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Quartic 3-Fold: Pfaffians, Vector Bundles, and Half-Canonical Curves

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Introduction

This paper is a part of the study of moduli spaces of vector bundles with small Chern classes on certain Fano 3-folds. We investigate the moduli component of kernel bundles on a quartic 3-fold, defined similarly to that of [IM; MT] for the case of a cubic 3-fold. Our work received a strong pulse with the publication of Beauville's paper [B], which allowed us to simplify some arguments and put our results in a more general framework of Pfaffian hypersurfaces.

In [MT], it was proved that the Abel–Jacobi map of the family of normal elliptic quintics lying on a general cubic 3-fold V factors through a moduli component of stable rank-2 vector bundles on V with Chern numbers $c_1 = 0$ and $c_2 = 2$ and whose general point represents a vector bundle obtained by Serre's construction from an elliptic quintic. The elliptic quintics mapped to a point of the moduli component vary in a 5-dimensional projective space inside the Hilbert scheme of curves, and the map from the moduli component to the intermediate Jacobian is quasi-finite. In [IM], this moduli component was identified with the variety of representations of V as a linear section of the Pfaffian cubic in $\mathbb{P}^{14}$ and it was proved that the degree of the quasi-finite map is 1, so the moduli component is birational to the intermediate Jacobian $J^2(X)$. According to [D], the moduli space $M_V(2; 0, 2)$ is irreducible, so its unique component is the one just described.

In the present paper, we prove that a generic quartic 3-fold X admits a 7-dimensional family of essentially different representations as the Pfaffian of an 8×8 skew-symmetric matrix of linear forms. Thanks to [B], this provides a 7-dimensional family of arithmetically Cohen–Macaulay (ACM for short) vector bundles on X, obtained as the bundles of kernels of the 8×8 skew-symmetric matrices of rank 6 representing points of X. We show that this family is a smooth open set M_X in the moduli space of stable vector bundles $M_X(2; 3, 14) \simeq M_X(2; -1, 6)$. The ACM property means the vanishing of the intermediate cohomology $H^i(X, \mathcal{E}(j))$ for all $i = 1, 2$ with $j \in \mathbb{Z}$.

We also give a precise geometric characterization of the ACM curves arising as schemes of zeros of sections of the kernel vector bundles. According to Beauville, they are half-canonical ACM curves of degree 14 in $\mathbb{P}^4$; we show that they are linear sections of the rank-4 locus $Z \subset \mathbb{P}(\wedge^2\mathbb{C}^7)$ in the projectivized space of the

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7×7 skew-symmetric matrices. Linear sections of Z have already arisen in the literature: Rødland [R] studied the sections $\mathbb{P}^6 \cap Z$, which are Calabi–Yau 3-folds. We show that such curves fill out open sets of smooth points of the Hilbert schemes of X (of dimension 14) and of $\mathbb{P}^4$ (of dimension 56). We show also that the isomorphism classes of smooth members of this family fill out a 32-dimensional moduli component $\overset{\circ}{\mathcal{M}}{}^4_{15}$ of curves of genus 15 with a theta-characteristic linear series of dimension 4.

Next we study the Abel–Jacobi map of the ACM half-canonical curves of genus 15 in X. It factors through M_X via Serre's construction: the fibers over points of M_X are $\mathbb{P}^7$, and the resulting map from M_X to $J^2(X)$ is quasi-finite and nonramified; hence its image is 7-dimensional. The role of these half-canonical curves is similar to that of normal elliptic quintics in the case of the cubic 3-fold V, where the Abel–Jacobi map factors through the instanton moduli space with fibers $\mathbb{P}^5$ and with a 5-dimensional image; since $\dim J^2(V) = 5$, the image is an open subset of $J^2(V)$. The result for a quartic 3-fold is somewhat weaker: here we do not know whether the degree of the quasi-finite map is 1 or whether M_X is irreducible. Moreover, as $7 = \dim M_X < 30 = \dim J^2(X)$, we cannot conclude (as in the case of a cubic 3-fold) that the image of M_X is an open subset of an Abelian variety; we can only state that every component of it, and hence of M_X itself, has a nonnegative Kodaira dimension.

In Section 1 we prove that a generic quartic 3-fold is Pfaffian with the same method used by Adler in his Appendix to [AR] for a cubic 3-fold: Take a particular Pfaffian quartic and prove that the differential of the Pfaffian map from the family of all the 8×8 skew-symmetric matrices of linear forms to the family of quinary quartics is of maximal rank. We also prove basic facts about M_X: stability, dimension 7, and smoothness.

Section 2 treats half-canonical ACM curves of genus 15 on X and in $\mathbb{P}^4$. Section 3 applies the technique of the tangent bundle sequence of Clemens and Griffiths [CG] and Welters [W] to the calculation of the differential of the Abel–Jacobi map for the family of the above half-canonical curves C. It identifies the kernel of the differential with $H^1(\mathcal{N}_{C/\mathbb{P}^4}(-1))^\vee$, and we prove that it has dimension 7.

