

QUASI-CONTINUOUS FUNCTIONS ASSOCIATED WITH A HUNT PROCESS

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ABSTRACT. A stochastic characterization is given for quasi-continuous functions associated with Hunt processes.

Introduction. Quasi-continuous functions have been studied in various frameworks of potential theory, especially Dirichlet spaces (cf. [1, 4, 5]). The main object of this paper is to establish the relation between the quasi-continuity and the quasi left continuity of processes. Under rather strong duality assumptions, it also appears that the quasi-continuous functions are the functions which are finely and cofinely continuous.

1. Definitions. Let X_t be a Hunt process defined on a compact space E containing a cemetery Δ with a reference measure m we may assume to be bounded. In the sequel we shall use mostly the notations of [2], and all functions and sets will be implicitly assumed nearly Borel.

A bounded sequence of functions f_n is said to be converging *quasi-uniformly* towards f iff for some $\alpha > 0$ there exists a sequence g_n of α -excessive functions decreasing to 0 m a.s. (and therefore quasi everywhere (cf. [2, Proposition 3.2, p. 280])), such that $|f - f_n|$ is smaller than g_n .

This definition is independent of α . Indeed, for any $\beta > 0$, $g_n + (\alpha - \beta)U^\beta g_n$ is β excessive, decreases to 0 q.e. and dominates $|f - f_n|$.

One can give an alternate and more intuitive definition of the quasi-uniform convergence.

PROPOSITION 1. *A bounded sequence f_n converges quasi-uniformly iff there exists a sequence G_m of open sets such that*

- (a) f_n converges uniformly on every set $E - G_m$;
- (b) the capacitary potentials $e_{G_m}^\alpha = E_x(e^{-\alpha T_{G_m}})$ are decreasing to 0 m a.s. (and therefore quasi everywhere), for $\alpha > 0$.

REMARK. The property (b) is independent of α and equivalent to (b') the stopping times T_{G_m} increase to $+\infty P_\mu$ a.s. for every initial distribution μ charging no polar set.

PROOF. The sufficiency is easy to check. If f_n is uniformly bounded by M , for each m , $|f_n - f| \leq 2^{-m} + 2Me_{G_m}^\alpha$ for n large enough, and the sequence of α excessive functions $2^{-m} + 2Me_{G_m}^\alpha$ decreases a.s. to 0.

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To prove the converse, it is sufficient to construct a sequence of finely open sets G_m (cf. [2, Theorem 11.2, p. 62]). Set $n_j = \inf\{n, m(g_n) \leq 2^{-2^j}\}$ and $F_k = \bigcap_{j \geq k} \{g_{n_j} \leq 2^{-j}\}$. If $A_j = \{g_{n_j} > 2^{-j}\}$, $2^j g_{n_j} \geq 1_{A_j}$ and so $2^j g_{n_j} \geq e_{A_j}^\alpha$. Hence $m(e_{A_j}^\alpha) \leq 2^{+j} m(g_{n_j}) \leq 2^{-j}$. Thus, $m(e_{E-F_k}^\alpha) \leq \sum_{j \geq k} m(e_{A_j}^\alpha) \leq 2^{-(k-1)}$ and therefore $e_{E-F_k}^\alpha$ decreases to 0 q.e. as k tends to $+\infty$. Clearly f_n converges uniformly on each F_k .

We shall say that a bounded function f is *quasi-continuous* iff it is a quasi-uniform limit of continuous functions. One checks easily that a quasi-uniform limit f of quasi-continuous functions f_n is quasi-continuous. (Find f'_n continuous and h_n α -excessive such that $|f_n - f'_n| \leq h_n$ and $m(h_n) < 2^{-n}$. $|f - f'_n| \leq \sum_{m \geq n} h_m + |f - f_n|$ decreases to 0 quasi-uniformly.)

We shall say that a stochastic process is *quasi-continuous* iff it is right continuous and quasi left continuous (i.e. continuous on strictly increasing sequences of stopping times, i.e. such that its predictable projection is left continuous). In particular, if f is a function on E , the process $f(X_t)$, $t > 0$, is quasi-continuous iff it is right continuous and the process $f(X_{t-})$ (which is the predictable projection of $f(X_t)$ by [7]) is left continuous.

