

QUASI-FREE ACTIONS OF FINITE GROUPS ON THE CUNTZ ALGEBRA \mathcal{O}_∞

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Abstract. We show that any faithful quasi-free actions of a finite group on the Cuntz algebra \mathcal{O}_∞ are mutually conjugate, and that they are asymptotically representable.

1. Introduction. The Cuntz algebra \mathcal{O}_n , $n = 2, 3, \dots, \infty$, is the universal C^* -algebra generated by isometries $\{s_i\}_{i=1}^n$ with mutually orthogonal ranges, satisfying $\sum_{i=1}^n s_i s_i^* = 1$ if n is finite. It is well known that the two algebras \mathcal{O}_2 and \mathcal{O}_∞ , among the others, play special roles in the celebrated classification theory of Kirchberg algebras (see [15], [18]).

An action α of a group G on \mathcal{O}_n is said to be *quasi-free* if $\alpha_g(\mathcal{H}_n) = \mathcal{H}_n$ for all $g \in G$, where \mathcal{H}_n is the closed linear span of the generators $\{s_i\}_{i=1}^n$. We restrict our attention to finite G throughout this note. To develop a G -equivariant version of the classification theory, it is expected that G -actions on \mathcal{O}_2 with the Rohlin property and the quasi-free G -actions on \mathcal{O}_∞ would play similar roles as \mathcal{O}_2 and \mathcal{O}_∞ do in the case without group actions. Since we have already had a good understanding of the former thanks to [4], our task in this note is to investigate the latter, the quasi-free G -actions on \mathcal{O}_∞ .

The space \mathcal{H}_n has a Hilbert space structure with inner product $t^*s = \langle s, t \rangle 1$, and a quasi-free G -action α gives a unitary representation $(\pi_\alpha, \mathcal{H}_\alpha)$, where $\pi_\alpha(g)$ is the restriction of α_g to \mathcal{H}_α . It is known that the association $\alpha \mapsto \pi_\alpha$ gives a one-to-one correspondence between the quasi-free G -actions on \mathcal{O}_n and the unitary representations of G in \mathcal{H}_n . The conjugacy class of α depends on the unitary equivalence class of $(\pi_\alpha, \mathcal{H}_n)$, at least a priori. Indeed, it really does when n is finite, and this can be seen by computing the K -groups of the crossed product (see, for example, [2], [4], [5], [11]). However, when $n = \infty$, the pair $(\mathcal{O}_\infty, \alpha)$ is KK_G -equivalent to the pair (\mathbb{C}, id) , and there is no way to differentiate the quasi-free actions as far as K -theory is concerned.

One of the purposes of this note is to show that any two faithful quasi-free G -actions on \mathcal{O}_∞ are indeed mutually conjugate for every finite group G (Corollary 5.2). Our main technical result is Theorem 4.1, an equivariant version of Lin-Phillips's result [10, Theorem 3.3], and Corollary 5.2 follows from it via Theorem 5.1, an equivariant version of Kirchberg-Phillips's \mathcal{O}_∞ theorem [7, Theorem 3.15].

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Using Theorem 4.1, we also show that quasi-free actions are asymptotically representable for any finite group G , which is another purpose of this note. The notion of asymptotic representability for group actions was introduced by the second-named author, and it is found to be important in the recent development of the classification of group actions on C^* -algebras (see [6], [12]).

The reader is referred to [18] for the basic properties and classification results for Kirchberg algebras. We denote by \mathbf{K} the set of compact operators on a separable infinite dimensional Hilbert space. For a C^* -algebra A , we denote by \tilde{A} and $M(A)$ the unitization and the multiplier algebra of A respectively. When A is unital, we denote by $U(A)$ the unitary group of A . For a homomorphism $\rho : A \rightarrow B$ between C^* -algebras A, B , we denote by $K_*(\rho)$ the homomorphism from $K_*(A)$ to $K_*(B)$ induced by ρ . We denote by $A \otimes B$ the minimal tensor product of A and B .

This work originated from the first-named author’s unpublished preprint [3], where the idea of developing an equivariant version of Lin-Phillips’s argument was introduced. Some results in this note are also obtained by N. C. Phillips, and the authors would like to thank him for informing of it.