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1. Generic Quartic 3-Fold is Pfaffian

Let X be a smooth quartic 3-fold. It is well known that $\operatorname{Pic}(X)$ is isomorphic to $\mathbb{Z}$ and generated by the class of the hyperplane section H, and that the group of algebraic 1-cycles modulo topological equivalence is also isomorphic to $\mathbb{Z}$ but generated by the class of a line $l \subset X$. For two integers k, α we will denote by $M_X(2; k, \alpha)$ the moduli space of *stable* rank-2 vector bundles $\mathcal{E}$ on X with Chern classes $c_1 = k[H]$ and $c_2 = \alpha[l]$. We will identify the Chern classes with integers in using the generators $[H]$ and $[l]$ of the corresponding groups of algebraic cycles. We have $[H]^2 = 4[l]$.

By the definition of the Chern classes and by Riemann–Roch–Hirzebruch, for $\mathcal{E} \in M_X(2; k, \alpha)$ we have

$$c_1(\mathcal{E}(n)) = c_1(\mathcal{E}) + 2n[H] = (k + 2n)[H],$$

$$c_2(\mathcal{E}(n)) = c_2(\mathcal{E}) + n[H]c_1(\mathcal{E}) + n^2[H]^2 = (\alpha + 4kn + 4n^2)[l],$$

$$\chi(\mathcal{E}) = \tfrac{2}{3}k^3 - \tfrac{1}{2}k\alpha + k^2 - \tfrac{1}{2}\alpha + \tfrac{7}{3}k + 2.$$

A rank-2 torsion-free sheaf $\mathcal{E}$ on X is *normalized* if $c_1(\mathcal{E}) = k[H]$ with $k = 0$ or $k = -1$. We can make $\mathcal{E}$ normalized by replacing it with a suitable twist $\mathcal{E}(n)$.

The following lemma is well known (see [Ha; Ko]).

LEMMA 1.1. *Let $\mathcal{E}$ be a normalized rank-2 reflexive sheaf on a nonsingular projective variety X with $\operatorname{Pic}(X) \simeq \mathbb{Z}$. Then it is stable if and only if $h^0(\mathcal{E}) = 0$.*

Let now E be an 8-dimensional vector space over $\mathbb{C}$. Fix a basis $e_0, \dots, e_7$ for E; then $e_{ij} = e_i \wedge e_j$ for $0 \le i < j \le 7$ form a basis for the Plücker space $\wedge^2 E$. Let x_{ij} be the corresponding (Plücker) coordinates. The embedding of the Grassmannian $G = G(2, E)$ in $\mathbb{P}^{27} = \mathbb{P}(\wedge^2 E)$ is precisely the locus of rank-2 skew-symmetric 8×8 matrices M with elements x_{ij} above the diagonal. Let $G \subset \Omega \subset \Xi \subset \mathbb{P}^{27}$ be the filtration of $\mathbb{P}^{27}$ by the rank of M, that is, $\Omega = \{M \mid \operatorname{rk} M \le 4\}$ and $\Xi = \{M \mid \operatorname{rk} M \le 6\}$. Then G, $\Omega \setminus G$, $\Xi \setminus \Omega$, and $\mathbb{P}^{27} \setminus \Xi$ are orbits of $\mathrm{PGL}(8)$, acting via $\wedge^2$ of its standard representation (see e.g. [KS]), and we have $G = \operatorname{Sing} \Omega$ with $\dim G = 12$ and $\Omega = \operatorname{Sing} \Xi$ with $\dim \Omega = 21$. The equation for Ξ is $\operatorname{Pf}(M) = 0$, where Pf stands for the Pfaffian of a skew-symmetric matrix. We will call Ξ the *Pfaffian hypersurface* of $\mathbb{P}^{27}$.

Let $H \subset \mathbb{P}^{27}$ be a 4-dimensional linear subspace that is not contained in Ξ. Then the intersection $H \cap \Xi$ will be called a Pfaffian quartic 3-fold. Since $\operatorname{codim}_\Xi \Omega = 5$, the linear section $H \cap \Xi$ is nonsingular for general H. Suppose that a quartic 3-fold $X \subset \mathbb{P}^4$ has two different representations, $\phi_1 \colon X \xrightarrow{\sim} H_1 \cap \Xi$ and $\phi_2 \colon X \xrightarrow{\sim} H_2 \cap \Xi$, as linear sections of Ξ. We will call them *equivalent* if $\phi_2 \circ \phi_1^{-1}$ is the restriction of a transformation from $\mathrm{PSL}(8)$.