2. Stochastic characterisation of quasi-continuous functions. The main purpose of this paper is to prove the following:

THEOREM 1. *For a bounded function f , the following properties are equivalent:*

- (A) f is quasi-continuous,
- (B) $f(X_t)$ is a.s. quasi-continuous on $(0 + \infty)$,
- (C) $\alpha U_\alpha f$ converges towards f quasi-uniformly as $\alpha \uparrow +\infty$.

REMARK. The assumption that m is a reference measure can be removed if the initial distributions are restricted to those which do not charge m -polar sets. This applies in particular to the case of processes associated to Dirichlet spaces, where the probabilities P_μ are canonically defined only for such initial measures (cf. [6, 8]).

Equivalence of C and B.

(a) *Necessity.* It is well known that $U_\alpha f(X_t)$ is quasi-continuous. We only have to prove that the stochastic quasi-continuity is conserved in a quasi-uniform limit. With the notations of the definition, if T_m is a sequence of stopping time converging towards a stopping time $T > 0$, we have to prove that $f(X_{T_m}) - f(X_T)$ converges to 0 a.s.

By truncation, we can suppose that the stopping times are in some compact interval of $(0 + \infty)$. Then it is clear that $Z_M = \sup_{m \geq M} |f(X_{T_m}) - f(X_T)|$ is, for all n , dominated by

$$\sup_{m \geq M} |f_n(X_{T_m}) - f_n(X_T)| + g_n(X_T) + \sup_{m \geq M} g_n(X_{T_m}).$$

$$E(Z_M > \epsilon) \leq E\left(\sup_{m \geq M} |f_n(X_{T_m}) - f_n(X_T)| > \epsilon\right) + E\left(g_n(X_T) + \sup_{m \geq M} g_n(X_{T_m}) > \epsilon\right).$$

The second term can be made arbitrarily small by choosing n large enough, using the maximal inequality for the supermartingale $e^{-\alpha t}g_n(X_t)$. Then the second term can be made arbitrarily small by choosing M , using the quasi-continuity of $f_n(X_t)$.

(b) *Sufficiency.* (i) Assuming $f(X_t)$, $t > 0$, stochastically quasi-continuous, let us show that $\alpha U_\alpha f \rightarrow f$ quasi-uniformly. We shall in fact prove that $P_t f \rightarrow f$ quasi-uniformly.

$$\begin{aligned} \sup_{u \leq t} |f(x) - P_u f(x)| &\leq \sup_{u \leq t} E_x(|f(X_0) - f(X_u)|) \\ &\leq E_x \left(\sup_{u \leq t} |f(X_u) - f(X_0)| \right). \end{aligned}$$

We shall denote this latter function by $\phi_t(x)$. We have to prove that $\phi_t(x) \downarrow 0$ quasi-uniformly as $t \downarrow 0$.

(ii) Let us show that $\phi_t(X_s)$ is quasi-continuous P_μ a.s. for any initial distribution μ . As a first step, let us prove that the process

$$A_s^t = \sup_{s \leq u \leq t+s} |f(X_u) - f(X_s)|$$

is quasi-continuous in s . If T and S are two stopping times, with $T < S$ on $\{S < \infty\}$, one easily checks that

$$|A_T^t - A_S^t| \leq \inf \left(\sup_{T \leq u \leq S} |f(X_u) - f(X_T)|, \sup_{T+t \leq u \leq S+t} |f(X_u) - f(X_{T+t})| \right).$$

Therefore, since $f(X_t)$ is quasi-continuous, $|A_T^t - A_S^t|$ converges to zero as $T \uparrow S$ or as $S \downarrow T$.

(iii) Let us prove the quasi-continuity of $\phi_t(X_s)$ on $(0, u)$. For a sequence T_n of stopping times increasing or decreasing towards T ,

$$\lim_{n \rightarrow \infty} (\phi_t(X)_{T_n}) = \lim E(A_{T_n}^t) = E(A_T^t) = E(\phi_t(X)_T).$$

By Theorem 28 of Chapter IV of [3], this proves the right continuity and the existence of left limits. If T is predictable, $E(\phi_t(X)_{T-}) = E(\phi_t(X)_T)$. For any $A \in \mathcal{F}_T$, T_A is also predictable (since $\mathcal{F}_{T-} = \mathcal{F}_T$). If S_n is an announcing sequence for T_A and $S'_n = S_n \wedge u$, $\lim_{n \rightarrow \infty} \Phi_t(X)_{S'_n} = \Phi_t(X)_{T-1_A} + \Phi_t(X_u)1_{A^c}$. Therefore applying the result to $T_A \wedge u$, we get that $E(\Phi_t(X)_{T-1_A}) = E(\Phi_t(X)_T 1_A)$, which allows us to conclude:

(iv) As $t \downarrow 0$, $\phi_t(x) \downarrow 0$ quasi everywhere.

