QUARTIC DOUBLE SOLIDS WITH ICOSAHEDRAL SYMMETRY

IVAN CHELTSOV, VICTOR PRZYJALKOWSKI, CONSTANTIN SHRAMOV

ABSTRACT. We study quartic double solids admitting icosahedral symmetry.

1. INTRODUCTION

To study possible embeddings of a finite group G into the Cremona group

$$\mathrm{Cr}_n(\mathbb{C}) = \mathrm{Bir}(\mathbb{P}^n),$$

one should first describe all three-dimensional G-Mori fiber spaces (see [8, Definition 1.1.5]), and then to decide which of these G-Mori fiber spaces are rational and which are not. To describe all such embeddings up to conjugation, one should also describe G-birational maps between the resulting rational G-Mori fiber spaces. A priori, all three problems (classification, rationality and conjugation) can be solved for a given group G. Actually, this has been done only for $n = 2$ (see [13]) and for some very special finite groups for $n = 3$.

In [15], Prokhorov managed to find all finite *simple* non-abelian subgroups of $\mathrm{Cr}_3(\mathbb{C})$. He proved that the six groups $\mathfrak{A}_5$, $\mathrm{PSL}_2(\mathbb{F}_7)$, $\mathfrak{A}_6$, $\mathrm{SL}_2(\mathbb{F}_8)$, $\mathfrak{A}_7$, and $\mathrm{PSp}_4(\mathbb{F}_3)$ are the only non-abelian finite simple subgroups of $\mathrm{Cr}_3(\mathbb{C})$. The former three of these six groups actually admit embeddings to $\mathrm{Cr}_2(\mathbb{C})$, and $\mathfrak{A}_5$ is also realized as a subgroup of $\mathrm{Cr}_1(\mathbb{C}) = \mathrm{PGL}_2(\mathbb{C})$, while the latter three groups are new three-dimensional artefacts.

The groups $\mathrm{SL}_2(\mathbb{F}_8)$, $\mathfrak{A}_7$, and $\mathrm{PSp}_4(\mathbb{F}_3)$ do not act faithfully on three-dimensional conic bundles and del Pezzo fibrations. Thus, the only G-Mori fiber spaces with

$$G \in \{\mathrm{SL}_2(\mathbb{F}_8), \mathfrak{A}_7, \mathrm{PSp}_4(\mathbb{F}_3)\}$$

are G-Fano threefolds, i.e., Fano threefolds with terminal singularities such that all G-invariant Weil divisors on them are $\mathbb{Q}$-rationally equivalent to multiples of the anticanonical ones. Prokhorov classified all such threefolds in [15]. This together with [2] and [7, Corollary 1.22] implies that $\mathrm{Cr}_3(\mathbb{C})$ contains two (unique, respectively) subgroups isomorphic to $\mathrm{PSp}_4(\mathbb{F}_3)$ ($\mathrm{SL}_2(\mathbb{F}_8)$ or $\mathfrak{A}_7$, respectively) up to conjugation.

The papers [6] and [7] describe several non-conjugate embeddings of the groups $\mathrm{PSL}_2(\mathbb{F}_7)$ and $\mathfrak{A}_6$ into $\mathrm{Cr}_3(\mathbb{C})$ using a similar approach, although a complete answer is not known in this two cases. Similarly, the book [8] is devoted to the icosahedral subgroups in $\mathrm{Cr}_3(\mathbb{C})$. In particular, it describes three non-conjugate embeddings of the group $\mathfrak{A}_5$ into $\mathrm{Cr}_3(\mathbb{C})$, all arising from some $\mathfrak{A}_5$-Mori fiber spaces.

Since quartic double solids are known to have interesting geometrical properties, it is interesting to study those of them who admit an icosahedral symmetry. Rational quartic double solids of this kind are of special interest, since they would provide embeddings of the group $\mathfrak{A}_5$ into $\mathrm{Cr}_3(\mathbb{C})$. In this paper, we describe all quartic double solids with an action of $\mathfrak{A}_5$, and study their rationality using the results obtained in the prequel [5]. In particular, we construct one more embedding

$$\mathfrak{A}_5 \hookrightarrow \mathrm{Cr}_3(\mathbb{C}),$$

and we show that this embedding is not conjugate to any of the three embeddings described in the book [8].

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2. Quartic double solids with an action of $\mathfrak{A}_5$

Let $\tau \colon X \to \mathbb{P}^3$ be double cover branched over a (possibly reducible) reduced quartic surface S.

Remark 2.1. Recall from [21, Theorem 5] and [23, Corollary 4.7(b)] that X is irrational provided that the surface S is smooth. Therefore, we will be mostly interested in the cases when S is singular.

Suppose that X admits a faithful action of the icosahedral group $\mathfrak{A}_5$.

Remark 2.2. Since τ is given by the linear system $|-\frac{1}{2}K_X|$, we see that the action of $\mathfrak{A}_5$ descents to its action on $\mathbb{P}^3$. Vice versa, every $\mathfrak{A}_5$-action on $\mathbb{P}^3$ that leaves the quartic surface S invariant can be lifted to the corresponding double cover.
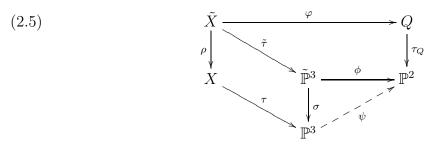
Up to conjugation, the group

$$\mathrm{Aut}(\mathbb{P}^3) \cong \mathrm{PGL}_4(\mathbb{C})$$

contains five subgroups isomorphic to $\mathfrak{A}_5$ (cf. [3, Chapter VII]). Namely, denote by I the trivial representation of $\mathfrak{A}_5$. Let W_3 be one of the two irreducible three-dimensional representations of $\mathfrak{A}_5$, and let W_4 be the irreducible four-dimensional representation of $\mathfrak{A}_5$. Let U_2 and U_2' be two non-isomorphic two-dimensional representations of the central extension $2.\mathfrak{A}_5$ of the group $\mathfrak{A}_5$, and let $U_4 \cong \mathrm{Sym}^3(U_2)$ be the faithful four-dimensional irreducible representation of $2.\mathfrak{A}_5$. Then $\mathbb{P}^3$ equipped with a faithful action of the group $\mathfrak{A}_5$ can be identified with one of the following projective spaces: $\mathbb{P}(W_4)$, $\mathbb{P}(I \oplus W_3)$, $\mathbb{P}(U_2 \oplus U_2)$, $\mathbb{P}(U_2 \oplus U_2')$, or $\mathbb{P}(U_4)$.

Remark 2.3. Computing the symmetric powers $\mathrm{Sym}^2(U_2 \oplus U_2)$ and $\mathrm{Sym}^4(U_2 \oplus U_2)$, we see that the only $\mathfrak{A}_5$-invariant quartic surface in $\mathbb{P}(U_2 \oplus U_2)$ is not reduced (it is the unique $\mathfrak{A}_5$-invariant quadric taken with multiplicity two). Similarly, we see that there are no $\mathfrak{A}_5$-invariant quartic surfaces in $\mathbb{P}(U_2 \oplus U_2')$ at all.

Since the quartic surface S is reduced, our $\mathbb{P}^3$ can be identified with either $\mathbb{P}(U_4)$, or $\mathbb{P}(I \oplus W_3)$, or $\mathbb{P}(W_4)$. Let us start with the case $\mathbb{P}^3 = \mathbb{P}(U_4)$.

Theorem 2.4. *Suppose that $\mathbb{P}^3 = \mathbb{P}(U_4)$. Then $\mathfrak{A}_5$-invariant quartic surfaces in $\mathbb{P}^3$ form a pencil $\mathcal{P}$. This pencil contains exactly two (isomorphic) singular surfaces. Let $\mathcal{S}$ be one of these surfaces. Then the singular locus of $\mathcal{S}$ is a twisted cubic curve $\mathscr{C}$. Moreover, if $S = \mathcal{S}$, then there exists a commutative diagram*

(2.5)

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\;\varphi\;} & Q \\
\rho \downarrow \;\;\;\; \searrow^{\tilde{\tau}} & & \downarrow \tau_Q \\
X \;\;\;\;\;\;\; & \tilde{\mathbb{P}}^3 \xrightarrow{\;\phi\;} & \mathbb{P}^2 \\
\;\;\;\; \searrow^{\tau} & \downarrow \sigma \;\; \nearrow_{\psi} & \\
& \mathbb{P}^3 &
\end{array}
$$

Here the morphism σ is a blow up of the curve $\mathscr{C}$, the morphism ρ is a blow up of the preimage of the curve $\mathscr{C}$ on X, the morphism $\tilde{\tau}$ is a double cover branched over the proper transform of the surface $\mathcal{S}$ on $\tilde{\mathbb{P}}^3$, the rational map ψ is given by the linear system of quadrics in $\mathbb{P}^3$ passing through $\mathscr{C}$, the surface Q is a smooth quadric, τ_Q is a double cover branched over the unique $\mathfrak{A}_5$-invariant conic in $\mathbb{P}^2$, and ϕ and φ are $\mathbb{P}^1$-bundles. In particular, in this case X is rational.

Proof. Restrict U_4 to the subgroups of $2.\mathfrak{A}_5$ isomorphic to $2.\mathfrak{A}_4$, $2.\mathrm{D}_{10}$, $2.\mathfrak{S}_3$, and $2.\mathrm{D}_4$, where $2.\mathrm{D}_{2n}$ denotes the central extension of the dihedral group D_{2n} of order $2n$. We see that none of these respresentations has one-dimensional subrepresentations. Thus, $\mathbb{P}^3$ does not contain $\mathfrak{A}_5$-orbits of lengths 5, 6, 10, and 15.

One has $\mathrm{Sym}^2(U_4) = W_3 \oplus W_3' \oplus W_4$, where W_3' is the irreducible representation of $\mathfrak{A}_5$ that is not isomorphic to W_3. Denote by $\mathcal{Q}$ and $\mathcal{Q}'$ the linear system of quadrics in $\mathbb{P}^3$ that correspond to W_3 and W_3', respectively. Since $\mathbb{P}^3$ does not contain $\mathfrak{A}_5$-orbits of lengths less or equal to eight, we see that the base loci of $\mathcal{Q}$ and $\mathcal{Q}'$ contain $\mathfrak{A}_5$-invariant curves $\mathscr{C}$ and $\mathscr{C}'$, respectively. The degrees of these curves must be less than four. Since U_4 is an irreducible representation, we see that both $\mathscr{C}$ and $\mathscr{C}'$ are twisted cubic curves. This also implies that the base loci of $\mathcal{Q}$ and $\mathcal{Q}'$ are exactly the curves $\mathscr{C}$ and $\mathscr{C}'$, respectively.

