Introduction to QCD

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(Notes taken by Ying Cui and Youngshin Kwon)

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Further Readings

- Introductory:
  - Quarks and Leptons: An Introductory Course in Modern Particle Physics
    by *F. Halzen* and *A. D. Martin*
- Advanced:
  - An Introduction to Quantum Field Theory
    by *M. E. Peskin* and *D. V. Schroeder*
  - Foundations of Quantum Chromodynamics
    by *T. Muta*
  - The Theory of Quark and Gluon Interactions
    by *F. J. Yndurian*
  - Dynamics of the Standard Model
    by *J. F. Donoghue*, *E. Golowich* and *B. R. Holstein*
  - The Structure of the Nucleon
    by *A. W. Thomas* and *W. Weise*

Preparations and Conventions

- We use units such that "$c = \hbar = 1$"

- Space-time four vector x:

$$\text{contravariant form: } x^\mu = (t,\, \vec{x})^T$$

$$\text{covariant form: } x_\mu = (t,\, -\vec{x})^T = g_{\mu\nu}x^\nu$$

  with $\mu = 0, 1, 2, 3$ and metric tensor defined as

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- Inner product:

$$a \cdot b = a_\mu b^\mu = a_0 b^0 - \vec{a} \cdot \vec{b}$$

  where, and also from now on, we use the Einstein summation convention.

- Four-gradient:

$$\frac{\partial}{\partial x^\mu} \equiv \partial_\mu = \left(\frac{\partial}{\partial t},\, \vec{\nabla}\right)^T$$

$$\frac{\partial}{\partial x_\mu} \equiv \partial^\mu = \left(\frac{\partial}{\partial t},\, -\vec{\nabla}\right)^T$$

- d'Alembert operator:

$$\Box \equiv \partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$$

- Four-momenta:

$$p^\mu = (p^0,\, \vec{p}\,)^T = (E,\, \vec{p}\,)^T$$

$$p^2 \equiv p_\mu p^\mu = E^2 - \vec{p}^{\,2}$$

- Dirac- and Pauli-matrices:

$$\gamma^\mu = (\gamma^0,\, \vec{\gamma}\,)^T \;; \qquad \gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix},\; \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\gamma_5 = \gamma^5 \equiv i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

- Useful properties of Dirac- and Pauli-matrices:

$$\sigma^{\mu\nu} \equiv \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right] = \frac{i}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right)$$

$$\left\{\gamma^\mu, \gamma^\nu\right\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

$$\sigma_i \cdot \sigma_j = i\,\epsilon_{ijk}\,\sigma_k$$

$$\left[\sigma_i, \sigma_j\right] = 2i\,\epsilon_{ijk}\,\sigma_k$$

$$\left\{\sigma_i, \sigma_j\right\} = 2\delta_{ij}$$

where ϵ_{ijk} is the totally antisymmetric tensor with $\epsilon_{123} = +1$.

1.1. Quarks in Hadrons and Concept of "Color"

- e^+e^- annihilation into hadrons

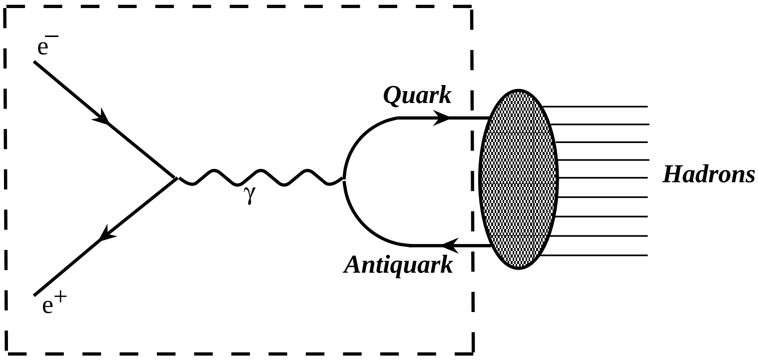

FIG. 1.1: Feynman diagram of $e^+e^- \to hadrons$: The reaction is considered to be proceed through pair production of quark-antiquark as indicated inside the dashed box.