We can now complete the proof with the following "Dini lemma":

LEMMA. *If a bounded sequence f_n decreases to 0 quasi everywhere, and if each process $f_n(X_t)$ is quasi-continuous, then the convergence is quasi-uniform.*

Set $F_n = \{f_n \geq 2^{-k}\}$. It is enough to prove that $\lim_{n \rightarrow \infty} m(e_\alpha^{F_n}) = 0$ for some $\alpha > 0$. Indeed, if $m(e_\alpha^{F_{N(k)}}) < 2^{-k}$, $f_{N(k)} \leq 2^{-k} + e_\alpha^{F_{N(k)}}$ and a fortiori, $f_{N(k)} \leq g_k$, with $g_k = \sum_k^\infty (e_\alpha^{F_{N(p)}} + 2^{-p})$. But g_k is a sequence of α excessive functions decreasing to 0 a.s. since $m(g_k) \leq 2^{-k+2}$. Define $T_n = T_{F_n}$; since $f(X_t)$ is right continuous and f_n decreasing, clearly $f_n(X_{T_p}) \geq 2^{-k}$ almost surely on $\{T_p < \infty\}$ for all $p \geq n$. Set $T = \lim_{n \rightarrow \infty} \uparrow T_n$. By the quasi left continuity, it is

clear that $f_n(X_T) \geq 2^{-k}$ a.s. on $\{T < \infty\}$, for all n . Therefore $\lim_{n \rightarrow \infty} T_n = +\infty$ almost surely. It follows that $m(e_{P_n}^\alpha) = \lim \downarrow E_m(e^{-\alpha T_n}) = 0$.

REMARK. It is easy to verify on the second part of the proof that a function f is quasi-continuous as soon as $f(X_t)$ is quasi-continuous on $(0 + \infty) P_m$ a.s.

Equivalence of A and B. The stochastic quasi-continuity of continuous functions is obvious (since X_t is quasi left continuous) and it has been observed that this property is stable under quasi-uniform limit.

Let B be the Banach space obtained by completion of the space of stochastic quasi-continuous functions with respect to the norm $\|f\| = \inf(m(g), g \text{ 1-excessive, } g \geq |f|)$. B is contained in $L^1(m)$ and the quasi-uniform convergence in E implies the convergence in norm.

Since $|f| \leq |g|$ implies $\|f\| \leq \|g\|$, any continuous linear form on B is the difference of two positive linear forms. By Lemma 1 and Daniell's theorem any positive linear form on B extends into a positive measure on E , charging no polar sets. Therefore, it is null as soon as it is orthogonal to all continuous functions, and by the Hahn Banach theorem, any quasi-continuous function g is limit in norm of a sequence of continuous functions g_n . A fortiori it is the limit in norm of the sequence \tilde{g}_n , \tilde{g}_n being the truncation of g_n by the upper and lower bounds of g . Taking a subsequence n_K such that $\sum \|g - \tilde{g}_{n_K}\| < \infty$, one checks easily that \tilde{g}_{n_K} converges towards g quasi-uniformly.

REMARK. In the case of the deterministic translation process on the real line, quasi-continuous functions appear to be the continuous functions, as finely continuous functions are the right continuous functions.