The lines in $\mathbb{P}^3$ that are tangent to the curves $\mathscr{C}$ and $\mathscr{C}'$ sweep out quartic surfaces $\mathcal{S}$ and $\mathcal{S}'$, respectively. These surfaces are $\mathfrak{A}_5$-invariant. The singular loci of $\mathcal{S}$ and $\mathcal{S}'$ are the curves $\mathscr{C}$ and $\mathscr{C}'$, respectively. In particular, the surfaces $\mathcal{S}$ and $\mathcal{S}'$ are different. Their singularities along these curves are locally isomorphic to a product of $\mathbb{A}^1$ and a cusp.

Suppose that $S = \mathcal{S}$. Let us construct the commutative diagram (2.5). Taking a blow up $\sigma \colon \tilde{\mathbb{P}}^3 \to \mathbb{P}^3$ of the curve $\mathscr{C}$, we obtain an $\mathfrak{A}_5$-equivariant $\mathbb{P}^1$-bundle

$$
\phi \colon \tilde{\mathbb{P}}^3 \to \mathrm{Sym}^2(\mathscr{C}) \cong \mathbb{P}^2
$$

that is a projectivization of a stable rank two vector bundle $\mathcal{E}$ on $\mathbb{P}^2$ with $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = 2$ defined by the exact sequence

$$
0 \longrightarrow \mathcal{O}_{\mathbb{P}_2}(-1)^{\oplus 2} \longrightarrow \mathcal{O}_{\mathbb{P}_2}^{\oplus 4} \longrightarrow \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}_2}(1) \longrightarrow 0,
$$

see [20, Application 1] for details.

The fibers of ϕ are proper transforms of the secant or tangent lines to $\mathscr{C}$. Moreover, the proper transforms of the tangent lines to $\mathscr{C}$ are mapped by ϕ to the points of the unique $\mathfrak{A}_5$-invariant conic C in $\mathbb{P}^2$. Let $\tau_Q \colon Q \to \mathbb{P}^2$ be a double cover branched over C. Then Q is a smooth quadric surface. A preimage on X of a secant line to $\mathscr{C}$ splits as a union of two smooth rational curves, while a preimage of a tangent line to $\mathscr{C}$ is contained in the

ramification locus of τ. This shows the existence of the commutative diagram (2.5) and the rationality of X. The same construction applies to the case $S = \mathcal{S}'$.

Denote by $\mathcal{P}$ the pencil generated by $\mathcal{S}$ and $\mathcal{S}'$. Computing $\mathrm{Sym}^4(U_4)$, we see that $\mathcal{P}$ contains all $\mathfrak{A}_5$-invariant quartic surfaces in $\mathbb{P}^3$. Since $\mathscr{C}$ is projectively normal, there is an exact sequence of $2.\mathfrak{A}_5$-representations

$$0 \to H^0(\mathcal{O}_{\mathbb{P}^3}(4) \otimes \mathcal{I}_{\mathscr{C}}) \to H^0(\mathcal{O}_{\mathbb{P}^3}(4)) \to H^0(\mathcal{O}_{\mathscr{C}} \otimes \mathcal{O}_{\mathbb{P}^3}(4)) \to 0,$$

where $\mathcal{I}_{\mathscr{C}}$ is the ideal sheaf of $\mathscr{C}$. The $2.\mathfrak{A}_5$-representation $H^0(\mathcal{O}_{\mathscr{C}} \otimes \mathcal{O}_{\mathbb{P}^3}(4))$ contains a one-dimensional subrepresentation corresponding to the unique $\mathfrak{A}_5$-orbit of length 12 in $\mathscr{C} \cong \mathbb{P}^1$. This shows that $\mathcal{P}$ contains a surface that does not pass through $\mathscr{C}$, so that $\mathscr{C}$ is not contained in the base locus of $\mathcal{P}$. Similarly, we see that $\mathscr{C}'$ is not contained in the base locus of $\mathcal{P}$.

Now we suppose that $S \neq \mathcal{S}$ and $S \neq \mathcal{S}'$. Let us show that S is smooth. Assume that this is not the case, i.e. S is singular. Note that S is irreducible, because $\mathbb{P}^3$ contains no $\mathfrak{A}_5$-invariant surfaces of degree less than four.

We claim that S has isolated singularities. Indeed, suppose that S is singular along some $\mathfrak{A}_5$-invariant curve Z. Taking a general plane section of S, we see that the degree of Z is at most three. Thus, Z is a twisted cubic curve, so that either $Z = \mathscr{C}$ or $Z = \mathscr{C}'$. Since neither of these curves is contained in the base locus of $\mathcal{P}$, this would imply that either $S = \mathcal{S}$ or $S = \mathcal{S}'$. The latter is not the case by assumption.

We see that the singularities of S are isolated. Hence, S contains at most two non-Du Val singular points (cf. [10]). This follows from [22, Theorem 1] or from Shokurov's famous [19, Theorem 6.9] applied to the minimal resolution of the surface S. Since the set of all non-Du Val singular points of the surface S must be $\mathfrak{A}_5$-invariant, we see that S has none of them, because U_4 is an irreducible representation of the group $2.\mathfrak{A}_5$. Thus, all singularities of S are Du Val.

By [8, Lemma 6.7.3(iii)], the surface S is nodal, the set $\mathrm{Sing}(S)$ consists of one $\mathfrak{A}_5$-orbit, and

$$\big|\mathrm{Sing}(S)\big| \in \big\{5, 6, 10, 12, 15\big\}.$$

Since $\mathbb{P}^3$ does not contain $\mathfrak{A}_5$-orbits of lengths 5, 6, 10, and 15, we see that $\mathrm{Sing}(S)$ is an $\mathfrak{A}_5$-orbit Σ_{12} of length 12. Therefore, S does not contain other $\mathfrak{A}_5$-orbits of length 12 by [8, Lemma 6.7.3(iv)].

Since $\mathscr{C}$ is not contained in the base locus of $\mathcal{P}$, and $\mathscr{C}$ is contained in $\mathcal{S}$, we see that $\mathscr{C} \not\subset S$. Since $S \cdot \mathscr{C} = 12$ and Σ_{12} is the only $\mathfrak{A}_5$-orbit of length at most 12 in $\mathscr{C} \cong \mathbb{P}^1$, we have $S \cap \mathscr{C} = \Sigma_{12}$. Thus,

$$12 = S \cdot \mathscr{C} \geqslant \sum_{P \in \Sigma_{12}} \mathrm{mult}_P(S) = 2|\Sigma_{12}| = 24,$$

which is absurd. $\qquad\square$

Remark 2.6. Let us use notation of Theorem 2.4. Denote by C the base locus of the pencil $\mathcal{P}$. Then C is an irreducible curve with 24 cusps, and its normalization has genus 9. Indeed, let $\rho \colon \hat{\mathcal{S}} \to \mathcal{S}$ be the normalization of the surface $\mathcal{S}$, let $\hat{\mathscr{C}}$ be the preimage of the curve $\mathscr{C}$ via ρ, and let $\hat{C}$ be the preimage of the curve C via ρ. Then the action of $\mathfrak{A}_5$ lifts to $\hat{\mathcal{S}}$, one has $\hat{\mathcal{S}} \cong \mathbb{P}^1 \times \mathbb{P}^1$, and $\rho^*(\mathcal{O}_{\mathscr{C}} \otimes \mathcal{O}_{\mathbb{P}^3}(1))$ is a divisor of bi-degree $(1, 2)$. This shows that $\hat{\mathscr{C}}$ is a divisor of bi-degree $(1, 1)$, and $\hat{C}$ is a divisor of bi-degree $(4, 8)$. In particular, the action of $\mathfrak{A}_5$ on $\hat{\mathcal{S}}$ is diagonal by [8, Lemma 6.4.3(i)], so that $\hat{\mathscr{C}}$ is irreducible by [8, Lemma 6.4.4(i)]. Note that the curve $\hat{\mathscr{C}}$ is singular. Indeed, denote by
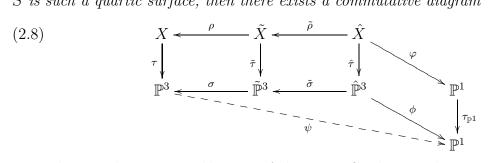
$\mathcal{S}'$ the second singular surface in $\mathcal{P}$. Then its singular locus is a twisted cubic curve $\mathscr{C}'$. Since $\mathscr{C}'$ is not contained in $\mathcal{S}$, we see that the intersection $\mathcal{S} \cap \mathscr{C}'$ is an $\mathfrak{A}_5$-orbit Σ_{12} of length 12. Similarly, we see that the intersection $\mathcal{S}' \cap \mathscr{C}$ is also an $\mathfrak{A}_5$-orbit Σ'_{12} of length 12. These $\mathfrak{A}_5$-orbits Σ_{12} and Σ'_{12} are different, since two different twisted cubic curves cannot have twelve points in common. Since C is the scheme theoretic intersection of the surfaces $\mathcal{S}$ and $\mathcal{S}'$, it must be singular at every point of $\Sigma_{12} \cup \Sigma'_{12}$. Denote by $\hat{\Sigma}_{12}$ and $\hat{\Sigma}'_{12}$ the preimages via ρ of the $\mathfrak{A}_5$-orbits Σ_{12} and Σ'_{12}, respectively. Then $\hat{C}$ is singular in every point of $\hat{\Sigma}'_{12}$. Moreover, $\hat{C}$ is smooth away of $\hat{\Sigma}'_{12}$, because its arithmetic genus is 21, and the surface $\hat{\mathcal{S}}$ does not contain $\mathfrak{A}_5$-orbits of length less than 12. On the other hand, we have
$$\hat{C} \cap \hat{\mathscr{C}} = \hat{\Sigma}_{12},$$
because $\hat{C} \cdot \mathscr{C}' = 12$ and $\hat{\Sigma}_{12} \subset \hat{C}$. This shows that C is an irreducible curve whose only singularities are the points of $\Sigma_{12} \cup \Sigma'_{12}$, and each such point is a cusp of the curve C.

Now let us deal with the case $\mathbb{P}^3 = \mathbb{P}(I \oplus W_3)$.

