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Research Article

Quarter-Symmetric Nonmetric Connection on P-Sasakian Manifolds

Abul Kalam Mondal[1] and U. C. De[2]

[1] *Department of Mathematics, Dum Dum Motijheel Rabindra Mahavidyalaya, 208/B/2, Dum Dum Road, West Bengal, Kolkata 700074, India*
[2] *Department of Pure Mathematics, University of Calcutta 35, Ballygaunge Circular Road, West Bengal, Kolkata 700019, India*

Correspondence should be addressed to Abul Kalam Mondal, kalam_ju@yahoo.co.in

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The object of the present paper is to study a quarter-symmetric nonmetric connection on a P-Sasakian manifold. In this paper we consider the concircular curvature tensor and conformal curvature tensor on a P-Sasakian manifold with respect to the quarter-symmetric nonmetric connection. Next we consider second-order parallel tensor with respect to the quarter-symmetric non-metric connection. Finally we consider submanifolds of an almost paracontact manifold with respect to a quarter-symmetric non-metric connection.

1. Introduction

In 1975, Golab [1] defined and studied quarter-symmetric connection in a differentiable manifold with affine connection.

A linear connection $\tilde{\nabla}$ on an n-dimensional Riemannian manifold (M, g) is called a quarter-symmetric connection [1] if its torsion tensor T of the connection $\tilde{\nabla}$

$$T(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y] \tag{1.1}$$

satisfies

$$T(X, Y) = \eta(Y)\phi X - \eta(X)\phi Y, \tag{1.2}$$

where η is a 1 form and ϕ is a $(1, 1)$ tensor field.

In particular, if $\phi(X) = X$, then the quarter-symmetric connection reduces to a semisymmetric connection [2]. Thus the notion of quarter-symmetric connection generalizes the notion of the semisymmetric connection.

If, moreover, a quarter-symmetric connection $\tilde{\nabla}$ satisfies the condition

$$\left(\tilde{\nabla}_X g\right)(Y, Z) = 0 \tag{1.3}$$

for all $X, Y, Z \in T(M)$, where $T(M)$ is the Lie algebra of vector fields of the manifold M, then $\tilde{\nabla}$ is said to be a quarter-symmetric metric connection; otherwise it is said to be a quarter-symmetric nonmetric connection.

After Golab [1], Rastogi [3, 4] continued the systematic study of quarter-symmetric metric connection.

In 1980, Mishra and Pandey [5] studied quarter-symmetric metric connection in Riemannian, Kaehlerian, and Sasakian manifolds.

In 1982, Yano and Imai [6] studied quarter-symmetric metric connection in Hermitian and Kaehlerian manifolds.

In 1991, Mukhopadhyay et al. [7] studied quarter-symmetric metric connection on a Riemannian manifold (M, g) with an almost complex structure ϕ.

In 1997, Biswas and De [8] studied quarter-symmetric metric connection on a SP-Sasakian manifold. In 2000, Ali and Nivas [9] studied quarter-symmetric connection on submanifolds of a manifold. Also in 2008, Sular et al. [10] studied quarter-symmetric metric connection in a Kenmotsu manifold.

Let $\overline{M}$ be a submanifold of an almost paracontact metric manifold M with a positive definite metric g. Let the induced metric on $\overline{M}$ also be denoted by g. The usual Gauss and Weingarten formulae are given, respectively, by

$$\nabla_X Y = \overline{\nabla}_X Y + h(X, Y), \quad X, Y \in T\left(\overline{M}\right), \tag{1.4}$$

$$\nabla_X N = -A_N X + \overline{\nabla}^{\perp}_X N, \quad N \in T^{\perp}\left(\overline{M}\right), \tag{1.5}$$

where $\overline{\nabla}$ is the induced Riemannian connection on $\overline{M}$, h is the second fundamental form of the immersion, and $-A_N X$ and $\overline{\nabla}^{\perp}_X N$ are the tangential and normal parts of $\nabla_X N$. From (1.4) and (1.5) one gets

$$g(h(X, Y), N) = g(A_N X, Y). \tag{1.6}$$

The submanifold $\overline{M}$ of an almost paracontact manifold M is called invariant (resp. anti-invariant) if for each point $p \in \overline{M}$, $\phi T_p(\overline{M}) \subset T_p(\overline{M})$ (resp., $\phi T_p\ (\overline{M}) \subset T^{\perp}_p\ (\overline{M})$. The submanifold is called totally umbilical if $h(X, Y) = g(X, Y)H$, for all $X, Y \in T(\overline{M})$, where H is the mean curvature vector defined by $H = (1/n)\ \sum\{h(e_i, e_i)\}$, where $\{e_i\}$ is an orthonormal basis of $T(\overline{M})$. The submanifold is called totally geodesic if $h(X, Y) = 0$ for all $X, Y \in T(\overline{M})$.