PROPOSITION 1.2. *A generic quartic 3-fold admits a 7-parameter family of non-equivalent representations as linear sections of the Pfaffian hypersurface in $\mathbb{P}^{27}$.*

Proof. We use the same argument as in [AR, Thm. (47.3)]. The family of quartic 3-folds in $\mathbb{P}^4$ is parameterized by $\mathbb{P}^{69}$ and that of the Pfaffian representations of quartic 3-folds by an open set in the variety $\operatorname{Lin}(\mathbb{P}^4, \mathbb{P}^{27})$ of linear morphisms between the two projective spaces. We shall therefore specify one particular quartic 3-fold $X_0 = \{F_0 = 0\}$ that admits a Pfaffian representation $F_0 = \operatorname{Pf}(M_0)$; then we will show that the differential of the map $f \colon \operatorname{Lin}(\mathbb{P}^4, \mathbb{P}^{27}) \dashrightarrow \mathbb{P}^{69}$ at M_0 is surjective, and this will imply that f is dominant.

Let

$$M_0 = \begin{bmatrix} 0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_1 & 0 \\ -x_1 & 0 & 0 & x_5 & 0 & 0 & -x_3 & -x_1 \\ -x_2 & 0 & 0 & x_1 & x_1 & 0 & 0 & -x_4 \\ -x_3 & -x_5 & -x_1 & 0 & x_2 & 0 & 0 & 0 \\ -x_4 & 0 & -x_1 & -x_2 & 0 & x_3 & x_1 & 0 \\ -x_5 & 0 & 0 & 0 & -x_3 & 0 & x_4 & x_2 \\ -x_1 & x_3 & 0 & 0 & -x_1 & -x_4 & 0 & x_5 \\ 0 & x_1 & x_4 & 0 & 0 & -x_2 & -x_5 & 0 \end{bmatrix}$$

and

$$\begin{aligned} F_0 = \mathrm{Pf}(M_0) &= x_1^3x_2 - x_1^3x_3 + x_2^3x_3 - x_1x_2x_3^2 - x_1x_2^2x_4 + x_1^2x_3x_4 \\ &\quad + x_1x_2x_3x_4 + x_3^3x_4 - x_1^2x_4^2 + x_1x_2x_4^2 + x_1^3x_5 - x_1^2x_2x_5 \\ &\quad - x_1x_2^2x_5 - x_1^2x_3x_5 + x_1x_3x_4x_5 + x_2x_3x_4x_5 + x_4^3x_5 \\ &\quad + x_2x_3x_5^2 - x_1x_4x_5^2 + x_1x_5^3. \end{aligned}$$

A point $M \in \mathrm{Lin}(\mathbb{P}^4, \mathbb{P}^{27})$ is the proportionality class of an 8×8 skew-symmetric matrix of linear forms l_{ij} and is given by its $5 \cdot 28 = 140$ homogeneous coordinates (a_{ijk}) such that $l_{ij} = \sum_k a_{ijk}x_k$ $(0 \le i < j \le 8,\ 1 \le k \le 5)$. We have $\partial f(M)/\partial a_{ijk} = x_k \,\mathrm{Pf}_{ij}(M)$, where $\mathrm{Pf}_{ij}(M)$ denotes the Pfaffian of the 6×6 matrix obtained by deleting the ith and jth rows and the ith and jth columns of M.

Computation by the Macaulay 2 program [GS] shows that, for the matrix M_0, the 140 quartic forms $x_k \,\mathrm{Pf}_{ij}(M_0)$ generate the whole 70-dimensional space of quinary quartic forms; hence f is of maximal rank at M_0. One can also easily make Macaulay 2 verify that X_0 is in fact nonsingular, though this is not essential for our proof.

It remains to verify that the generic fiber of the induced map

$$\bar{f} \colon \mathrm{PGL}(5)\backslash \mathrm{Lin}(\mathbb{P}^4, \mathbb{P}^{27}) / \mathrm{PGL}(8) \dashrightarrow \mathrm{PGL}(5)\backslash \mathbb{P}^{69}$$

is 7-dimensional. By counting dimensions, one sees that this is equivalent to the fact that the stabilizer of a generic point of the Grassmannian

$$G(5, 28) = \mathrm{PGL}(5)\backslash \mathrm{Lin}(\mathbb{P}^4, \mathbb{P}^{27})$$

in $\mathrm{PGL}(8)$ is 0-dimensional.