Theorem 2.7. *Suppose that $\mathbb{P}^3 = \mathbb{P}(I \oplus W_3)$. Then there exists a unique $\mathfrak{A}_5$-invariant conic C in $\mathbb{P}^3$. All $\mathfrak{A}_5$-invariant quadric surfaces in $\mathbb{P}^3$ form a pencil $\mathcal{P}$. Two general quadrics from this pencil are tangent to each other along C. Moroever, any reduced $\mathfrak{A}_5$-invariant quartic surface $\mathbb{P}^3$ is a union of two different quadrics from $\mathcal{P}$. Furthermore, if S is such a quartic surface, then there exists a commutative diagram*

(2.8)

Here the morphism σ is a blow up of the conic C, the morphism ρ is a blow up of the preimage of C on X (which is the singular locus of X), the morphism $\tilde{\tau}$ is a double cover branched over the proper transform of the surface S on $\tilde{\mathbb{P}}^3$, the morphism $\tilde{\sigma}$ is a blow up of the intersection curve $\tilde{C}$ of the proper transforms on $\tilde{\mathbb{P}}^3$ of the irreducible components of S, the morphism $\tilde{\rho}$ is a blow up of the preimage of $\tilde{C}$ on $\tilde{X}$ (which is the singular locus of $\tilde{X}$), the morphism $\hat{\tau}$ is a double cover branched over the proper transform of the surface S on $\hat{\mathbb{P}}^3$, the rational map ψ is given by the pencil $\mathcal{P}$, the morphism $\tau_{\mathbb{P}^1}$ is a double cover, and general fibers of ϕ and φ are smooth quadric surfaces. In particular, in this case X is rational.

Proof. Straightforward and left to the reader. $\qquad\square$

We will deal with the remaining case $\mathbb{P}^3 = \mathbb{P}(W_4)$ in the next section.

3. Hashimoto's pencil of quartic surfaces

Recall that W_4 denotes the irreducible four-dimensional representation of $\mathfrak{A}_5$. Put $\mathbb{P}^3 = \mathbb{P}(W_4)$. Note that the sum of W_4 and the trivial $\mathfrak{A}_5$-representation I is the usual five-dimensional permutation representation of $\mathfrak{A}_5$. Let $x_0, \ldots, x_4$ be the coordinates in $I \oplus W_4$ permuted by $\mathfrak{A}_5$. Then W_4 is given in $I \oplus W_4$ by the equation
$$x_0 + x_1 + \ldots + x_4 = 0.$$

Thus, we can consider $x_0, \ldots, x_4$ as homogenous coordinates on $\mathbb{P}^3 \cong \mathbb{P}(W_4)$, keeping in mind that $x_0 = -(x_1 + \ldots + x_4)$.

For every positive i, put

$$F_i = x_0^i + x_1^i + x_2^i + x_3^i + x_4^i.$$

Then F_2 is the unique $\mathfrak{A}_5$-invariant polynomial of degree two in $x_0, \ldots, x_4$ modulo scaling. Similarly, modulo F_2 and scaling, F_3 is the unique $\mathfrak{A}_5$-invariant polynomial of degree three in $x_0, \ldots, x_4$. Finally, modulo F_2 and scaling, F_4 is the unique $\mathfrak{A}_5$-invariant polynomial of degree four in $x_0, \ldots, x_4$. Thus, any $\mathfrak{A}_5$-invariant quartic surface in $\mathbb{P}^3$ is given by

(3.1) $$F_4 = \lambda F_2^2$$

for some $\lambda \in \mathbb{C}$.

Remark 3.2. The quartic surfaces given by (3.1) have been studied by Kenji Hashimoto in [12]. In particular, he described all singular surfaces in this pencil.

The lengths of $\mathfrak{A}_5$-orbits in $\mathbb{P}^3$ are 5, 10, 12, 15, 20, 30, and 60 (see, for example, [8, Corollary 5.2.3]). Let Σ_5, Σ_{10}, Σ'_{10}, Σ_{15} be the $\mathfrak{A}_5$-orbits of the points

$$[-4 : 1 : 1 : 1 : 1], \ [0 : 0 : 0 : -1 : 1], \ [-2 : -2 : -2 : 3 : 3], \ [0 : -1 : -1 : 1 : 1],$$

respectively. Then $|\Sigma_5| = 5$, $|\Sigma_{10}| = |\Sigma'_{10}| = 10$ and $|\Sigma_{15}| = 15$.

Remark 3.3. The points $[0 : 0 : 0 : -1 : 1]$, $[0 : 0 : -1 : 0 : 1]$ and $[0 : 0 : -1 : 1 : 0]$ are collinear. Thus, there are ten lines in $\mathbb{P}^3$ such that each of them contains three points of Σ_{10}, and each point of Σ_{10} lies on exactly three of these lines. Similarly, the points $[-2 : -2 : -2 : 3 : 3]$, $[-2 : -2 : 3 : -2 : 3]$, $[-2 : -2 : 3 : 3 : -2]$ and $[3 : 3 : -2 : -2 : -2]$ are coplanar. Hence, there are ten planes in $\mathbb{P}^3$ such that each of them contains four points of Σ'_{10}, and each point of Σ'_{10} lies on exactly four of these planes. In particular, for each $\mathfrak{A}_5$-orbit $\Sigma \in \{\Sigma_{10}, \Sigma'_{10}\}$, there exists a plane $\Pi \subset \mathbb{P}^3$ that contains at least four points of Σ such that no three of them are collinear.
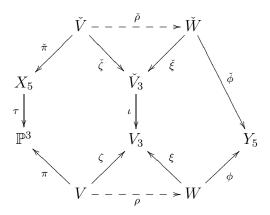
The $\mathfrak{A}_5$-orbits Σ_5, Σ_{10}, Σ'_{10}, and Σ_{15} are the only $\mathfrak{A}_5$-orbits of lengths 5, 10 and 15 in $\mathbb{P}^3$. Moreover, there are exactly two $\mathfrak{A}_5$-orbits Σ_{12} and Σ'_{12} in $\mathbb{P}^3$ of length 12. This is well-known (see, for example, [8, Corollary 5.2.3] and [8, Lemma 5.3.3]).

By [8, Lemma 5.3.3(xi)], the curve in $\mathbb{P}^3$ that is given by $F_2 = F_4 = 0$ is a smooth curve of genus 9. In particular, this curve does not contain the $\mathfrak{A}_5$-orbits Σ_5, Σ_{10}, Σ'_{10}, and Σ_{15} (see, for example, [8, Lemma 5.1.4]). Therefore, there exists a unique $\mathfrak{A}_5$-invariant quartic surface S_5 (respectively, S_{10}, S'_{10}, S_{15}) in $\mathbb{P}^3$ that contains the $\mathfrak{A}_5$-orbit Σ_5 (respectively, Σ_{10}, Σ'_{10}, Σ_{15}).

Proposition 3.4 ([12, Proposition 3.1]). *The surface S_5 (respectively, S_{10}, S'_{10}, S_{15}) is given by the equation* (3.1) *with $\lambda = \frac{13}{20}$ (respectively, $\lambda = \frac{1}{2}$, $\lambda = \frac{7}{30}$, $\lambda = \frac{1}{4}$). The surface S_5 (respectively, S_{10}, S'_{10}, S_{15}) has nodes at the points of the $\mathfrak{A}_5$-orbit Σ_5 (respectively, Σ_{10}, Σ'_{10}, Σ_{15}), and is smooth outside of it. Moreover, S_5, S_{10}, S'_{10} and S_{15} are the only singular surfaces given by* (3.1).

Denote by X_5 (respectively, X_{10}, X'_{10}, X_{15}) the double cover of $\mathbb{P}^3$ branched over the surface S_5 (respectively, S_{10}, S'_{10}, S_{15}). As we already mentioned in Remark 2.2, the action of the group $\mathfrak{A}_5$ lifts to each of these threefolds.

Theorem 3.5. *The threefold X_5 is $\mathbb{Q}$-factorial and irrational. Moreover, twisted cubics passing through Σ_5 define a commutative diagram*

$$
\begin{array}{ccc}
\check{V} & \dashrightarrow{\check{\rho}} & \check{W} \\
\downarrow{\check{\pi}} \; \swarrow{\check{\zeta}} & \searrow{\check{\xi}} & \\
X_5 \quad\quad \check{V}_3 & & \downarrow{\check{\phi}} \\
\downarrow{\tau} \quad\quad \downarrow{\iota} & & \\
\mathbb{P}^3 \quad\quad V_3 & & Y_5 \\
\swarrow{\pi} \; \nearrow{\zeta} & \nwarrow{\xi} & \nearrow{\phi} \\
V & \dashrightarrow{\rho} & W
\end{array}
$$

Here the morphism τ is a double cover branched over the surface S_5, the morphism π is the blow-up of Σ_5, the morphism $\check{\pi}$ is the blow-up of the singular locus of X_5, the morphism ζ is the contraction of the proper transforms of the ten lines $L_1, \ldots, L_{10}$ in $\mathbb{P}^3$ passing through pairs of points in Σ_5, the rational map ρ is a composition of Atiyah flops in these ten curves, the morphism $\check{\zeta}$ is the contraction of the proper transforms of the twenty curves that are mapped by τ to the lines $L_1, \ldots, L_{10}$, the rational map $\check{\rho}$ is a composition of Atiyah flops in these twenty curves, the variety V_3 is the Segre cubic hypersurface in $\mathbb{P}^4$, the variety $\check{V}_3$ is a complete intersection of the cone over V_3 and a quadric hypersurface in $\mathbb{P}^4$, the morphism ι is the natural double cover given by the projection of this cone from its vertex, and both morphisms ϕ and $\check{\phi}$ are conic bundles. In particular, the $\mathfrak{A}_5$-Mori fiber space $\check{\phi} \colon \check{W} \to Y_5$ is $\mathfrak{A}_5$-birational to X_5.

Proof. The threefold X_5 is $\mathbb{Q}$-factorial by [5, Theorem 1.8], and it is irrational by [5, Theorem 1.2]. The preimage on X_5 of a general twisted cubic that passes through Σ_5 is an irreducible (singular) rational curve. Thus, the existence of the commutative diagram follows from [14, Proposition 4.7]. $\qquad\square$

In the proof of [5, Theorem 4.2], we constructed a birational transformation of X_5 into a standard conic bundle over the smooth del Pezzo surface Y_5 of degree 5. Unlike the birational map $\check{\rho} \circ \check{\pi}^{-1}$ from Theorem 3.5, this transformation is not $\mathfrak{A}_5$-equivariant (it is only $\mathfrak{A}_4$-equivariant).