The paper is organized as follows. After recalling the basic properties of P-Sasakian manifolds in Section 3, we establish the relation between the Riemannian connection and the quarter-symmetric nonmetric connection. In Section 4, we study the curvature tensor, Ricci tensor, scalar curvature, and the first Bianchi identity with respect to the quarter-symmetric nonmetric connection. Section 5 deals with concircular and conformal curvature tensor on a P-Sasakian manifold with respect to the quarter-symmetric nonmetric connection and prove that if in a P-Sasakian manifold the concircular curvature tensor is invariant under quarter-symmetric nonmetric connection, then the Ricci tensors are equal with respect to the both connections and also prove if a P-Sasakian manifold is conformally flat with respect to the quarter-symmetric nonmetric connection, then the manifold is of quasiconstant curvature with respect to the Levi-Civita connection. In the next section we consider second-order parallel tensor with respect to the quarter-symmetric nonmetric connection. In the last section we consider submanifolds of an almost paracontact manifold with respect to a quarter-symmetric nonmetric connection and prove that on an anti-invariant submanifold of aa almost paracontact manifold with a quarter-symmetric nonmetric connection the induced quarter-symmetric non-connection and the induced Riemannian connection are equivalent. Finally, we prove that a submanifold of a P-Sasakian manifold with a quarter-symmetric nonmetric connection is also a P-Sasakian manifold with respect to the induced quarter-symmetric nonmetric connection.

2. P-Sasakian Manifold

An n-dimensional differentiable manifold M is said to admit an almost paracontact Riemannian structure (ϕ, ξ, η, g), [11] where ϕ is a $(1,1)$-tensor field, ξ is a vector field, η is a 1-form, and g is a Riemannian metric on M such that

$$\phi\xi = 0, \qquad \eta\phi = 0, \qquad \eta(\xi) = 1, \qquad g(\xi, X) = \eta(X), \tag{2.1}$$

$$\phi^2 X = X - \eta(X)\xi, \qquad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

for all vector fields $X, Y \in T(M)$. The equation $\eta(\xi) = 1$ is equivalent to $|\eta| \equiv 1$, and then ξ is just the metric dual of η, where g is the Riemannian metric on M. If (ϕ, ξ, η, g) satisfy the following equations:

$$d\eta = 0, \qquad \nabla_X \xi = \phi X, \tag{2.3}$$

$$(\nabla_X \phi)Y = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.4}$$

then M is called a para-Sasakian manifold or briefly a P-Sasakian manifold, [12, 13]. Especially, a P-Sasakian manifold M is called a special para-Sasakian manifold or briefly a SP-Sasakian manifold if M admits a 1-form η satisfying

$$(\nabla_X \eta)(Y) = -g(X, Y) + \eta(X)\eta(Y). \tag{2.5}$$

It is known that in a P-Sasakian manifold the following relation holds:

$$\begin{aligned} S(X, \xi) &= (1 - n)\eta(X), \\ \eta(R(X, Y)Z) &= g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \end{aligned} \tag{2.6}$$

for any vector fields $X, Y, Z \in T(M)$.

Let (M, g) be an n-dimensional Riemannian manifold. Then the concircular curvature tensor C^* and the Weyl conformal curvature tensor C are defined by [14]

$$C^*(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}\left(g(Y,Z)X - g(X,Z)Y\right), \tag{2.7}$$

$$\begin{aligned} C(X,Y)Z = {} & R(X,Y)Z - \frac{1}{n-2}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)LX - g(X,Z)LY\} \\ & + \frac{r}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\} \end{aligned} \tag{2.8}$$

for all $X, Y, Z \in T(M)$, respectively, where r is the scalar curvature of M, and L is the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S.

We observe immediately from the definition of the concircular curvature tensor that Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus one can think of the concircular curvature tensor as a measure of the failure of a Riemannian manifold to be of constant curvature. Also necessary and sufficient condition that a Riemannian manifold be reducible to a Euclidian space by a suitable concircular transformation is that its concircular curvature tensor vanishes. Also conformal curvature tensor plays an important role in differential geometry.