3. Various properties.

3.1. We shall say that a function f is *quasi finely continuous* iff it is finely continuous except on a polar set. An equivalent definition is that $f(X_t)$ is almost surely right continuous on $(0 + \infty)$ and this property needs only to be checked P_m a.s. (If f is finely continuous except on a polar set B , and if $f(X_t)$ is not P_μ a.s. right continuous on $(0 + \infty) P_\mu$ a.s. for some μ , by the section theorem there is stopping time T such that $f(X)_{T+} \neq f(X_T)$ and by the strong Markov property $X_T \in B$ P_μ a.s., which yields a contradiction. Conversely, if f is finely discontinuous on a nonpolar closed set A , by the strong Markov property, $f(X_t)$ is not P_m a.s. right continuous at T_A . T_A cannot be P_m a.s. infinite, since A is nonpolar, and therefore $m(e_A^\alpha)$ is positive.)

We shall say that a set is quasi finely open iff it is finely open except for a polar set, and quasi-open iff it is quasi finely open, and for any sequence T_n announcing any predictable stopping time T , $\{X_T \in U\} \subseteq \lim \inf \{X_{T_n} \in U\}$. A quasi closed set will be the complementary of a quasi open set.

PROPOSITION 2. *A bounded function f is quasi-continuous iff, for any open interval (a, b) , the set $\{x, a < f(x) < b\}$ is quasi-open.*

PROOF. The necessity is clear. To prove the sufficiency we may use reductio ad absurdum. Suppose a predictable time $T > 0$ and an announcing sequence T_n such that $\lim_n \sup |f(X_T) - f(X_{T_n})| \geq \epsilon > 0$ on a set A of positive probability. (We fix an initial measure.) There is at least an interval I of the form $(r - \epsilon/3, r + \epsilon/3)$,

r rational, such that the set $B = A \cap \{f(X_T) \in I\}$ has positive probability. Since $\{x, f(x) \in I\}$ is quasi-open, $f(X_{T_n}) \in I$ for n large enough and $\omega \in B$. But then,

$$|f(X_T) - f(X_{T_n})| < 2\epsilon/3$$

and we get a contradiction.

COROLLARY. *The following properties are equivalent:*

- (a) *Every quasi finely continuous function is quasi-continuous.*
- (b) *Every quasi finely open set is quasi-open.*
- (c) *Every bounded α -excessive function f is regular, i.e. such that $f(X_t)$ is quasi left continuous.*

PROOF. Clearly (a) follows from (b) by the proposition and (c) follows from (a). (b) follows from (c) by Theorem 4.5, p. 86 in [2]. In particular, this corollary applies to the case of classical potential theory and Brownian motion. It also applies to the case of Dirichlet spaces.

REMARK. With the argument given in Lemma 1, one easily checks that if the intersection of a decreasing family of quasi-closed sets is polar, they decrease quasi-uniformly to 0.

3.2. PROPOSITION 3 (BALAYAGE ON REGULAR QUASI-CLOSED SETS). *If M is a regular and quasi-closed set and f a quasi-continuous function, then $P_{T_M} f$ is quasi-continuous. In particular, M is a projective set.*

PROOF. One easily checks that the process $f(X_{T_M^t})$ (with $T_M^t = T_M \circ \theta_t + t$) is quasi-continuous on $(0 + \infty)$.

Its optional projection $P_{T_M} f(X_t)$ is therefore quasi-continuous (cf. step (iii) in the proof of the first equivalence).

3.3 Quasi-continuity and cofine continuity. In this section, we shall assume the existence of a Hunt process \hat{X}_t , with reference measure m , such that U_α and \hat{U}_α are in duality with respect to m . (Recall that m is assumed to be bounded.)

THEOREM 2. *A function is quasi-continuous iff it is quasi finely and quasi cofinely continuous.*

PROOF. Let ρ_t be the time reversal operator defined on $\{\omega, \zeta(\omega) > t\}$ by $X_s(\rho_t \omega) = X_{(t-s)-}(\omega)$ if $s < t$.

$$X_s(\rho_t \omega) = \delta \quad \text{if } s \geq t.$$

One checks easily on cylindric sets that, for any measurable set of paths A in $\hat{\mathcal{F}}_t$

$$E_m(\rho_t^{-1}(A)1_{\{\zeta > t\}}) = \hat{E}_m(A1_{\{\zeta > t\}}).$$

Taking $A = \{\omega, f(X_u(\omega)) \text{ is right continuous on } (0t)\}$ and letting t take all positive values, it is clear that $f(\hat{X}_u)$ is \hat{P}_m a.s. right continuous iff $f(X_{u-})$ is P_m a.s. left continuous.

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