

## QUASI-EINSTEIN TOTALLY REAL SUBMANIFOLDS OF THE NEARLY KÄHLER 6-SPHERE

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**Abstract.** We investigate Lagrangian submanifolds of the nearly Kähler 6-sphere. In particular we investigate Lagrangian quasi-Einstein submanifolds of the 6-sphere. We relate this class of submanifolds to certain tubes around almost complex curves in the 6-sphere.

**1. Introduction.** In this paper, we investigate 3-dimensional totally real submanifolds  $M^3$  of the nearly Kähler 6-sphere  $S^6$ . A submanifold  $M^3$  of  $S^6$  is called totally real if the almost complex structure  $J$  on  $S^6$  interchanges the tangent and the normal space. It has been proven by Ejiri ([E1]) that such submanifolds are always minimal and orientable. In the same paper, he also classified those totally real submanifolds with constant sectional curvature. Note that 3-dimensional Einstein manifolds have constant sectional curvature. Here, we will investigate the totally real submanifolds of  $S^6$  for which the Ricci tensor has an eigenvalue with multiplicity at least 2. In general, a manifold  $M^n$  whose Ricci tensor has an eigenvalue of multiplicity at least  $n - 1$  is called quasi-Einstein.

The paper is organized as follows. In Section 2, we recall the basic formulas about the vector cross product on  $\mathbf{R}^7$  and the almost complex structure on  $S^6$ . We also relate the standard Sasakian structure on  $S^5$  with the almost complex structure on  $S^6$ . Then, in Section 3, we derive a necessary and sufficient condition for a totally real submanifold of  $S^6$  to be quasi-Einstein. Using this condition, we deduce from [C], see also [CDVV1] and [DV], that totally real submanifolds  $M$  with  $\delta_M = 2$  are quasi-Einstein. Here,  $\delta_M$  is the Riemannian invariant defined by

$$\delta_M(p) = \tau(p) - \inf K(p),$$

where  $\inf K$  is the function assigning to each  $p \in M$  the infimum of  $K(\pi)$ ,  $K(\pi)$  denoting the sectional curvature of a 2-plane  $\pi$  of  $T_p M$ , where  $\pi$  runs over all 2-planes in  $T_p M$  and  $\tau$  is the scalar curvature of  $M$  defined by  $\tau = \sum_{i < j} K(e_i \wedge e_j)$ . Totally real submanifolds of  $S^6$  with  $\delta_M = 2$  have been classified in [DV]. Essentially, these submanifolds are either local lifts of holomorphic curves in  $CP^2$  or tubes with radius  $\pi/2$  in the direction of  $NN^2$ , where  $N^2$  is a non-totally geodesic almost complex curve and  $NN^2$  denotes the vector bundle whose fibres are planes orthogonal to the first osculating space of  $N^2$ .

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In Section 4, we then construct some other examples of 3-dimensional quasi-Einstein totally real submanifolds by considering also tubes with different radii. More specifically, we prove

**THEOREM 1.** *Let  $\phi : N^2 \rightarrow S^6$  be an almost complex curve in  $S^6(1)$  without totally geodesic points. Denote by  $UN^2$  the unit tangent bundle of  $N^2$ . Define*

$$\psi_\gamma : UN^2 \rightarrow S^6 : v \mapsto \cos \gamma \phi + \sin \gamma v \times \frac{\alpha(v, v)}{\|\alpha(v, v)\|},$$

where  $\alpha$  denotes the second fundamental form of the surface  $N^2$ . Then  $\psi$  is an immersion on an open dense subset of  $UN^2$ . Moreover  $\psi$  is totally real if and only if either

- (1)  $\gamma = \pi/2$ , or
- (2)  $\cos^2 \gamma = 5/9$  and  $N^2$  is a superminimal surface.

Further, in both cases the immersion defines a quasi-Einstein metric on  $UN^2$  and if (1) holds, then with respect to this metric  $\delta_{UN^2} = 2$  and if (2) holds, then  $\delta_{UN^2} < 2$ .

The above theorem also generalizes results obtained by Ejiri [E2], who only considered tubes around superminimal almost complex curves and who omitted Case (1).

Next, in Section 5, we prove the following converse:

**THEOREM 2.** *Let  $F : M^3 \rightarrow S^6$  be a totally real immersion of a 3-dimensional quasi-Einstein manifold. Then either  $\delta_{M^3} = 2$  or there exists an open dense subset  $W$  of  $M$  such that each point  $p$  of  $W$  has a neighborhood  $W_1$  such that either*

- (1)  $F(W_1) = \psi_\gamma(UN^2)$ , where  $N^2$  is a superminimal linearly full almost complex curve in  $S^6$ , and  $\psi_\gamma$  with  $\cos^2 \gamma = 5/9$  is as defined in Theorem 1, or
- (2)  $F(W_1)$  is an open subset of  $\tilde{\psi}(S^3)$ , where  $\tilde{\psi}$  is as defined in Section 4.

Case (2) can be considered as a limit case of the previous one, by taking for  $N^2$  a totally geodesic almost complex curve. Note also that Theorem 2 together with the classification theorems of [DV] provides a complete classification of the totally real quasi-Einstein submanifolds of  $S^6$ .

**2. The vector cross product and the almost complex structure on  $S^6$ .** We give a brief exposition of how the standard nearly Kähler structure on  $S^6$  arises in a natural manner from the Cayley multiplication. For further details about the Cayley numbers and their automorphism group  $G_2$ , we refer the reader to [W] and [HL].

The multiplication on the Cayley numbers  $\mathcal{O}$  may be used to define a vector cross product on the purely imaginary Cayley numbers  $\mathbf{R}^7$  using the formula

$$(2.1) \quad u \times v = (1/2)(uv - vu),$$

while the standard inner product on  $\mathbf{R}^7$  is given by

$$(2.2) \quad (u, v) = -(1/2)(uv + vu).$$

It is now elementary to show that

$$(2.3) \quad u \times (v \times w) + (u \times v) \times w = 2(u, w)v - (u, v)w - (w, v)u,$$

and that the triple scalar product  $(u \times v, w)$  is skew symmetric in  $u, v, w$ , see [HL] for proofs.

Conversely, denoting by  $\text{Re}(\mathcal{O})$  (respectively  $\text{Im}(\mathcal{O})$ ) the real (respectively imaginary) Cayley numbers, the Cayley multiplication on  $\mathcal{O}$  is given in terms of the vector cross product and the inner product by

$$(2.4) \quad \begin{aligned} (r + u)(s + v) &= rs - (u, v) + rv + su + (u \times v), \\ r, s &\in \text{Re}(\mathcal{O}), \quad u, v \in \text{Im}(\mathcal{O}). \end{aligned}$$

In view of (2.1), (2.2) and (2.4), it is clear that the group  $G_2$  of automorphisms of  $\mathcal{O}$  is precisely the group of isometries of  $\mathbf{R}^7$  preserving the vector cross product.