A Riemannian manifold of quasiconstant curvature was given by Chen and Yano [15] as a conformally flat manifold with the curvature tensor R of type $(0,4)$ which satisfies the condition

$$\begin{aligned} R(X,Y,Z,W) = {} & a\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \\ & + b\{g(Y,Z)T(X)T(W) - g(X,Z)T(Y)T(W) \\ & + g(X,W)T(Y)T(Z) - g(Y,W)T(X)T(Z)\}, \end{aligned} \tag{2.9}$$

where $R(X,Y,Z,W) = g(R(X,Y)Z,W)$, a, b are scalars, T is a nonzero 1-form defined by $T(X) = g(X,\rho)$, and ρ is a unit vector field.

It can be easily seen that if the curvature tensor is of the form (2.9), then the manifold is conformally flat. If $b = 0$, then it reduces to a manifold of constant curvature.

An n-dimensional P-Sasakian manifold is said to be η-Einstein if the Ricci tensor S satisfies

$$S = ag + b\eta \otimes \eta, \tag{2.10}$$

where a and b are smooth function on the manifold. If $b = 0$, then the manifold reduces to an Einstein manifold.

3. Relation between the Riemannian Connection and the Quarter-Symmetric Nonmetric Connection

Let $\tilde{\nabla}$ be a linear connection and ∇ be a Riemannian connection of a P-Sasakian manifold M such that

$$\tilde{\nabla}_X Y = \nabla_X Y + U(X, Y), \tag{3.1}$$

where U is a tensor of type $(1,2)$. For $\tilde{\nabla}$ to be a quarter-symmetric connection in M, we have [1]

$$U(X, Y) = \frac{1}{2}\left[T(X, Y) + T'(X, Y) + T'(Y, X)\right], \tag{3.2}$$

where

$$g\left(T'(X, Y), Z\right) = g(T(Z, X), Y). \tag{3.3}$$

From (1.2) and (3.3) we get

$$T'(X, Y) = g(\phi Y, X)\xi - \eta(X)\phi Y \tag{3.4}$$

and using (1.2) and (3.4) in (3.2) we obtain

$$U(X, Y) = -\eta(X)\phi Y. \tag{3.5}$$

Hence a quarter-symmetric connection $\tilde{\nabla}$ in a P-Sasakian manifold is given by

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y. \tag{3.6}$$

Conversely, we show that a linear connection $\tilde{\nabla}$ on a P-Sasakian manifold defined by

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y \tag{3.7}$$

determines a quarter-symmetric connection.

Using (3.7) the torsion tensor of the connection $\tilde{\nabla}$ is given by

$$\begin{aligned} T(X, Y) &= \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y] \\ &= \eta(Y)\phi X - \eta(X)\phi Y. \end{aligned} \tag{3.8}$$

The above equation shows that the connection $\tilde{\nabla}$ is quarter-symmetric [1].

Also we have

$$\left(\tilde{\nabla}_X g\right)(Y,Z) = Xg(Y,Z) - g\left(\tilde{\nabla}_X Y, Z\right) - g\left(Y, \tilde{\nabla}_X Z\right)$$
$$= \eta(X)\left[g(\phi Y, Z) + g(\phi Z, Y)\right] \tag{3.9}$$
$$= 2\eta(X)g(\phi Y, Z).$$

In virtue of (3.8) and (3.9) we conclude that $\tilde{\nabla}$ is a quarter-symmetric nonmetric connection. Therefore (3.6) is the relation between the Riemannian connection and the quarter-symmetric connection on a P-Sasakian manifold.

4. Curvature Tensor of a P-Sasakian Manifold with Respect to the Quarter-Symmetric Nonmetric Connection

We define the curvature tensor of a P-Sasakian manifold with respect to the quarter-symmetric nonmetric connection $\tilde{\nabla}$ by

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z. \tag{4.1}$$

Using (3.7) we obtain

$$\tilde{R}(X,Y)Z = R(X,Y)Z - (\nabla_X \eta)(Y)\phi Z + (\nabla_Y \eta)(X)\phi Z$$
$$- \eta(Y)(\nabla_X \phi)Z + \eta(X)(\nabla_Y \phi)Z, \tag{4.2}$$

which in view of (2.4) and (2.5) yields

$$\tilde{R}(X,Y)Z = R(X,Y)Z + \{\eta(Y)g(X,Z) - \eta(X)g(Y,Z)\}\xi$$
$$+ \{\eta(Y)X - \eta(X)Y\}\eta(Z). \tag{4.3}$$