Theorem 3.6. *The threefolds X_{10} and X'_{10} are $\mathbb{Q}$-factorial, and X_{10} is not stably rational.*

Proof. The $\mathbb{Q}$-factoriality of X_{10} (respectively, X'_{10}) is equivalent to the fact that Σ_{10} (respectively, Σ'_{10}) is not contained in a quadric surface in $\mathbb{P}^3$, see [9, §3] or [11]. The latter condition is very easy to see explicitly by solving the system of linear equations. However, it can also be easily proved without any computations. Indeed, suppose that there are quadrics in $\mathbb{P}^3$ passing through Σ_{10}. Note that Σ_{10} does not lie on the quadric given by $F_2 = 0$ by [8, Lemma 5.3.3(vi)]. The latter equation defines the unique $\mathfrak{A}_5$-invariant quadric in $\mathbb{P}^3$. Thus, the intersection of all quadrics passing through Σ_{10} is either an $\mathfrak{A}_5$-invariant set of at most eight points, or an $\mathfrak{A}_5$-invariant curve Z of degree at most four. The former case is clearly impossible because the set Σ_{10} contains more than eight points. In the latter case one has $\deg(Z) = 4$, because there are no $\mathfrak{A}_5$-invariant curves of degree at most three in $\mathbb{P}^3$ by [8, Lemma 5.3.3(ix)]. Thus, Z is a complete intersection of two quadrics. On the other hand, a direct computation shows that $\mathrm{Sym}^2(W_4)$ does

not contain two-dimensional subrepresentations of $\mathfrak{A}_5$. The obtained contradiction shows that Σ_{10} is not contained in any quadric surface in $\mathbb{P}^3$. Similarly, we see that Σ'_{10} is not contained in any quadric surface in $\mathbb{P}^3$ either. Hence, both threefolds X_{10} and X'_{10} are $\mathbb{Q}$-factorial.

Let us show that X_{10} is not stably rational. To do this, pick a point $O \in \Sigma_{10}$, denote by τ the double cover $X_{10} \to \mathbb{P}^3$ that is branched at S_{10}, and denote by P the point in X_{10} that is mapped to the point O by the double cover τ. Then there exists a commutative diagram

$$\begin{array}{ccc} X_{10} & \xrightarrow{\ \tau\ } & \mathbb{P}^3 \\ {\scriptstyle f_P}\big\uparrow & & \big\downarrow{\scriptstyle p_O} \\ \tilde{X}_{10} & \xrightarrow{\ \pi\ } & \mathbb{P}^2 \end{array}$$

where the morphism p_O is the linear projection from the point O, the morphism f_P is the blow up of the point P, and π is a conic bundle.

Let us describe the degeneration curve C of the conic bundle π. We may assume that $O = [0 : 0 : 0 : -1 : 1]$. Plugging $x_0 = -x_1 - x_2 - x_3 - x_4$ into the equation (3.1) with $\lambda = \frac{1}{2}$, and considering the affine equation of S_{10} in the chart $x_4 \neq 0$ with the new coordinates

$$y_1 = \frac{x_1}{x_4}, \quad y_2 = \frac{x_2}{x_4}, \quad y_3 = \frac{x_3}{x_4} - 1,$$

we see that $O = (0, 0, 0)$, and the chart of surface S_{10} in $\mathbb{A}^3$ is given by

$$(3.7) \quad (y_1+y_2+y_3+2)^4+y_1^4+y_2^4+(y_3+1)^4+1 = \frac{1}{2}\left((y_1+y_2+y_3+2)^2+y_1^2+y_2^2+(y_3+1)^2+1\right)^2$$

Then every line L in $\mathbb{A}^3$ passing through O is given by

$$(y_1, y_2, y_3) = t(z_1, z_2, z_3)$$

for some $(z_1, z_2, z_3) \neq (0, 0, 0)$, where t is a parameter. Plugging this parametric equation into (3.7), dividing the resulting equation by t^2, and taking the discriminant of the resulting quadratic equation, we see that the equation of C in $\mathbb{P}^2$ is

$$z_3(z_3(z_1 + z_2) + z_1^2 + z_1 z_2 + z_2^2)(z_2 z_3^2 + z_1 z_3^2 + z_1^2 z_3 + 5 z_1 z_2 z_3 + 4 z_1^2 z_2 + z_2^2 z_3 + 4 z_1 z_2^2) = 0,$$

where we consider z_1, z_2 and z_3 as homogeneous coordinates on $\mathbb{P}^2$. Thus C is a union of a line ℓ that is given by $z_3 = 0$, a smooth conic γ that is given by

$$z_3(z_1 + z_2) + z_1^2 + z_1 z_2 + z_2^2 = 0,$$

and a smooth cubic curve ζ that is given by

$$z_2 z_3^2 + z_1 z_3^2 + z_1^2 z_3 + 5 z_1 z_2 z_3 + 4 z_1^2 z_2 + z_2^2 z_3 + 4 z_1 z_2^2 = 0.$$

The line ℓ intersects the curves γ and ζ transversally. Moreover, the curves γ and ζ are tangent at three points $[0 : 1 : -1]$, $[1 : 0 : -1]$, and $[0 : 0 : 1]$, so that $\gamma \cup \zeta$ has three tacnodes at these points (cf. Remark 3.3 and [5, Proposition 3.2(i)]). Furthermore, no point in Σ_{10} is mapped to a point in $\ell \cap \gamma$ by p_O, because all points in Σ_{10} are defined over $\mathbb{Q}$, and the two points of the intersections $\ell \cap \gamma$ are $[-1 + \sqrt{3} : 2 : 0]$ and $[-1 + \sqrt{3} : 2 : 0]$.

By [5, Theorem 4.2(i),(ii)], there exists a commutative diagram

$$
\begin{array}{ccc}
V & \dashrightarrow^{\rho} & \tilde{X}_{10} \\
{\scriptstyle\nu}\downarrow & & \downarrow{\scriptstyle\pi} \\
U & \xrightarrow{\quad\varrho\quad} & \mathbb{P}^2
\end{array}
$$

Here V is a smooth projective threefold, U is a smooth surface, the relative Picard group of V over U has rank 1, and ϱ is a birational morphism that factors as

$$
U \xrightarrow{\varrho'_t} U_t \xrightarrow{\varrho_t} U_n \xrightarrow{\varrho_n} \mathbb{P}^2,
$$

where the morphism ϱ_n is a blow up of the three points of $\zeta \cap \ell$, the morphism ϱ_t is a blow up of the three points of $\gamma \cap \zeta$, and ϱ'_t is a blow up of the three intersection points of the proper transforms of the curves γ and ζ on the surface U_t.

Let Δ be the degeneration curve of the conic bundle ν. Then Δ is the proper transform of the curve C by [5, Theorem 4.2(iii)]. Thus, the curve Δ is not connected, so that $H^3(V, \mathbb{Z})$ has non-trivial 2-torsion by [24, Theorem 2] (see also [1, Proposition 3]). So the threefold X_{10} is not stably rational by [1, Proposition 1]. $\square$

We do not know whether X'_{10} is rational or not.

Remark 3.8. The proof of irrationality of X_{10} is not applicable to X'_{10}. Indeed, arguing as in the proof of Theorem 3.6, we obtain a commutative diagram

$$
\begin{array}{ccc}
X'_{10} & \xrightarrow{\quad\tau\quad} & \mathbb{P}^3 \\
{\scriptstyle f_P}\uparrow & & \vdots{\scriptstyle p_O} \\
\tilde{X}'_{10} & \xrightarrow{\quad\pi\quad} & \mathbb{P}^2
\end{array}
$$

where the morphism τ is the double cover branched at S'_{10}, the rational map p_O is the linear projection from a point $O = [-2 : -2 : -2 : 3 : 3]$ of Σ'_{10}, the morphism f_P is the blow up of the point in X'_{10} that is mapped to O, and π is a conic bundle. Denote by C the degeneration curve of the conic bundle π. Making computations similar to those in the proof of Theorem 3.6, we see that C can be given by

$$
\begin{aligned}
& -16z_1^6 - 16z_2^6 - 13z_2^2z_3^4 - 13z_1^2z_3^4 - 42z_2^3z_3^3 - 61z_2^4z_3^2 - \\
& \quad -42z_1^3z_3^3 - 61z_1^4z_3^2 + 12z_1^4z_2^2 + 104z_1^3z_3^3 + 12z_1^2z_2^4 - 48z_1^5z_2 - 48z_1^5z_3 - 48z_1z_2^5 - \\
& \quad -48z_2^5z_3 + 93z_1^2z_2^2z_3^2 - 26z_1^3z_2z_3^2 - 12z_1^2z_2z_3^3 - 12z_1z_2^2z_3^3 - 72z_1^4z_2z_3 + 120z_1^2z_2^3z_3 - \\
& \quad\quad\quad\quad\quad -72z_1z_2^4z_3 - 26z_1z_2^3z_3^2 - 10z_1z_2z_3^4 + 120z_1^3z_2^2z_3 = 0,
\end{aligned}
$$

where z_1, z_2 and z_3 are homogeneous coordinates on $\mathbb{P}^3$. Using this equation, one can check that C is an irreducible nodal curve with exactly nine nodes. It follows from [5, Theorem 4.2(i),(ii)] that there exists a commutative diagram

$$
\begin{array}{ccc}
V & \dashrightarrow^{\rho} & \tilde{X}'_{10} \\
{\scriptstyle\nu}\downarrow & & \downarrow{\scriptstyle\pi} \\
U & \xrightarrow{\quad\varrho\quad} & \mathbb{P}^2
\end{array}
$$

where V is a smooth projective threefold, U is a smooth surface, the relative Picard group of V over U has rank 1, and ϱ is a blow up of nine nodes of C. Moreover, the

degeneration curve of the conic bundle ν is the proper transform of the curve C by [5, Theorem 4.2(iii)]. Thus, Δ is an irreducible smooth elliptic curve in $|-2K_U|$, which implies that the group $H^3(V, \mathbb{Z})$ is trivial (see, for example, [24, Theorem 2]). In particular, the intermediate Jacobian of V is trivial, and the approach of [1] does not work in this case either. Shokurov's famous [18, Conjecture 10.3] predicts that X'_{10} is irrational.

Remark 3.9. On [11, p. 354], it is claimed that the resolution of singularities of any $\mathbb{Q}$-factorial nodal quartic double solid with ten nodes has non-trivial torsion in the third integral cohomology group. The computations in Remark 3.8 show that X'_{10} is a counter-example to this claim.

Remark 3.10. In the proof of Theorem 3.6 and in Remark 3.8, we refer to [24, Theorem 2]. Note that the notation of this theorem is a bit non-standard. Namely, the second summand on the right hand side in [24, (13)] is torsion free (see [24, Lemma 5] and [24, Lemma 7] for a detailed computation).

