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Quasi-Exactly-Solvable Problems and $sl(2)$ Algebra

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Abstract. Recently discovered quasi-exactly-solvable problems of quantum mechanics are shown to be related to the existence of the finite-dimensional representations of the group $SL(2,Q)$, where $Q=R,C$. It is proven that the bilinear form $h=a_{\alpha\beta}J^\alpha J^\beta+b_\alpha J^\alpha$ (J^α stand for the generators) allows one to generate a set of quasi-exactly-solvable problems of different types, including those that are already known. We get, in particular, problems in which the spectral Riemannian surface containing an infinite number of sheets is split off one or two finite-sheet pieces. In the general case the transition $h\to H=-\frac{d^2}{dx^2}+V(x)$ is realized with the aim of the elliptic functions. All known exactly-solvable quantum problems with known spectrum and factorized Riemannian surface can be obtained in this approach.

Let us consider the spectral problem for the Schrödinger operator:

$$H\psi=E\psi\,,\qquad H=-\frac{d^2}{dx^2}+V(x)\,,\tag{1}$$

where x belongs either to the interval $(-\infty,\infty)$ or to $[0,\infty)$.

Recently some number of cases was discovered where the first N eigenvalues and corresponding eigenfunctions were found explicitly. These problems were called quasi-exactly-solvable [1] (see also [2]). The list of the potentials which we have in mind is presented in Table 1 (see below). The known energy eigenvalues (and eigenfunctions) form a separated Riemannian surface consisting of a finite number of sheets in the space of potential parameters. This surface is separate from the rest of the Riemannian surface corresponding to unknown eigenvalues and eigenfunctions and having infinitely many sheets. These problems are quasi-exactly-solvable ones of the first type. There are quasi-exactly-solvable problems of the second type for which a set of N potentials differing by the values of parameters and related by analytic continuation have the same eigenvalue of i-th eigenstate in i-th potential $(i=1,2,\ldots,N)$[1].

[1] These problems as well can be considered as a generalized Schrödinger equation [see below, Eq. (7)]

Table 1. Quasi-exactly-solvable problems and corresponding "quantum tops." For problems (I), (II), (III), (VII), (VIII), (IX) $N+1=2j$; for (IV), (V), (VI) $[k/2]=2j$, $v=0$ for odd states and $v=1$ for even states. For (X) four types of solutions are known: with period $2\pi/\alpha$ and parity $(-1)^{v+1}$, $v=0,1$ $\mu=0$ and with period $4\pi/\alpha$ and parity $(-1)^{\frac{\mu+1}{2}}$, $\mu=\pm 1$ (in this case $v=1/2$). For definitions, see text