A relation between the curvature tensor of M with respect to the quarter-symmetric nonmetric connection $\tilde{\nabla}$ and the Riemannian connection ∇ is given by the relation (4.3). So from (4.3) and (2.3) we have

$$\tilde{R}(X,\xi)Y = R(X,\xi)Y + g(X,Y) - 2\eta(X)\eta(Y)\xi + \eta(Y)X, \tag{4.4}$$

$$\tilde{R}(X,Y)\xi = 2\{\eta(Y)X - \eta(X)Y\}. \tag{4.5}$$

Taking inner product of (4.3) with W we have

$$\tilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + \{\eta(Y)g(X,Z) - \eta(X)g(Y,Z)\}\eta(W)$$
$$+ \{\eta(Y)g(X,W) - \eta(X)g(Y,W)\}\eta(Z), \tag{4.6}$$

where $\tilde{R}(X,Y,Z,W) = g(\tilde{R}(X,Y,Z),W)$.

From (4.6) we can state the following.

Proposition 4.1. *If the manifold is of constant curvature with respect to the Levi-Civita connection, then the manifold is of quasiconstant curvature with respect to the quarter-symmetric nonmetric connection.*

Also from (4.6) clearly

$$\tilde{R}(X,Y,Z,W) = -\tilde{R}(Y,X,Z,W), \tag{4.7}$$

but

$$\tilde{R}(X,Y,Z,W) \neq -\tilde{R}(X,Y,W,Z). \tag{4.8}$$

From (4.3) it is obvious that

$$\tilde{R}(X,Y)Z + \tilde{R}(Y,Z)X + \tilde{R}(Z,X)Y = 0. \tag{4.9}$$

Hence we can state that the curvature tensor with respect to the quarter-symmetric nonmetric connection satisfies first Bianchi identity.

Contracting (4.6) over X and W, we obtain

$$\tilde{S}(Y,Z) = S(Y,Z) - g(Y,Z) + n\eta(Y)\eta(Z), \tag{4.10}$$

where $\tilde{S}$ and S is the Ricci tensors of the connection $\tilde{\nabla}$ and ∇, respectively. So in a P-Sasakian manifold the Ricci tensor with respect to the quarter-symmetric nonmetric connection is symmetric. Also if M is Einstein or η-Einstein with respect to the Riemannian connection, then M is η-Einstein with respect to the quarter-symmetric nonmetric connection.

Again contracting (4.10) we have $\tilde{r} = r$, where $\tilde{r}$ and r are the scalar curvature of the connection $\tilde{\nabla}$ and ∇, respectively. So we have the following.

Proposition 4.2. *For a P-Sasakian manifold M with the quarter-symmetric metric connection $\tilde{\nabla}$*

(a) *the curvature tensor $\tilde{R}$ is given by* (4.6),

(b) *the Ricci tensor $\tilde{S}$ is given by* (4.10),

(c) *the first Bianchi identity is given by* (4.8),

(d) $\tilde{r} = r$,

(e) *the Ricci tensor $\tilde{S}$ is symmetric,*

(f) *if M is Einstein or η-Einstein with respect to the Riemannian connection, then M is η-Einstein with respect to the quarter-symmetric nonmetric connection.*

5. Concircular and Conformal Curvature Tensor on a P-Sasakian Manifold with Respect to the Quarter-Symmetric Nonmetric Connection

We define the concircular curvature tensor $\widetilde{C}^*$ and conformal curvature tensor $\widetilde{C}$ on a P-Sasakian manifold with respect to the quarter-symmetric nonmetric connection $\widetilde{\nabla}$ by

$$\widetilde{C}^*(X,Y)Z = \widetilde{R}(X,Y)Z - \frac{\widetilde{r}}{n(n-1)}\left(\widetilde{g}(Y,Z)X - \widetilde{g}(X,Z)Y\right), \tag{5.1}$$

$$\begin{aligned}\widetilde{C}(X,Y)Z = \widetilde{R}(X,Y)Z &- \frac{1}{n-2}\left\{\widetilde{S}(Y,Z)X - \widetilde{S}(X,Z)Y + g(Y,Z)\widetilde{L}X - g(X,Z)\widetilde{L}Y\right\} \\ &+ \frac{\widetilde{r}}{(n-1)(n-2)}\left\{\widetilde{g}(Y,Z)X - \widetilde{g}(X,Z)Y\right\}\end{aligned} \tag{5.2}$$

for all $X, Y, Z \in T(M)$, respectively, where $\widetilde{r}$ is the scalar curvature, and $\widetilde{L}$ is the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor $\widetilde{S}$ with respect to quarter-symmetric nonmetric connection.