Take a generic 4-dimensional linear subspace $H \subset \mathbb{P}^{27}$. Then the quartic 3-fold $X = H \cap \Xi$ is generic and hence $\mathrm{Aut}(X)$ is trivial. Thus the stabilizer G_H of H in $\mathrm{PGL}(8)$ acts trivially on X and hence on H. This implies the triviality of G_H by (5.3) of [B]. □

Let now $\mathcal{K}$ be the kernel bundle on Ξ whose fiber at $x \in \Xi$ is $\ker x$. Thus $\mathcal{K}$ is a rank-2 vector subbundle of the trivial rank-8 vector bundle $E_\Xi = E \otimes_{\mathbb{C}} \mathcal{O}_\Xi$ over $\Xi_0 = \Xi \setminus \Omega$. Let $\phi \colon X \to H \cap \Xi$ be a representation of a nonsingular quartic 3-fold $X \subset \mathbb{P}^4$ as a linear section of Ξ. Giving ϕ is equivalent to specifying a skew-symmetric 8×8 matrix M of linear forms such that the equation of X is $\mathrm{Pf}(M) = 0$. Such a representation yields a rank-2 vector bundle $\mathcal{E} = \mathcal{E}_\phi$ on X that is defined by $\mathcal{E} = \phi^*\mathcal{K}$. According to [B, Prop. 8.2], the scheme of zeros of any section $s \neq 0$ of $\mathcal{E}$ is an arithmetically Cohen–Macaulay 1-dimensional scheme C of degree 14 that is not contained in any quadric hypersurface and such that its canonical bundle $\omega_C \simeq \mathcal{O}_C(2)$. Varieties satisfying the latter condition are usually called half-canonical. Moreover, $\mathcal{E}$ is also ACM and has a resolution of the form

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-1)^8 \xrightarrow{M} \mathcal{O}_{\mathbb{P}^4}^8 \to \mathcal{E} \to 0. \tag{1}$$

This implies, in particular, that two Pfaffian representations ϕ_1, ϕ_2 are equivalent if and only if the corresponding vector bundles $\mathcal{E}_1, \mathcal{E}_2$ are isomorphic. By (8.1) in [B], $\mathcal{E}$ can be given also by Serre's construction as the middle term of the extension

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{I}_{C,X}(3) \to 0, \tag{2}$$

where $\mathcal{I}_{C,X}$ denotes the ideal sheaf of C in X. Thus, the following assertion holds.

Corollary 1.3. *A generic quartic* 3*-fold* $X \subset \mathbb{P}^4$ *has a* 7*-dimensional family of isomorphism classes of rank-*2 *ACM vector bundles* $\mathcal{E}$ *with* $\det \mathcal{E} \simeq \mathcal{O}(3)$ *and* $h^0(\mathcal{E}) = 8$, *and these bundles are characterized by one of the following equivalent properties*:

- (i) $\mathcal{E}$ *as a sheaf on* $\mathbb{P}^4$ *possesses a resolution of the form* (1) *with a skew-symmetric matrix of linear forms* M;
- (ii) *the scheme of zeros of any section* $s \neq 0$ *of* $\mathcal{E}$ *is an ACM half-canonical curve* C *of degree* 14 *and arithmetic genus* 15 *that is not contained in any quadric hypersurface in* $\mathbb{P}^4$;
- (iii) $\mathcal{E}$ *can be obtained by Serre's construction from a curve* $C \subset X$ *as in* (ii).

In fact, the vector bundles under consideration are stable, so the 7-parameter family just described is a part of the moduli space of vector bundles.

Theorem 1.4. *Let* X *be a generic quartic* 3*-fold, and let* $M_X(2; -1, 6)$ *be the moduli space of stable rank-*2 *vector bundles* $\mathcal{G}$ *on* X *with Chern classes* $c_1 = -[H]$ *and* $c_2 = 6[l]$, *where* $[l] \in H^2(X, \mathbb{Z})$ *is the class of a line. Then the isomorphism classes of the ACM vector bundles of the form* $\mathcal{G} = \mathcal{E}(-2)$, *where* $\mathcal{E}$ *are vector bundles introduced in Corollary 1.3, form an irreducible open subset* M_X *of dimension* 7 *in the nonsingular locus of* $M_X(2; -1, 6)$.

Proof.

Stability. If $\mathcal{E}$ is given by the extension (2), then twisting by $\mathcal{O}_X(-2)$ and using $h^0(\mathcal{I}_{C,X}(k)) = 0$ for $k \leq 2$ ((ii) of Lemma 1.3) yields that $h^0(\mathcal{E}(-2)) = 0$. The stability follows from Lemma 1.1.

Smoothness and dimension. The stability implies that $\mathcal{E}$ is simple; that is, $h^0(\mathcal{E}^\vee \otimes \mathcal{E}) = 1$. Hence the tangent space $T_{[\mathcal{E}]}M_X(2; -1, 6)$ at $[\mathcal{E}]$ is identified with $\mathrm{Ext}^1(\mathcal{E}, \mathcal{E}) = H^1(X, \mathcal{E}^\vee \otimes \mathcal{E})$. By [B, (8.9)], the vanishing of $h^2(\mathcal{E}nd_0(\mathcal{E}))$ follows from the fact that the map f, introduced in the proof of Proposition 1.2, is dominant. Hence $M_X(2;-1,6)$ is smooth at $[\mathcal{E}]$ of local dimension $\dim_{[\mathcal{E}]} M_X(2;-1,6) = h^1(\mathcal{E}^\vee \otimes \mathcal{E})$.