An ordered orthonormal basis  $e_1, \dots, e_7$  is said to be a  $G_2$ -frame if

$$(2.5) \quad e_3 = e_1 \times e_2, \quad e_5 = e_1 \times e_4, \quad e_6 = e_2 \times e_4, \quad e_7 = e_3 \times e_4.$$

For example, the standard basis of  $\mathbf{R}^7$  is a  $G_2$ -frame. Moreover, if  $e_1, e_2, e_4$  are mutually orthogonal unit vectors with  $e_4$  orthogonal to  $e_1 \times e_2$ , then  $e_1, e_2, e_4$  determine a unique  $G_2$ -frame  $e_1, \dots, e_7$  and  $(\mathbf{R}^7, \times)$  is generated by  $e_1, e_2, e_4$  subject to the relations

$$(2.6) \quad e_i \times (e_j \times e_k) + (e_i \times e_j) \times e_k = 2\delta_{ik}e_j - \delta_{ij}e_k - \delta_{jk}e_i.$$

Therefore, for any  $G_2$ -frame, we have the following multiplication table:

$\times$	$e_1$	$e_2$	$e_3$	$e_4$	$e_5$	$e_6$	$e_7$
$e_1$	0	$e_3$	$-e_2$	$e_5$	$-e_4$	$-e_7$	$e_6$
$e_2$	$-e_3$	0	$e_1$	$e_6$	$e_7$	$-e_4$	$-e_5$
$e_3$	$e_2$	$-e_1$	0	$e_7$	$-e_6$	$e_5$	$-e_4$
$e_4$	$-e_5$	$-e_6$	$-e_7$	0	$e_1$	$e_2$	$e_3$
$e_5$	$e_4$	$-e_7$	$e_6$	$-e_1$	0	$-e_3$	$e_2$
$e_6$	$e_7$	$e_4$	$-e_5$	$-e_2$	$e_3$	0	$-e_1$
$e_7$	$-e_6$	$e_5$	$e_4$	$-e_3$	$-e_2$	$e_1$	0

The standard nearly Kähler structure on  $S^6$  is then obtained as follows.

$$Ju = x \times u, \quad u \in T_x S^6, \quad x \in S^6.$$

It is clear that  $J$  is an orthogonal almost complex structure on  $S^6$ . In fact,  $J$  is a nearly Kähler structure in the sense that the (2,1)-tensor field  $G$  on  $S^6(1)$  defined by

$$G(X, Y) = (\tilde{\nabla}_X J)(Y),$$

where  $\tilde{\nabla}$  is the Levi-Civita connection on  $S^6(1)$ , is skew-symmetric. A straightforward computation also shows that

$$G(X, Y) = X \times Y - \langle x \times X, Y \rangle x.$$

For more information on the properties of the Cayley multiplication,  $J$  and  $G$ , we refer to [Ca2], [BVW] and [DVV].

Finally, we recall the explicit construction of a Sasakian structure on  $S^5(1)$  starting from  $C^3$  and its relation with the nearly Kähler structure on  $S^6$ . For more details about Sasakian

structures we refer the reader to [B]. We consider  $S^5$  as the hypersphere in  $S^6 \subset \mathbb{R}^7$  given by the equation  $x_4 = 0$  and define

$$j : S^5(1) \rightarrow \mathbb{C}^3 : (x_1, x_2, x_3, 0, x_5, x_6, x_7) \mapsto (x_1 + ix_5, x_2 + ix_6, x_3 + ix_7).$$

Then at a point  $p$  the structure vector field is given by

$$\xi(p) = (x_5, x_6, x_7, 0, -x_1, -x_2, -x_3) = e_4 \times p,$$

and for a tangent vector  $v = (v_1, v_2, v_3, 0, v_5, v_6, v_7)$ , orthogonal to  $\xi$ , we have

$$\phi(v) = (-v_5, -v_6, -v_7, 0, v_1, v_2, v_3) = v \times e_4.$$

Also,  $\phi\xi(p) = 0 = (e_4 \times p) \times e_4 - \langle (e_4 \times p) \times e_4, p \rangle p$ , from which we deduce for any tangent vector  $w$  to  $S^5$  that

$$(2.7) \quad \phi(w) = w \times e_4 - \langle w \times e_4, p \rangle p.$$

**3. A pointwise characterization.** Let  $M^3$  be a totally real submanifold of  $S^6$ . From [E1], we know that  $M^3$  is minimal and that for tangent vector fields  $X$  and  $Y$ ,  $G(X, Y)$  is a normal vector field on  $M^n$ . Moreover,  $\langle h(X, Y), JZ \rangle$  is symmetric in  $X, Y$  and  $Z$ . Denote by  $S$  the Ricci tensor of  $M^3$  defined by

$$S(Y, Z) = \text{trace}\{X \mapsto R(X, Y)Z\},$$

and denote by Ric the associated 1-1 tensor field, i.e.,

$$\langle \text{Ric}(Y), Z \rangle = S(Y, Z).$$

Let  $p \in M$  and assume that  $p$  is not a totally geodesic point of  $M^3$ . Then, we know from [V] that there exists an orthonormal basis  $\{e_1, e_2, e_3\}$  at the point  $p$  such that

$$\begin{aligned} h(e_1, e_1) &= \lambda_1 J e_1, & h(e_2, e_2) &= \lambda_2 J e_1 + a J e_2 + b J e_3, \\ h(e_1, e_2) &= \lambda_2 J e_2, & h(e_2, e_3) &= b J e_2 - a J e_3, \\ h(e_1, e_3) &= \lambda_3 J e_3, & h(e_3, e_3) &= \lambda_3 J e_1 - a J e_2 - b J e_3, \end{aligned}$$

where  $\lambda_1 + \lambda_2 + \lambda_3 = 0$ ,  $\lambda_1 > 0$ ,  $\lambda_1 - 2\lambda_2 \geq 0$  and  $\lambda_1 - 2\lambda_3 \geq 0$ . If  $\lambda_2 = \lambda_3$ , we can choose  $e_2$  and  $e_3$  such that  $b = 0$ . Then by a straightforward computation, we have the following lemma:

LEMMA 3.1. *Let  $\{e_1, e_2, e_3\}$  be the basis constructed above. Then*

$$(S(e_i, e_j)) = \begin{pmatrix} 2 - \lambda_1^2 - \lambda_2^2 - \lambda_3^2 & -(\lambda_2 - \lambda_3)a & -(\lambda_2 - \lambda_3)b \\ -(\lambda_2 - \lambda_3)a & 2 - 2\lambda_2^2 - 2a^2 - 2b^2 & 0 \\ -(\lambda_2 - \lambda_3)b & 0 & 2 - 2\lambda_3^2 - 2a^2 - 2b^2 \end{pmatrix}.$$

Remark that if  $\lambda_2 = \lambda_3$ , it immediately follows from Lemma 3.1 that  $M$  is quasi-Einstein.

LEMMA 3.2. *Let  $M^3$  be a 3-dimensional totally real submanifold of  $S^6$  with the second fundamental form  $h$ . Then the Ricci tensor  $S$  has a double eigenvalue at a point  $p$  of  $M^3$*

if and only if  $p$  is a totally geodesic point or there exists an orthonormal basis  $\{e_1, e_2, e_3\}$  of  $T_pM$  such that either

- (1)  $h(e_1, e_1) = \lambda J e_1, \quad h(e_2, e_2) = -\lambda J e_1,$   
 $h(e_1, e_2) = -\lambda J e_2, \quad h(e_2, e_3) = 0,$   
 $h(e_1, e_3) = 0, \quad h(e_3, e_3) = 0,$
- (2)  $h(e_1, e_1) = 2\lambda J e_1, \quad h(e_2, e_2) = -\lambda J e_1 + a J e_2,$   
 $h(e_1, e_2) = -\lambda J e_2, \quad h(e_2, e_3) = -a J e_3,$   
 $h(e_1, e_3) = -\lambda J e_3, \quad h(e_3, e_3) = -\lambda J e_1 - a J e_2,$

where  $\lambda$  is a non-zero number.

PROOF. If  $p$  is a totally geodesic point of  $M^3$ , there is nothing to prove. Hence, we may assume that  $p$  is not totally geodesic and we can use the basis  $\{e_1, e_2, e_3\}$  constructed above. So we see that

$$\begin{aligned} Ric(e_2) &= -(\lambda_2 - \lambda_3)ae_1 + 2(1 - \lambda_2^2 - a^2 - b^2)e_2, \\ \langle Ric(e_1), e_3 \rangle &= -(\lambda_2 - \lambda_3)b. \end{aligned}$$

Since  $M^3$  is quasi-Einstein, we know that  $e_2, Ric(e_2)$  and  $Ric(Ric(e_2))$  have to be linearly dependent. Hence the above formulas imply that

$$ab(\lambda_2 - \lambda_3)^2 = 0.$$

If  $\lambda_2 = \lambda_3$ , we see that  $\{e_1, e_2, e_3\}$  satisfies Case (2) of Lemma 3.2 by rechoosing  $e_2$  and  $e_3$  if necessary. Therefore, we may assume that  $\lambda_2 \neq \lambda_3$ . Then, if necessary by interchanging  $e_2$  and  $e_3$ , we may assume that  $b = 0$ .