Using (2.7) and (4.2), (5.1) reduces to

$$\begin{aligned}\widetilde{C}^*(X,Y)Z = C^*(X,Y)Z &- \left(g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\right) \\ &+ \left(\eta(Y)X - \eta(X)Y\right)\eta(Z).\end{aligned} \tag{5.3}$$

Now if we consider $\widetilde{C}^* = C^*$, then from (5.3) we have

$$g(X,Y) = n\eta(X)\eta(Y). \tag{5.4}$$

Using (5.4) in (4.10) we have

$$\widetilde{S}(X,Y) = S(X,Y). \tag{5.5}$$

So we can state the following.

Theorem 5.1. *If in a P-Sasakian manifold the concircular curvature tensor is invariant under quarter-symmetric nonmetric connection, then the Ricci tensors are equal with respect to both the connections.*

Let us suppose that $\widetilde{C}^*(\xi, X) \cdot \widetilde{S} = 0$, and then we get

$$\widetilde{S}\left(\widetilde{C}^*(\xi,X)Y, Z\right) + \widetilde{S}\left(Y, \widetilde{C}^*(\xi,X)Z\right) = 0, \tag{5.6}$$

which in view of (5.3) gives

$$\begin{aligned}&\widetilde{S}(C^{*}(\xi,X)Y,Z)+\left(2\eta(X)\eta(Y)-g(X,Y)\right)\widetilde{S}(\xi,Z)-\eta(Y)\widetilde{S}(X,Z)\\&\quad+\widetilde{S}(Y,C^{*}(\xi,X)Z)+\left(2\eta(X)\eta(Z)-g(X,Z)\right)\widetilde{S}(\xi,Y)-\eta(Z)\widetilde{S}(X,Y)=0.\end{aligned}\tag{5.7}$$

So by the use of (2.6), (5.7) yields

$$\frac{r}{n(n-1)}\widetilde{S}(X,Y)=0.\tag{5.8}$$

From this either $r=0$ or, $\widetilde{S}(X,Y)=0$. Now $\widetilde{S}=0$ implies

$$S(X,Y)=g(X,Y)-n\eta(X)\eta(Y).\tag{5.9}$$

The converse is trivial.

So we can state the following.

Theorem 5.2. *An n-dimensional P-Sasakian manifold with nonzero scalar curvature satisfies the condition $\widetilde{C}^{*}(\xi,X)\cdot\widetilde{S}=0$ if and only if the manifold is an η-Einstein manifold of the form* (5.9).

Also using (4.3) and (4.10), (5.2) reduces to

$$\begin{aligned}\widetilde{C}(X,Y,Z,W)=\;&R(X,Y,Z,W)+\left(g(X,Z)\eta(Y)-g(Y,Z)\eta(X)\right)\eta(W)\\&+\left(\eta(Y)g(X,W)-\eta(X)g(Y,W)\right)\eta(Z)-\frac{1}{n-2}\\&\times\left\{\left(s(Y,Z)-g(Y,Z)+n\eta(Y)\eta(Z)\right)g(X,W)\right.\\&\quad-\left(S(X,Z)-g(X,Z)+n\eta(X)\eta(Z)\right)g(Y,W)\\&\quad+\left(S(X,W)-g(X,W)+n\eta(X)\eta(W)\right)g(Y,Z)\\&\quad\left.-\left(S(Y,W)-g(Y,W)+n\eta(Y)\eta(W)\right)g(X,Z)\right\}\\&+\frac{\widetilde{r}}{(n-1)(n-2)}\left\{\widetilde{g}(Y,Z)X-\widetilde{g}(X,Z)Y\right\}.\end{aligned}\tag{5.10}$$

Using (2.8) in (5.10) we obtain

$$\begin{aligned}\widetilde{C}(X,Y,Z,W)=\;&C(X,Y,Z,W)+\frac{2}{n-2}\left(g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\right)\\&+\left(1+\frac{n}{n-2}\right)\left\{g(X,Z)\eta(Y)\eta(W)-g(Y,Z)\eta(X)\eta(W)\right.\\&\quad\left.+g(X,W)\eta(Y)\eta(Z)-g(Y,W)\eta(X)\eta(Z)\right\}\end{aligned}\tag{5.11}$$

which is the relation between conformal curvature tensor C with respect to Riemannian connection and $\tilde{C}$ with respect to the quarter-symmetric nonmetric connection.

