

$$\begin{aligned} J_m(k^2) &= \frac{1}{(2\pi\mathrm{i})^2}\frac{1}{\Gamma(\mu)\Gamma(\nu)}\iint_{-\mathrm{i}\infty}^{\mathrm{i}\infty} ds\,dt(m^2)^{s+t}\Gamma(-s)\Gamma(-t)\Gamma(\mu+s)\Gamma(\nu+t) \\ &\quad\times\int\frac{d\,\omega}{(\omega^2)^{\mu+s}[(k-\omega)^2]^{\nu+t}}. \end{aligned} \tag{B.4}$$

Inserting the general result for the massless integral, derived in Ref. [30]:

$$\int\frac{d\,\omega}{[\omega^2]^\mu[(k-\omega)^2]^\nu} = \frac{[k^2]^{2-\mu-\nu-\varepsilon}}{(4\pi)^{2-\varepsilon}}\frac{\Gamma(\mu+\nu+\varepsilon-2)}{\Gamma(\mu)\Gamma(\nu)}\frac{\Gamma(2-\mu-\varepsilon)\Gamma(2-\nu-\varepsilon)}{\Gamma(4-\mu-\nu-2\varepsilon)}, \tag{B.5}$$

we get for the integral J_m:

$$\begin{aligned} J_m(k^2) &= \frac{[k^2]^{2-\nu-\mu-\varepsilon}}{(4\pi)^{2-\varepsilon}} \frac{1}{(2\pi\mathrm{i})^2} \frac{1}{\Gamma(\mu)\Gamma(\nu)} \iint\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty} ds\, dt \left(\frac{m^2}{k^2}\right)^{s+t} \Gamma(-s)\Gamma(-t) \\ &\quad \times\Gamma(2-\varepsilon-\mu-s)\Gamma(2-\varepsilon-\nu-t) \frac{\Gamma(\mu+\nu+s+t-2+\varepsilon)}{\Gamma(4-2\varepsilon-\mu-\nu-s-t)}. \end{aligned} \tag{B.6}$$

With the change of variable $t = 2-\varepsilon-\mu-\nu-u-s$ (for such a replacement, the left and right poles of the Γ function are simply interchanged and therefore the condition of separating the poles is not contradicted) we obtain:

$$\begin{aligned} J_m(k^2) &= \frac{[m^2]^{2-\mu-\nu-\varepsilon}}{(4\pi)^{2-\varepsilon}} \frac{1}{(2\pi\mathrm{i})} \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty} du \left(\frac{m^2}{k^2}\right)^{-u} \frac{\Gamma(-u)}{\Gamma(2-\varepsilon+u)} \\ &\quad \times \frac{1}{(2\pi\mathrm{i})} \int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty} ds \Gamma(-s)\Gamma(2-\varepsilon-\mu-s)\Gamma(-2+\varepsilon+\nu+\mu+u+s)\Gamma(\mu+u+s). \end{aligned} \tag{B.7}$$

To evaluate the integral over s we use the Barnes Lemma:

$$\frac{1}{(2\pi\mathrm{i})} \int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty} ds\ \Gamma(a+s)\Gamma(b+s)\Gamma(c-s)\Gamma(d-s) = \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)} \tag{B.8}$$

and for the integral Eq. (B.7) it follows immediately that

$$\begin{aligned} J_m(k^2) &= \frac{[m^2]^{2-\mu-\nu-\varepsilon}}{(4\pi)^{2-\varepsilon}} \frac{1}{(2\pi\mathrm{i})} \frac{1}{\Gamma(\mu)\Gamma(\nu)} \\ &\quad \int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty} du \left(\frac{m^2}{k^2}\right)^{-u} \frac{\Gamma(-u)\Gamma(\mu+u)\Gamma(\nu+u)\Gamma(\mu+\nu-2+\varepsilon+u)}{\Gamma(\mu+\nu+2u)}. \end{aligned} \tag{B.9}$$

Closing the integration contour on the right we have:

$$\begin{aligned} J_m(k^2) &= \frac{[m^2]^{2-\mu-\nu-\varepsilon}}{(4\pi)^{2-\varepsilon}} \frac{1}{(2\pi\mathrm{i})} \frac{1}{\Gamma(\mu)\Gamma(\nu)} \\ &\quad (2\pi\mathrm{i}) \sum_{j=0}^{\infty} \left(-\frac{m^2}{k^2}\right)^{-j} \frac{1}{j!} \frac{\Gamma(\mu+j)\Gamma(\nu+j)\Gamma(\mu+\nu-2+\varepsilon+j)}{\Gamma(\mu+\nu+2j)}. \end{aligned} \tag{B.10}$$

With the help of the duplication formula

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right), \tag{B.11}$$

we can rewrite J_m as:

$$J_m(k^2) = \frac{[m^2]^{2-\mu-\nu-\varepsilon}}{(4\pi)^{2-\varepsilon}} \frac{\Gamma(\mu+\nu-2+\varepsilon)}{\Gamma(\mu+\nu)} \sum_{j=0}^{\infty} \left(-\frac{m^2}{k^2}\right)^{-j} \frac{1}{2^{2j}} \frac{1}{j!} \frac{\Gamma(\mu+j)}{\Gamma(\mu)} \frac{\Gamma(\nu+j)}{\Gamma(\nu)} \frac{\Gamma(\mu+\nu-2+\varepsilon+j)}{\Gamma(\mu+\nu-2+\varepsilon)} \frac{\Gamma(\frac{\mu+\nu}{2})}{\Gamma(\frac{\mu+\nu}{2}+j)} \frac{\Gamma(\frac{\mu+\nu+1}{2})}{\Gamma(\frac{\mu+\nu+1}{2}+j)}. \tag{B.12}$$