	Quasi-exactly-solvable potentials	"Top Hamiltonians"
I	$a^2e^{-2\alpha x}-a[2b+\alpha(2N+1)]e^{-\alpha x}+c(2b-\alpha)e^{\alpha x}+c^2e^{2\alpha x}$ $z=e^{-\alpha x}$, $g=ae^{-\alpha x}/\alpha+bx+ce^{\alpha x}/\alpha$, $\varrho=\alpha^{-1}$	$-\alpha J^+J^-+2aJ^+-[\alpha(2j+1)+2b]J^0-2\alpha cJ^-$
II	$d^2e^{-4\alpha x}+2ade^{-3\alpha x}+[a^2+2d(b+\alpha+2N\alpha)]e^{-2\alpha x}$ $+(2ab+\alpha a+\lambda)e^{-\alpha x}$ $E=-b^2$, $z=e^{-\alpha x}$, $g=de^{-2\alpha x}/2\alpha+ae^{-\alpha x}/\alpha-bx$, $\varrho=\alpha^{-1}e^{\alpha x}$	$-\alpha J^0J^--2dJ^+-2aJ^0-[\alpha(j+1)+2b]J^-$
III	$d^2e^{4\alpha x}+2ade^{3\alpha x}+[a^2+2d(b-\alpha)]e^{2\alpha x}+(2ab-\alpha a+\lambda)e^{\alpha x}$ $E=-b^2-\alpha k(\alpha+2b)$, $z=e^{-\alpha x}$, $g=-de^{2\alpha x}/2\alpha-ae^{\alpha x}/\alpha-bx$, $\varrho=\alpha^{-1}e^{-\alpha x}$	$-\alpha J^+J^0-(2b+3\alpha j)J^+-2cJ^0-2dJ^-$
IV	$-[a(a+\alpha)+\alpha k(\alpha k+\alpha+2a)]\,\mathrm{ch}^{-2}\alpha x-c(c+2\alpha-2a)\,\mathrm{ch}^2\alpha x+c^2\,\mathrm{ch}^4\alpha x$ $z=\mathrm{ch}^{-2}\alpha x$, $g=-c\,\mathrm{ch}\,2\alpha x/4\alpha-a/\alpha\ln\mathrm{ch}\,\alpha x$, $\varrho=\alpha^{-1}$	$4\alpha J^+J^0-4\alpha J^+J^-+2[\alpha(6j+1)+2a]J^+$ $-4[\alpha(2j+v)+a-c]J^0+4cJ^-$
V	$-b^2\,\mathrm{ch}^{-6}\alpha x+b[2a+3b+\alpha(2k+3)]\,\mathrm{ch}^{-4}\alpha x$ $-[(a+3b)(a+b+\alpha)+2k\alpha b+\lambda]\,\mathrm{ch}^{-2}\alpha x$ $E=-(a+b)^2$, $z=\mathrm{ch}^{-2}\alpha x$, $g=b\,th^2\alpha x/2\alpha-(a+b)\ln\mathrm{ch}\,\alpha x/2$, $\varrho=\alpha^{-1}\,\mathrm{ch}^2\alpha x$	$4\alpha J^+J^--4\alpha J^0J^--4bJ^++2[\alpha(4j+2v+1)+2a+4b]J^0$ $-4(a+2b+\alpha v)J^-$
VI	$a^2x^6+2abx^4+[b^2-(2k+3)a]x^2$ $z=x^2$, $g=ax^4/4+bx^2/2$, $\varrho=1$	$-4J^0J^-+4aJ^++4bJ^0-2(2j+2-v)J^-$
VII	$a^2r^6+2abr^4+[b^2-(4N+2l+d+2-2c)a]r^2+c(c-2l-d+2)r^{-2}$ $z=r^2$, $g=ar^4/4+br^2/2+(c-l)\ln r$, $\varrho=1$	$-4J^0J^-+4aJ^++4bJ^0-2(2j+2l+d-2c)J^-$
VIII	$b^2r^2+2abr-[a(2l+d-1-2c)+\lambda]r^{-1}+c(c-2l-d+2)r^{-2}$ $z=r$, $g=ar+br^2/2+(c-l)\ln r$, $\varrho=r$	$-J^0J^-+2bJ^++2aJ^0-(2l+d-1-2c+j)J^-$
IX	$b^2r^{-4}+b(2c-2l-d+3)r^{-3}$ $+[c(c-2l-d+2)+2ab+\lambda]r^{-2}-a(2N+2l+d-1-2c)r^{-1}$ $z=r$, $g=ar-br^{-1}+(c-l)\ln r$, $\varrho=r^2$	$-J^+J^-+2aJ^+-(2j+2l+d-1-2c)J^0+2bJ^-$
X	$-a^2\,\mathrm{Cos}^2\alpha x-a\alpha k\,\mathrm{Cos}\,\alpha x$ $z=\mathrm{Cos}\,\alpha x$, $g=a\,\mathrm{Cos}\,\alpha x/\alpha$, $\varrho=1$	$\alpha^2J^+J^--\alpha^2J^-J^-+2a\alpha J^++\alpha^2(2j+3-2v)J^0$ $-\alpha(2a+\mu\alpha)J^-$

In this work we will show that the quasi-exactly-solvable problems are connected with some finite-dimensional representations of the group $SL(2, Q)$, where $Q = R$ or C. In other words, each problem in Table 1 corresponds to a certain quantum "top" of the spin $j = (N-1)/2$ in an external constant magnetic field[2]. Moreover, such an arbitrary quantum "top" in a constant magnetic field generates a family of quasi-exactly-solvable problems.

1. To begin with consider the Schrödinger equation (1). Table 1 presents the potentials $V(x)$ for all known quasi-exactly-solvable problems, both one-dimensional and spherically-symmetric.

We consider a finite-dimensional representation of the $SL(2, Q)$ group with spin j in a space of polynomials with the basis functions $z^{j+m}(j \geqq m \geqq -j)$. The generators of the group have the form [3]

$$J^+ = z^2 \frac{d}{dz} - 2jz, \qquad J^0 = z\frac{d}{dz} - j, \qquad J^- = \frac{d}{dz}, \tag{2}$$

and $j(j+1)$ is the eigenvalue of the Casimir operator. Let us express the hamiltonian of the quasi-exactly-solvable problem in terms of the generators (2). For simplicity, we will consider, as an example, problem VI from the Table 1 where the first $N+1$ even states are known. The wave function can be written in the following form:

$$\psi(x) = \varphi_N(x^2)\exp\left\{-\frac{ax^4}{4} - \frac{bx^2}{2}\right\}. \tag{3}$$