Recall that a normal variety X with an action of the finite group G is said to be $G\mathbb{Q}$-factorial if any G-invariant Weil divisor on X is a $\mathbb{Q}$-Cartier divisor.

Theorem 3.11. *The threefold X_{15} is not $\mathbb{Q}$-factorial, it is $\mathfrak{A}_5\mathbb{Q}$-factorial, and it is rational.*

Proof. The threefold X_{15} is not $\mathbb{Q}$-factorial by [5, Corollary 1.7]. Its rationality follows from [5, Theorem 1.3] or [16, Theorem 8.1].

Let us show that X_{15} is $\mathfrak{A}_5\mathbb{Q}$-factorial. Put $O = [0 : -1 : -1 : 1 : 1]$, and denote by Γ its stabilizer. Then $O \in \Sigma_{15}$, and $\Gamma \cong (\mathbb{Z}/2\mathbb{Z})^2$. To prove that X_{15} is $\mathfrak{A}_5\mathbb{Q}$-factorial, it is enough to show that the Γ-invariant local class group of the point O is trivial.

The quartic S_{15} is given by the equation $R = 0$, where

$$R = x_0^4 + x_1^4 + x_2^2 + x_3^2 + x_4^4 - \frac{1}{4}\left(x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2\right)^2,$$

and $x_0 = -(x_1 + x_2 + x_3 + x_4)$. Let σ be the element of $\mathfrak{A}_5$ acting by

$$\sigma(x_0) = x_0, \ \sigma(x_1) = x_3, \ \sigma(x_2) = x_4, \ \sigma(x_3) = x_1, \ \sigma(x_4) = x_2.$$

Then $\sigma \in \Gamma$. Consider new homogeneous coordinates $y_1, \ldots, y_4$ in $\mathbb{P}^3$ such that

$$x_1 = y_1 - y_3 - y_4, \ x_2 = y_2 - y_4, \ x_3 = y_1 + y_3 + y_4, \ x_4 = y_2 + y_4.$$

Then $x_0 = -2(y_1 + y_2)$, and the point O is $[0 : 0 : 0 : 1]$ in the new coordinates. One has

$$\sigma(y_1) = y_1, \ \sigma(y_2) = y_2, \ \sigma(y_3) = -y_3, \ \sigma(y_4) = -y_4.$$

Write

$$R = R_2(y_1, y_2, y_3)y_4^2 + R_3(y_1, y_2, y_3)y_4 + R_4(y_1, y_2, y_3),$$

where R_i is a form of degree i. Then

$$R_2(y_1, y_2, y_3) = 4y_3^2 - 16y_1y_2.$$

The threefold X_{15} is given in the weighted projective space $\mathbb{P}(1, 1, 1, 1, 2)$ with weighted homogeneous coordinates y_1, y_2, y_3, y_4 and w by equation $w^2 = R$, and σ acts trivially on w. Identify the point $O \in \mathbb{P}^3$ with the unique point of X_{15} that is mapped to O by the double cover morphism. Then the tangent cone to X_{15} at O is a cone over a quadric surface B given by equation

$$16y_1y_2 = 4y_3^2 - w^2$$

in a three-dimensional projective space with coordinates y_1, y_2, y_3 and w. The two lines $y_1 = w - 2y_3 = 0$ and $y_1 = w + 2y_3 = 0$ are contained in two different pencils of lines on B. They are interchanged by the involution σ, which implies that the Γ-invariant local class group of the point O is trivial. $\square$

Since X_{15} is rational, the $\mathfrak{A}_5$-action on X_{15} gives an embedding

$$\mathfrak{A}_5 \hookrightarrow \mathrm{Cr}_3(\mathbb{C}).$$

In the next section we will see that X_{15} is $\mathfrak{A}_5$-birationally superrigid (see [8, Definition 3.1.1]), so that the latter embedding is not conjugate to the three embeddings described in [8, §1.4].

4. $\mathfrak{A}_5$-BIRATIONAL SUPERRIGIDITY

Let us use notation of Section 3. Let S be a quartic surface in $\mathbb{P}^3$ that is given by (3.1), and let $\tau \colon X \to \mathbb{P}^3$ be a double cover branched over S. By Remark 2.2, the threefold X is faithfully acted on by the group $\mathfrak{A}_5$. Denote by H the class of the pull-back of the plane in $\mathbb{P}^3$ via τ.

Remark 4.1. Every $\mathfrak{A}_5$-invariant Weil divisor on X is rationally equivalent to a multiple of H. This follows from Theorems 3.5, 3.6, 3.11, and the Lefschetz theorem.

Thus, X is $\mathfrak{A}_5$-Fano threefold in the sense of [16]. The goal of this section is to prove the following result.

Theorem 4.2. *The threefold X is $\mathfrak{A}_5$-birationally superrigid if and only if $X \neq X_5$.*

Corollary 4.3. *The group $\mathrm{Cr}_3(\mathbb{C})$ contains at least four non-conjugate subgroups isomorphic to $\mathfrak{A}_5$.*

Proof. Since the threefold X_{15} is rational by Theorem 3.11, the required assertion follows from Theorem 4.2, [8, Remark 1.2.1], [8, Example 1.3.9], and [8, Theorem 1.4.1]. $\square$

Corollary 4.4. *The group $\mathrm{Cr}_3(\mathbb{C})$ contains at least three non-conjugate subgroups isomorphic to $\mathfrak{S}_5$.*

Proof. Note that W_4 is a restriction to $\mathfrak{A}_5$ of the representation of the group $\mathfrak{S}_5$. Consider the corresponding action of the group $\mathfrak{S}_5$ on $\mathbb{P}^3$. Since S_{15} is $\mathfrak{S}_5$-invariant, X_{15} is also acted on by $\mathfrak{S}_5$ (cf. Remark 2.2). Moreover, the quadric $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ is also $\mathfrak{S}_5$-invariant. Thus, we have three rational Fano threefolds acted on by $\mathfrak{S}_5$. They are $\mathbb{P}^3$, $Q \times \mathbb{P}^1$ (with a trivial action on the second factor), and X_{15}. Since X_{15} is $\mathfrak{A}_5$-birationally superrigid by Theorem 4.2, it is also $\mathfrak{S}_5$-birationally superrigid. Hence, there are no $\mathfrak{S}_5$-birational maps $X_{15} \dashrightarrow \mathbb{P}^3$ and $X_{15} \dashrightarrow Q \times \mathbb{P}^1$. On the other hand, the abelian subgroup

$$(\mathbb{Z}/2\mathbb{Z})^2 \subset \mathfrak{A}_5 \subset \mathfrak{S}_5$$

fixes a point in $\Sigma_{15} \subset \mathbb{P}^3$ and does not have fixed points in Q. Therefore, there is no $\mathfrak{S}_5$-birational map $\mathbb{P}^3 \dashrightarrow Q \times \mathbb{P}^1$ by [17, Proposition A.4] (see also [8, Theorem 1.1.1]). $\square$

We already described $\mathfrak{A}_5$-orbits in $\mathbb{P}^3$ of small length in Section 3. To prove Theorem 4.2, we must also describe $\mathfrak{A}_5$-invariant curves in $\mathbb{P}^3$ of degree less than eight. As we will see a bit later, they all lie in the surface given by $F_3 = 0$, and one of them also lies in the surface given by $F_2 = 0$. Thus, we need to take a closer look at these surfaces.

Denote by Q the surface in $\mathbb{P}^3$ that is given by $F_2 = 0$. Then Q is smooth, so that $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$. The surface Q does not contain the $\mathfrak{A}_5$-orbits Σ_5, Σ_{10}, Σ'_{10}, and Σ_{15}, and

it does contain the $\mathfrak{A}_5$-orbits Σ_{12} and Σ'_{12}, see [8, Lemma 5.3.3(vi)]. Moreover, since Q contains only two $\mathfrak{A}_5$-orbits of length 12, it follows from [8, Lemma 6.4.3(i)] that the $\mathfrak{A}_5$-action on Q is twisted diagonal, i.e. Q can be identified with $\mathbb{P}(U_2) \times \mathbb{P}(U'_2)$.

Denote by S_3 the surface in $\mathbb{P}^3$ that is given by $F_3 = 0$, and denote by $\mathcal{B}_6$ the curve in $\mathbb{P}^3$ that is given by $F_2 = F_3 = 0$. Then S_3 is a smooth surface known as the *Clebsch diagonal cubic surface*, and $\mathcal{B}_6$ is a smooth irreducible curve of genus four known as the *Bring's curve*.

Lemma 4.5. *The curve $\mathcal{B}_6$ is the only $\mathfrak{A}_5$-invariant curve in Q of degree less than eight. It contains the $\mathfrak{A}_5$-orbits Σ_{12} and Σ'_{12}. Moreover, the set $\Sigma_{12} \cup \Sigma'_{12}$ is cut out on $\mathcal{B}_6$ by the equation $F_4 = 0$.*

Proof. By [8, Lemma 5.1.5], the curve $\mathcal{B}_6$ contains Σ_{12} and Σ'_{12}, and does not contain other $\mathfrak{A}_5$-orbits of length less than 30. On the other hand, $\mathcal{B}_6$ is not contained in the surface given by $F_4 = 0$, because the curve in $\mathbb{P}^3$ that is given by $F_2 = F_4 = 0$ is a smooth curve of genus 9. Therefore, the equation $F_4 = 0$ cuts out a subset of $\mathcal{B}_6$ that consists of

$$4\deg(\mathcal{B}_6) = 24$$

points (counted with multiplicities). Hence, we see that $\Sigma_{12} \cup \Sigma'_{12}$ is cut out on $\mathcal{B}_6$ by the equation $F_4 = 0$.

Let C be a curve in Q of degree $d < 8$. Then $d \geqslant 4$ by [8, Lemma 5.3.3(ix)]. Let us show that $C = \mathcal{B}_6$. If C is contained in S_3, then $C = \mathcal{B}_6$ by construction. Thus, we may assume that this is not the case. This implies that $d \neq 6$ and $d \neq 7$, because

$$S_3 \cdot \Gamma = 3d$$

and the lengths of $\mathfrak{A}_5$-orbits in $\mathbb{P}^3$ are 5, 10, 12, 15, 20, 30, and 60. Hence, either $d = 4$ or $d = 5$.