The sum is clearly a representation of the hypergeometric ${}_3F_2(a,b,c;d,e;z)$ (see, for example, Ref. [65]) and we finally obtain:

$$\begin{aligned} J_m(k^2) &= \frac{[m^2]^{2-\mu-\nu-\varepsilon}}{(4\pi)^{2-\varepsilon}} \frac{\Gamma(\mu+\nu-2+\varepsilon)}{\Gamma(\mu+\nu)} \\ &\quad \times {}_3F_2\left(\mu,\nu,\mu+\nu-2+\varepsilon;\frac{\mu+\nu}{2},\frac{\mu+\nu+1}{2};-\frac{k^2}{4m^2}\right). \end{aligned} \tag{B.13}$$

A trivial computation shows that the result Eq. (B.13) is consistent with the known results in the limit $m = 0$. All we have to do is to invert the argument of the hypergeometric according to the formula (found, for example, in Ref. [119])

$$\begin{aligned} {}_3F_2(a_1,a_2,a_3;b_1,b_2;z) &= \frac{\Gamma(b_1)\Gamma(b_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \\ &\times \left\{ \frac{\Gamma(a_1)\Gamma(a_2-a_1)\Gamma(a_3-a_1)}{\Gamma(b_1-a_1)\Gamma(b_2-a_1)} (-z)^{-a_1} \right. \\ &\times {}_3F_2\left(a_1,a_1-b_1+1,a_1-b_2+1;a_1-a_2+1,a_1-a_3+1;\frac{1}{z}\right) \\ &+ \frac{\Gamma(a_2)\Gamma(a_1-a_2)\Gamma(a_3-a_2)}{\Gamma(b_1-a_2)\Gamma(b_2-a_2)} (-z)^{-a_2} \\ &\times {}_3F_2\left(a_2,a_2-b_1+1,a_2-b_2+1;-a_1+a_2+1,a_2-a_3+1;\frac{1}{z}\right) \\ &+ \frac{\Gamma(a_3)\Gamma(a_1-a_3)\Gamma(a_2-a_3)}{\Gamma(b_1-a_3)\Gamma(b_2-a_3)} (-z)^{-a_3} \\ &\left. \times {}_3F_2\left(a_3,a_3-b_1+1,a_3-b_2+1;-a_1+a_3+1,-a_2+a_3+1;\frac{1}{z}\right) \right\}. \end{aligned} \tag{B.14}$$

The same method can be applied to the other noncovariant integrals with different denominator structures that appear in the text and the general expressions for these read:

$$\begin{aligned} \int \frac{d\omega}{[\omega^2]^\mu[(k-\omega)^2+m^2]^\nu} &= \frac{[m^2]^{2-\mu-\nu-\varepsilon}}{(4\pi)^{2-\varepsilon}} \frac{\Gamma(2-\mu-\varepsilon)\Gamma(\mu+\nu+\varepsilon-2)}{\Gamma(\nu)\Gamma(2-\varepsilon)} \\ &\quad \times {}_2F_1\left(\mu,\mu+\nu+\varepsilon-2;2-\varepsilon;-\frac{k^2}{m^2}\right), \end{aligned} \tag{B.15}$$

$$\begin{aligned} \int \frac{d\omega}{[\vec{\omega}^2]^\mu[(k-\omega)^2+m^2]^\nu} &= \frac{[m^2]^{2-\mu-\nu-\varepsilon}}{(4\pi)^{2-\varepsilon}} \frac{\Gamma(\frac{3}{2}-\mu-\varepsilon)\Gamma(\mu+\nu+\varepsilon-2)}{\Gamma(\nu)\Gamma(3/2-\varepsilon)} \\ &\quad \times {}_2F_1\left(\mu,\mu+\nu+\varepsilon-2;3/2-\varepsilon;-\frac{\vec{k}^2}{m^2}\right). \end{aligned} \tag{B.16}$$

Differentiation with respect to k_4 or k_i gives rise to expressions for integrals with more complicated numerator structure. For completeness, we list here the first order ε expansion of the integrals arising into the one-loop perturbative expressions considered in this work:

$$\int \frac{d\,\omega}{(\omega^2+m^2)[(k-\omega)^2+m^2]} = \frac{[m^2]^{-\varepsilon}}{(4\pi)^{2-\varepsilon}}\left\{\frac{1}{\varepsilon}-\gamma+2 - \sqrt{1+\frac{4m^2}{k^2}}\ln\left(\frac{\sqrt{1+\frac{4m^2}{k^2}}+1}{\sqrt{1+\frac{4m^2}{k^2}}-1}\right)+\mathcal{O}(\varepsilon)\right\}, \quad \text{(B.17)}$$