Suppose now that  $a = 0$ . Hence  $e_1, e_2$  and  $e_3$  are eigenvectors of  $Ric$ . Since we assumed that  $\lambda_2 \neq \lambda_3$ , we see that (if necessary after interchanging  $e_2$  and  $e_3$ , which is allowed in this case since  $a$  and  $b$  both vanish)  $M^3$  is quasi-Einstein if and only if

$$2 - \lambda_1^2 - \lambda_2^2 - \lambda_3^2 = 2 - 2\lambda_2^2,$$

which reduces to

$$-2\lambda_1^2 - 2\lambda_1\lambda_2 = 0.$$

Hence, since  $\lambda_1 \neq 0$ , we see that  $\lambda_2 = -\lambda_1$  and  $\lambda_3 = 0$ . Thus  $\{e_1, e_2, e_3\}$  is a basis as described in Case (1) of Lemma 3.2.

Finally, we consider the case that  $\lambda_2 \neq \lambda_3$  and  $a \neq 0$ . Since  $a \neq 0$ , we see that  $M^3$  is quasi-Einstein if and only if  $2 - 2\lambda_3^2 - 2a^2$  is a double eigenvalue of  $S$ . This is the case if and only if

$$\det \begin{pmatrix} \lambda_3^2 - \lambda_1^2 - \lambda_2^2 + 2a^2 & (\lambda_3 - \lambda_2)a \\ (\lambda_3 - \lambda_2)a & 2(\lambda_3^2 - \lambda_2^2) \end{pmatrix} = 0.$$

Since  $\lambda_2 \neq \lambda_3$  and  $\lambda_3 = -\lambda_1 - \lambda_2$ , this is the case if and only if

$$\det \begin{pmatrix} 2\lambda_1\lambda_2 + 2a^2 & -(\lambda_1 + 2\lambda_2)a \\ a & -2\lambda_1 \end{pmatrix} = 0,$$

i.e., if and only if

$$(3.1) \quad 4\lambda_1^2\lambda_2 = -3\lambda_1a^2 + 2\lambda_2a^2.$$

Now, we consider the following change of basis

$$\begin{aligned} u_1 &= \frac{1}{\sqrt{a^2 + 4\lambda_1^2}}(ae_1 - 2\lambda_1e_2), \\ u_2 &= \frac{1}{\sqrt{a^2 + 4\lambda_1^2}}(2\lambda_1e_1 + ae_2), \\ u_3 &= e_3. \end{aligned}$$

Then, using (3.1), we have

$$\begin{aligned} h(ae_1 - 2\lambda_1e_2, ae_1 - 2\lambda_1e_2) &= (a^2\lambda_1 + 4\lambda_1^2\lambda_2)Je_1 + (-4a\lambda_1\lambda_2 + 4a\lambda_1^2)Je_2 \\ &= -2(\lambda_1 - \lambda_2)a(aJe_1 - 2\lambda_1Je_2), \\ h(ae_1 - 2\lambda_1e_2, e_3) &= a(\lambda_1 - \lambda_2)Je_3, \\ h(2\lambda_1e_1 + ae_2, e_3) &= (2\lambda_1\lambda_3 - a^2)Je_3, \\ h(ae_1 - 2\lambda_1e_2, 2\lambda_1e_1 + ae_2) &= (2a\lambda_1^2 - 2a\lambda_1\lambda_2)Je_1 + (a^2\lambda_2 - 2a^2\lambda_1 - 4\lambda_1^2\lambda_2)Je_2 \\ &= a(\lambda_1 - \lambda_2)(2\lambda_1Je_1 + aJe_2). \end{aligned}$$

Using now the minimality of  $M$ , together with the fact that  $\langle h(X, Y), JZ \rangle$  is totally symmetric it follows that the basis  $\{u_1, u_2, u_3\}$  satisfies Case (2) of Lemma 3.2.

REMARK 3.3. An elementary computation shows that if Case (1) of Lemma 3.2 is satisfied, the Ricci tensor has eigenvalues  $2(1 - \lambda^2)$ ,  $2(1 - \lambda^2)$  and  $2$ , while if Case (2) is satisfied its eigenvalues are  $2 - 6\lambda^2$ ,  $2 - 2\lambda^2 - 2a^2$  and  $2 - 2\lambda^2 - 2a^2$ .

REMARK 3.4. Submanifolds satisfying Case (1) of Lemma 3.2 are exactly those totally real submanifolds of  $S^6$  which satisfy Chen’s equality (see [CDVV1], [CDVV2] and [DV]). A complete classification of these submanifolds was obtained in [DV].

**4. Examples of totally real submanifolds.**

EXAMPLE 4.1. We recall from [DVV] the following example: Consider the unit sphere

$$S^3 = \{(y_1, y_2, y_3, y_4) \in \mathbf{R}^4 \mid y_1^2 + y_2^2 + y_3^2 + y_4^2 = 1\}$$

in  $\mathbf{R}^4$ . Let  $X_1, X_2$  and  $X_3$  be the vector fields defined by

$$\begin{aligned} X_1(y_1, y_2, y_3, y_4) &= (y_2, -y_1, y_4, -y_3), \\ X_2(y_1, y_2, y_3, y_4) &= (y_3, -y_4, -y_1, y_2), \\ X_3(y_1, y_2, y_3, y_4) &= (y_4, y_3, -y_2, -y_1). \end{aligned}$$

Then  $X_1, X_2$  and  $X_3$  form a basis of tangent vector fields to  $S^3$ . Moreover, we have  $[X_1, X_2] = 2X_3$ ,  $[X_2, X_3] = 2X_1$  and  $[X_3, X_1] = 2X_2$ . Inspired by [M], we define a metric  $\langle \cdot, \cdot \rangle_1$  on  $S^3$  such that  $X_1, X_2$  and  $X_3$  are orthogonal and such that  $\langle X_2, X_2 \rangle_1 = \langle X_3, X_3 \rangle_1 = 8/3$  and

$\langle X_1, X_1 \rangle_1 = 4/9$ . Then  $E_1 = (3/2)X_1$ ,  $E_2 = (\sqrt{3}/2\sqrt{2})X_2$  and  $E_3 = -(\sqrt{3}/2\sqrt{2})X_3$  form an orthonormal basis on  $S^3$ . We denote the Levi-Civita connection of  $\langle \cdot, \cdot \rangle_1$  by  $\nabla$ . We recall from [DVV] that there exists an isometric totally real immersion  $\psi$  from  $(S^3, \langle \cdot, \cdot \rangle_1)$  given by

$$\psi : S^3(1) \rightarrow S^6(1) : (y_1, y_2, y_3, y_4) \mapsto (x_1, x_2, x_3, x_4, x_5, x_6, x_7),$$

where

$$\begin{aligned} x_1 &= (1/9)(5y_1^2 + 5y_2^2 - 5y_3^2 - 5y_4^2 + 4y_1), \\ x_2 &= -(2/3)y_2, \\ x_3 &= (2\sqrt{5}/9)(y_1^2 + y_2^2 - y_3^2 - y_4^2 - y_1), \\ x_4 &= (\sqrt{3}/9\sqrt{2})(-10y_3y_1 - 2y_3 - 10y_2y_4), \\ x_5 &= (\sqrt{3}\sqrt{5}/9\sqrt{2})(2y_1y_4 - 2y_4 - 2y_2y_3), \\ x_6 &= (\sqrt{3}\sqrt{5}/9\sqrt{2})(2y_1y_3 - 2y_3 + 2y_2y_4), \\ x_7 &= (\sqrt{3}/9\sqrt{2})(10y_1y_4 + 2y_4 - 10y_2y_3). \end{aligned}$$

Its connection is given by

$$\begin{aligned} \nabla_{E_1}E_1 &= 0, & \nabla_{E_2}E_2 &= 0, & \nabla_{E_3}E_3 &= 0, \\ \nabla_{E_1}E_2 &= -(11/4)E_3, & \nabla_{E_1}E_3 &= (11/4)E_2, & \nabla_{E_2}E_3 &= -(1/4)E_1, \\ \nabla_{E_2}E_1 &= (1/4)E_3, & \nabla_{E_3}E_1 &= -(1/4)E_2, & \nabla_{E_3}E_2 &= (1/4)E_1, \end{aligned}$$

and its second fundamental form satisfies

$$\begin{aligned} h(E_1, E_1) &= (\sqrt{5}/2)JE_1, & h(E_3, E_1) &= -(\sqrt{5}/4)JE_3, \\ h(E_1, E_2) &= -(\sqrt{5}/4)JE_2, & h(E_3, E_2) &= 0, \\ h(E_2, E_2) &= -(\sqrt{5}/4)JE_1, & h(E_3, E_3) &= -(\sqrt{5}/4)JE_1. \end{aligned}$$

Hence  $\tilde{\psi}$  is quasi-Einstein.