Comparing the total cross section with that of the elementary process $e^+e^- \to \mu^+\mu^-$ which is the analogue in pure QED:

$$R = \frac{\sigma\big(e^+e^- \to Hadrons\big)}{\sigma\big(e^+e^- \to \mu^+\mu^-\big)} = \sum_q \frac{\sigma\big(e^+e^- \to q\bar{q}\,\big)}{\sigma\big(e^+e^- \to \mu^+\mu^-\big)} = N_c \sum_q Z_q^2 \tag{1.1}$$

with

$$\begin{aligned} \sigma\big(e^+e^- \to \mu^+\mu^-\big) &= \frac{4\pi\alpha_e^2}{3s} \\ \sigma\big(e^+e^- \to q\bar{q}\,\big) &= \frac{4\pi\alpha_e^2}{3s} N_c Z_q^2 \end{aligned} \tag{1.2}$$

where s is the center of mass energy squared and $\alpha_e = \frac{e^2}{4\pi} \simeq \frac{1}{137}$ is the fine structure constant. The hypothetical introduction of the "color" freedom ($N_c = 3$) gives consistent explanation of experimental results as shown in Fig. 1.2.

q	u	d	c	s	t	b
Z_q	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$

TABLE 1.1: The electric charge Z_q of each quark flavor in unit e

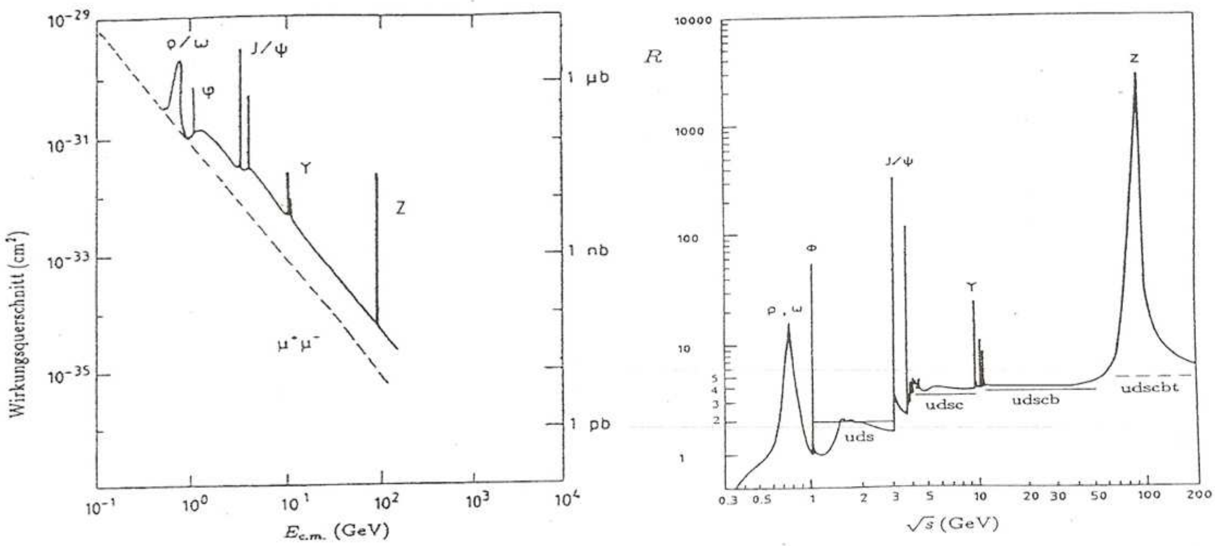

FIG. 1.2: The prediction of the quark model on the total cross section (left) and the ratio R (right) with $N_c = 3$. The typical vector meson resonances are represented.

- Spectroscopy and quark models

  According to the quark theory Δ^{++}, a particle of spin 3/2, should consist of three u quarks with parallel spins if in a state of maximal spin projection:

$$|\Delta^{++}, m_J = 3/2\rangle = |\, u \uparrow u \uparrow u \uparrow\rangle, \tag{1.3}$$

  while the Pauli exclusion principle forbids three identical fermions in the same ground state. Therefore it was suggested that each quark has an additional degree of freedom (three "colors"), thus avoiding violation of the Pauli exclusion principle.

$$|\Delta^{++}, m_J = 3/2\rangle = \frac{1}{\sqrt{6}} \sum_{ijk} \epsilon_{ijk} \,|\, u_i \uparrow u_j \uparrow u_k \uparrow\rangle \tag{1.4}$$

1. 2. Quarks as Dirac-Fields

- Quarks are spin 1/2 particles.
- They exist in 6 species called *flavors*.
- Each quark carries a 3-fold intrinsic degree of freedom (*color*).

- Representation of quarks in terms of fields

$$\psi(x) \equiv \big(\psi_{\alpha i}(x)\big) = \begin{pmatrix} \psi_{ui}(x) \\ \psi_{di}(x) \\ \psi_{si}(x) \\ \vdots \\ \psi_{\alpha i}(x) \end{pmatrix} \tag{1.5}$$

where $\alpha = u, d, s, c, b, t$ and $i = 1, 2, 3$ are flavor and color indices respectively.