$$\int \frac{d\,\omega}{\omega^2[(k-\omega)^2+m^2]} = \frac{[m^2]^{-\varepsilon}}{(4\pi)^{2-\varepsilon}}\left\{\frac{1}{\varepsilon}-\gamma+2 - \left(1+\frac{m^2}{k^2}\right)\ln\left(1+\frac{k^2}{m^2}\right)+\mathcal{O}(\varepsilon)\right\}, \quad \text{(B.18)}$$

$$\int \frac{d\,\omega\,\omega_i}{\omega^2[(k-\omega)^2+m^2]} = k_i\frac{[m^2]^{-\varepsilon}}{(4\pi)^{2-\varepsilon}}\left\{\frac{1}{2\varepsilon}-\gamma 2+1 + \frac{1}{2}\frac{m^2}{k^2}-\frac{1}{2}\left(1+\frac{m^2}{k^2}\right)^2\ln\left(1+\frac{k^2}{m^2}\right)+\mathcal{O}(\varepsilon)\right\}, \quad \text{(B.19)}$$

$$\int \frac{d\,\omega}{\vec{\omega}^2[(k-\omega)^2+m^2]} = \frac{[m^2]^{-\varepsilon}}{(4\pi)^{2-\varepsilon}}\left\{\frac{2}{\varepsilon}-2\gamma+8 - 2\sqrt{1+\frac{m^2}{\vec{k}^2}}\ln\left(\frac{\sqrt{1+\frac{m^2}{\vec{k}^2}}+1}{\sqrt{1+\frac{m^2}{\vec{k}^2}}-1}\right)+\mathcal{O}(\varepsilon)\right\}. \quad \text{(B.20)}$$

Appendix C.

Checking the nonstandard integrals

One way to check analytically the results for the nonstandard integrals introduced in Chapter 3, A_m and A_m^4, Eq. (3.65) and Eq. (3.76), respectively, is to make an expansion around $k_4^2 = 0$ and evaluate the resulting integrals with the help of the Schwinger parametrization. Let us first consider the integral A_m, Eq. (3.30). Using Schwinger parameters [120] (see also [121]), we can rewrite the denominator factors as exponential functions to give:

$$\begin{aligned} A_m &= \int_0^\infty d\alpha\, d\beta\, d\gamma \int d\omega\, \exp\left\{-(\alpha+\beta)\omega_4^2 + 2\beta k_4\omega_4 - \beta k_4^2 - (\alpha+\beta+\gamma)\vec{\omega}^2 \right. \\ &\quad \left. +2\beta\vec{k}\cdot\vec{\omega} - \beta\vec{k}^2 - \beta m^2\right\}. \end{aligned} \tag{C.1}$$

Applying similar reasoning as in Ref. [30], we come to the following parametric form of the integral (recall that $x = k_4^2$, $y = \vec{k}^2$):

$$\begin{aligned} A_m &= \frac{(x+y+m^2)^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}}\Gamma(1+\varepsilon) \\ &\quad \times \int_0^1 d\beta \int_0^{1-\beta} \frac{d\alpha}{(\alpha+\beta)^{1/2}} \left[\frac{\alpha\beta}{(\alpha+\beta)}\frac{x}{(x+y+m^2)} + \frac{\beta(1-\beta)y + \beta m^2}{x+y+m^2}\right]^{-1-\varepsilon}. \end{aligned} \tag{C.2}$$

For general values of x, the integral above cannot be solved because of the highly nontrivial denominator factor. Since there can be no singularities at $x = 0$ (this would invalidate the Wick rotation which, as discussed in Chapter 3, does hold in this case), we can safely make an expansion around this point and then integrate. To first order in powers of x we have:

$$\begin{aligned} A_m &\overset{x\to 0}{=} \frac{(x+y+m^2)^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}}\Gamma(1+\varepsilon)\int_0^1 d\beta\int_0^{1-\beta} d\alpha\,(\alpha+\beta)^{-1/2}\left\{\left[\beta\frac{m^2+y(1-\beta)}{m^2+y}\right]^{-1-\varepsilon}\right. \\ &\quad \left. -\left[\beta\frac{m^2+y(1-\beta)}{m^2+y}\right]^{-2-\varepsilon}\left[\frac{\alpha\beta}{(m^2+y)(\alpha+\beta)} - \beta\frac{m^2+y(1-\beta)}{(m^2+y)^2}\right](1+\varepsilon)x\ +\mathcal{O}(x^2)\right\}. \end{aligned} \tag{C.3}$$

After performing the integration we get:

$$\begin{aligned} A_m(x,y) &\overset{x\to 0}{=} \frac{(x+y+m^2)^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}}(-2)\left\{\frac{1}{\varepsilon}\Gamma(1+\varepsilon)\,{}_2F_1\left(-\varepsilon, 2+\varepsilon; 1-\varepsilon; -\frac{y}{m^2+y}\right)\right. \\ &\quad \left. -\frac{x}{m^2+y} + \sqrt{1+\frac{m^2}{y}}\ln\left(\frac{\sqrt{1+\frac{m^2}{y}}+1}{\sqrt{1+\frac{m^2}{y}}-1}\right) + \mathcal{O}(x^2) + \mathcal{O}(\varepsilon)\right\}. \end{aligned} \tag{C.4}$$