The curve Γ is a divisor of bi-degree (a, b) on $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, where a and b are non-negative integers. Without loss of generality, we may assume that $a \leqslant b$. Since
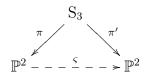
$$a + b = d \in \{4, 5\},$$

the pair (a, b) must be one of the following: $(0, 4)$, $(1, 3)$, $(2, 2)$, $(0, 5)$, $(1, 4)$, or $(2, 3)$. The cases $(0, 4)$ and $(0, 5)$ are impossible by [8, Lemma 6.4.1]. Moreover, Q contains no $\mathfrak{A}_5$-invariant effective divisors of bi-degree $(1, 3)$ and $(1, 4)$ by [8, Lemma 6.4.11(o)], because the $\mathfrak{A}_5$-action on Q is twisted diagonal. Thus, either $(a, b) = (2, 2)$ or $(a, b) = (2, 3)$. By [8, Lemma 6.4.3(ii)], the quadric Q contains a smooth rational curve $C_{1,7}$ that is a divisor of of bi-degree $(1, 7)$. Therefore,

$$\Gamma \cdot C_{1,7} = 7a + b \in \{16, 18\},$$

which is impossible, because the lengths of $\mathfrak{A}_5$-orbits in $C_{1,7} \cong \mathbb{P}^1$ are 12, 20, 30, and 60. $\qquad\square$

Let us describe four more $\mathfrak{A}_5$-invariant sextic curves contained in the surface S_3. Recall from [8, Lemma 6.3.3] that S_3 contains two $\mathfrak{A}_5$-invariant curves $\mathcal{L}_6$ and $\mathcal{L}'_6$ such that each of them is a disjoint union of 6 lines, and there is a commutative diagram

$$
\begin{array}{ccc}
 & S_3 & \\
{\scriptstyle \pi}\swarrow & & \searrow{\scriptstyle \pi'} \\
\mathbb{P}^2 \dashrightarrow{\scriptstyle \varsigma} & & \mathbb{P}^2
\end{array}
$$

where π (respectively, π') is an $\mathfrak{A}_5$-birational morphism that contracts the lines of $\mathcal{L}_6$ (respectively, the lines of $\mathcal{L}'_6$) to the unique $\mathfrak{A}_5$-orbit of length 6 in $\mathbb{P}^2$. One has

$$\mathcal{L}_6 + \mathcal{L}'_6 \sim -4K_{S_3}$$

by construction. By [8, Lemma 6.3.12(iii)], each curve $\mathcal{L}_6$ and $\mathcal{L}'_6$ contains a unique $\mathfrak{A}_5$-orbit of length 12. Moreover, the intersection $\mathcal{L}_6 \cap \mathcal{L}'_6$ is an $\mathfrak{A}_5$-orbit of length 30 by [8, Lemma 6.3.12(vii)]. Thus, without loss of generality, we may assume that $\Sigma_{12} \subset \mathcal{L}_6$ and $\Sigma'_{12} \subset \mathcal{L}'_6$.

Similarly, denote by $\mathcal{C}$ (respectively, by $\mathcal{C}'$) the smooth rational curve in S_3 that is a proper transform of the unique $\mathfrak{A}_5$-invariant conic in $\mathbb{P}^2$ via π (respectively, via π'). Then

$$\mathcal{C} + \mathcal{C}' \sim -4K_{S_3},$$

and both curves $\mathcal{C}$ and $\mathcal{C}'$ are smooth rational sextic curves. By construction, one has $\mathcal{L}_6 \cap \mathcal{C} = \varnothing$ and $\mathcal{L}'_6 \cap \mathcal{C}' = \varnothing$. By [8, Lemma 6.3.17], one has $\mathcal{L}_6 \cap \mathcal{C}' = \Sigma_{12}$, $\mathcal{L}'_6 \cap \mathcal{C} = \Sigma'_{12}$, and $\mathcal{C} \cap \mathcal{C}'$ is an $\mathfrak{A}_5$-orbit of length 30.

Lemma 4.6. *The $\mathfrak{A}_5$-orbits of general points of the curves $\mathcal{B}_6$, $\mathcal{C}$, $\mathcal{C}'$, $\mathcal{L}_6$, and $\mathcal{L}'_6$ are of length 60. These curves are the only $\mathfrak{A}_5$-invariant curves in $\mathbb{P}^3$ of degree less than eight.*

Proof. The stabilizers in $\mathfrak{A}_5$ of general points of the curves $\mathcal{B}_6$, $\mathcal{C}$ and $\mathcal{C}'$ are trivial, because these curves are irreducible and none of them is contained in a plane in $\mathbb{P}^3$. Thus, the $\mathfrak{A}_5$-orbits of general points of the curves $\mathcal{B}_6$, $\mathcal{C}$, and $\mathcal{C}'$ are of length 60. The stabilizer of each irreducible component of the curve $\mathcal{L}_6$ acts faithfully on it by [8, Corollary 5.2.3(v)]. This implies that the $\mathfrak{A}_5$-orbit of a general point of (an irreducible component of) the curve $\mathcal{L}_6$ is of length 60. Similarly, the $\mathfrak{A}_5$-orbit of a general point of the curve $\mathcal{L}'_6$ is also of length 60.

Let us show that $\mathcal{B}_6$, $\mathcal{C}$, $\mathcal{C}'$, $\mathcal{L}_6$, and $\mathcal{L}'_6$ are the only $\mathfrak{A}_5$-invariant curves in $\mathbb{P}^3$ of degree less than eight. Let Γ be an $\mathfrak{A}_5$-invariant curve Γ in $\mathbb{P}^3$ of degree d. If $\Gamma \subset S_3$, then the required assertion follows from [8, Theorem 6.3.18]. Thus, we may assume that $\Gamma \not\subset S_3$. One has $\Gamma \not\subset Q$ by Lemma 4.5.

By [8, Lemma 5.3.3(ix)], one has $d \geqslant 3$. Hence either $d = 4$ or $d = 5$, because

$$S_3 \cdot \Gamma = 3d$$

and the lengths of $\mathfrak{A}_5$-orbits in $\mathbb{P}^3$ are 5, 10, 12, 15, 20, 30, and 60. Thus, one has

$$Q \cdot \Gamma = 2d \in \{8, 10\}.$$

This is a contradiction, because the lengths of $\mathfrak{A}_5$-orbits in the quadric Q are at least 12. $\square$

Corollary 4.7. *There exists a unique $\mathfrak{A}_5$-invariant quartic surface $S_{\mathcal{L}_6}$ (respectively, $S_{\mathcal{C}}$) in $\mathbb{P}^3$ that contains the curve $\mathcal{L}_6$ (respectively, $\mathcal{C}$). The surface $S_{\mathcal{L}_6}$ contains the curve $\mathcal{L}'_6$ and*

$$\mathcal{L}_6 + \mathcal{L}'_6 = S_3|_{S_{\mathcal{L}_6}}.$$

The surface $S_{\mathcal{C}}$ contains the curve $\mathcal{C}'$ and

$$\mathcal{C} + \mathcal{C}' = S_3|_{S_{\mathcal{C}}}.$$

Moreover, the surface $S_{\mathcal{L}_6}$ (respectively, $S_{\mathcal{C}}$) is the unique $\mathfrak{A}_5$-invariant quartic surface in $\mathbb{P}^3$ that contains the curve $\mathcal{L}'_6$ (respectively, $\mathcal{C}'$). Furthermore, both surfaces $S_{\mathcal{C}}$ and $S_{\mathcal{L}_6}$ are smooth.

Proof. Existence of the surfaces $S_{\mathcal{C}}$ and $S_{\mathcal{L}_6}$ follows from Lemma 4.6, because none of the curves $\mathcal{C}$ and $\mathcal{L}_6$ is contained in Q by Lemma 4.5. Moreover, the curve $\mathcal{C}$ (respectively, $\mathcal{L}_6$) is contained in the smooth locus of the surface $S_{\mathcal{C}}$ (respectively, $S_{\mathcal{L}_6}$) by Proposition 3.4. Thus, $\mathcal{C}^2 = -2$ on the surface $S_{\mathcal{C}}$, and $\mathcal{L}_6^2 = -12$ on the surface $S_{\mathcal{L}_6}$. Now it follows from [8, Lemma 6.7.3(i),(ii)] that both surfaces $S_{\mathcal{C}}$ and $S_{\mathcal{L}_6}$ are smooth.

One has

$$S_3|_{S_{\mathcal{C}}} = \mathcal{C} + \Omega,$$

where Ω is an effective $\mathfrak{A}_5$-invariant divisor on $S_{\mathcal{C}}$ of degree 6. By Lemma 4.6, the divisor Ω is one of the curves $\mathcal{B}_6$, $\mathcal{C}$, $\mathcal{C}'$, $\mathcal{L}_6$, or $\mathcal{L}_6'$. By Lemma 4.5, one has $\Omega \neq \mathcal{B}_6$. If $\Omega = \mathcal{C}$, then

$$-2 = \Omega \cdot \mathcal{C} = (S_3|_{S_{\mathcal{C}}} - \mathcal{C}) \cdot \mathcal{C} = 18 - \mathcal{C}^2 = 20,$$

which is absurd. Similarly, if $\Omega = \mathcal{L}_6$, then

$$0 = \Omega \cdot \mathcal{C} = (S_3|_{S_{\mathcal{C}}} - \mathcal{C}) \cdot \mathcal{C} = 20,$$

which is absurd. Finally, if $\Omega = \mathcal{L}_6'$, then $\Omega \cdot \mathcal{C} = 24$ by [8, Lemma 6.3.17], so that

$$24 = \Omega \cdot \mathcal{C} = (S_3|_{S_{\mathcal{C}}} - \mathcal{C}) \cdot \mathcal{C} = 20,$$

which is again absurd. Therefore, one has $\Omega = \mathcal{C}'$. Similarly, we see that

$$\mathcal{L}_6 + \mathcal{L}_6' = S_3|_{S_{\mathcal{L}_6}}.$$

By [8, Lemma 5.3.3(xi)], the curve in $\mathbb{P}^3$ that is given by $F_2 = F_4 = 0$ is a smooth curve of genus 9. This implies all uniqueness assertions of the corollary. $\square$

To prove Theorem 4.2, we also need the following technical result.

Lemma 4.8. *Let Σ be an $\mathfrak{A}_5$-orbit on $\mathbb{P}^3$. Put*

$$m = \begin{cases} 2 \text{ if } \Sigma = \Sigma_5, \\ 3 \text{ if } \Sigma = \Sigma_{10}, \text{ or } \Sigma = \Sigma_{10}', \text{ or } \Sigma = \Sigma_{15}, \\ 4 \text{ if } \Sigma = \Sigma_{12} \text{ or } \Sigma = \Sigma_{12}', \\ 10 \text{ if } |\Sigma| = 20, \\ 15 \text{ if } |\Sigma| = 30, \\ 30 \text{ if } |\Sigma| = 60. \end{cases}$$

Let $\mathcal{M}$ be the linear system consisting of all surfaces in $\mathbb{P}^3$ of degree m that pass through the set Σ. Then $\mathcal{M}$ does not have base curves and fixed components.