Now substitute (3) in (1). Then we get the following spectral problem:

$$h\varphi_N(x^2) = E\varphi_N(x^2), \qquad h = -\frac{d^2}{dx^2} + 2x(ax^2+b)\frac{d}{dx} - (4Nax^2 - b). \tag{4}$$

Now let us make a change of variables, $z = x^2$; we get

$$h\varphi_N(z) = \left(-4z\frac{d^2}{dz^2} + 2(2az^2 + 2bz - 1)\frac{d}{dz} - (4Naz - b)\right)\varphi_N(z). \tag{5}$$

One can easily rewrite (5) in terms of the generators (2),

$$h = -4J^0J^- + 4aJ^+ + 4bJ^0 - 2(2j+1)J^- + 4az(2j-N) + b(4j+1). \tag{6}$$

If $N = 2j$ the operator (6) is reduced to a combination of the generators of the $SL(2, Q)$ group acting in the above mentioned space of polynomials in z. Thus, we reduce the problem to a description of the quantum "top" in a constant magnetic field. One can handle all other quasi-exactly-solvable problems (see Table 1) in an analogous way. Two important facts are worth emphasizing. First, the representation in terms of generators of $SL(2, Q)$ is possible only for $N = 2j$[3]. Second, all quantum "tops" emerging in this way are non-physical, although at $j = N/2$ they have real spectra coinciding with those of the corresponding quasi-exactly-solvable problems. Notice that for the problems of the second type in passing from

[2] In general, these quantum "tops" have no physical meaning [see (8)]. Our "top" Hamiltonian means the quadratic element of universal enveloping algebra $sl(2)$

[3] Thus, the spin j "top" solutions can be expressed in terms of the spin j representations of the $SL(2, Q)$ group

H to h there appears a non-trivial factor $\varrho(x): H \to \varrho^{-1} h$. This is a manifestation of the fact that the second type problems can be considered as generalized spectral problems [1],

$$H\psi = \lambda \varrho(x) \psi , \tag{7}$$

where $\varrho(x)$ is a weight function.

2. Passing to the inverse problem, consider an arbitrary "top" in an external constant magnetic field

$$h\varphi = \varepsilon\varphi , \qquad h = - \sum_{\substack{j,\, i = +,0,- \\ j \geq i}} a_{ij} J^i J^j + \sum_{i = +,0,-} b_i J^i , \tag{8}$$

where J^i are the generators of the $SL(2)$ group. We restrict ourselves to the case, when a_{ij}, b_i are arbitrary *real* numbers. It is clear that the problem under consideration has a polynomial in z solutions of the power n, provided

$$n = 2j . \tag{9}$$

The eigenvalues ε in Eq. (8) are the real roots of a certain algebraic equation of the n-th power. If the coefficient a_{++} in (8) vanishes[4], there are additional polynomial solutions of the power

$$n' = j - k , \tag{10}$$

where $k = b_+/a_{+0}$ is an integer or half-integer.

Clearly, if j is an integer k must also be an integer and if j is a half-integer k must be a half-integer ($k \leqq j$). In the general case the equation $h\varphi = \varepsilon\varphi$ can be transformed, by a change of variables and a change of function φ to the generalized Schroedinger equation (7) with a certain Hamiltonian H.[5] So, in the new variable the spin j "top" polynomial (in initial variable) solutions are orthogonal in a certain weight. To this end we first rewrite Eq. (8) in the differential form [substituting (2) into (8)]:

$$- P_4(z) \frac{d^2\varphi}{dz^2} + P_3(z) \frac{d\varphi}{dz} + (P_2 - \varepsilon)\varphi = 0 . \tag{11}$$

Here P_i are the polynomials of the i-th power:

$$P_4 = a_{++} z^4 + a_{+0} z^3 + (a_{+-} + a_{00}) z^2 + a_{0-} z + a_{--} ,$$

$$P_3 = 2(2j-1) a_{++} z^3 + [(3j-1) a_{+0} + b_+] z^2 + [2j(a_{+-} + a_{00}) + a_{00} + b_0] z + j a_{0-} + b_- ,$$

$$- P_2 = 2j(2j-1) a_{++} z^2 + 2jz(j a_{+0} + b_+) + a_{00} j^2 + b_0 j .$$

Equation (11) is of the Fuchs type. Now let us introduce a new variable $x = x(z)$ and a new function $\psi = \varphi \exp(-g)$. Taking

$$2g = \int \left(\frac{P_3}{P_4} - \frac{x''}{x'} \right) dz \tag{13}$$

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[4] In this case the Lamé equation in algebraic form emerges [see below (11), (12)]