Since $\operatorname{rk}\mathcal{E} = 2$, we have $\mathcal{E}^\vee \simeq \mathcal{E} \otimes (\det \mathcal{E})^{-1} \simeq \mathcal{E}(-3)$. By Serre duality, $h^3(\mathcal{E}^\vee \otimes \mathcal{E}) = h^0(\mathcal{E}^\vee \otimes \mathcal{E}(-1)) = 0$. By (1), $h^0(\mathcal{E}(-3)) = \chi(\mathcal{E}(-3)) = 0$. Together with the ACM property for $\mathcal{E}$, this gives $h^i(\mathcal{E}(-3)) = 0$ for all $i \in \mathbb{Z}$. Now, from (2) tensored by $\mathcal{E}(-3)$, we obtain the isomorphisms

$$H^i(\mathcal{E}^\vee \otimes \mathcal{E}) = H^i(\mathcal{E} \otimes \mathcal{E}(-3)) = H^i(\mathcal{E} \otimes \mathcal{I}_C) \quad \forall i \in \mathbb{Z}. \tag{3}$$

Further, the restriction sequence

$$0 \to \mathcal{E} \otimes \mathcal{I}_C \to \mathcal{E} \to \mathcal{E}\big|_C \to 0 \tag{4}$$

yields $\chi(\mathcal{E} \otimes \mathcal{I}_C) = \chi(\mathcal{E}) - \chi(\mathcal{E}\big|_C) = 8 - 14 = -6$, so $h^1(\mathcal{E}^\vee \otimes \mathcal{E}) = 7$ and we are done. □

2. Curves of Degree 14 and Genus 15 in $\mathbb{P}^4$

Let $X = \{F = 0\}$ be a generic quartic 3-fold in $\mathbb{P}^4$, and let $X = H \cap \Xi$ (so the $\mathbb{P}^4$ is identified with H) be a Pfaffian representation for X. For the sake of functoriality, we should have defined Ξ as embedded in $\mathbb{P}(\wedge^2(E^\vee))$ so that the points $x \in X$ could be interpreted as alternating bilinear forms of rank 6 on E, whilst $G = G(2,8) \subset \mathbb{P}(\wedge^2 E)$; to avoid this dichotomy, we will work in coordinates and identify E with $E^\vee$. Let $\mathcal{E}$ be the corresponding rank-2 vector bundle and C the scheme of zeros of a section $s \neq 0$ of $\mathcal{E}$. Let $H_{14,15}$ (resp., $H^X_{14,15}$) denote the union of the components of the Hibert scheme of curves in $\mathbb{P}^4$ (resp., in X) whose generic points represent a curve C as before. For generic s, the curve C is nonsingular.

Similarly to the previous section, introduce the rank filtration on the 7×7 skew-symmetric matrices: $G' = G(2,7) \subset Z \subset \mathbb{P}^{20} = \mathbb{P}(\wedge^2(\mathbb{C}^7))$. According to [R], we have $\dim G' = 10$, $\deg G' = 42$, $\omega_{G'} = \mathcal{O}_{G'}(-7)$, $\dim Z = 17$, $\deg Z = 14$, and $\omega_Z = \mathcal{O}_Z(-14)$. We will identify G' with a subvariety of G for the standard inclusion $\mathbb{C}^7 \subset \mathbb{C}^8$.

PROPOSITION 2.1. *The following assertions hold.*

(i) $h^0(\mathcal{N}_{C/X}) = 14$ *and* $h^1(\mathcal{N}_{C/X}) = 0$; *hence* $H^X_{14,15}$ *is smooth at* $[C]$ *of local dimension* 14.

(ii) $h^0(\mathcal{N}_{C/\mathbb{P}^4}) = 56$ *and* $h^1(\mathcal{N}_{C/\mathbb{P}^4}) = 0$; *hence* $H_{14,15}$ *is smooth at* $[C]$ *of local dimension* 56.

(iii) *C can be identified with a section of the rank-4 locus Z of 7×7 skew-symmetric matrices by a 4-dimensional linear subspace $L \subset \mathbb{P}^{20}$.*

Proof. (i) The restriction sequence (4) yields $h^2(\mathcal{E} \otimes \mathcal{I}_C) = h^1(\mathcal{E}\big|_C)$. We proved in Theorem 1.4 the vanishing of $h^2(\mathcal{E} \otimes \mathcal{I}_C) = h^2(\mathcal{E}nd_0(\mathcal{E}))$. As C is the scheme of zeros of a section of $\mathcal{E}$, we have $\mathcal{E}\big|_C \simeq \mathcal{N}_{C/X}$. Thus we obtain $h^1(\mathcal{N}_{C/X}) = 0$. By Riemann–Roch, $h^0(\mathcal{N}_{C/X}) = 14$ and we are done.