2. Preliminaries for G - C^* -algebras. We fix a finite group G . By a G - C^* -algebra (A, α) , we mean a C^* -algebra A with a fixed G -action α . We denote by A^G the fixed point algebra

$$\{a \in A ; \alpha_g(a) = a \text{ for all } g \in G\}.$$

We denote by $\{\lambda_g^\alpha\}_{g \in G}$ the implementing unitary representation of G in the crossed product $A \rtimes_\alpha G$. For a finite dimensional (not necessarily irreducible) unitary representation (π, H_π) of G , we introduce a homomorphism

$$\hat{\alpha}_\pi : A \rtimes_\alpha G \rightarrow (A \rtimes_\alpha G) \otimes B(H_\pi),$$

which is a part of the dual coaction of α , by $\hat{\alpha}_\pi(a) = a \otimes 1$ for $a \in A$, and $\hat{\alpha}_\pi(\lambda_g^\alpha) = \lambda_g^\alpha \otimes \pi(g)$ for $g \in G$. We denote by \hat{G} the unitary dual of G , and by $Z\hat{G}$ the representation ring of G . Then identifying $K_*(A \rtimes_\alpha G)$ with $K_*((A \rtimes_\alpha G) \otimes B(H_\pi))$, we get a $Z\hat{G}$ -module structure of $K_*(A \rtimes_\alpha G)$ from $K_*(\hat{\alpha}_\pi)$.

Let

$$e_\alpha = \frac{1}{\#G} \sum_{g \in G} \lambda_g^\alpha,$$

which is a projection in $(A \rtimes_\alpha G) \cap A^G$. We denote by j_α the homomorphism from A^G into $A \rtimes_\alpha G$ defined by $j_\alpha(x) = xe_\alpha$. If A is simple and α is outer, that is, α_g is outer for every $g \in G \setminus \{e\}$, then $K_*(j_\alpha)$ is an isomorphism from $K_*(A^G)$ onto $K_*(A \rtimes_\alpha G)$. If A is purely infinite and simple, and α is outer, then A^G and $A \rtimes_\alpha G$ are purely infinite and simple.

A G -homomorphism φ from a G - C^* -algebra (A, α) into another G - C^* -algebra (B, β) is a homomorphism from A into B intertwining the two G -actions α and β . Such φ gives rise to an element in the equivariant KK -group $KK_G(A, B)$, which is denoted by $KK_G(\varphi)$. We denote by $\text{Hom}_G(A, B)$ the set of nonzero G -homomorphisms from (A, α) into (B, β) . Two

actions α and β are said to be conjugate if there exists an invertible element in $\text{Hom}_G(A, B)$. Two G -homomorphisms $\varphi, \psi \in \text{Hom}_G(A, B)$ are said to be G -unitarily equivalent if there exists a unitary $u \in M(B)^G$ satisfying $\varphi(x) = u\psi(x)u^*$ for all $x \in A$. They are said to be G -asymptotically unitarily equivalent if there exists a norm continuous family of unitaries $\{u(t)\}_{t \geq 0}$ in $M(B)^G$ satisfying

$$\lim_{t \rightarrow \infty} \|\varphi(x) - \text{Ad } u(t) \circ \psi(x)\| \text{ for all } x \in A.$$

If they satisfy the same condition with a sequence of unitaries $\{u_n\}_{n=1}^\infty$ in $M(B)^G$ instead of the continuous family, they are said to be G -approximately unitarily equivalent.

For a free ultrafilter $\omega \in \beta N \setminus N$ and a G - C^* -algebra (A, α) , we use the following notation:

$$c_\omega(A) = \{(x_n) \in \ell^\infty(N, A) ; \lim_{n \rightarrow \omega} \|x_n\| = 0\},$$

$$A^\omega = \ell^\infty(N, A) / c_\omega(A).$$

As usual, we often omit the quotient map from $\ell^\infty(N, A)$ onto A^ω . We regard A as a C^* -subalgebra of A^ω consisting of the constant sequences, and we set $A_\omega = A^\omega \cap A'$. We denote by α^ω and α_ω the G -actions on A^ω and A_ω induced by α respectively, and we regard $(A^\omega, \alpha^\omega)$ and $(A_\omega, \alpha_\omega)$ as G - C^* -algebras.

LEMMA 2.1. *Let G be a finite group, and let (A, α) be a G - C^* -algebra. We assume that A is unital, purely infinite, and simple, and α is outer. Let $\omega \in \beta N \setminus N$.*

- (1) *A^ω is purely infinite and simple, and α^ω is outer.*
- (2) *If A is a Kirchberg algebra, A_ω is purely infinite and simple, and α_ω is outer.*