EXAMPLE 4.2. Here, we will consider tubes in the direction of the orthogonal complement of the first osculating space on an almost complex curve. In [E2], N. Ejiri already showed that a tube with radius  $\cos^2 \gamma = 5/9$  on a superminimal almost complex curve defines a totally real submanifold of  $S^6$ , and in [DV] it was shown that a tube with radius  $\pi/2$  on any almost complex curve defines a totally real submanifold  $M$  with  $\delta_M = 2$ .

An immersion  $\tilde{\phi} : N \rightarrow S^6(1)$  is called almost complex if  $J$  preserves the tangent space, i.e.,  $J_p\tilde{\phi}_*(T_pN) = \tilde{\phi}_*(T_pN)$ . It is well-known that such immersions are always minimal, and as indicated in [BVW] there are essentially 4 types of almost complex immersions in  $S^6(1)$ , namely, those which are

- (I) linearly full in  $S^6(1)$  and superminimal,
- (II) linearly full in  $S^6(1)$  but not superminimal,
- (III) linearly full in some totally geodesic  $S^5(1)$  in  $S^6(1)$  (and thus by [Ca1] necessarily not superminimal),
- (IV) totally geodesic.

Now, let  $\bar{\phi} : N \rightarrow S^6(1)$  be an almost complex curve. We denote its position vector in  $\mathbb{R}^7$  also by  $\bar{\phi}$ . For the proof of elementary properties of such surfaces, we refer to [S]. Here, we simply recall that for tangent vector fields  $X$  and  $Y$  to  $N$  and for a normal vector field  $\eta$ , we have

$$(4.1) \quad \alpha(X, JY) = J\alpha(X, Y),$$

$$(4.2) \quad A_{J\eta} = JA_\eta = -A_\eta J,$$

$$(4.3) \quad \nabla_X^\perp J\eta = G(X, \eta) + J\nabla_X^\perp \eta,$$

$$(4.4) \quad (\nabla\alpha)(X, Y, JZ) = J(\nabla\alpha)(X, Y, Z) + G(\bar{\phi}_*X, \alpha(Y, Z)),$$

where  $\alpha$  denotes the second fundamental form of the immersion and the pull-back of  $J$  to  $N$  is also denoted by  $J$ .

Next, if necessary, by restricting ourselves to an open dense subset of  $N$ , we may assume that  $N$  does not contain any totally geodesic points. Let  $p \in N$  and  $V$  be an arbitrary unit tangent vector field defined on a neighborhood  $W$  of  $p$ . We define a local non vanishing function  $\mu = \|\alpha(V, V)\|$  and an orthogonal tangent vector field  $U$  such that  $\bar{\phi}_*U = J\bar{\phi}_*V = \bar{\phi} \times \bar{\phi}_*V$ . Then, using the properties of the vector cross product, it is easy to see that  $F_1 = \bar{\phi}$ ,  $F_2 = \bar{\phi}_*V$ ,  $F_3 = J\bar{\phi}_*V$ ,  $F_4 = \alpha(V, V)/\mu$ ,  $F_5 = \alpha(V, JV)/\mu = J\alpha(V, V)/\mu = F_1 \times F_4$ ,  $F_6 = F_2 \times \alpha(V, V)/\mu$  and  $F_7 = F_3 \times \alpha(V, V)/\mu$  form a  $G_2$ -frame and hence satisfy the multiplication table as defined in Section 2.

Since  $F_4, \dots, F_7$  form a basis for the normal space along  $N$ , it is clear that we can write any normal vector field as a linear combination of these basis vector fields. Thus there exist functions  $a_1, \dots, a_4$  such that

$$(4.5) \quad (\nabla\alpha)(V, V, V) = \mu(a_1F_4 + a_2F_5 + a_3F_6 + a_4F_7).$$

Then using (4.4) and the multiplication table, we get that

$$(4.6) \quad (\nabla\alpha)(V, V, U) = \mu(-a_2F_4 + a_1F_5 + (1 + a_4)F_6 - a_3F_7).$$

From (4.5) and (4.6), it is immediately clear that

- (1)  $N$  is an almost complex curve of Type (I) if and only if  $a_3 = 0$  and  $a_4 = -1/2$ .
- (2)  $N$  is an almost complex curve of Type (III) if and only if  $a_4 + a_3^2 + a_4^2 = 0$ .

Introducing local functions  $\mu_1$  and  $\mu_2$  on  $N$  by

$$\nabla_V V = \mu_1 U, \quad \nabla_U U = \mu_2 V, \quad \nabla_V U = -\mu_1 V, \quad \nabla_U V = -\mu_2 U,$$

it follows from (4.5) and (4.6) that  $a_1 = V(\mu)/\mu$  and  $a_2 = -U(\mu)/\mu$ .

Now, in order to construct explicitly the totally real immersion from the unit tangent bundle, we recall a technical lemma from [DV].

LEMMA 4.1. Denote by  $D$  the standard connection on  $\mathbf{R}^7$ . Then, we have

$$\begin{aligned} D_V(\mu F_4) &= \mu(-\mu F_2 + a_1 F_4 + (a_2 + 2\mu_1)F_5 + a_3 F_6 + a_4 F_7), \\ D_U(\mu F_4) &= \mu(\mu F_3 - a_2 F_4 + (a_1 - 2\mu_2)F_5 + (1 + a_4)F_6 - a_3 F_7), \\ D_V(\mu F_5) &= \mu(-\mu F_3 - (a_2 + 2\mu_1)F_4 + a_1 F_5 + (1 + a_4)F_6 - a_3 F_7), \\ D_U(\mu F_5) &= \mu(-\mu F_2 - (a_1 - 2\mu_2)F_4 - a_2 F_5 - a_3 F_6 - a_4 F_7), \\ D_V(\mu F_6) &= \mu(-a_3 F_4 - (a_4 + 1)F_5 + a_1 F_6 + (a_2 + 3\mu_1)F_7), \\ D_U(\mu F_6) &= \mu(-(a_4 + 1)F_4 + a_3 F_5 - a_2 F_6 + (a_1 - 3\mu_2)F_7), \\ D_V(\mu F_7) &= \mu(-a_4 F_4 + a_3 F_5 - (a_2 + 3\mu_1)F_6 + a_1 F_7), \\ D_U(\mu F_7) &= \mu(a_3 F_4 + a_4 F_5 + (3\mu_2 - a_1)F_6 - a_2 F_7). \end{aligned}$$

PROOF OF THEOREM 1. We define a map

$$\bar{\psi} : UN \rightarrow S^6(1) : v_p \mapsto \cos \gamma \bar{\phi}(p) + \sin \gamma \bar{\phi}_*(v) \times \frac{\alpha(v, v)}{\|\alpha(v, v)\|}.$$

