Quartic spline solution of a third order singularly perturbed boundary value problem

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Abstract

Singularly perturbed boundary value problems are solved using various techniques. The spline of degree four is used for the approximate solution of a third order self adjoint singularly perturbed boundary value problem. Convergence analysis is given and the method is proved to be second order convergent. Two examples numerically illustrate the method.

Contents

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1 Introduction

The solution of a singularly perturbed boundary value problem exhibits a multiscale character. There is a thin transition layer, where the solution varies swiftly, while away from the layer the solution behaves regularly and varies gradually, therefore many complications are faced in solving singularly perturbed boundary value problems using standard numerical methods. In recent years, a large number of special purpose methods have been established to provide accurate results.

Consider the self adjoint singularly perturbed boundary value problem of the form

$$\begin{gathered} Ly(x) = -\varepsilon y^{(3)}(x) + p(x)y(x) = f(x), \quad p(x) \geqslant 0\,, \\ y(0) = \alpha_0\,, \quad y(1) = \alpha_1\,, \quad y^{(1)}(0) = \alpha_2\,, \end{gathered} \tag{1}$$

or

$$\begin{gathered} Ly(x) = -\varepsilon y^{(3)}(x) + p(x)y(x) = f(x), \quad p(x) \geqslant 0\,, \\ y(0) = \alpha_0\,, \quad y(1) = \alpha_1\,, \quad y^{(2)}(0) = \alpha_3\,, \end{gathered} \tag{2}$$

where α_0, α_1, α_2 and α_3 are constants and ε is a small positive parameter ($0 < \varepsilon \leqslant 1$), also $f(x)$ and $p(x)$ are smooth functions. In this problem, we take $p(x) = p =$ constant. Singularly perturbed problems usually crop up in chemical reactions, quantum mechanics, fluid mechanics, optical control, etc. Three principal approaches are available to solve such problems, namely, finite difference methods, finite element methods and spline approximation methods.

The existence and uniqueness of singularly perturbed boundary value problems (BVPs) was discussed by Howers, Kelevedjiev, and Roos et al. [3, 5, 10]. A numerical method for a class of second order singularly perturbed two point boundary value problems on a uniform mesh using a compressed spline was proposed by Mohanty and Jha [8]. Quartic non-polynomial spline functions were used to develop a class of numerical methods for solving self adjoint second order singularly perturbed two point boundary value problems by Tirmizi et al. [13]. Difference schemes were developed for the numerical solution of second order two point singularly perturbed boundary value problems using a tension spline by Kadalbajoo and Patidar [4]. A sextic spline has been used to solve second order singularly perturbed boundary value problems by Khan et al. [6] and the method proved to be fifth order accurate. The method has been presented to solve a class of second order singularly perturbed two point boundary value problems for certain ordinary differential equations having singular coefficients [14]. A numerical technique has been derived for a class of singularly perturbed two point boundary value problems on an uniform mesh using polynomial cubic spline by Rashidinia [9] and was given to be second order convergent. A fourth order finite difference scheme based on a nonuniform mesh for a class of singular two point perturbed boundary value problem was described by Kumar [7] and shown to be of order four. This method has also solved a class of third order singularly perturbed boundary value problems [1]. The singularly perturbed boundary value problem for quasilinear third order ordinary differential equations involving two small parameters has been considered by Su-rang et al. [11] and asymptotic solutions constructed by the method of two step expansions. The derivation of solutions to a singular two point boundary value problem for third order nonlinear differential systems by employing the method of descent has been discussed by Du [2]. Numerical solutions of fifth order two point singularly perturbed boundary value problems have been considered by Syam and Attili [12].

This paper is organized in four sections. Section 2 determines the consistency relation and end condition required for the solution of BVP (1) and (2). Section 3 discusses the convergence analysis of the quartic spline method. Finally, Section 4 discusses the results of two numerical examples.