- Each of the $\psi_{\alpha i}(x)$ satisfies a Dirac equation in case of free quarks

$$\big[i\gamma_\mu \partial^\mu - \mathbf{m}\big]\psi(x) = 0 \tag{1.6}$$

with the mass matrix

$$\mathbf{m} = \begin{pmatrix} m_u & 0 & 0 & 0 & 0 & 0 \\ 0 & m_d & 0 & 0 & 0 & 0 \\ 0 & 0 & m_s & 0 & 0 & 0 \\ 0 & 0 & 0 & m_c & 0 & 0 \\ 0 & 0 & 0 & 0 & m_b & 0 \\ 0 & 0 & 0 & 0 & 0 & m_t \end{pmatrix} \tag{1.7}$$

- Explicit representation (spin projection $s = \pm\frac{1}{2}$)

$$\boxed{\psi(x) = \sum_s \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_p}\,\Big[b(p,s)\,u_s(p)\,e^{-ip\cdot x} + d^\dagger(p,s)\,v_s(p)\,e^{ip\cdot x}\Big]} \tag{1.8}$$

where $E_p = \sqrt{\vec{p}^{\,2} + m_q^2}$.

- Definition: State vector of a given quark with spin $s = \pm\frac{1}{2}$ and 4-momentum p^μ;

$$\boxed{\begin{aligned} \text{quark :} \quad & |p,\, s\rangle_{\alpha i} = b^\dagger_{\alpha i}(p,s)|0\rangle \\ \text{antiquark :} \quad & \overline{|p,\, s\rangle}_{\alpha i} = d^\dagger_{\alpha i}(p,s)|0\rangle \end{aligned}} \tag{1.9}$$

where the vacuum $|0\rangle$ is defined as: $b|0\rangle = d|0\rangle = 0$ with:

$$\begin{cases} b^\dagger & \text{: creation operator for a quark} \\ b & \text{: annihilation operator for a quark} \\ d^\dagger & \text{: creation operator for an antiquark} \\ d & \text{: annihilation operator for an antiquark.} \end{cases}$$

- Anticommutation rules for creation and annihilation operators

$$\left\{b_{\alpha i}(p,s), b^\dagger_{\beta j}(p',s')\right\} = \left\{d_{\alpha i}(p,s), d^\dagger_{\beta j}(p',s')\right\} = 2E_p(2\pi)^3\delta^3(\vec{p}-\vec{p}\,')\delta_{\alpha\beta}\delta_{ij}\delta_{ss'} \quad (1.10)$$

otherwise vanish, *e.g.*

$$\left\{b^\dagger, b^\dagger\right\} = \left\{b, b\right\} = \left\{d^\dagger, d^\dagger\right\} = \left\{d, d\right\} = 0 \quad (1.11)$$

- Normalization of state vector

$$\langle\, p'\, s' \,|\, p\, s\,\rangle = 2E_p(2\pi)^3\delta^3(\vec{p}-\vec{p}\,')\delta_{ss'} \quad (1.12)$$

- Digression: Lorentz invariant phase space

$$\begin{aligned} \int \frac{d^4p}{(2\pi)^4}\, 2\pi\, \delta(p^2-m^2) &= \int \frac{dE}{2\pi}\frac{d^3p}{(2\pi)^3}\, 2\pi\, \delta(E^2-\vec{p}^{\,2}-m^2) \\ &= \int \frac{dE}{2\pi}\int \frac{d^3p}{(2\pi)^3}\, 2\pi\, \frac{\delta(E-\sqrt{\vec{p}^{\,2}+m^2})}{2E} \\ &= \int \frac{d^3p}{(2\pi)^3 2E_p} \end{aligned} \quad (1.13)$$

where $E_p = \sqrt{\vec{p}^{\,2}+m^2}$

- Dirac equations for particle and antiparticle

$$\begin{aligned} \left(\gamma_\mu p^\mu - m\right)u_s(p) &= 0 \\ \left(\gamma_\mu p^\mu + m\right)v_s(p) &= 0 \end{aligned} \quad (1.14)$$

Free Dirac spinors

$$\begin{aligned} u_s(p) &= \sqrt{E_p+m}\begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\,\chi_s \end{pmatrix} \\ v_s(p) &= \underbrace{\eta\,(-1)^{\frac{1}{2}-s}}_{\text{phase free}}\,\sqrt{E_p+m}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\,\chi_{-s} \\ \chi_{-s} \end{pmatrix}, \end{aligned} \quad (1.15)$$

where $\chi_{s=\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\chi_{s=-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively.