In the above formula, we isolated the hypergeometric term in order to evaluate the ε expansion separately. For this purpose, we differentiate the hypergeometric function with respect to the parameters. In general, differentiation of ${}_2F_1(a,b;c;z)$ with respect to, e.g. the parameter b, gives (similar expressions are obtained for differentiation with respect to a,c):

$$ {}_2F_1^{(0,1,0,0)}\left(a,b;c;z\right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k \Psi(b+k)}{(c)_k} \frac{z^k}{k!} - \Psi(b) {}_2F_1\left(a,b;c;z\right), \tag{C.5} $$

where $\Psi(k)$ is the digamma function and $(a)_k$ is the so-called Pochhammer symbol (defined, for instance, in the standard textbook [119]):

$$ (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}. \tag{C.6} $$

With the help of formula Eq. (C.5), we obtain:

$$ {}_2F_1\left(-\varepsilon, 2+\varepsilon; 1-\varepsilon; z\right) = 1 + \varepsilon \ln(1-z) + \mathcal{O}(\varepsilon). \tag{C.7} $$

Inserting this back into Eq. (C.4), we can write down the result for the integral A_m (to first order in powers of x):

$$ \begin{aligned} A_m(x,y) \overset{x\to 0}{=} & \frac{(x+y+m^2)^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}} \left\{ -\frac{2}{\varepsilon} + 2\gamma + 2\left[-\ln\left(\frac{m^2}{m^2+y}\right) + \frac{x}{m^2+y}\right] \right. \\ & \left. -2\sqrt{1+\frac{m^2}{y}} \ln\left(\frac{\sqrt{1+\frac{m^2}{y}}+1}{\sqrt{1+\frac{m^2}{y}}-1} \right) + \mathcal{O}(x^2) + \mathcal{O}(\varepsilon) \right\}, \end{aligned} \tag{C.8} $$

which agrees explicitly with the corresponding expansion of the result given in Eq. (3.65).

We now turn to the integral A_m^4, given by Eq. (3.31). The parametric form has the expression:

$$ \begin{aligned} A_m^4 = & k_4 \frac{(x+y+m^2)^{-1-\varepsilon}\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}} \\ & \times \int_0^1 d\beta \int_0^{1-\beta} d\alpha \, \frac{\beta}{(\alpha+\beta)^{3/2}} \left[\frac{\alpha\beta}{(\alpha+\beta)} \frac{x}{(x+y+m^2)} + \frac{\beta(1-\beta)y + \beta m^2}{x+y+m^2} \right]^{-1-\varepsilon} \end{aligned} \tag{C.9} $$

Calculations similar to the integral A_m bring us to the following result (to first order in x):

$$ \begin{aligned} A_m^4(x,y) \overset{x\to 0}{=} & k_4 \frac{(x+y+m^2)^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}} \left\{ 2\sqrt{1+\frac{m^2}{y}} \ln\left(\frac{\sqrt{1+\frac{m^2}{y}}+1}{\sqrt{1+\frac{m^2}{y}}-1} \right) \right. \\ & -\frac{2}{3}\frac{x}{y}\left[1 + \left(\frac{m^2}{y} - 2\right) \ln\frac{m^2}{m^2+y} - \frac{2}{\sqrt{1+\frac{m^2}{y}}} \ln\left(\frac{\sqrt{1+\frac{m^2}{y}}+1}{\sqrt{1+\frac{m^2}{y}}-1} \right) \right] \\ & \left. +2\left(1+\frac{m^2}{y}\right) \ln\left(\frac{m^2}{m^2+y}\right) + \mathcal{O}(x^2) + \mathcal{O}(\varepsilon) \right\}, \end{aligned} \tag{C.10} $$

again in agreement with the expansion of the result given in Eq. (3.76).

Another useful check comes from the study of the mass differential equation, i.e., the mass is in this case regarded as a variable.[1]

With I^n given by Eq. (3.32), the derivative with respect to the mass reads:

$$m\frac{\partial I^n}{\partial m} = \int \frac{d\omega\ \omega_4^n}{\omega^2[(k-\omega)^2+m^2]\vec{\omega}^2}\left\{-2\frac{m^2}{(k-\omega)^2+m^2}\right\}. \tag{C.11}$$

From the relations Eq. (3.33), Eq. (3.34) and Eq. (C.11) we get the following relation:

$$k_4\frac{\partial I^n}{\partial k_4} + k_k\frac{\partial I^n}{\partial k_k} + m\frac{\partial I^n}{\partial m} = (d+n-5)I^n. \tag{C.12}$$

Using the same procedures as described in Chapter 3, we can then derive a differential equation for the integral in terms of the mass:

$$m^2\frac{\partial I^n}{\partial m^2} = (d+n-4)\frac{m^2}{k^2+m^2}I^n - \frac{m^2}{k^2+m^2}\int \frac{d\omega\ \omega_4^n}{[(k-\omega)^2+m^2]^2\vec{\omega}^2}. \tag{C.13}$$