2 Consistency relations

To develop the consistency relations the following fourth degree spline is considered:

$$Q_i(x) = a_i(x-x_i)^4 + b_i(x-x_i)^3 + c_i(x-x_i)^2 + d_i(x-x_i) + e_i \qquad (3)$$

defined on $[a,b]$, where $x \in [x_i, x_{i+1}]$ with equally spaced knots, $x_i = a + ih$, $i = 0, 1, \ldots, N$, and $h = (b-a)/N$. Using the following notations

$$\begin{aligned} Q_i(x_i) &= y_i\,, & Q_i(x_{i+1}) &= y_{i+1}\,, \\ Q_i^{(3)}(x_i) &= m_i\,, & Q_i^{(3)}(x_{i+1}) &= m_{i+1}\,, \\ Q_i^{(1)}(x_i) &= n_i\,, \end{aligned}$$

the coefficients in (3) are determined to be

$$\begin{aligned} a_i &= (m_i - m_{i+1})/24h\,, \\ b_i &= m_i/6\,, \\ c_i &= -(3h^3 m_i + h^3 m_{i+1} + 24hn_i + 24y_i - 24y_{i+1})/24h^2, \\ d_i &= n_i\,, \\ e_i &= y_i\,. \end{aligned}$$

Applying the first and second derivative continuities at knots, that is, $Q_{i-1}^{(\mu)}(x_i) = Q_i^{(\mu)}(x_i)$, for $\mu = 1$ and 2, the following relations are derived:

$$\frac{1}{12h}\left[h^3 m_{i-1} + h^3 m_i - 12(hn_{i-1} + hn_i + 2y_{i-1} - 2y_i)\right] = 0\,,$$

$$\frac{1}{12h^2}\left[h^3 m_{i-1} + 8h^3 m_i + h^3 m_{i+1} - 24hn_{i-1} + 24hn_i \right.$$
$$\left. - 24y_{i-1} + 48y_i - 24y_{i+1}\right] = 0\,,$$

which leads the following consistency relation in terms of m_i and y_i

$$m_{i-2} + 11m_{i-1} + 11m_i + m_{i+1} = \frac{24}{h^3}(-y_{i-2} + 3y_{i-1} - 3y_i + y_{i+1})\,, \quad (4)$$

for $i = 2, 3, \ldots, N-1$. Using Equation (1), Equation (4) is

$$(24\varepsilon + ph^3)y_{i-2} + (-72\varepsilon + 11ph^3)y_{i-1} + (72\varepsilon + 11ph^3)y_i$$
$$+ (-24\varepsilon + ph^3)y_{i+1} = h^3(f_{i-2} + 11f_{i-1} + 11f_i + f_{i+1}), \quad (5)$$

for $i = 2, 3, \ldots, N-1$. Since the above system consists of $(N-2)$ equations with $(N-1)$ unknowns, one more equation is required. The following relation describing truncation error is used in this regard:

$$T_0 = -h^3(a_0 m_{-1} + a_1 m_0 + a_2 m_1)$$
$$+ \left(a_3 y_{-1} + a_4 y_0 + a_5 y_1 + a_6 y_2 + a_7 h y_0^{(1)}\right). \quad (6)$$

Moreover, the values of y_{-1} and m_{-1} can be calculated and are

$$y_{-1} = (5y_0 - 10y_1 + 10y_2 - 5y_3 + y_4),$$
$$m_{-1} = (5m_0 - 10m_1 + 10m_2 - 5m_3 + m_4).$$

Using Taylor series of the right-hand side of Equation (6) along with the coefficients of y_0, $hy_0^{(1)}$, $h^2y_0^{(2)}$, $h^3y_0^{(3)}$, $h^4y_0^{(4)}$, the value of a_is are calculated as

$$a_0 = 1\,, \quad a_1 = -\frac{5}{3}\,, \quad a_2 = \frac{3}{2}\,, \quad a_3 = -\frac{1}{2}\,,$$
$$a_4 = 3\,, \quad a_5 = -\frac{7}{2}\,, \quad a_6 = 1\,, \quad a_7 = 1\,.$$

Using the values of a_is, y_{-1} and m_{-1} in Equation (6), the required end condition is

$$\left(\frac{3}{2}\varepsilon+\frac{17}{2}ph^3\right)y_1-(4\varepsilon+10ph^3)y_2+\left(\frac{5}{2}\varepsilon+5ph^3\right)y_3-\left(\frac{1}{2}\varepsilon+ph^3\right)y_4 \\ -h^3\left(-\frac{10}{3}f_0+\frac{17}{2}f_1-10f_2+5f_3-f_4\right)-\left(-\frac{1}{2}\varepsilon+\frac{10}{3}ph^3\right)\alpha_0 \\ -\varepsilon h\alpha_2+O(h^5)=0\,. \tag{7}$$