Suppose that the P-Sasakian manifold is conformally flat with respect to the quarter-symmetric nonmetric connection, that is, $\tilde{C}(X,Y,Z,W)=0$. Now from (5.2) we get

$$\tilde{R}(X,Y)Z = \frac{1}{n-2}\left\{\tilde{S}(Y,Z)X-\tilde{S}(X,Z)Y+g(Y,Z)\tilde{L}X-g(X,Z)\tilde{L}Y\right\} - \frac{\tilde{r}}{(n-1)(n-2)}\{\tilde{g}(Y,Z)X-\tilde{g}(X,Z)Y\}. \tag{5.12}$$

Putting $Y=W=\xi$ in (5.12) and using (4.6) we obtain

$$\tilde{S}(X,Y)=\left(\frac{r}{n-1}-n+3\right)g(X,Y)-\left(\frac{r}{n-1}+2\right)\eta(X)\eta(Y). \tag{5.13}$$

Putting this value in (5.12) we have

$$\begin{aligned}\tilde{R}(X,Y,Z,W) = {}& \frac{1}{n-2}\left(\frac{r}{n-1}-2n+6\right)\{g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\} \\ &+\frac{1}{n-2}\left(-\frac{r}{n-1}-2\right)\{g(Y,Z)\eta(X)\eta(W)-g(X,Z)\eta(Y)\eta(W) \\ &+g(X,W)\eta(Y)\eta(Z)-g(Y,W)\eta(X)\eta(Z)\}.\end{aligned} \tag{5.14}$$

From this we obtain the following.

Theorem 5.3. *If a P-Sasakian manifold is conformally flat with respect to the quarter-symmetric nonmetric connection, then the manifold is of quasiconstant curvature with respect to the Levi-Civita connection.*

6. Second-Order Parallel Tensor on P-Sasakian Manifold with Respect to the Quarter-Symmetric Nonmetric Connection

Definition 6.1. A tensor α of second order is said to be a second-order parallel tensor if $\nabla\alpha=0$ where ∇ denotes the operator of covariant differentiation with respect to the Riemannian connection.

In [16] De proves that on a P-Sasakian manifold a second-order symmetric parallel tensor is a constant multiple of the associated metric tensor. In this section we consider a second-order parallel tensor with respect to the quarter-symmetric nonmetric connection defined as $\tilde{\nabla}\alpha=0$.

Then it follows that

$$\alpha\left(\tilde{R}(W,X)Y,Z\right)+\alpha\left(Y,\tilde{R}(W,X)Z\right)=0, \tag{6.1}$$

for arbitrary vector fields W,X,Y,Z on M.

Substitution of $W = Z = \xi$ in (6.1) which gives us

$$\alpha\left(\widetilde{R}(\xi, X)Y, \xi\right) + \alpha\left(Y, \widetilde{R}(\xi, X)\xi\right) = 0. \tag{6.2}$$

Using (4.4), (6.2) yields

$$\alpha(X, \xi)\eta(Y) = \alpha(\xi, \xi)\{g(X, Y) - \eta(X)\eta(Y)\}. \tag{6.3}$$

Putting $Y = \xi$ in the above we get

$$\alpha(X, \xi) = 0. \tag{6.4}$$

Differentiating (6.4) covariantly along Y, we get

$$\alpha(\nabla_Y X, \xi) + \alpha(X, \nabla_Y \xi) = 0. \tag{6.5}$$

From the help of (2.2) and (2.3) we get

$$\alpha(X, Y) = 0. \tag{6.6}$$

Hence we can state the following.

Theorem 6.2. *On a P-Sasakian manifold there is no nonzero second order parallel tensor with respect to the quarter-symmetric nonmetric connection.*

As an immediate corollary we can state the following.

Corollary 6.3. *There does not exist a Ricci symmetric* $(\nabla S = 0)$ *P-sasakian manifold with respect to the quarter-symmetric nonmetric connection.*

7. Submanifolds of an Almost Paracontact Manifold with Respect to a Quarter-Symmetric Nonmetric Connection

We define quarter-symmetric nonmetric connection by (3.7). Now if $\widetilde{\nabla}'$ is the induced connection on submanifold from the connection $\widetilde{\nabla}$, then we have

$$\widetilde{\nabla}_X Y = \widetilde{\nabla}'_X Y + m(X, Y), \tag{7.1}$$

where m is the second fundamental form of $\widetilde{M}$ in M.