Using the above vector fields, we can write  $v_p = \cos(t/3)V + \sin(t/3)U$ ; and an easy computation shows that the map  $\bar{\psi}$  can be locally parameterized by

$$(4.7) \quad \bar{\psi}(q, t) = \cos \gamma F_1(q) + \sin \gamma (\cos t F_6(q) + \sin t F_7(q)),$$

where  $q \in W$  and  $t \in \mathbf{R}$ . Since the case with  $\gamma = \pi/2$  was already treated in [DV], we restrict ourselves here to the case that  $\cos \gamma \neq 0$ . We immediately see that

$$(4.8) \quad \bar{\psi}_* \left( \frac{\partial}{\partial t} \right) = \sin \gamma (-\sin t F_6 + \cos t F_7).$$

Using Lemma 4.1, we then obtain that

$$\begin{aligned} \bar{\psi}_* &= \cos \gamma D_V F_1 + \sin \gamma (\cos t D_V F_6 + \sin t D_V F_7) \\ &= \cos \gamma F_2 + \sin \gamma (-\cos t (V(\mu)/\mu) F_6) \\ (4.9) \quad &+ \cos t (-a_3 F_4 - (a_4 + 1)F_5 + a_1 F_6 + (a_2 + 3\mu_1)F_7) - \sin t (V(\mu)/\mu) F_7 \\ &+ \sin t (-a_4 F_4 + a_3 F_5 - (a_2 + 3\mu_1)F_6 + a_1 F_7) \\ &= \cos \gamma F_2 + \sin \gamma ((-a_3 \cos t - a_4 \sin t) F_4 \\ &+ (a_3 \sin t - (a_4 + 1) \cos t) F_5) + (3\mu_1 - (U(\mu)/\mu)) \bar{\psi}_* \left( \frac{\partial}{\partial t} \right). \end{aligned}$$

Using similar computations, we also get that

$$(4.10) \quad \begin{aligned} \bar{\psi}_*(U) &= \cos \gamma F_3 + \sin \gamma ((a_3 \sin t - (1 + a_4) \cos t) F_4 \\ &+ (a_3 \cos t + a_4 \sin t) F_5) + (-3\mu_2 + (V(\mu)/\mu)) \bar{\psi}_* \left( \frac{\partial}{\partial t} \right). \end{aligned}$$

From (4.8), (4.9) and (4.10), we see that  $\bar{\psi}$  is an immersion at every point  $(q, t)$ .

Now, we put

$$\begin{aligned} X &= V - (3\mu_1 - (U(\mu)/\mu))\frac{\partial}{\partial t}, \\ Y &= U - (-3\mu_2 + (V(\mu)/\mu))\frac{\partial}{\partial t}. \end{aligned}$$

A straightforward computation, using the multiplication table of Section 2, then shows that

$$\begin{aligned} \bar{\psi} \times \bar{\psi}_* \left( \frac{\partial}{\partial t} \right) &= -\sin^2 \gamma F_1 + \cos \gamma \sin \gamma (\cos t F_6 + \sin t F_7), \\ \bar{\psi} \times \bar{\psi}_*(X) &= \sin^2 \gamma (a_3 \cos 2t + a_4 \sin 2t + (1/2) \sin 2t) F_2 \\ &\quad + (\cos^2 \gamma + \sin^2 \gamma (a_3 \sin 2t - a_4 \cos 2t - \cos^2 t)) F_3 \\ &\quad + \cos \gamma \sin \gamma (-a_3 \sin t + (a_4 + 2) \cos t) F_4 \\ &\quad + \cos \gamma \sin \gamma (-a_3 \cos t - (a_4 - 1) \sin t) F_5. \end{aligned}$$

Consequently,  $\bar{\psi}$  is a totally real immersion if and only if

$$\langle \bar{\psi} \times \bar{\psi}_*(X), \bar{\psi}_*(Y) \rangle = 0,$$

i.e., if and only if

$$\begin{aligned} \cos \gamma (\cos^2 \gamma + \sin^2 \gamma (3a_3 \sin 2t - a_4 \cos 2t - \cos^2 t - a_3^2 \\ - (a_4 + 2)(1 + a_4) \cos^2 t - a_4(a_4 - 1) \sin^2 t)) = 0. \end{aligned}$$

Hence, since we assumed  $\cos \gamma \neq 0$ , we find that

$$3a_3 \sin^2 \gamma \sin 2t - 3(a_4 + 1/2) \sin^2 \gamma \cos 2t + \cos^2 \gamma - \sin^2 \gamma (a_4 + a_3^2 + a_4^2 + 3/2) = 0.$$

Since the above formula has to be satisfied for every value of  $t$ , we deduce that  $a_3 = 0$ ,  $a_4 = -1/2$  and  $\cos^2 \gamma = 5/9$ . Hence  $N^2$  is a superminimal almost complex curve in  $S^6$  and the radius of the tube satisfies  $\cos^2 \gamma = 5/9$ . Using the above values for  $a_3$  and  $a_4$ , we then obtain that

$$\begin{aligned} \bar{\psi}_*(X) &= \cos \gamma F_2 + (1/2) \sin \gamma (\sin t F_4 - \cos t F_5), \\ \bar{\psi}_*(Y) &= \cos \gamma F_3 - (1/2) \sin \gamma (\cos t F_4 + \sin t F_5), \\ J\bar{\psi}_* \left( \frac{\partial}{\partial t} \right) &= -\sin^2 \gamma F_1 + \cos \gamma \sin \gamma (\cos t F_6 + \sin t F_7), \\ J\bar{\psi}_*(X) &= (1/3) F_3 + (3/2) \cos \gamma \sin \gamma (\cos t F_4 + \sin t F_5), \\ J\bar{\psi}_*(Y) &= -(1/3) F_2 + (3/2) \cos \gamma \sin \gamma (\sin t F_4 - \cos t F_5). \end{aligned}$$

Therefore, by a straightforward computation, we obtain that

$$\begin{aligned} D_{\frac{\partial}{\partial t}} \bar{\psi}_*(X) &= (1/2) \sin \gamma (\cos t F_4 + \sin t F_5) = -(1/2)((1/3)\bar{\psi}_*(Y) - \cos \gamma J\bar{\psi}_*(X)), \\ D_{\frac{\partial}{\partial t}} \bar{\psi}_*(Y) &= -(1/2) \sin \gamma (-\sin t F_4 + \cos t F_5) = (1/2)((1/3)\bar{\psi}_*(X) + \cos \gamma J\bar{\psi}_*(Y)), \\ D_{\frac{\partial}{\partial t}} \bar{\psi}_* \left( \frac{\partial}{\partial t} \right) &= -\sin \gamma (\cos t F_6 + \sin t F_7) = -\left( (4/9)\bar{\psi} + \cos \gamma J\bar{\psi}_* \left( \frac{\partial}{\partial t} \right) \right). \end{aligned}$$

So, if we put  $E_1 = (3/2)(\partial/\partial t)$ ,  $E_2 = (\sqrt{3}/\sqrt{2})X$  and  $E_3 = (\sqrt{3}/\sqrt{2})Y$  we see that  $E_1, E_2$  and  $E_3$  form an orthonormal basis of the tangent space to  $UN$  and

$$\begin{aligned} h(E_1, E_1) &= -(3/2) \cos \gamma J \bar{\psi}_*(E_1), \\ h(E_1, E_2) &= (3/4) \cos \gamma J \bar{\psi}_*(E_2), \\ h(E_1, E_3) &= (3/4) \cos \gamma J \bar{\psi}_*(E_3). \end{aligned}$$

Since  $UN$  is totally real (and thus minimal) and  $\langle h(X, Y), JZ \rangle$  is totally symmetric in  $X, Y$  and  $Z$ , the above formulas and Lemma 3.1 imply that  $\bar{\psi}$  is quasi-Einstein. Since the first normal space is 3-dimensional, with respect to the induced metric we have  $\delta_{UN} < 2$  (see [C]). Hence  $UN$  satisfies Case (2) of Lemma 3.2.

**5. Proof of Theorem 2.** Throughout this section we will assume that  $F : M^3 \rightarrow S^6$  is a totally real immersion which is quasi-Einstein. Unless otherwise indicated, we will identify  $M^3$  with its image in  $S^6$ .