[5] It is worth noting that the interpretation of some sorts of Schrödinger operators in terms of $SU(2)$ generators given in differential form was discussed earlier (see e.g. [6])

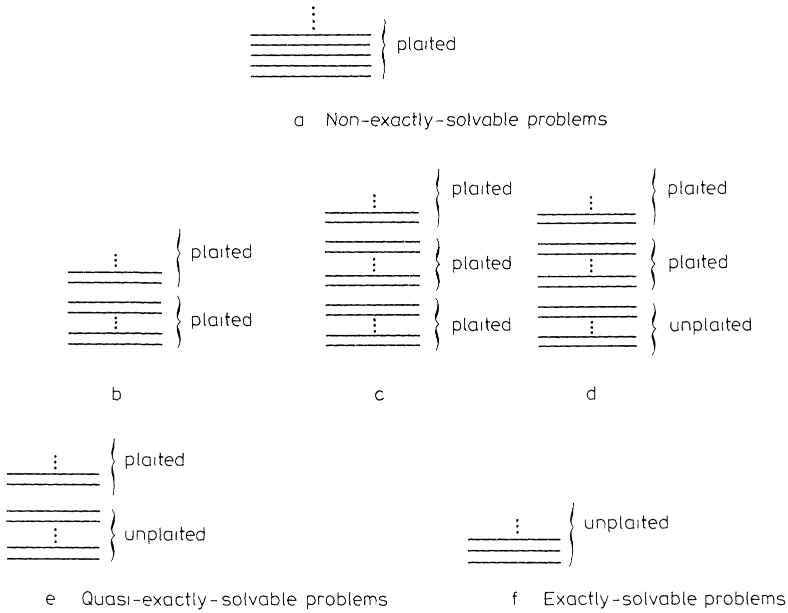

Fig. 1. The different types of Riemannian surface pictures in spectral problems

and dividing by $P_4 \cdot (x')^2$ both parts of the equation emerging in this way, we arrive at the Schrödinger-type equation

$$-\frac{d^2\psi}{dx^2} + \left[g'^2 - g'' + \frac{P_2 - \varepsilon}{P_4 \cdot (x')^2}\right] \psi = 0, \tag{14}$$

where the role of the weight function [see Eq. (7)] is played by $\varrho = 1/(P_4 \cdot x'^2)$. When $\varrho = 1$, i.e.

$$x = \pm \int \frac{dz}{\sqrt{P_4}}, \tag{15}$$

we get the standard Schrödinger equation in which we know $N = 2j + 1$ solutions in the form of polynomials of the N-th power in z multiplied by $e^{-\varphi}$. These solutions are orthogonal with respect to the weight function $e^{-2\varphi}$. The eigenvalues plait forming the Riemann surface of $(N+1)$ sheets with respect to every parameter of the original problem (8) (see Fig. 1b). Thus we get the quasi-exactly-solvable problems of the first type. The second type problems emerge either when

$$P_4 \cdot (x')^2 = z + \alpha, \qquad x = \pm \int \sqrt{\frac{z+d}{P_4}}\, dz \tag{16}$$

and the role of energy in the standard Schrödinger equation is played by $\lambda \equiv 2j(ja_{+0} + b_+)$, or when

$$P_4 \cdot (x')^2 = z^2 + \beta z + \gamma, \qquad x = \pm \int \sqrt{\frac{z^2 + \beta z + \gamma}{P_4}}\, dz \tag{17}$$

and the role of energy is played by $\lambda \equiv 2j(2j-1)a_{++}$. Here α, β, γ are parameters. In the general case of arbitrary weight function $\varrho(z)$ we obtain the generalized Schrödinger equation (7) in $x = \pm \int dz/\sqrt{\varrho P_4(z)}$.

A special situation takes place when [cf. Eq. (10)]

$$a_{++}=0\,, \qquad b_{+}=(j-n')a_{+0}\,. \tag{18}$$

At a fixed value of j there are two types of solutions of Eq. (8) of the polynomial form: of the power $n=2j$ and power n'. These solutions coincide at $n=n'$. If $n>n'$ all solutions of the second type are actually a subset of the solutions of the first type. The second type solutions form their own $(n'+1)$-sheet Riemann surface. The first type solutions also form a separate Riemann surface with the $(n-n')$ sheets (see Fig. 1c). If $n<n'$, all first type solutions are a subset of the second type solutions. The first type solutions form an $(n+1)$-sheet Riemann surface split out $(n'+1)$-sheet surface of all known solutions. Let us emphasize that in this case the solution of the problem of a spin j quantum "top" contains contributions from representations with the spin $j'=n'/2$. The essence of the phenomenon is in the fact that the spin j "top" (8) can be rewritten in terms of the spin j "top" provided the condition (18) is fulfilled.