For $X \in T(\overline{M})$ and $N \in T^{\perp}(\overline{M})$, we put

$$\phi X = PX + QX, \qquad PX \in T\left(\overline{M}\right),\ QX \in T^{\perp}\left(\overline{M}\right), \tag{7.2}$$

$$\phi N = tN + sN, \qquad tN \in T\left(\overline{M}\right),\ sN \in T^{\perp}\left(\overline{M}\right). \tag{7.3}$$

Using (7.2), (1.4), and (3.7) from (7.1) we have

$$\tilde{\nabla}'_X Y + m(X,Y) = \overline{\nabla}_X Y + h(X,Y) - \eta(X)PY - \eta(X)QY. \tag{7.4}$$

Now equating tangential and normal parts, we have

$$\tilde{\nabla}'_X Y = \overline{\nabla}_X Y - \eta(X)PY, \tag{7.5}$$

$$m(X,Y) = h(X,Y) - \eta(X)QY. \tag{7.6}$$

From (7.1) we obtain

$$\tilde{\nabla}_X Y = \tilde{\nabla}'_X Y + h(X,Y) - \eta(X)QY. \tag{7.7}$$

From (7.7) the torsion tensor with respect to the induced quarter-symmetric nonmetric connection is given by

$$T(X,Y) = \eta(Y)PX - \eta(X)PY. \tag{7.8}$$

Also using (7.7) we have

$$\left(\tilde{\nabla}'_X g\right)(Y,Z) = \left(\tilde{\nabla}_X g\right)(Y,Z). \tag{7.9}$$

Hence we have the following.

Theorem 7.1. *The connection induced on a submanifold of an almost paracontact manifold with a quarter-symmetric nonmetric connection is also a quarter-symmetric nonmetric connection.*

From (7.5), it follows that if the submanifold is anti-invariant, that is, $PY = 0$, then we have the following.

Corollary 7.2. *On an anti-invariant submanifold of an almost paracontact manifold with a quarter-symmetric nonmetric connection the induced quarter-symmetric non-connection and the induced Riemannian connection are equivalent.*

Let $\{e_1, e_2, \ldots, e_n\}$ be an orthogonal basis of $T(\overline{M})$, where $e_n = \xi$.
From (7.6), we obtain

$$m(e_i, e_i) = h(e_i, e_i) - \eta(e_i)Q(e_i). \tag{7.10}$$

Since $Q(e_i) = 0$, summing up for $i = 1, 2, \ldots, n$ and dividing by n we obtain

$$H = \frac{1}{n}\sum\{h(e_i, e_i)\} = \frac{1}{n}\sum\{m(e_i, e_i)\}, \tag{7.11}$$

that is, the mean curvature of the submanifold $\overline{M}$ with respect to the Riemannian connection coincides with that of $\overline{M}$ with respect to the quarter symmetric nonmetric connection.

From (7.6), we have

$$h(X,Y) - m(X,Y) = \eta(X)QY. \tag{7.12}$$

If $\overline{M}$ is totally umbilical with respect to both the Riemannian connection and the quarter symmetric nonmetric connection, then, with the hep of (7.11), from (7.12) we have

$$m(X,Y) = g(X,Y)H = h(X,Y). \tag{7.13}$$

So, from (7.12) we get for all $X, Y \in T(\overline{M})$,

$$\eta(X)QY = 0. \tag{7.14}$$

Putting $Y = \xi$ in (7.14) we obtain that $QY = 0$, for all $X \in T(\overline{M})$, which implies that $\overline{M}$ is an invariant submanifold. The converse is trivial. So we have the following.

Theorem 7.3. *If $\overline{M}$ is totally umbilical with respect to both the connections, then $\overline{M}$ is invariant. Conversely, if $\overline{M}$ is invariant, then $\overline{M}$ is totally umbilical (resp., totally geodesic) with respect to quarter-symmetric connection if and only if M is totally umbilical (resp., totally geodesic) with respect to the Riemannian connection.*

Let us consider that the ambient manifold M is a P-Sasakian manifold. Using (3.6) we have

$$\begin{aligned}\left(\overline{\nabla}_X\phi\right)Y &= \overline{\nabla}_X\phi Y - \phi\left(\overline{\nabla}_X Y\right)\\ &= \nabla_X(\phi Y) - \eta(X)\phi(\phi Y) - \phi(\nabla_X Y - \eta(X)\phi Y)\\ &= (\nabla_X\phi)Y.\end{aligned} \tag{7.15}$$

Therefore we have the following.