Similarly, the end condition for system (2) is

$$\left(16\varepsilon+\frac{103}{12}ph^3\right)y_1-(19\varepsilon+10ph^3)y_2+(10\varepsilon+5ph^3)y_3 \\ -(2\varepsilon+ph^3)y_4-h^3\left(-\frac{43}{12}f_0+\frac{103}{12}f_1-10f_2+5f_3-f_4\right) \\ -\left(5\varepsilon+\frac{43}{12}ph^3\right)\alpha_0+\varepsilon h^2\alpha_3+O(h^5)=0\,. \tag{8}$$

3 Convergence of the method

The system of Equations (5) and (7) provides the required quartic spline solution of BVP (1), which can be written in the matrix form

$$AY-h^3DF=C\,, \tag{9}$$

where

$$A=\begin{bmatrix} (\frac{3}{2}\varepsilon+\frac{17}{2}ph^3) & -(4\varepsilon+10ph^3) & (\frac{5}{2}\varepsilon+5ph^3) & -(\frac{1}{2}\varepsilon+ph^3) & \\ 11ph^3-72\varepsilon & \delta_2 & \delta_1 & & \\ ph^3+24\varepsilon & 11ph^3-72\varepsilon & \delta_2 & \delta_1 & \\ & \ddots & \ddots & \ddots & \\ & ph^3+24\varepsilon & 11ph^3-72\varepsilon & \delta_2 & \delta_1 \\ & & ph^3+24\varepsilon & 11ph^3-72\varepsilon & \delta_2 \end{bmatrix},$$

with $\delta_1 = ph^3 - 24\varepsilon$, $\delta_2 = 11ph^3 + 72\varepsilon$,

$$D = \begin{bmatrix} \frac{17}{2} & -10 & 5 & -1 & & & \\ 11 & 11 & 1 & & & & \\ 1 & 11 & 11 & 1 & & & \\ & \ddots & \ddots & \ddots & \ddots & & \\ & & 1 & 11 & 11 & 1 \\ & & & 1 & 11 & 11 \end{bmatrix},$$

$C = (c_1, c_2, \ldots, c_{N-1})^T$, $Y = (y_1, y_2, \ldots, y_{N-1})^T$ and $F = (f_1, f_2, \ldots, f_{N-1})^T$. Also

$$\begin{aligned} c_1 &= \Big(-\frac{1}{2}\varepsilon + \frac{10}{3}ph^3\Big)\alpha_0 - \frac{10}{3}h^3 f_0 - \varepsilon h y_0^{(1)}, \\ c_2 &= (-24\varepsilon - ph^3)\alpha_0 + h^3 f_0 \, , \\ c_i &= 0 \, , \qquad i = 3, 4, \ldots, N-2 \, , \\ c_{N-1} &= (24\varepsilon - ph^3)\alpha_1 + h^3 f_N \, . \end{aligned}$$

The exact solution is defined as $\bar{Y} = [y(x_1), y(x_2), \ldots, y(x_{N-1})]$, then Equation (9) is rewritten as

$$A\bar{Y} - h^3 DF = T + C \, , \tag{10}$$

where $T = [t_1, t_2, \ldots, t_{N-1}]^T$ with

$$\begin{aligned} t_1 &= -\frac{181}{120}\epsilon h^5 y^{(5)}\,(\zeta_1), \quad x_0 < \zeta_1 < x_4 \, , \\ t_i &= 2\varepsilon h^5 y^{(5)}\,(\zeta_i), \quad x_{i-2} < \zeta_i < x_{i+1} \, , \quad i = 2, 3, \ldots, N-1 \, . \end{aligned} \tag{11}$$