Proof. If $\Sigma = \Sigma_5$, then $\mathcal{M}$ does not have fixed components by [8, Lemma 5.3.3(vi)], and $\mathcal{M}$ does not have base curves, because $\mathbb{P}^3$ does not have $\mathfrak{A}_5$-invariant curves of degree less than 6 by Lemma 4.6. Similarly, if $\Sigma = \Sigma_{10}$ (respectively, $\Sigma = \Sigma_{10}'$, $\Sigma = \Sigma_{15}$), then Σ is a singular locus of the nodal quartic surface S_{10} (respectively, S_{10}', S_{15}). Since the singular locus of the surface S_{10} (respectively, S_{10}', S_{15}) is cut out by cubics, we see that the base locus of $\mathcal{M}$ is Σ in the latter case. If $\Sigma = \Sigma_{12}$ or $\Sigma = \Sigma_{12}'$, then $\mathcal{M}$ does not have base curves and fixed components by Lemma 4.5. Thus, we may assume that $|\Sigma| \geqslant 20$, so that $m = \frac{|\Sigma|}{2}$.

Suppose that Σ is not contained in the surface Q. Let $P = [a_0 : a_1 : a_2 : a_3 : a_4]$ be a point in Σ. Then $F_2(a_0, a_1, a_2, a_3, a_4) \neq 0$. Put

$$\alpha = \frac{F_5^2(a_0, a_1, a_2, a_3, a_4)}{F_2^5(a_0, a_1, a_2, a_3, a_4)}, \quad \beta = \frac{F_4(a_0, a_1, a_2, a_3, a_4)}{F_2^2(a_0, a_1, a_2, a_3, a_4)}, \quad \gamma = \frac{F_3^2(a_0, a_1, a_2, a_3, a_4)}{F_2^3(a_0, a_1, a_2, a_3, a_4)}.$$

Then the system

$$\begin{cases} F_5^2 = \alpha F_2^5, \\ F_4 = \beta F_2^2, \\ F_3^2 = \gamma F_2^3, \end{cases}$$

has finitely many solutions in $\mathbb{P}^3$, because the system of equations $F_2 = F_3 = F_4 = F_5$ has no solution in $\mathbb{P}^3$. Using the forms $F_5^2 - \alpha F_2^5$, $F_4 - \beta F_2^2$, and $F_3^2 - \gamma F_2^3$, we can produce three surfaces in $\mathcal{M}$ that have only finitely many common points. This shows that the base locus of $\mathcal{M}$ is zero-dimensional as requested.

To complete the proof, we may assume that $\Sigma \subset Q$. Note that $\mathcal{M}$ contains divisors of the form $Q + R$, where R is any surface in $\mathbb{P}^3$ of degree $m - 2$. Thus, the base locus of $\mathcal{M}$ is contained in Q. Denote by T the hyperplane section of Q. Let $\mathcal{M}_Q$ be the linear system consisting of all curves in $|mT|$ that pass through Σ. Then $\mathcal{M}_Q$ is not empty, because

$$h^0(Q, \mathcal{O}_Q(mT)) = (m + 1)^2 > |\Sigma|.$$

Moreover, every curve in $\mathcal{M}_Q$ is cut out on Q by a surface in $\mathcal{M}$, because we have a surjection

$$H^0\left(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m)\right) \twoheadrightarrow H^0\left(Q, \mathcal{O}_Q(mT)\right).$$

Thus, $\mathcal{M}_Q$ is a restriction of the linear system $\mathcal{M}$ to the surface Q, and Q is not a fixed surface of $\mathcal{M}$. In particular, the base loci of $\mathcal{M}$ and $\mathcal{M}_Q$ are the same.

Suppose that the base locus of $\mathcal{M}_Q$ contains a curve. Since $\mathcal{M}_Q$ is $\mathfrak{A}_5$-invariant, the base locus of $\mathcal{M}_Q$ contains an $\mathfrak{A}_5$-invariant curve. Let us denote it by Z. Then Z is a divisor of bi-degree (a, b) on $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, where $a \leqslant m$ and $b \leqslant m$. One has

$$(m - a + 1)(m - b + 1) = h^0(Q, \mathcal{O}_Q(mT - Z)) \geqslant h^0(Q, \mathcal{O}_Q(mT)) - |\Sigma| = (m + 1)^2 - |\Sigma|,$$

which implies that

$$|\Sigma| \geqslant am + mb + (a + b) - ab.$$

Without loss of generality, we may assume that $a \leqslant b$. This gives $2m = |\Sigma| \geqslant mb$, so that $a \leqslant b \leqslant 2$. Thus, the degree of the curve Z is $a + b \leqslant 4$, which contradicts Lemma 4.6. $\square$

Now we are ready to prove Theorem 4.2.

Proof of Theorem 4.2. By Theorem 3.5, the threefold X_5 is not $\mathfrak{A}_5$-birationally superrigid. Thus, we may assume that $X \neq X_5$. Suppose that X is not $\mathfrak{A}_5$-birationally superrigid. Then it follows from [8, Corollary 3.3.3] that there exits an $\mathfrak{A}_5$-invariant mobile linear system $\mathcal{D}$ on X such that the singularities of the log pair $\left(X, \frac{2}{n}\mathcal{D}\right)$ are not canonical, where n is a positive integer that is defined by $\mathcal{D} \sim nH$.

Let Z be an irreducible subvariety of X that is a center of canonical singularities of the log pair $\left(X, \frac{2}{n}\mathcal{D}\right)$, see [8, Definition 2.4.1]. Then either Z is a point, or Z is an irreducible curve. Let $\{Z_1, \ldots, Z_r\}$ be the $\mathfrak{A}_5$-orbit of $Z = Z_1$. If Z is a curve, then

$$(4.9) \qquad \qquad \qquad \mathrm{mult}_{Z_i}(D) > \frac{n}{2}$$

for each Z_i and any general surface D in $\mathcal{D}$, see, for example, [8, Lemma 2.4.4]. If Z is a smooth point, then

$$(4.10) \qquad \qquad \qquad \mathrm{mult}_{Z_i}(D_1 \cdot D_2) > n^2$$

for each Z_i and any two general surfaces D_1 and D_2 in $\mathcal{D}$, see, for example, [8, Theorem 2.5.2]. We will use (4.9) to show that Z is not a curve, then we will use (4.10) to show

that Z is not a smooth point of X. Finally we will use [4, Theorem 1.7.20] to exclude the case when Z is a singular point of X.

Suppose that Z is a curve. Put $\Sigma = Z_1 + \ldots + Z_r$. For general surfaces D_1 and D_2 in $\mathcal{D}$, the inequality (4.9) gives

$$2n^2 = H \cdot D_1 \cdot D_2 \geqslant \sum_{i=1}^{r} \mathrm{mult}_{Z_i} \left(D_1 \cdot D_2 \right) H \cdot Z_i \geqslant$$

$$\geqslant \sum_{i=1}^{r} \mathrm{mult}_{Z_i} \left(D_1 \right) \mathrm{mult}_{Z_i} \left(D_2 \right) H \cdot Z_i > \frac{n^2}{4} H \cdot \Sigma,$$

which implies that $H \cdot \Sigma < 8$. Applying Lemma 4.6, we see that

$$H \cdot \Sigma = 6,$$

and $\tau(\Sigma)$ is one of the following curves: $\mathcal{B}_6$, $\mathcal{C}$, $\mathcal{C}'$, $\mathcal{L}_6$ or $\mathcal{L}'_6$. Since $H \cdot \Sigma = 6$ and $\tau(\Sigma)$ is a curve of degree 6, the double cover τ induces an isomorphism $\Sigma \to \tau(\Sigma)$.

We claim that $\tau(\Sigma) \neq \mathcal{B}_6$. Indeed, $\mathcal{B}_6$ is not contained in S, and

$$\mathcal{B}_6 \cap S = \Sigma_{12} \cup \Sigma'_{12}$$

by Lemma 4.5. This shows that $\mathcal{B}_6$ intersects the surface S transversally in 24 points. Hence, the preimage of the curve $\mathcal{B}_6$ via τ is a smooth irreducible curve that is a double cover of $\mathcal{B}_6$ branched over $\Sigma_{12} \cup \Sigma'_{12}$. In particular, the preimage of the curve $\mathcal{B}_6$ via τ is not isomorphic to $\mathcal{B}_6$, so that $\tau(\Sigma) \neq \mathcal{B}_6$.

The threefold X is a quartic hypersurface in the weighted projective space $\mathbb{P}(1, 1, 1, 1, 2)$ with weighted homogeneous coordinates x_1, x_2, x_3, x_4 and w defined by equation

$$w^2 = F_4 - \lambda F_2^2,$$

where we put $x_0 = -(x_1 + \ldots + x_4)$. Denote by $\mathcal{P}$ the pencil that is cut out on X by $\alpha w = \beta F_2$, where $[\alpha : \beta] \in \mathbb{P}^1$.

Let $Y([\alpha : \beta])$ be a surface in $\mathcal{P}$ corresponding to the point $[\alpha : \beta]$. If $[\alpha : \beta] \neq [0 : 1]$, then $Y([\alpha : \beta])$ is mapped isomorphically by τ to the surface in $\mathbb{P}^3$ that is given by

$$\left(1 - \frac{\beta^2}{\alpha^2} \right) F_2 = \frac{1}{4} F_4.$$

If $[\alpha : \beta] = [0 : 1]$, then τ induces a double cover $Y([\alpha : \beta]) \to Q$ branched over a curve that is cut out by $F_4 = 0$ on the quadric Q. This curve is smooth by [8, Lemma 5.3.3(xi)]. In particular, $Y([\alpha : \beta])$ is either a smooth $K3$ surface or a nodal $K3$ surface by Proposition 3.4.

Let P be a general point in Γ. Then its $\mathfrak{A}_5$-orbit is of length 60, because the $\mathfrak{A}_5$-orbit of the point $\tau(P)$ is of length 60 by Lemma 4.6. Let Y be a surface in $\mathcal{P}$ such that $P \in Y$. Then $\Sigma \subset Y$, since otherwise we would have

$$12 = Y \cdot \Sigma > 60.$$

Hence, we have $\tau(\Sigma) \subset \tau(Y)$, which implies that $\tau(Y) \neq Q$, because none of the curves $\mathcal{C}$, $\mathcal{C}'$, $\mathcal{L}_6$ and $\mathcal{L}'_6$ is contained in Q by Lemma 4.5. Thus, $\tau(Y)$ is a (possibly nodal) $\mathfrak{A}_5$-invariant quartic surface, and τ induces an isomorphism $Y \to \tau(Y)$.