Proposition 7.4. *If M is a P-Sasakian manifold admitting a quarter-symmetric nonmetric connection, then M is also a P-Sasakian manifold with respect to the quarter-symmetric nonmetric connection.*

Also induced quarter-symmetric connection is given by (7.5), and using this relation we have

$$\begin{aligned}\left(\tilde{\nabla}_X\phi\right)Y &= \tilde{\nabla}_X\phi Y - \phi\left(\tilde{\nabla}_X Y\right)\\ &= \tilde{\nabla}'_X(\phi Y) - \eta(X)P(\phi(Y)-) - \phi\left(\tilde{\nabla}'_X Y - \eta(X)PY\right)\\ &= \left(\tilde{\nabla}'_X\phi\right)Y - \eta(X)P^2Y + \eta(X)\phi(PY)\\ &= \left(\tilde{\nabla}'_X\phi\right)Y - \eta(X)P^2Y + \eta(X)P^2Y\\ &= \left(\tilde{\nabla}'_X\phi\right)Y.\end{aligned} \tag{7.16}$$

Therefore we have the following.

Theorem 7.5. *A submanifold of a P-Sasakian manifold with a quarter-symmetric nonmetric connection is also a P-Sasakian manifold with respect to the induced quarter-symmetric nonmetric connection.*

References

[1] S. Golab, "On semi-symmetric and quarter-symmetric linear connections," *Tensor: New Series*, vol. 29, no. 3, pp. 249–254, 1975.

[2] A. Friedmann and J. A. Schouten, "Über die Geometrie der halbsymmetrischen Übertragungen," *Mathematische Zeitschrift*, vol. 21, no. 1, pp. 211–223, 1924.

[3] S. C. Rastogi, "On quarter-symmetric metric connection," *Comptes Rendus de l'Académie Bulgare des Sciences*, vol. 31, no. 7, pp. 811–814, 1978.

[4] S. C. Rastogi, "On quarter-symmetric metric connections," *Tensor: New Series*, vol. 44, no. 2, pp. 133–141, 1987.

[5] R. S. Mishra and S. N. Pandey, "On quarter symmetric metric F-connections," *Tensor: New Series*, vol. 34, no. 1, pp. 1–7, 1980.

[6] K. Yano and T. Imai, "Quarter-symmetric metric connections and their curvature tensors," *Tensor: New Series*, vol. 38, pp. 13–18, 1982.

[7] S. Mukhopadhyay, A. K. Roy, and B. Barua, "Some properties of a quarter symmetric metric connection on a Riemannian manifold," *Soochow Journal of Mathematics*, vol. 17, no. 2, pp. 205–211, 1991.

[8] S. C. Biswas and U. C. De, "Quarter-symmetric metric connection in an SP-Sasakian manifold," *Communications, Faculty of Sciences. University of Ankara Series A*, vol. 46, no. 1-2, pp. 49–56, 1997.

[9] S. Ali and R. Nivas, "On submanifolds immersed in a manifold with quarter symmetric connection," *Rivista di Matematica della Università di Parma*, vol. 6, no. 3, pp. 11–23, 2000.

[10] S. Sular, C. Özgür, and U. C. De, "Quarter-symmetric metric connection in a Kenmotsu manifold," *SUT Journal of Mathematics*, vol. 44, no. 2, pp. 297–306, 2008.

[11] I. Satō, "On a structure similar to the almost contact structure," *Tensor: New Series*, vol. 30, no. 3, pp. 219–224, 1976.

[12] T. Adati and T. Miyazawa, "On P-Sasakian manifolds satisfying certain conditions," *Tensor: New Series*, vol. 33, no. 2, pp. 173–178, 1979.

[13] I. Sato and K. Matsumoto, "On P-Sasakian manifolds satisfying certain conditions," *Tensor: New Series*, vol. 33, no. 2, pp. 173–178, 1979.

[14] K. Yano and M. Kon, *Structures on Manifolds*, vol. 3, World Scientific, Singapore, 1984.

[15] B.-Y. Chen and K. Yano, "Hypersurfaces of a conformally flat space," *Tensor: New Series*, vol. 26, pp. 318–322, 1972.

[16] U. C. De, "Second order parallel tensors on P-Sasakian manifolds," *Publicationes Mathematicae Debrecen*, vol. 49, no. 1-2, pp. 33–37, 1996.