First, we remark that if  $\delta_M = 2$ , there is nothing to prove. Next, we assume that  $M^3$  is Einstein. Since a 3-dimensional Einstein manifold has constant sectional curvatures, it follows from [E1] that a neighborhood of  $p$  is  $G_2$ -congruent with an open part of the image of the totally real immersion of  $S^3(1/16)$  in  $S^6(1)$  as described in [E1] (see also [DVV]). From [E2], we also know that we can consider this image as the tube with radius  $\gamma$ , with  $\cos^2 \gamma = 5/9$  on the almost complex curve with constant Gaussian curvature  $1/6$ . This completes the proof in this case.

Next assume that  $p \in M$  such that Ric has an eigenvalue with multiplicity 2 and  $\delta_M(p) \neq 2$ . Since in a neighborhood of  $p$ ,  $M$  is quasi-Einstein, but not Einstein, there exist local orthonormal vector fields  $\{E_1, E_2, E_3\}$  such that  $E_1$  spans the 1-dimensional eigenspace and  $\{E_2, E_3\}$  span the 2-dimensional eigenspace. Hence, applying Lemma 3.2, we see that there exist local functions  $\lambda, a$  and  $b$  such that

$$\begin{aligned} h(E_1, E_1) &= 2\lambda J E_1, & h(E_2, E_2) &= -\lambda J E_1 + a J E_2, \\ h(E_1, E_2) &= -\lambda J E_2, & h(E_2, E_3) &= -a J E_3, \\ h(E_1, E_3) &= -\lambda J E_3, & h(E_3, E_3) &= -\lambda J E_1 - a J E_2. \end{aligned}$$

If necessary by changing the sign of  $E_3$ , we may assume that  $G(E_1, E_2) = J E_3, G(E_2, E_3) = J E_1$  and  $G(E_3, E_1) = J E_2$ . We now introduce local functions  $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$  by

$$\begin{aligned} \nabla_{E_1} E_1 &= a_1 E_2 + a_2 E_3, & \nabla_{E_1} E_2 &= -a_1 E_1 + a_3 E_3, & \nabla_{E_1} E_3 &= -a_2 E_1 - a_3 E_2, \\ \nabla_{E_2} E_1 &= b_1 E_2 + b_2 E_3, & \nabla_{E_2} E_2 &= -b_1 E_1 + b_3 E_3, & \nabla_{E_2} E_3 &= -b_2 E_1 - b_3 E_2, \\ \nabla_{E_3} E_1 &= c_1 E_2 + c_2 E_3, & \nabla_{E_3} E_2 &= -c_1 E_1 + c_3 E_3, & \nabla_{E_3} E_3 &= -c_2 E_1 - c_3 E_2. \end{aligned}$$

**LEMMA 5.1.** *The function  $\lambda$  satisfies  $\lambda = \sqrt{5}/4$ . Moreover,  $a_1 = a_2 = c_2 = b_1 = 0$  and  $b_2 = -c_1 = 1/4$ .*

PROOF. A straightforward computation shows that

$$\begin{aligned} (\nabla h)(E_2, E_1, E_1) &= 2E_2(\lambda)JE_1 - 2\lambda JE_3 + 4\lambda(b_1JE_2 + b_2JE_3), \\ (\nabla h)(E_1, E_2, E_1) &= -E_1(\lambda)JE_2 - \lambda JE_3 - \lambda(-a_1JE_1) \\ &\quad + 2\lambda a_1JE_1 - a_1(-\lambda JE_1 + aJE_2) - a_2(-aJE_3). \end{aligned}$$

Hence, it follows from the Codazzi equation  $(\nabla h)(E_2, E_1, E_1) = (\nabla h)(E_1, E_2, E_1)$  that

$$(5.1) \quad E_2(\lambda) = 2\lambda a_1,$$

$$(5.2) \quad E_1(\lambda) = -4\lambda b_1 - aa_1,$$

$$(5.3) \quad 4b_2 = 1 + (a/\lambda)a_2.$$

Similarly, we obtain from the Codazzi equation  $(\nabla h)(E_3, E_1, E_1) = (\nabla h)(E_1, E_3, E_1)$  that

$$(5.4) \quad E_3(\lambda) = 2\lambda a_2,$$

$$(5.5) \quad E_1(\lambda) = -4\lambda c_2 + aa_1,$$

$$(5.6) \quad 4c_1 = -1 + (a/\lambda)a_2.$$

Comparing (5.5) and (5.2), we get that

$$(5.7) \quad c_2 - b_1 = (a/2\lambda)a_1.$$

A straightforward computation, using (5.1) and (5.4), then shows that

$$(5.8) \quad (\nabla h)(E_1, E_2, E_3) = aa_2JE_1 + (3aa_3 + a - a_2\lambda)JE_2 - (a_1\lambda + E_1(a))JE_3,$$

$$(5.9) \quad (\nabla h)(E_2, E_1, E_3) = (4\lambda b_2 - \lambda)JE_1 + b_2aJE_2 + (b_1a - 2\lambda a_1)JE_3,$$

$$(5.10) \quad (\nabla h)(E_3, E_1, E_2) = (4\lambda c_1 + \lambda)JE_1 - (ac_1 + 2\lambda a_2)JE_2 + ac_2JE_3.$$

Therefore, using the Codazzi equations and (5.3), (5.6) and (5.7), we get that

$$((a^2/2\lambda) + 2\lambda)a_2 = 0, \quad ((a^2/2\lambda) + 2\lambda)a_1 = 0.$$

Hence  $a_1 = a_2 = 0$  and we deduce from the previous equations that

$$c_2 = b_1, \quad c_1 = -1/4, \quad b_2 = 1/4.$$

This implies that the function  $\lambda$  is a solution of the following system of differential equations:

$$E_1(\lambda) = -4\lambda b_1, \quad E_2(\lambda) = 0, \quad E_3(\lambda) = 0,$$

Since  $[E_2, E_3] = -(1/2)E_1 - b_3E_2 - c_3E_3$ , it immediately follows from the integrability conditions that  $b_1 = 0$  and hence  $\lambda$  is a constant.

To compute the actual value of  $\lambda$  we use the Gauss equation. We have

$$R(E_1, E_2)E_1 = -E_2 + \lambda^2 E_2 + 2\lambda^2 E_2 = (3\lambda^2 - 1)E_2.$$

On the other hand, we have

$$\begin{aligned} R(E_1, E_2)E_1 &= \nabla_{E_1}\nabla_{E_2}E_1 - \nabla_{E_2}\nabla_{E_1}E_1 - \nabla_{[E_1, E_2]}E_1 \\ &= \nabla_{E_1}((1/4)E_3) - (a_3 - 1/4)\nabla_{E_3}E_1 \\ &= -(1/4)a_3E_2 + (1/4)(a_3 - 1/4)E_2 = -(1/16)E_2. \end{aligned}$$

Hence

$$\lambda^2 = 5/16.$$

Since  $\lambda$  is positive, this completes the proof of the lemma.