By Corollary 4.7, the surface $\tau(Y)$ is the surface $S_\mathcal{C}$ (respectively, $S_{\mathcal{L}_6}$) in the case when $\tau(\Sigma)$ is one of the curves $\mathcal{C}$ or $\mathcal{C}'$ (respectively, $\mathcal{L}_6$ or $\mathcal{L}'_6$). In particular, the surface $\tau(Y)$ is smooth.

By Corollary 4.7, the surface Y contains an $\mathfrak{A}_5$-invariant curve Σ' such that $\Sigma' \neq \Sigma$, the curves Σ' and Σ are isomorphic, and

$$\Sigma + \Sigma' \sim 3H|_Y.$$

Indeed, we have

$$\mathcal{L}_6 + \mathcal{L}_6' = S_3|_{S_{\mathcal{L}_6}}$$

and

$$\mathcal{C} + \mathcal{C}' = S_3|_{S_{\mathcal{C}}}$$

by Corollary 4.7. Thus, if $\tau(\Sigma) = \mathcal{C}$ (respectively, $\tau(\Sigma) = \mathcal{C}'$, $\tau(\Sigma) = \mathcal{L}_6$, $\tau(\Sigma) = \mathcal{L}_6'$), then we can let Σ' to be the preimage on Y of the curve $\mathcal{C}'$ (respectively, $\mathcal{C}$, $\mathcal{L}_6'$, $\mathcal{L}_6$).

By construction, we have $\Sigma^2 = (\Sigma')^2 < 0$. On the other hand, we have

$$\mathcal{M}|_Y = m\Sigma + m'\Sigma' + \Omega \sim nH|_Y,$$

where m and m' are non-negative integers, and Ω is an effective divisor on the surface Y whose support does not contain the curves Σ and Σ'. Since $\Sigma + \Sigma' \sim 3H|_Y$, we have

$$(3m - n)\Sigma + (3m' - n)\Sigma' + 3\Omega \sim 0.$$

Moreover, one has $3m - n > 0$, because $m > \frac{n}{2}$ by (4.9). Hence, we obtain $3m' - n < 0$, so that

$$0 \leqslant (3m - n)\Sigma \cdot \Sigma' + 3\Omega \cdot \Sigma' = (n - 3m')\left(\Sigma'\right)^2 < 0,$$

which is absurd. The obtained contradiction shows that Z is not a curve.

We see that Z is a point. Denote by Ξ its $\mathfrak{A}_5$-orbit. Define an integer m as in Lemma 4.8. Let $\mathcal{M}$ be the linear system consisting of all surfaces in $|mH|$ passing through Ξ. Then $\mathcal{M}$ does not have base curves and fixed components by Lemma 4.8. Thus, if Z is a smooth point of X, then (4.10) gives

$$n^2|\Xi| \geqslant 2mn^2 = M \cdot D_1 \cdot D_2 \geqslant \sum_{i=1}^{r} \operatorname{mult}_{Z_i}(D_1 \cdot D_2) > n^2|\Xi|$$

for a general surface M in $\mathcal{M}$, and two general surfaces D_1 and D_2 in $\mathcal{D}$. Therefore, we see that Z is a singular point of X. By Proposition 3.4, either $X = X_{10}$, or $X = X_{10}'$, or $X = X_{15}$. Put $r = |\Xi|$, so that either $r = 10$ or $r = 15$.

Let $f \colon W \to X$ be a blow up of Ξ, and let $E_1, \ldots, E_r$ be the f-exceptional surfaces. Denote by $\widetilde{\mathcal{D}}$ be the proper transform of the linear system $\mathcal{D}$ via f. Let $\widetilde{D}_1$ and $\widetilde{D}_2$ be two general surfaces in $\widetilde{\mathcal{D}}$. Then

$$\widetilde{D}_1 \sim \widetilde{D}_2 \sim f^*(nH) - m\sum_{i=1}^{r} E_i$$

for some positive integer m. Denote by $\widetilde{\mathcal{M}}$ the proper transform of the linear system $\mathcal{M}$ via f. Let $\widetilde{M}$ be a general surface in $\widetilde{\mathcal{M}}$. Then

$$\widetilde{M} \sim f^*(3H) - t\sum_{i=1}^{r} E_i$$

for some positive integer t. Actually, one can show that $t = 1$, but we will not use this. Since $\mathcal{M}$ does not have base curves and fixed components, the divisor $\widetilde{M}$ is nef. Thus, we have

$$0 \leqslant \widetilde{M} \cdot \widetilde{D}_1 \cdot \widetilde{D}_2 = 6n^2 - 2rm^2t \leqslant 6n^2 - 2rm^2,$$

which implies that $m \leqslant \frac{n\sqrt{3}}{\sqrt{r}}$. On the other hand, $m > \frac{n}{2}$ by [4, Theorem 1.7.20]. This gives $r = 10$, so that either $\tau(\Xi) = \Xi_{10}$ or $\tau(\Xi) = \Xi'_{10}$.

By Remark 3.3, there exists a plane $\Pi \subset \mathbb{P}^3$ that contains at least four points of $\tau(\Xi)$ such that no three of them are collinear. Let C be a general conic in Π that passes through these four points. Then its preimage on X via τ splits as a union of two smooth rational curves. Denote by $\widetilde{C}$ the proper transform of one of these curves on W via f. Then $\widetilde{C}$ is not contained in the support of $\widetilde{D}_1$. On the other hand, we have

$$\widetilde{D}_1 \cdot \widetilde{C} = \left(f^*(nH) - m \sum_{i=1}^{r} E_i \right) \cdot \widetilde{C} = 2n - m \sum_{i=1}^{r} E_i \cdot \widetilde{C} \leqslant 2n - 4m,$$

which implies that $m \leqslant \frac{n}{2}$. This is again impossible, because we already know that $m > \frac{n}{2}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

References

[1] M. Artin, D. Mumford, *Some elementary examples of unirational varieties which are not rational*, Proc. Lond. Math. Soc. **25** (1972), 75–95.

[2] A. Beauville, *Non-rationality of the symmetric sextic Fano threefold*, Geometry and arithmetic, 57–60, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2012.

[3] H. Blichfeldt, *Finite Collineation Groups*, University of Chicago Press, Chicago, 1917.

[4] I. Cheltsov, *Birationally rigid Fano varieties*, Russian Math. Surveys **60** (2005), 875–965.

[5] I. Cheltsov, V. Przyjalkowski, C. Shramov, *Which quartic double solids are rational?*, preprint (2015).

[6] I. Cheltsov, C. Shramov, *Three embeddings of the Klein simple group into the Cremona group of rank three*, Transformation Groups **17** (2012), 303–35.

[7] I. Cheltsov, C. Shramov, *Five embeddings of one simple group*, Transactions of the AMS **366** (2014), 1289–1331.

[8] I. Cheltsov, C. Shramov, *Cremona groups and the icosahedron*, CRC Press, 2015.

[9] H. Clemens, *Double solids*, Adv. in Math. **47** (1983), 107–230.

[10] A. Degtyarev, *Classification of surfaces of degree four having a nonsimple singular point*, Mathematics of the USSR-Izvestiya **35** (1990), 607–627.

[11] S. Endraß, *On the divisor class group of double solids*, Manuscripta Mathematica **99** (1999), 341–358.

[12] K. Hashimoto, *Period map of a certain K3 family with an S₅-action*, J. Reine Angew. Math. **652** (2011), 1–65.

[13] I. Dolgachev, V. Iskovskikh, *Finite subgroups of the plane Cremona group*, Progr. Math. **269** (2009), 443–548.

[14] Yu. Prokhorov, *Fields of invariants of finite linear groups*, Cohomological and geometric approaches to rationality problems, 245–273, *Progr. Math.* **282**, Birkhäuser Boston, Inc., Boston, MA, 2010.

[15] Yu. Prokhorov, *Simple finite subgroups of the Cremona group of rank 3*, J. Alg. Geom. **21** (2012), 563–600.

[16] Yu. Prokhorov, *G-Fano threefolds, I*, Adv. Geom. **13** (2013), 389–418.

[17] Z. Reichstein, B. Youssin, Essential dimensions of algebraic groups and a resolution theorem for *G*-varieties, with an appendix by J. Kollár and E. Szabo, *Canad. J. Math.* **52** (2000), no. 5, 1018–1056.

[18] V. Shokurov, *Prym varieties: theory and applications*, Izv. Akad. Nauk SSSR Ser. Mat. **47** (1983), 785–855.

[19] V. Shokurov, *3-fold log flips*, Russian Acad. Sci. Izv. Math. **40** (1993), 95–202.

[20] M. Szurek, J. Wiśniewski, *Fano bundles of rank 2 on surfaces*, Compositio Math. **76** (1990), 295–305.

[21] A. Tihomirov, *Singularities of the theta-divisor of the intermediate Jacobian of the double $\mathbb{P}^3$ of index two*, Izv. Akad. Nauk SSSR Ser. Mat. **46** (1982), no. 5, 1062–1081.

[22] Y. Umezu, *On normal projective surfaces with trivial dualizing sheaf*, Tokyo J. Math. **4** (1981), 343–354.

[23] C. Voisin, *Sur la jacobienne intermédiaire du double solide d'indice deux*, Duke Math. J. **57** (1988), 629–646.

[24] A. Zagorski, *Three-dimensional conical fibrations*, Math. Notes **21** (1977), 420–427.

Ivan Cheltsov
School of Mathematics, The University of Edinburgh, Edinburgh EH9 3JZ, UK.
Laboratory of Algebraic Geometry, GU-HSE, 7 Vavilova street, Moscow, 117312, Russia.
Faculty of Mechanics and Mathematics, Moscow State University, Moscow, 119991, Russia.
`I.Cheltsov@ed.ac.uk`

Victor Przyjalkowski
Steklov Institute of Mathematics, 8 Gubkina street, Moscow 119991, Russia.
Laboratory of Algebraic Geometry, GU-HSE, 7 Vavilova street, Moscow, 117312, Russia
`victorprz@mi.ras.ru`

Constantin Shramov
Steklov Institute of Mathematics, 8 Gubkina street, Moscow 119991, Russia.
Laboratory of Algebraic Geometry, GU-HSE, 7 Vavilova street, Moscow, 117312, Russia
`costya.shramov@gmail.com`