- Normalization of spinors

$$\begin{aligned} u_s^\dagger(p)\, u_{s'}(p) &= 2E_p\, \delta_{ss'} \\ v_s^\dagger(p)\, v_{s'}(p) &= 2E_p\, \delta_{ss'}. \end{aligned} \tag{1.16}$$

1. 3. Quark Currents

- Dirac current density of quarks:

$$\begin{aligned} J^\mu(x) &= \bar{\psi}(x)\, \gamma^\mu\, \psi(x) \\ J^0(x) &= \bar{\psi}(x)\, \gamma^0\, \psi(x) = \psi^\dagger(x)\psi(x) \equiv \rho(x) \\ \vec{J}(x) &= \bar{\psi}(x)\, \vec{\gamma}\, \psi(x) = \psi^\dagger(x)\, \gamma^0 \vec{\gamma}\, \psi(x) = \psi^\dagger(x)\, \vec{\alpha}\, \psi(x) \end{aligned} \tag{1.17}$$

where $\bar{\psi} = \psi^\dagger \gamma^0$ and $\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$

- Continuity equation:

$$\partial_\mu J^\mu(x) = \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \tag{1.18}$$

- Electromagnetic quark current:

$$J^\mu_{e.m.}(x) = \bar{\psi}(x) Q \gamma^\mu \psi(x) \tag{1.19}$$

with quark charges, $Q = \begin{cases} \pm\frac{2}{3}e \text{ for } u,\, c,\, t \text{ or } \bar{u},\, \bar{c},\, \bar{t} \\ \mp\frac{1}{3}e \text{ for } d,\, s,\, b \text{ or } \bar{d},\, \bar{s},\, \bar{b} \end{cases}.$

1. 4. Lagrangian Density (*Lagrangian*) of free quarks

$$\boxed{\mathcal{L}_0(x) = \bar{\psi}(x)\big[i\gamma_\mu \partial^\mu - m\big]\psi(x)} \tag{1.20}$$

- Generalized variables: fields ψ, $\partial_\mu \psi$, $\bar{\psi}$ and $\partial_\mu \bar{\psi}$.

- Action for free quark:

$$S_0 = \int d^4x \, \mathcal{L}_0(x) = S_0\big[\psi,\, \partial_\mu\psi\,;\, \cdots\big] \tag{1.21}$$

▷ Stationary action principle: $\delta\, S_0 = 0$.

- Euler-Lagrange equations:

$$\begin{aligned} \frac{\partial \mathcal{L}_0}{\partial \psi} - \partial^\mu \frac{\partial \mathcal{L}_0}{\partial(\partial^\mu \psi)} = 0 \\ \frac{\partial \mathcal{L}_0}{\partial \bar{\psi}} - \partial^\mu \frac{\partial \mathcal{L}_0}{\partial\big(\partial^\mu \bar{\psi}\big)} = 0. \end{aligned} \tag{1.22}$$

- Dirac equations from Euler-Lagrange equations:

$$\begin{aligned} \frac{\partial \mathcal{L}_0}{\partial \bar{\psi}} = 0 \;\Rightarrow\; \big[i\gamma_\mu \partial^\mu - m\big]\psi(x) = 0 \\ \frac{\partial \mathcal{L}_0}{\partial \psi} = \partial^\mu \frac{\partial \mathcal{L}_0}{\partial\big(\partial^\mu \psi\big)} \;\Rightarrow\; \bar{\psi}(x)\big[i\gamma_\mu \partial^\mu - m\big] = 0 \end{aligned}$$

1. 5. Hamiltonian Density (*Hamiltonian*)

- Canonical conjugate field:

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \tag{1.23}$$

$$\mathcal{L}_0 = \bar{\psi}\Big[i\gamma_0 \frac{\partial}{\partial t} + i\vec{\gamma}\cdot\vec{\nabla} - m\Big]\psi \;\Rightarrow\; \pi = i\bar{\psi}\gamma_0 = i\psi^\dagger \tag{1.24}$$

- Canonical form of *Hamiltonian* as Legendre transform from *Lagrangian*:

$$\begin{aligned} \mathcal{H}(x) &= \pi\dot{\psi} - \mathcal{L}(x) \\ &= \psi^\dagger\big[-i\vec{\alpha}\cdot\vec{\nabla} + \beta m\big]\psi \\ &= \psi^\dagger i\frac{\partial}{\partial t}\psi, \end{aligned} \tag{1.25}$$

where $\vec{\alpha} \equiv \gamma_0\vec{\gamma}$ and $\beta \equiv \gamma_0$.

- Dirac equation in *Hamiltonian* form

$$\big[-i\vec{\alpha}\cdot\vec{\nabla} + \beta m\big]\psi(x) = i\frac{\partial}{\partial t}\psi(x) \tag{1.26}$$