Starting with the case $n=0$ where $I^0 \equiv A_m$, we see that by inserting the solution, Eq. (3.65), we have that in the limit $\varepsilon \to 0$

$$m^2\frac{\partial A_m}{\partial m^2} + (1+2\varepsilon)\frac{m^2}{k^2+m^2}A_m = -\frac{m^2(x+y+m^2)^{-2-\varepsilon}}{(4\pi)^{2-\varepsilon}}\frac{1}{y\sqrt{1+\frac{m^2}{y}}}\ln\left(\frac{\sqrt{1+\frac{m^2}{y}}+1}{\sqrt{1+\frac{m^2}{y}}-1}\right) + \mathcal{O}(\varepsilon). \tag{C.14}$$

In terms of Schwinger parameters, the explicit integral of Eq. (C.13) reads:

$$\frac{m^2}{k^2+m^2}\int \frac{d\omega}{[(k-\omega)^2+m^2]^2\vec{\omega}^2} = \frac{m^2}{x+y+m^2}\frac{\Gamma(1+\varepsilon)}{(4\pi)^{2-\varepsilon}}\int_0^1 d\alpha\,(1-\alpha)^{-1/2-\varepsilon}(m^2+\alpha y)^{-1-\varepsilon} \tag{C.15}$$

and for $m^2 \neq 0$ indeed

$$\frac{m^2}{k^2+m^2}\int \frac{d\omega}{[(k-\omega)^2+m^2]^2\vec{\omega}^2} = \frac{m^2(x+y+m^2)^{-2-\varepsilon}}{(4\pi)^{2-\varepsilon}}\frac{1}{y\sqrt{1+\frac{m^2}{y}}}\ln\left(\frac{\sqrt{1+\frac{m^2}{y}}+1}{\sqrt{1+\frac{m^2}{y}}-1}\right) + \mathcal{O}(\varepsilon), \tag{C.16}$$

showing that the mass differential equation is satisfied. In fact, in the differential equation Eq. (C.13), there is a potential ambiguity arising from the ordering of the limits $m^2 \to 0$ and $\varepsilon \to 0$. Namely, for $m^2 = 0$, the right-hand side of Eq. (C.14) vanishes as $m^2 \ln m^2$, whereas the parametric form of the integral in Eq. (C.15) goes like m^2/ε. However, this problem is not manifest because of multiplication with the overall factor m^2 in the differential equation. Since the solution of the mass differential equation is in principle formally derived as the integral over m^2 and $m^2 = 0$ is the only the limit of this integral,

[1] Recall that when deriving the differential equations presented in Chapter 3, the mass has been treated as a parameter, and only the differentiation with respect to k_4 and k_i has been considered.

the ambiguity encountered may be regarded as an integrable singularity and presents no problem.

Turning now to the case $n = 1$ where $I^1 \equiv A^4_m$, we first extract the overall k_4 factor as before by defining $A^4 = k_4 \overline{A}_m$ such that the differential equation is

$$m^2 \frac{\partial \overline{A}_m}{\partial m^2} = -2\varepsilon \frac{m^2}{k^2 + m^2} \overline{A}_m - \frac{m^2}{k^2 + m^2} \int \frac{d\,\omega}{[(k-\omega)^2 + m^2]^2 \vec{\omega}^2}. \tag{C.17}$$

Notice that in the integral term we have used the identities

$$\int \frac{d\,\omega\ \omega_4}{[(k-\omega)^2 + m^2]^2 \vec{\omega}^2} = \int \frac{d\,\omega\ (k_4 - \omega_4)}{[\omega^2 + m^2]^2 \left(\vec{k} - \vec{\omega}\right)^2} = k_4 \int \frac{d\,\omega}{[(k-\omega)^2 + m^2]^2 \vec{\omega}^2}. \tag{C.18}$$

Now, for $m^2 \neq 0$, the integral term of Eq. (C.17) is finite as $\varepsilon \to 0$; however, the $m^2 = 0$ limit is again ambiguous but as above this can be regarded as an integrable singularity. Also, when $m^2 = 0$, $\overline{A}$ is known to be ε finite (it is the massless integral considered in Ref. [30]). This means that as $\varepsilon \to 0$ we have the simple integral expression

$$m^2 \frac{\partial \overline{A}_m}{\partial m^2} = -\frac{1}{x + y + m^2} \frac{1}{(4\pi)^{2-\varepsilon}} \frac{1}{y\sqrt{1 + \frac{m^2}{y}}} \ln \left(\frac{\sqrt{1 + \frac{m^2}{y}} + 1}{\sqrt{1 + \frac{m^2}{y}} - 1} \right). \tag{C.19}$$

Knowing the solution, Eq. (3.76), it is straightforward to show that when $m^2 = 0$ the original massless integral from Ref. [30] is reproduced and that the derivative of the massive solution satisfies the above.

Appendix D.