(ii) We have $h^1(\mathcal{N}_{C/X}) = 0$. The normal bundle sequence

$$0 \to \mathcal{N}_{C/X} \to \mathcal{N}_{C/\mathbb{P}^4} \to \mathcal{O}_C(4) \to 0$$

implies the vanishing of $h^1(\mathcal{N}_{C/\mathbb{P}^4})$. By Riemann–Roch, $h^0(\mathcal{N}_{C/\mathbb{P}^4}) = 56$.

(iii) The sections of $\mathcal{E}$ are naturally identified with elements of $E^\vee$ via the embedding of $\mathcal{E}$ into the trivial rank-8 vector bundle $E_X = E \otimes \mathcal{O}_X$. Let $\mathrm{Cl}\colon\ \Xi \setminus \Omega \to G = G(2, 8)$ be the classifying map that sends each $x \in \Xi \setminus \Omega$ to the projectivized kernel of x, considered as a point of G, with Cl_X the restriction of Cl to X. We can choose the coordinates in E in such a way that $s = x_7$. Hence $C = \mathrm{Cl}_X^{-1}(\sigma_{11}(\mathbb{P}^6))$, where $\mathbb{P}^6$ is the hyperplane $\{x_7 = 0\}$ in $\mathbb{P}^7 = \mathbb{P}(E)$ and $\sigma_{11}(\mathbb{P}^6) = G' \subset G$ is the Schubert subvariety of all the lines contained in the hyperplane. We can also write $C = \mathrm{Cl}^{-1}(G') \cap H$. The closure of the 24-fold $\mathrm{Cl}^{-1}(G')$ in Ξ is defined by the seven cubic Pfaffians $\mathrm{Pf}_{r7}(x)$, $0 \le r \le 6$.

As cubic forms, the Pfaffians $\mathrm{Pf}_{r7}(x)$, $0 \le r \le 6$, do not depend on the variables x_{p7}, $0 \le p \le 7$. Therefore $\mathrm{Cl}^{-1}(G')$ is isomorphic to the cone $C(Z)$ with vertex (or ridge) $\bar{\mathbb{P}}^6 = \langle e_{07}, \dots, e_{67}\rangle$ and base

$$Z = \{x' : \mathrm{Pf}_{07}\, x' = \cdots = \mathrm{Pf}_{67}\, x' = 0\} \subset \mathbb{P}(\wedge^2\langle e_0, \dots, e_6\rangle);$$

here $x' = (x_{pq})_{0 \le p,q \le 6}$ is the eighth principal adjoint matrix of the matrix x (i.e., x' is obtained from x by deleting its last column and row). It is well known that the vanishing of the principal minors of order $2n$ of a skew-symmetric $(2n+1) \times (2n+1)$ matrix is equivalent to the vanishing of all its minors of order $2n$, so Z is the locus of 7×7 skew-symmetric matrices of rank 4. The projection $\pi : \mathbb{P}^{27} \dashrightarrow \mathbb{P}^{20}$ with center $\bar{\mathbb{P}}^6$ maps isomorphically (for generic H) the intersection $H \cap C(Z)$ to $L \cap Z$, where $L = \pi(H)$. This finishes the proof. □

Let $\mathcal{M}_g$ denote the moduli space of smooth curves of genus g and let $\mathcal{M}_g^r$ be the subvariety of $\mathcal{M}_g$ parameterizing half-canonical curves with a theta-characteristic D such that $\dim|D| = r$.

Corollary 2.2. *The following assertions hold.*

(i) *$H_{14,15}$ is irreducible of dimension* 56.

(ii) *For generic $\mathcal{L} \in \mathrm{Lin}(\mathbb{P}^4, \mathbb{P}^{20})$, the stabilizer of $\mathcal{L}$ in $\mathrm{PGL}(7)$, acting on the right, is finite; the natural map $\mathrm{Lin}(\mathbb{P}^4, \mathbb{P}^{20})/\,\mathrm{PGL}(7) \dashrightarrow H_{14,15}$ is generically finite.*

(iii) *The natural map $g\colon \mathrm{PGL}(5)\backslash \mathrm{Lin}(\mathbb{P}^4, \mathbb{P}^{20})/\,\mathrm{PGL}(7) \dashrightarrow \mathcal{M}_{15}^4$ is generically finite, and its image is a 32-dimensional irreducible component $\mathring{\mathcal{M}}_{15}^4$ of $\mathcal{M}_{15}^4$.*

Proof. (i) Indeed, $H_{14,15}$ is the image of $\mathrm{Lin}(\mathbb{P}^4, \mathbb{P}^{20})$.