Now, in order to complete the proof of the theorem, we have to make a distinction between  $M$  and its image under  $F$  in  $S^6$ . First, we consider the case that  $a$  is identically zero in a neighborhood of the point  $p$ . Then, we have the following lemma:

LEMMA 5.2. *There exists an orthonormal basis  $\{E_1, E_2, E_3\}$  with  $G(E_1, E_2) = JE_3$ ,  $G(E_2, E_3) = JE_1$  and  $G(E_3, E_1) = JE_2$ , defined on a neighborhood of the point  $p$  such that*

$$\begin{aligned} h(E_1, E_1) &= (\sqrt{5}/2)JE_1, & h(E_2, E_2) &= -(\sqrt{5}/4)JE_1, \\ h(E_1, E_2) &= -(\sqrt{5}/4)JE_2, & h(E_2, E_3) &= 0, \\ h(E_1, E_3) &= -(\sqrt{5}/4)JE_3, & h(E_3, E_3) &= -(\sqrt{5}/4)JE_1. \end{aligned}$$

Moreover, they satisfy

$$\begin{aligned} \nabla_{E_1}E_1 &= 0, & \nabla_{E_1}E_2 &= -(11/4)E_3, & \nabla_{E_1}E_3 &= (11/4)E_2, \\ \nabla_{E_2}E_1 &= (1/4)E_3, & \nabla_{E_2}E_2 &= 0, & \nabla_{E_2}E_3 &= -(1/4)E_1, \\ \nabla_{E_3}E_1 &= -(1/4)E_2, & \nabla_{E_3}E_2 &= (1/4)E_1, & \nabla_{E_3}E_3 &= 0. \end{aligned}$$

PROOF. We take the local orthonormal basis  $\{E_1, E_2, E_3\}$  constructed in the previous lemma. Clearly, this basis already satisfies the first condition. Since  $a_1 = a_2 = c_2 = b_1 = 0$  and  $b_2 = -c_1 = 1/4$ , the Gauss equations  $\langle R(E_1, E_2)E_2, E_3 \rangle = 0$ ,  $\langle R(E_1, E_3)E_3, E_2 \rangle = 0$  and  $\langle R(E_2, E_3)E_3, E_2 \rangle = 21/16$  reduce to

$$(5.11) \quad E_1(b_3) - E_2(a_3) - c_3(a_3 - 1/4) = 0,$$

$$(5.12) \quad -E_1(c_3) + E_3(a_3) - b_3(a_3 - 1/4) = 0,$$

$$(5.13) \quad E_3(b_3) - E_2(c_3) - (1/2)a_3 - b_3^2 - c_3^2 = 11/8.$$

Now, we use the following transformation of the local frame  $\{E_1, E_2, E_3\}$ :

$$\begin{aligned} U_1 &= E_1, \\ U_2 &= \cos \theta E_2 + \sin \theta E_3, \\ U_3 &= -\sin \theta E_2 + \cos \theta E_3. \end{aligned}$$

where  $\theta$  is an arbitrary locally defined function on  $M$ . It is immediately clear that  $\{U_1, U_2, U_3\}$  satisfies the conditions of the lemma if and only if the function  $\theta$  satisfies the following system of differential equations:

$$\begin{aligned} d\theta(E_1) + a_3 + 11/4 &= 0, \\ d\theta(E_2) + b_3 &= 0, \\ d\theta(E_3) + c_3 &= 0, \end{aligned}$$

i.e.,  $d\theta = -(a_3 + 11/4)\theta_1 - b_3\theta_2 - c_3\theta_3$ , where  $\{\theta_1, \theta_2, \theta_3\}$  is that dual basis of  $\{E_1, E_2, E_3\}$ . Now, this system locally has a solution if and only if the 1-form  $\omega = (a_3 + 11/4)\theta_1 + b_3\theta_2 +$

$c_3\theta_3$  is closed. One can easily verify that  $d\omega = 0$  is equivalent with (5.11), (5.12) and (5.13). □

The proof now follows from the Cartan-Ambrose-Hicks Theorem and the uniqueness theorem for totally real immersions.

Finally, we deal with the case that  $a(p) \neq 0$ . Then, we have the following lemma:

LEMMA 5.3. *Let  $\{E_1, E_2, E_3\}$  be the local orthonormal frame constructed before. Then we have*

$$E_1(a) = 0 \quad \text{and} \quad a_3 = -1/4.$$

PROOF. We look again at the proof of Lemma 5.2. From (5.8), (5.9), (5.10) and the Codazzi equation, we get  $E_1(a) = 0$  and  $3a(1/4 + a_3) = 0$ . Since  $a(p) \neq 0$ , this completes the proof.

From now on we will make a distinction between  $M$  and its image under  $F$  in  $S^6$ . We will also write explicitly  $J_p v$  as  $p \times v$ , since we will be using the almost complex structure at different points of  $S^6$ . Let us recall that we have a local basis  $\{E_1, E_2, E_3\}$  on  $U$  such that

$$\begin{aligned} \nabla_{E_1} E_1 &= 0, & \nabla_{E_1} E_2 &= -(1/4)E_3, & \nabla_{E_1} E_3 &= (1/4)E_2, \\ \nabla_{E_2} E_1 &= (1/4)E_3, & \nabla_{E_2} E_2 &= b_3 E_3, & \nabla_{E_2} E_3 &= -(1/4)E_1 - b_3 E_2, \\ \nabla_{E_3} E_1 &= -(1/4)E_2, & \nabla_{E_3} E_2 &= (1/4)E_1 + c_3 E_3, & \nabla_{E_3} E_3 &= -c_3 E_2; \\ h(E_1, E_1) &= (\sqrt{5}/2)F \times F_\star E_1, & h(E_2, E_2) &= -(\sqrt{5}/4)F \times F_\star E_1 + aF \times F_\star E_2, \\ h(E_1, E_2) &= -(\sqrt{5}/4)F \times F_\star E_2, & h(E_2, E_3) &= -aF \times F_\star E_3, \\ h(E_1, E_3) &= -(\sqrt{5}/4)F \times F_\star E_3, & h(E_3, E_3) &= -(\sqrt{5}/4)F \times F_\star E_1 - aF \times F_\star E_2; \\ F_\star E_1 \times F_\star E_2 &= F \times F_\star E_3, \\ F_\star E_2 \times F_\star E_3 &= F \times F_\star E_1, \\ F_\star E_3 \times F_\star E_1 &= F \times F_\star E_2. \end{aligned}$$

We now define a mapping  $G : U \rightarrow S^6$ , where  $U$  is a neighborhood of  $p$ , by

$$G(q) = (\sqrt{5}/3)F(q) + (2/3)F \times F_\star(E_1(q)).$$

Then, using the above formulas, we find that

$$\begin{aligned} D_{E_1} G &= (\sqrt{5}/3)F_\star(E_1) + (2/3)F \times h(E_1, E_1) = 0, \\ D_{E_2} G &= (\sqrt{5}/3)F_\star(E_2) + (2/3)(F_\star(E_2) \times F_\star(E_1) + F \times F_\star(\nabla_{E_2} E_1) + F \times h(E_2, E_1)) \\ &= (\sqrt{5}/2)F_\star(E_2) - (1/2)F \times F_\star(E_3), \\ D_{E_3} G &= (\sqrt{5}/3)F_\star(E_3) + (2/3)(F_\star(E_3) \times F_\star(E_1) + F \times F_\star(\nabla_{E_3} E_1) + F \times h(E_3, E_1)) \\ &= (\sqrt{5}/2)F_\star(E_3) + (1/2)F \times F_\star(E_2), \end{aligned}$$

from which it follows that  $G$  is not an immersion.

Using [Sp, Vol. 1, p. 204], we can identify a neighborhood of  $p$  with a neighborhood  $I \times W_1$  of the origin in  $\mathbf{R}^3$  (with coordinates  $(t, u, v)$ ) such that  $p = (0, 0, 0)$  and  $E_1 = \partial/\partial t$ .

Then there exist functions  $\alpha_1$  and  $\alpha_2$  on  $W_1$  such that  $E_2 + \alpha_1 E_1$  and  $E_3 + \alpha_2 E_1$  form a basis for the tangent space to  $W_1 \subset U$  at the point  $q = (0, u, v)$ .