Moreover,

$$A(\bar{Y} - Y) = AE = T \, , \tag{12}$$

$$E = \bar{Y} - Y = (e_1, e_2, \ldots, e_{N-1})^T. \tag{13}$$

To determine the error bound the row sums $S_1, S_2, \ldots, S_{N-1}$ of matrix A are calculated:

$$\begin{aligned} S_1 &= \sum_j a_{1,j} = -\frac{1}{2}\varepsilon + \frac{5}{2}ph^3, \\ S_2 &= \sum_j a_{2,j} = -24\varepsilon + 23ph^3, \\ S_i &= \sum_j a_{i,j} = 24ph^3, \qquad i = 2, 3, \ldots, N-3\,, \\ S_{N-1} &= \sum_j a_{N-1,j} = 24\varepsilon + 23ph^3. \end{aligned} \tag{14}$$

Since the matrix A is irreducible and monotone, A^{-1} exists and its elements are nonnegative. Hence, from Equation (12),

$$E = A^{-1}T\,. \tag{15}$$

Also, from the theory of matrices

$$A^{-1}A = I_{(N-1)\times(N-1)}\,,$$

where

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,n-1} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \ldots & a_{n-1,n-1} \end{bmatrix},$$

$$A^{-1} = \begin{bmatrix} a^{-1}_{1,1} & a^{-1}_{1,2} & \ldots & a^{-1}_{1,n-1} \\ a^{-1}_{2,1} & a^{-1}_{2,2} & \ldots & a^{-1}_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a^{-1}_{n-1,1} & a^{-1}_{n-1,2} & \ldots & a^{-1}_{n-1,n-1} \end{bmatrix}$$

$$\text{and} \quad I = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix}.$$

Since each row sum of matrix $I_{(N-1)\times(N-1)} = 1$ and $A^{-1}A = I_{(N-1)\times(N-1)}$, therefore each row sum of $A^{-1}A$ equals 1; that is,

$$\begin{aligned} a_{1,1}^{-1}(a_{1,1} + a_{1,2} + \cdots + a_{1,n-1}) + a_{1,2}^{-1}(a_{2,1} + a_{2,2} + a_{2,n-1}) \\ + \cdots + a_{1,n-1}^{-1}(a_{n-1,1} + a_{n-1,2} + \cdots + a_{n-1,n-1}) &= 1\,, \\ a_{1,1}^{-1}S_1 + a_{1,2}^{-1}S_2 + \cdots + a_{1,n-1}^{-1}S_{N-1} &= 1\,, \end{aligned}$$

which is written in compact form as

$$\sum_{i=1}^{N-1} a_{k,i}^{-1}S_i = 1\,, \quad k = 1, 2, \ldots, N-1\,. \tag{16}$$

Defining $S_j = \min S_i$, then from Equation (16),

$$1 \geqslant S_j\left(a_{k,1}^{-1} + a_{k,2}^{-1} + \cdots + a_{k,N-1}^{-1}\right).$$

It follows that

$$\sum_{i=1}^{N-1} a_{k,i}^{-1} \leqslant 1/S_j = 1/(h^3 B_{i_0}), \tag{17}$$

where

$$B_{i_0} = (1/h^3)S_j \; > 0\,, \quad 1 \leqslant i_0 \leqslant N-1\,.$$

From Equation (12), the error terms can be written as

$$e_j = \sum_{i=1}^{N-1} a_{j,i}^{-1}T_i\,, \quad j = 1, 2, \ldots, N-1\,.$$

From Equation (11) and Equation (17), it can be proved that

$$|e_j| \leqslant Kh^2/B_{i_0}\,, \quad j = 1, 2, \ldots, N-1\,,$$

where K is constant and independent of h. It follows that

$$\|E\| = O(h^2).$$

Similarly, the method developed for the system (5) and (8) is also second order convergent. The result is summarized in the following theorem

Theorem 1. *Let $\bar{Y}(x)$ be the exact solution of the system (1) or (2) and let y_i, $i = 0, 1, \ldots, N$, be the exact solution of (9) then $\|E\| = O(h^2)$.*

4 Numerical results

Example 2. The following boundary value problems are considered

$$\begin{aligned} &-\epsilon y^{(3)}(x) + y(x) = f(x), \quad x \in [0, 1] \\ &y(0) = 0\,, \quad y(1) = 0\,, \quad y^{(1)}(0) = 0\,, \end{aligned} \tag{18}$$

and

$$\begin{aligned} &-\epsilon y^{(3)}(x) + y(x) = f(x), \quad x \in [0, 1] \\ &y(0) = 0\,, \quad y(1) = 0\,, \quad y^{(2)}(0) = 0\,, \end{aligned} \tag{19}$$

where

$$\begin{aligned} f(x) = 6\varepsilon(1-x)^5x^3 - 6\varepsilon^2 \big[&6(1-x)^5 - 90(1-x)^4x \\ &+ 180(1-x)^3x^2 - 60(1-x)^2x^3\big]\,. \end{aligned}$$

The analytic solution of system (18) or (19) is

$$y(x) = 6x^3\varepsilon(1-x)^5.$$

The maximum errors (in absolute values) associated with y_i, for the system (18) are summarized in Table 1.