One-loop non-covariant vertex integrals

In order to extract the divergence of the non-covariant three-point integrals appearing in the quark-gluon vertex functions presented in Chapter 3, we use a method based on the Schwinger parametrization [120]. As an example, consider the divergent integral:

$$I = \int d\omega \, \frac{\omega_i \omega_j}{\omega^2 (\vec{\omega} + \vec{k}_3)^2 (\omega - k_1)^2}. \tag{D.1}$$

In principle, one can also consider massive quark propagators. i.e. integrals of the form $1/\omega^2[(\vec{\omega} + \vec{k}_3)^2][(\omega - k_1)^2 - m^2]$. However, as will shortly become clear, the mass factor does not contribute to the divergent part and hence in the following we will set $m = 0$. Using the Schwinger parametrization, the formula Eq. (D.1) can be rewritten as:

$$\begin{aligned} I = \int_0^\infty d\alpha \, d\beta \, d\gamma \int d\omega \, \left[\frac{1}{2} \frac{\delta_{ij}}{\alpha + \beta + \gamma} + \frac{(\beta k_{3i} - \gamma k_{1i})(\beta k_{3j} - \gamma k_{1j})}{(\alpha + \beta + \gamma)^2} \right] \\ \times \exp \left\{ -(\alpha + \gamma)\omega_4^2 - (\alpha + \beta + \gamma)\vec{\omega}^2 + 2\gamma \, \omega_4 k_4 - 2\vec{\omega} \cdot (\beta \vec{k}_3 - \gamma \vec{k}_1) - \beta \vec{k_3}^2 - \gamma k_1^2 \right\}. \end{aligned} \tag{D.2}$$

We start by considering the first term (the factor $\delta_{ij}/2$ has been left aside):

$$\begin{aligned} I_0 \;=\; & \int_0^\infty d\alpha \, d\beta \, d\gamma \int d\omega \, \frac{1}{\alpha + \beta + \gamma} \exp \left\{ -(\alpha + \gamma)\omega_4^2 - (\alpha + \beta + \gamma)\vec{\omega}^2 + 2\gamma \omega_4 k_4 \right. \\ & \left. -2\vec{\omega} \cdot (\beta \vec{k}_3 - \gamma \vec{k}_1) - \beta \vec{k_3}^2 - \gamma k_1^2 \right\} \end{aligned} \tag{D.3}$$

(in fact, it turn out that the second term in Eq. (D.2) is convergent and therefore not interesting for our purpose). After making a shift

$$\vec{\omega} \to \vec{\omega} - \frac{\beta \vec{k}_3 - \gamma \vec{k}_1}{\alpha + \beta + \gamma}, \;\; \omega_4 \to \omega_4 + \frac{\gamma k_{14}}{\alpha + \gamma} \tag{D.4}$$

and performing the momentum integration, we are left with the parametric integral

$$\begin{aligned} I_0 \;=\; & \frac{1}{(4\pi)^{2-\varepsilon}} \int_0^\infty d\alpha \, d\beta \, d\gamma \frac{1}{(\alpha + \beta + \gamma)^{5/2-\varepsilon}} \frac{1}{(\alpha + \gamma)^{1/2}} \\ & \times \exp \left\{ \frac{(\beta \vec{k}_3 - \gamma \vec{k}_1)^2}{\alpha + \beta + \gamma} + \frac{\gamma^2 k_{14}^2}{\alpha + \gamma} - \beta \vec{k}_3^2 - \gamma k_1^2 \right\}. \end{aligned} \tag{D.5}$$

Following the usual procedure (also used in the Appendix C to check the nonstandard two-point integrals), we insert the identity $1 = \int_0^\infty d\lambda \, \delta(\lambda - \alpha - \beta - \gamma)$, make the rescaling

$\alpha \to \alpha\lambda, \beta \to \beta\lambda, \gamma \to \gamma\lambda$ and perform the integral over λ. After rearranging the terms, we get:

$$I_0 = \frac{\Gamma(\varepsilon)}{(4\pi)^{2-\varepsilon}} \int_0^1 d\alpha\, d\beta\, d\gamma \delta(1-\alpha-\beta-\gamma) \frac{1}{(\alpha+\gamma)^{1/2}} \times \left\{ \beta(1-\beta)\vec{k}_3^2 + \gamma(1-\gamma)\vec{k}_1^2 + 2\beta\gamma \vec{k}_1 \cdot \vec{k}_3 + \frac{\alpha\gamma k_{14}^2}{\alpha+\gamma} \right\}^{-\varepsilon}. \quad \text{(D.6)}$$

In the above equation, we see that the integrand is convergent in the limit $\varepsilon \to 0$, and the divergence of I_0 is only given by the overall factor $\Gamma(\varepsilon)$ and thus in the exponent we can set $\varepsilon = 0$. It also becomes clear that the mass term (which would enter the argument of the exponential) cannot not influence this result, and hence our initial setting with $m = 0$ is justified. The result is:

$$I_0 = \frac{1}{(4\pi)^2} \frac{2}{3} \frac{1}{\varepsilon} + \text{finite}. \quad \text{(D.7)}$$

By a similar calculation, one can show that the second term in Eq. (D.2) is convergent and therefore the overall divergent structure of the original integral is

$$I = \frac{1}{(4\pi)^2} \frac{\delta_{ij}}{3\varepsilon} + \text{finite}. \quad \text{(D.8)}$$

A straightforward example is the covariant integral

$$I_1 = \int \frac{d\omega\ \omega_i \omega_l}{\omega^2 (k_1-\omega)^2 (k_2+\omega)^2}. \quad \text{(D.9)}$$