An analysis of Eq. (8) shows that the quasi-exactly-solvable problems with no plaiting of the first $N=2j+1$ eigenvalues and the explicit expression for these eigenvalues emerge in two cases:

$$a_{++}=a_{+0}=b_{+}=0 \tag{19}$$

or

$$a_{0-}=a_{--}=b_{-}=0\,. \tag{20}$$

In general, the spectrum (more exactly, the first $N+1$ levels) is described by a polynomial of the second power in the number of a state

$$E_{j,k}=a_{+-}k(2j+1-k)-a_{00}(j-k)^2+b_0(k-j)\,, \qquad k=0,1,\ldots,2j\,, \tag{21}$$

or, in more conventional terms [cf. (2)] $k=j+m$,

$$E_{j,m}=-a_{+-}(j+m)(m-1-j)-a_{00}m^2+b_0 m\,, \qquad m=-j,\ldots,0,\ldots,j\,. \tag{22}$$

If $a_{00}=-a_{+-}$ and $b_0=0$, we obtain the standard answer for sphere top[6]. The potential corresponding to Eqs. (19) and (20) can be readily obtained in an explicit form by substituting Eq. (13) in (14).

Since the condition (19) automatically implies Eq. (18), the spin j "top" (8) with parameters (19) can be rewritten in terms of the arbitrary spin j' "top." Therefore, obviously, one can find the whole spectrum and, hence, *the "top" (8) with the condition (19) generates the exactly-solvable problems.* In this case all levels are unplaited while spin j plays the role of a parameter.

The "top" (8) with the parameters (20) in the general case generates the quasi-exactly-solvable problems in which the first $N=2j+1$ states are unplaited while the rest of the spectrum is plaited (see Figs. 1d and e). If, along with Eq. (20) the condition (18) takes place, we get the following situation. If $j'<j$ nothing is changed; $(2j+1)$ first unplaited sheets are split from the Riemann surface with the infinite number of sheets. If, however, $j'>j$, then we know $2j'+1$ states. The first

[6] In this case the term ma_{+-} vanishes, if we change in Eq. (8) the J^+J^- by anticommutator $\frac{1}{2}\{J^+J^-\}$. Instead of (14), we obtain the equation $j(j+1)\psi=\varepsilon\psi$

$2j+1$ sheets are unplaited. The next $(2j'-2j)$ sheets of the Riemannian surface of the states from $2j+2$ to $2j'+1$ are plaited.

3. Before the conclusion, let first make a remark. In our previous discussion we did not take into account the condition that all wavefunctions which we get should be normalizable. This requirement imposes some restrictions to the class of potentials obtained. Also some potentials might have singularities. All these questions will be discussed elsewhere.

Now let us note some interesting features of the approach presented:

(i) The quasi-exactly-solvable problems with the geometrical properties described above emerge both, in the standard Schrödinger equation (1), and in the generalized equation (7). (ii) The transition from the "top" to the Schrödinger equation is realized, in general, with the aid of the elliptic functions. (iii) For arbitrary $P_{4,3,2}$ in Eq. (11) the polynomial solutions exist only if the polynomials $P_{4,3,2}$, admit the representation (12), i.e. when Eq. (11) can be rewritten in the form (8).

In conclusion, I remark that a similar procedure for the search of multi-dimensional quasi-exactly-solvable problems, perhaps, can be developed, if, instead of the group $SL(2)$ we will use the generators of higher groups having finite-dimensional representations, e.g. $SL(n, Q)$. Along this line about the possible non-algebraic structure of the node surfaces in multi-dimensional quantal problems [4–5] must be recalled, when the variables in the Schrödinger equation are not separated. In general, the equation which appears can be naturally considered in a curved space. Another interesting situation arises, if graded algebra generators are exploited.

So, quantal quasi-exactly-solvable problems are generated by the quadratic elements of an enveloping universal $sl(2)$ algebra [see (8)]. There is an open question: are there quasi-exactly-solvable problems which cannot be represented in terms of $sl(2)$ generators?

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Note added in proof. Recently [7], n-zone Lamé equation was rewritten via $sl(2)$ generators. Eigenstates form $(2n+1)$-sheet Riemannian surface separated into four parts (three of them contain the same number of sheets).