(ii) This follows from the count of dimensions:

$$\dim \mathrm{Lin}(\mathbb{P}^4, \mathbb{P}^{20}) - \dim \mathrm{PGL}(7) = (5 \cdot 21 - 1) - (7^2 - 1) = 56 = \dim H_{14,15}.$$

(iii) According to Harris [H], if $r \le \frac{1}{2}(g-1)$ then the codimension of any component of $\mathcal{M}_g^r$ in $\mathcal{M}_g$ is at most $\frac{1}{2}r(r+1)$. Applying this to our case, we see that the dimension of every component of $\mathcal{M}_{15}^4$ is at least 32. Hence the component $\mathring{\mathcal{M}}_{15}^4$, containing the image of $H_{14,15}$, is of dimension ≥ 32. The dimension of $\mathrm{PGL}(5)\backslash^{\mathrm{Lin}(\mathbb{P}^4,\mathbb{P}^{20})}/\mathrm{PGL}(7)$ is 32, so it remains to show that g is dominant over $\mathring{\mathcal{M}}_{15}^4$.

Take a generic C from the image of g; C is a smooth ACM curve in $\mathbb{P}^4$. By the definition of $\mathcal{M}_g^r$, every small (analytic or étale) deformation of C is accompanied by a deformation of the theta-characteristic D embedding C into $\mathbb{P}^4$. Because the ACM property is generic, any generic small deformation of C is again in the image of g, and we are done. □

REMARK 2.32. In Corollary 2.2(ii), the stabilizer $G_{\mathcal{L}}$ of $\mathcal{L}$ might act by nontrivial automorphisms of C. Since $\mathrm{Aut}(C)$ is finite, the subgroup $H_{\mathcal{L}}$ fixing pointwise C, and hence $L = \mathcal{L}(\mathbb{P}^4)$, is of finite index in $G_{\mathcal{L}}$. The first assertion of (ii) is therefore equivalent to saying that $H_{\mathcal{L}}$ is finite. One can strengthen this assertion: The subgroup of $\mathrm{PGL}(2n+1)$ fixing pointwise a generic linear $\mathbb{P}^2 \subset \mathbb{P}(\wedge^2\mathbb{C}^{2n+1})$ for $n \geq 2$ is finite. This is easily reduced to the $2n$-dimensional case, stated in [B, (5.3)].

PROPOSITION 2.4. *Let $\mathring{H}_{14,15}^X \subset H_{14,15}^X$ be the locus of ACM half-canonical curves $C \subset X$ of degree 14 and arithmetic genus 15 not contained in any quadric hypersurface in $\mathbb{P}^4$, and let $M_X \subset M_X(2; -1, 6)$ be the open set defined in Theorem 1.4. Then the Serre construction defines a morphism $\phi\colon \mathring{H}_{14,15}^X \to M_X$ with fiber $\mathbb{P}^7$. Moreover, $\mathring{H}_{14,15}^X$ is isomorphic locally in the étale topology over M_X to a projectivized rank-8 vector bundle on M_X.*

Proof. It is easily seen that $\dim \mathrm{Ext}^1(\mathcal{I}_C(3), \mathcal{O}_X) = 1$ and so, given C, the Serre construction determines $\mathcal{E}$ uniquely. This yields ϕ as a set-theoretic map. An obvious relativization of the Serre construction shows that it is indeed a morphism.

Further, we have $h^0(\mathcal{E} \otimes \mathcal{I}_C) = 1$ by stability of $\mathcal{E}$ and (3), so the projective space $\mathbb{P}^7 = \mathbb{P}(H^0(\mathcal{E}))$ is injected into $H_{14,15}^X$ by sending each section $s \neq 0$ of $\mathcal{E}$ to its scheme of zeros. Hence the fibers of ϕ are set-theoretically 7-dimensional projective spaces. The proof of the last assertion of the proposition is completely similar to that of Lemma 5.3 in [MT]. □

3. Abel–Jacobi Map

We shall review briefly the Clemens–Griffiths technique for the calculation of the differential of the Abel–Jacobi map, following Welters [W, Sec. 2]. Let X be a nonsingular projective 3-fold with $h^{03} = 0$, and let $X \subset W$ be an embedding in a nonsingular (possibly noncomplete) 4-fold. Let $\Phi\colon B \to J^2(X)$ be the Abel–Jacobi map, where B is the base of a certain family of curves on X. The differential $d\Phi_{[Z]}$ at a point $[Z] \in B$, representing a curve Z, factors into the composition of the infinitesimal classifying map $T_{B,b} \to H^0(Z, \mathcal{N}_{Z/X})$ and of the universal "infinitesimal Abel–Jacobi map" $\psi_Z\colon H^0(Z, \mathcal{N}_{Z/X}) \to H^1(X, \Omega_X^2)^\vee = T_{J_1(X),0}$. The adjoint $\psi_Z^\vee$ is identified by the following commutative square:

$$\begin{array}{ccc} H^0(X, \mathcal{N}_{X/W} \otimes \omega_X) & \xrightarrow{R} & H^1(X, \Omega_X^2) \\ {\scriptstyle r_Z}\downarrow & & \downarrow{\scriptstyle \psi_Z^\vee} \\ H^0(Z, \mathcal{N}_{X/W} \otimes \omega_X\big|_Z) & \xrightarrow{\beta_Z} & H^0(Z, \mathcal{N}_{Z/X})^\vee. \end{array} \tag{5}$$