This can be calculated without explicitly writing out the parametric form, by using the following simple relation:

$$\int \frac{d\omega\ \omega_i (\omega+k_2)^2}{\omega^2 (k_1-\omega)^2 (k_2+\omega)^2} = k_2^2 \int \frac{d\omega\ \omega_i}{\omega^2 (k_1-\omega)^2 (k_2+\omega)^2} + 2k_\mu \int \frac{d\omega\ \omega_i \omega_\mu}{\omega^2 (k_1-\omega)^2 (k_2+\omega)^2} + \int \frac{d\omega\ \omega_i}{(k_1-\omega)^2 (k_2+\omega)^2}, \quad \text{(D.10)}$$

where the first term and the temporal part of the second term on the right hand side are convergent. Using the fact that the result has the form $\delta_{il} I^*$ (where I^* includes the factor $1/d$), it is simple to get the ε coefficient by combining the terms on the right hand side of Eq. (D.10). We obtain:

$$I_1 = \frac{\delta_{il}}{(4\pi)^2} \frac{1}{4\varepsilon} + \text{finite}. \quad \text{(D.11)}$$

Finally, let us examine the integral

$$I_2 = \int \frac{d\omega\ \omega_i \omega_l \omega_j \omega_k}{\omega^2 (k_1-\omega)^2 (k_2+\omega)^2 \vec{\omega}^2} = I^* (\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}). \quad \text{(D.12)}$$

As before, we rearrange the terms and obtain

$$2k_{2k} I^* (\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}) = \int \frac{d\omega\ \omega_i \omega_j \omega_l}{\omega^2 (k_1-\omega)^2 \vec{\omega}^2} - \int \frac{d\omega\ \omega_i \omega_j \omega_l}{\omega^2 (k_3+\omega)^2 (\vec{k}_2-\vec{\omega})^2} + \int d\omega\, \frac{\omega_i \omega_l k_{2j} + \omega_i \omega_j k_{2l} + \omega_l \omega_j k_{2l}}{\omega^2 (k_3+\omega)^2 (\vec{k}_2-\vec{\omega})^2} + \text{finite} \quad \text{(D.13)}$$

The last integral on the right hand side has been already evaluated and hence we only need to consider the integral

$$J = \int đ\omega \, \frac{\omega_i \omega_j \omega_l}{\omega^2 (\vec{k}_3 + \vec{\omega})^2 (k_1 - \omega)^2}. \tag{D.14}$$

This we evaluate by differentiating the parametric form of the integral Eq. (D.2) (i.e., the integral with the factor $\omega_i \omega_j$ in the numerator) with respect to k_{1l}. The result is

$$\begin{aligned} J &= \int đ\omega \, \{[\gamma(\delta_{ij} k_{1l} + \delta_{il} k_{1j} + \delta_{jl} k_{1i}) - \beta(\delta_{ij} k_{3l} + \delta_{il} k_{3j} + \delta_{jl} k_{3i})] \\ &\quad + \frac{(\gamma k_{1j} - \beta k_{3j})(\gamma k_{1i} - \beta k_{3i})(\gamma k_{1l} - \beta k_{3l})}{(\alpha + \beta + \gamma)^3} \Big\} \exp\{\dots\} \end{aligned} \tag{D.15}$$

(the argument of the exponential function is identical to the one of Eq. (D.2)). After making the usual manipulations, we extract the ε coefficient from the parametric integral and obtain:

$$I_2 = \frac{1}{(4\pi)^2} \frac{1}{20\varepsilon} (\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}) + \text{finite}. \tag{D.16}$$

We list here all the *divergent* (covariant and non-covariant) one-loop vertex functions used in the evaluation the Coulomb gauge quark-gluon vertices:

$$\int \frac{đ\omega \; \omega_i \omega_l}{\omega^2 (k_1 - \omega)^2 (k_2 + \omega)^2} = \frac{\delta_{il}}{(4\pi)^2} \frac{1}{4\varepsilon} + \text{finite} \tag{D.17}$$

$$\int \frac{đ\omega \; \omega_4^2}{\omega^2 (k_1 - \omega)^2 (k_2 + \omega)^2} = \frac{1}{(4\pi)^2} \frac{1}{4\varepsilon} + \text{finite} \tag{D.18}$$

$$\int \frac{đ\omega \; \omega_i \omega_l}{(k_1 - \omega)^2 (k_2 + \omega)^2 \vec{\omega}^2} = \frac{\delta_{il}}{(4\pi)^2} \frac{1}{3\varepsilon} + \text{finite} \tag{D.19}$$

$$\int \frac{đ\omega \; \omega_4^2}{(k_1 - \omega)^2 (k_2 + \omega)^2 \vec{\omega}^2} = \frac{1}{(4\pi)^2} \frac{1}{\varepsilon} + \text{finite} \tag{D.20}$$

$$\int \frac{đ\omega \; \omega_i \omega_l \omega_j \omega_k}{\omega^2 (k_1 - \omega)^2 (k_2 + \omega)^2 \vec{\omega}^2} = \frac{1}{(4\pi)^2} \frac{1}{20\varepsilon} (\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}) + \text{finite} \tag{D.21}$$

$$\int \frac{đ\omega \; \omega_4^2 \omega_j \omega_k}{\omega^2 (k_1 - \omega)^2 (k_2 + \omega)^2 \vec{\omega}^2} = \frac{\delta_{jk}}{(4\pi)^2} \frac{1}{12\varepsilon} + \text{finite}. \tag{D.22}$$

Appendix E.