Now since  $\nabla_{E_1} E_1 = 0$  and  $h(E_1, E_1) = (\sqrt{5}/2)F \times F_* E_1$ , it follows that the integral curve of  $E_1$  through the point  $F(q)$  is a circle with radius  $2/3$ , tangent vector  $F_* E_1(q)$  and normal vector  $(\sqrt{5}/3)F \times F_* E_1(q) - (2/3)F(q)$ . From this it is clear that  $F(U)$  can be reconstructed from  $W_1$  by

$$(5.14) \quad \begin{aligned} F(t, u, v) = & (\sqrt{5}/3)((\sqrt{5}/3)F(0, u, v) + (2/3)JF_* E_1(0, u, v)) \\ & + (2/3)(\sin(3t/2)E_1(0, u, v) - \cos(3t/2)((2/3)F(0, u, v) \\ & - (\sqrt{5}/3)F \times F_* E_1(0, u, v))) \end{aligned}$$

Now, we look at the restriction of the map  $G$  to  $W_1$ . Since

$$\begin{aligned} D_{E_2+\alpha_1 E_1} G &= (\sqrt{5}/2)F_*(E_2) - (1/2)F \times F_*(E_3), \\ D_{E_3+\alpha_2 E_1} G &= (\sqrt{5}/2)F_*(E_3) + (1/2)F \times F_*(E_2), \end{aligned}$$

we see that  $G$  is an immersion from  $W_1$  into  $S^6$ . Moreover, since

$$\begin{aligned} ((\sqrt{5}/3)F + (2/3)F \times F_* E_1) \times ((\sqrt{5}/2)F_* E_2 - (1/2)F \times F_* E_3) \\ = (\sqrt{5}/2)F_* E_3 + (1/2)F \times F_* E_2, \end{aligned}$$

we see that  $G$  is an almost complex immersion (and hence minimal). A straightforward computation now shows that

$$\begin{aligned} D_{E_2+\alpha_1 E_1}((\sqrt{5}/2)F_* E_2 - (1/2)F \times F_* E_3) \\ = f((\sqrt{5}/2)F_* E_3 + (1/2)F \times F_* E_2) - (3/2)G + a((\sqrt{5}/2)F \times F_* E_2 - (1/2)F_* E_3), \end{aligned}$$

where  $f$  is some function whose precise value is not essential. So, if we put  $X = E_2 + \alpha_1 E_1$  and  $Y = E_3 + \alpha_2 E_1$ , we see that  $X$  and  $Y$  are orthogonal with respect to the induced metric and have the same constant length  $\sqrt{3}/2$ . We also see that

$$h(X, X) = a((\sqrt{5}/2)F \times F_* E_2 - (1/2)F_* E_3).$$

Since  $G$  is an almost complex immersion, it follows that

$$h(X, Y) = h(X, G \times X) = G \times h(X, X).$$

Hence

$$h(X, Y) = a(-(1/2)F_* E_2 - ((\sqrt{5}/2)F \times F_* E_3)).$$

So, we see that the image of the tangent space and the first normal space to the almost complex immersion are spanned by  $F_* E_2(q)$ ,  $F_* E_3(q)$ ,  $F \times F_*(E_2)(q)$  and  $F \times F_* E_3(q)$ . Therefore, we get that its orthogonal complement in  $S^6$  is spanned by  $F_*(E_1)(q)$  and  $(2/3)F - (\sqrt{5}/3)F \times F_*(E_1)$ . Hence, the tube on the almost complex immersion  $G$  with radius  $\gamma$ , with  $\cos \gamma = \sqrt{5}/3$  in the direction of the orthogonal complement of the first osculating space is given by (5.14) and corresponds therefore to the original totally real immersion  $F$ . This completes the proof of Theorem 2.

REMARK 5.4. The above construction can also be applied to the totally real immersion of  $S^3$  into  $S^6$  constructed in Example 4.1. However, in that case, the resulting almost complex curve is totally geodesic, and hence it is impossible to define the first normal bundle. Taking coordinates

$$\begin{aligned} y_1 &= \cos(3t/2)z_1, \\ y_2 &= -\sin(3t/2)z_1, \\ y_3 &= \cos(3t/2)z_3 + \sin(3t/2)z_4, \\ y_4 &= -\sin(3t/2)z_3 + \cos(3t/2)z_4, \end{aligned}$$

we notice that  $\partial/\partial t$  corresponds with the vector field  $E_1$ . Since

$$\tilde{\psi}_*(E_1) = \begin{pmatrix} (2/3)y_2 \\ y_1 \\ -(\sqrt{5}/3)y_2 \\ -(\sqrt{3}/3\sqrt{2})y_4 \\ (\sqrt{15}/3\sqrt{2})y_3 \\ -(\sqrt{15}/3\sqrt{2})y_4 \\ (\sqrt{3}/3\sqrt{2})y_3 \end{pmatrix},$$

a straightforward computation shows that the resulting totally geodesic almost complex curve has components  $(u_1, \dots, u_7)$  given by

$$\begin{aligned} u_1 &= (\sqrt{5}/3)(y_1^2 + y_2^2 - y_3^2 - y_4^2), \\ u_2 &= 0, \\ u_3 &= (2/3)(y_1^2 + y_2^2 - y_3^2 - y_4^2), \\ u_4 &= -(2\sqrt{15}/3\sqrt{2})(y_1y_3 + y_2y_4), \\ u_5 &= (2\sqrt{3}/3\sqrt{2})(y_1y_4 - y_2y_3), \\ u_6 &= (2\sqrt{3}/3\sqrt{2})(y_1y_3 + y_2y_4), \\ u_7 &= -(2\sqrt{15}/3\sqrt{2})(y_1y_4 - y_2y_3). \end{aligned}$$

Therefore,  $\tilde{\psi}(S^3)$  can still be considered as some tube, given by (5.14), on a totally geodesic almost complex curve.

REMARK 5.5. It is clear that the examples satisfying Case (1) and (2) of Theorem 2 do not contain any totally geodesic points (the length of the second fundamental form is strictly greater than a positive constant) and that the eigenvalues of Ric are bounded by a constant strictly smaller than 2. Therefore, it follows from Lemma 3.2 that these examples can not be put together differentiably with examples  $M$  satisfying  $\delta_M = 2$ .

REMARK 5.6. We recall that a Riemannian manifold  $M$ , of any dimension  $n$ , is locally symmetric when its Riemann-Christoffel curvature tensor  $R$  is parallel, i.e.,  $\nabla R = 0$ , where  $\nabla$  is the Levi Civita connection of its metric  $\langle \cdot, \cdot \rangle$ , and that  $M$  is said to be semi-symmetric

when more generally,  $R \cdot R = 0$ , meaning that  $(R(X, Y) \cdot R)(X_1, X_2, X_3, X_4) = 0$  for all tangent  $X, Y, X_1, X_2, X_3, X_4$ , where  $R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}$  is the curvature operator of  $M$ . By pseudo-symmetric manifolds, we mean here the further generalisation of locally symmetric manifolds, namely those manifolds  $M$  for which  $R \cdot R = fQ(\langle \cdot, \cdot \rangle, R)$ , where  $f : M \rightarrow \mathbf{R}$  is a differentiable function and  $Q(\langle \cdot, \cdot \rangle, R)$  is defined by

$$Q(\langle \cdot, \cdot \rangle, R)(X, Y, X_1, X_2, X_3, X_4) = (X \wedge Y) \cdot R(X_1, X_2, X_3, X_4),$$

where  $(X \wedge Y)Z = \langle X, Z \rangle Y - \langle Y, Z \rangle X$ , for all vector fields  $Z$ . From the extrinsic as well as from the intrinsic point of view, this notion turns out to be natural generalisation of local- and semi-symmetry; for a survey on this see [Ver]. It is known that a 3-dimensional manifold is pseudo-symmetric if and only if it is quasi-Einstein. Therefore, Theorem 2 also provides a classification of all pseudo-symmetric 3-dimensional totally real submanifolds of  $S^6$ . The examples  $M$  with  $\delta_M = 2$  satisfy  $R \cdot R = Q(\langle \cdot, \cdot \rangle, R)$ , while the examples of Case (1) and (2) of Theorem 2 satisfy  $R \cdot R = (1/16)Q(\langle \cdot, \cdot \rangle, R)$ . So the pseudo-symmetry conditions for these submanifolds are realised with constant functions  $f$ , being respectively  $f = 1$  and  $f = 1/16$  (the values of which we observe to be precisely the only possibilities for  $K$  for totally real immersions of  $M^3$  into  $S^6(1)$  with constant sectional curvatures  $K$  [E1]).

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