Here r_Z is the map of restriction to Z, and the whole square (upon natural identifications) is the $H^0 \to H^1$ part of the commutative diagram of the long exact cohomology sequences associated to the following commutative diagram of sheaves:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_X^2 & \longrightarrow & \Omega_W^3 \otimes \mathcal{N}_{X/W} & \longrightarrow & \Omega_X^3 \otimes \mathcal{N}_{X/W} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega_X^3 \otimes \mathcal{N}_{Z/X} & \longrightarrow & \Omega_X^3 \otimes \mathcal{N}_{Z/W} & \longrightarrow & \Omega_X^3 \otimes \mathcal{N}_{X/W} \otimes \mathcal{O}_Z & \longrightarrow & 0. \end{array} \tag{6}$$

Specifying all this to the case where (a) X is a generic quartic 3-fold, (b) $Z = C \subset X$ is a generic curve from $H^X_{14,15}$, and (c) $W = \mathbb{P}^4$, we see that the dimensions in (5) form the array $\begin{pmatrix} 35 & 30 \\ 28 & 14 \end{pmatrix}$, that R and r_C are surjective, and that corank $\beta_C =$ corank $\psi_C^\vee = h^1(\mathcal{N}_{C/\mathbb{P}^4}(-1))$. Dualizing, we obtain the following.

Lemma 3.1. *For C, X as before,* $\dim\ker\psi_C = h^1(\mathcal{N}_{C/\mathbb{P}^4}(-1))$ *and* $\dim \operatorname{im} \psi_C = 14 - h^1(\mathcal{N}_{C/\mathbb{P}^4}(-1))$.

We have $\chi(\mathcal{N}_{C/\mathbb{P}^4}(-1)) = 14$ and so $h^0(\mathcal{N}_{C/\mathbb{P}^4}(-1)) = 14 + h^1(\mathcal{N}_{C/\mathbb{P}^4}(-1))$.

Lemma 3.2. $h^0(\mathcal{N}_{C/\mathbb{P}^4}(-1)) = 21$.

Proof. Obvious exact triples show that the assertion is equivalent to

$$h^2(\mathcal{I}^2_{C,\mathbb{P}^4}(3)) = 21, \qquad h^i(\mathcal{I}^2_{C,\mathbb{P}^4}(3)) = 0 \quad \forall i \neq 2.$$

The last equalities follow immediately from the resolution for $\mathcal{I}^2_{C,\mathbb{P}^4}(3)$, obtained from [R, (4)] by restriction to $L = \mathbb{P}^4 \subset \mathbb{P}^6$ and twisting by $\mathcal{O}(3)$:

$$0 \to 21\mathcal{O}_{\mathbb{P}^4}(-5) \to 48\mathcal{O}_{\mathbb{P}^4}(-4) \to 28\mathcal{O}_{\mathbb{P}^4}(-3) \to \mathcal{I}^2_{C,\mathbb{P}^4}(3) \to 0. \qquad \square$$

Lemmas 3.1 and 3.2 imply that the Abel–Jacobi map Φ has a 7-dimensional image in the 30-dimensional intermediate Jacobian $J^2(X)$ and 7-dimensional fibers. We can easily identify the irreducible components of the fiber. Indeed, by Proposition 2.4, each C is contained in a $\mathbb{P}^7 = \mathbb{P}(H^0(\mathcal{E})) \subset H^X_{14,15}$. Any rationally connected variety is contracted by the Abel–Jacobi map, so each one of its fibers is a union of these $\mathbb{P}^7$s. Since the dimension of the fiber is 7, the $\mathbb{P}^7$s are irreducible components of the fiber. Because they are fibers of ϕ, the irreducible components do not meet each other and so they are, in fact, connected components. Thus we have proved the following theorem.

Theorem 3.3. *Let X be a generic quartic* 3*-fold. Let $\mathring{H}^X_{14,15} \subset H^X_{14,15}$ be defined as in Proposition 2.4, and let $\Phi\colon \mathring{H}^X_{14,15} \to J^2(X)$ be the Abel–Jacobi map. Then* *the dimension of any component of* $\Phi(\mathring{H}^X_{14,15})$ *is equal to* 7, *and the fibers of* Φ *are the unions of finitely many disjoint* 7*-dimensional projective spaces. The natural map* $\psi\colon M_X \to J^2(X)$, *defined by* $\Phi = \psi \circ \phi$, *is quasi-finite and nonramified on* M_X.

We may immediately derive the following obvious corollary.

Corollary 3.4. *Every component of* M_X *has nonnegative Kodaira dimension*.

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