Temporal component of the quark-baryon vertex

In this Appendix we present the explicit derivation of the energy-dependent part of the Faddeev vertex, Eq. (5.84). We start with Eq. (5.83) and consider the first of the permutations of the energy integral:

$$I = -\int d\,k_0\,\frac{1}{\left[p_1^0 + k_0 - m - \mathcal{I}_r + \mathrm{i}\varepsilon\right]\left[p_2^0 - k_0 - m - \mathcal{I}_r + \mathrm{i}\varepsilon\right]}\Gamma_t(p_1^0 + k_0, p_2^0 - k_0, p_3^0). \tag{E.1}$$

Using

$$\frac{1}{[z + a + \mathrm{i}\varepsilon]\,[z + b + \mathrm{i}\varepsilon]} = \frac{1}{(b-a)}\left\{\frac{1}{z + a + \mathrm{i}\varepsilon} - \frac{1}{z + b + \mathrm{i}\varepsilon}\right\} \tag{E.2}$$

and shifting the integration variables, we find that the integral I, Eq. (E.1), depends only on the momentum p_3^0 (and implicitly on the bound state energy of the system P_0). Explicitly, it reads (using the symmetry of Γ_t):

$$I = -\frac{2}{\left[P_0 - p_3^0 - 2(m + \mathcal{I}_r)\right]}\int d\,k_0\,\frac{\Gamma_t(P_0 - p_3^0 + k_0, -k_0, p_3^0)}{\left[k_0 + P_0 - p_3^0 - m - \mathcal{I}_r + \mathrm{i}\varepsilon\right]} \tag{E.3}$$

Replacing this in the equation Eq. (5.83), we find:

$$\begin{aligned}\Gamma_t(p_1^0, p_2^0, p_3^0) &= -2g^2 C_B W_{\sigma\sigma}(|\vec{r}|)\sum_{i=1,2,3}\frac{1}{\left[P_0 - p_i^0 - 2(m + \mathcal{I}_r)\right]} \\ &\quad\times\int d\,k_0\,\frac{\Gamma_t(P_0 - p_i^0 + k_0, -k_0, p_i^0)}{\left[P_0 - p_i^0 + k_0 - m - \mathcal{I}_r + \mathrm{i}\varepsilon\right]}\end{aligned} \tag{E.4}$$

The form of the equation Eq. (E.4) suggests that the function Γ_t can be expressed as a symmetric sum

$$\Gamma_t(p_1^0, p_2^0, p_3^0) = f(p_1^0) + f(p_2^0) + f(p_3^0), \tag{E.5}$$

such that the integral equation for Γ_t (function of three variables) is reduced to an integral equation for the function f (of only one variable). The function $f(p_i^0)$ should be chosen such that the integral on the right hand side of the equation Eq. (E.4) generates a factor proportional to $\left[P_0 - p_i^0 - 2(m + \mathcal{I}_r)\right]$, to cancel the corresponding factor in the denominator. To examine this possibility, we impose the following condition:

$$\frac{1}{\left[P_0 - p_i^0 - 2(m + \mathcal{I}_r)\right]}\int d\,k_0\,\frac{f(P_0 - p_i^0 + k_0) + f(-k_0) + f(p_i^0)}{\left[P_0 - p_i^0 + k_0 - m - \mathcal{I}_r + \mathrm{i}\varepsilon\right]} = -\frac{\alpha\,\mathrm{i}}{P_0 - 3(m + \mathcal{I}_r)}f(p_i^0) \tag{E.6}$$

where α is a (dimensionless) positive constant which remains to be determined. Rearranging the terms to factorize the function $f(k_0)$, the above equation can be rewritten as

$$\int d\,k_0\, f(k_0) \left[\frac{1}{k_0 - m - \mathcal{I}_r + \mathrm{i}\varepsilon} + \frac{1}{P_0 - p_i^0 - k_0 - m - \mathcal{I}_r + \mathrm{i}\varepsilon} \right] = (-\mathrm{i}) \frac{(2\alpha - 1)P_0 - 2\alpha p_i^0 + (3 - 4\alpha)(m + \mathcal{I}_r)}{2[P_0 - 3(m + \mathcal{I}_r)]} f(p_i^0). \tag{E.7}$$

Then the most obvious ansatz for the function f is

$$f(k_0) = \frac{1}{(2\alpha - 1)P_0 - 2\alpha k_0 + (3 - 4\alpha)(m + \mathcal{I}_r) + \mathrm{i}\varepsilon} \tag{E.8}$$

such that on the right hand side of the equation Eq. (E.7) the numerator is cancelled by $f(p_i)$. The next step is to complete the integration on the on the left hand side, which gives (note that the ε prescription is chosen such that only the first term in the bracket survives – the integration must not give rise to any new terms containing the energy p_i^0):

$$\int d\,k_0\, f(k_0) \frac{1}{k_0 - m - \mathcal{I}_r + \mathrm{i}\varepsilon} = -\frac{\mathrm{i}}{(2\alpha - 1)P_0 + (3 - 6\alpha)(m + \mathcal{I}_r)}. \tag{E.9}$$

It is then straightforward to compare Eq. (E.7) and Eq. (E.9) and find that the equality is satisfied for $\alpha = 3/2$, leading to the expression for the vertex Γ_t used in the text.

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