The maximum errors (in absolute values) associated with y_i, for the system (19) using end condition (8) are summarized in Table 2.

Tables 1 and 2 confirm that if h is reduced by a factor $1/2$, n doubled, then $\|E\|$ is reduced by a factor $1/4$, which indicates that the present method gives second order results.

Table 1: Maximum absolute errors of y_i in problem (18).

ϵ	$n = 10$	$n = 20$	$n = 40$
$1/16$	2.9×10^{-3}	1.2×10^{-4}	6.4×10^{-6}
$1/32$	9.2×10^{-4}	3.8×10^{-5}	2.1×10^{-6}
$1/64$	1.4×10^{-4}	6.8×10^{-6}	4.6×10^{-7}

Table 2: Maximum absolute errors of y_i in problem (19).

ϵ	$n = 10$	$n = 20$	$n = 40$
$1/16$	1.3×10^{-2}	1.1×10^{-3}	7.8×10^{-5}
$1/32$	3.2×10^{-3}	2.7×10^{-4}	1.8×10^{-5}
$1/64$	3.4×10^{-4}	2.2×10^{-5}	1.1×10^{-6}

Example 3. The following boundary value problems are considered:

$$\begin{aligned} &-\varepsilon y^{(3)}(x) + y(x) = 81\varepsilon^2 \cos 3x + 3\varepsilon \sin 3x\,, \quad x \in [0,1], \\ &y(0) = 0\,, \quad y(1) = 3\varepsilon \sin 3\,, \quad y^{(1)}(0) = 9\varepsilon\,, \end{aligned} \tag{20}$$

and

$$\begin{aligned} &-\varepsilon y^{(3)}(x) + y(x) = 81\varepsilon^2 \cos 3x + 3\varepsilon \sin 3x\,, \quad x \in [0,1], \\ &y(0) = 0\,, \quad y(1) = 3\varepsilon \sin 3\,, \quad y^{(2)}(0) = 0\,. \end{aligned} \tag{21}$$

The analytic solution of the system (20) or (21) is

$$y(x) = 3\varepsilon \sin 3x\,.$$

The maximum errors (in absolute values) associated with y_i, for the system (20) are summarized in Table 3.

The maximum errors (in absolute values) associated with y_i, for the system (21) using end condition (8) are summarized in Table 4.

Table 3: Maximum absolute errors of y_i in problem (20).

ϵ	$n = 10$	$n = 20$	$n = 40$
$1/16$	6.9×10^{-5}	3.1×10^{-5}	5.4×10^{-6}
$1/32$	3.1×10^{-5}	1.8×10^{-5}	2.8×10^{-6}
$1/64$	4.9×10^{-5}	9.9×10^{-6}	1.4×10^{-7}

Table 4: Maximum absolute errors of y_i in problem (21).

ϵ	$n = 10$	$n = 20$	$n = 40$
$1/16$	2.5×10^{-3}	1.9×10^{-4}	1.4×10^{-5}
$1/32$	6.8×10^{-4}	5.7×10^{-5}	5.0×10^{-6}
$1/64$	1.2×10^{-4}	1.3×10^{-5}	1.6×10^{-6}

Tables 3 and 4 confirm that if h is reduced by a factor $1/2$, n doubled, then $\|E\|$ is reduced by a factor $1/4$, which indicates that the present method gives second order results.

5 Conclusion

The quartic spline method is developed for the approximate solution of a third order singularly perturbed boundary value problem. In addition to the boundary conditions corresponding to the first derivatives, the boundary conditions corresponding to the second derivatives are also considered. The method has been proved to be second order convergent. Two examples are considered for numerical illustration of the method. The results of these examples preserve the second order convergence.

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