PROOF. (1) It is easy to show that A^ω is purely infinite and simple, and so it suffices to show that if $\theta \in \text{Aut}(A)$ is outer, so is $\theta^\omega \in \text{Aut}(A^\omega)$ induced by θ . Assume that θ is outer and θ^ω is inner. Then there exists $u = (u_n) \in U(A^\omega)$ satisfying $\text{Ad } u = \theta^\omega$. We may assume that u_n is a unitary for all $n \in N$. Since A is purely infinite, there exist a sequence of nonzero projections $\{p_n\}_{n=1}^\infty$ in A and a sequence of complex numbers $\{c_n\}_{n=1}^\infty$ with $|c_n| = 1$ such that $\{p_n u_n p_n - c_n p_n\}_{n=1}^\infty$ converges to 0. By replacing u_n with $\overline{c_n} u_n$ if necessary, we may assume $c_n = 1$. Since θ is outer, Kishimoto's result [8, Lemma 1.1] shows that there exists a sequence of positive elements $a_n \in p_n A p_n$ with $\|a_n\| = 1$ such that $\{a_n \theta(a_n)\}_{n=1}^\infty$ converging to 0. This is a contradiction. Indeed, let $a = (a_n) \in A^\omega$, $p = (p_n) \in A^\omega$. On one hand we have $a \theta^\omega(a) = 0$, and on the other hand we have the following

$$a \theta^\omega(a) = a u a u^* = a p u p a u^* = a p a u^* = a^2 u^* \neq 0.$$

This shows that θ^ω is outer.

- (2) The statement follows from [7, Proposition 3.4] and [13, Lemma 2]. □

Now we state two results, which are equivariant versions of well-known results in the classification theory of nuclear C^* -algebras. We omit their proofs, which are verbatim modifications of the original ones. The first one is an equivariant version of [18, Corollary 2.3.4].

THEOREM 2.2. *Let G be a finite group, and let (A, α) and (B, β) be unital separable G - C^* -algebras. If there exist $\varphi \in \text{Hom}_G(A, B)$ and $\psi \in \text{Hom}_G(B, A)$ such that $\psi \circ \varphi$ is G -approximately unitarily equivalent to $\text{id}_{(A, \alpha)}$ and $\varphi \circ \psi$ is G -approximately unitarily equivalent to $\text{id}_{(B, \beta)}$, then the two actions α and β are conjugate.*

The following result is an equivariant version of [7, Proposition 3.13] (see also [18, Theorem 7.2.2]).

THEOREM 2.3. *Let G be a finite group, and let $(A, \alpha), (B, \beta)$ be unital separable G - C^* algebras. We regard the minimal tensor product $B \otimes B$ as a G - C^* -algebra with the diagonal action $\alpha \otimes \alpha$, and define $\rho_l, \rho_r \in \text{Hom}_G(B, B \otimes B)$ by $\rho_l(x) = x \otimes 1$ and $\rho_r(x) = 1 \otimes x$ for $x \in B$. We assume that ρ_l and ρ_r are G -approximately unitarily equivalent. Then if there exists a unital homomorphism in $\text{Hom}_G(B, A_\omega)$ with $\omega \in \beta N \setminus N$, the two G -actions α on A and $\alpha \otimes \beta$ on $A \otimes B$ are conjugate.*

3. Equivariant Rørdam’s theorem. The purpose of this section is to show the following theorem, which is an equivariant version of Rørdam’s theorem [17, Theorem 3.6], [18, Theorem 5.1.2].

THEOREM 3.1. *Let G be a finite group, let α be a quasi-free action of G on \mathcal{O}_n with finite n , and let (B, β) be a G - C^* -algebra. We assume that B is unital, purely infinite, and simple, and β is outer. For two unital G -homomorphisms $\varphi, \psi \in \text{Hom}_G(\mathcal{O}_n, B)$, we set*

$$u_{\psi, \varphi} = \sum_{i=1}^n \psi(s_i)\varphi(s_i)^* \in U(B^G).$$

We introduce an endomorphism $\Lambda_\varphi \in \text{End}(B^G)$ by

$$\Lambda_\varphi(x) = \sum_{i=1}^n \varphi(s_i)x\varphi(s_i^*), \quad x \in B^G.$$

Then the following conditions are equivalent.

- (1) *The G -homomorphisms φ and ψ are G -approximately unitarily equivalent.*
- (2) *The unitary $u_{\psi, \varphi}$ belongs to the closure of $\{v\Lambda_\varphi(v^*) \in U(B^G); v \in B^G\}$.*
- (3) *The K_1 -class $[u_{\psi, \varphi}] \in K_1(B^G)$ is in the image of $1 - K_1(\Lambda_\varphi)$.*
- (4) *The K_1 -class $K_1(j_\beta)([u_{\psi, \varphi}]) \in K_1(B \rtimes_\beta G)$ is in the image of $1 - K_1(\hat{\beta}\pi_\alpha)$.*
- (5) *The equality $KK_G(\varphi) = KK_G(\psi)$ holds in $KK_G(\mathcal{O}_n, B)$.*

PROOF. The equivalence of (1) and (2) follows from $\psi(s_i) = u_{\psi, \varphi}\varphi(s_i)$ and $v\varphi(s_i)v^* = v\Lambda_\varphi(v^*)\varphi(s_i)$.

The implication from (2) to (3) is trivial. In view of the proof of [17, Theorem 3.6], the implication from (3) to (2) is reduced to the Rohlin property of the shift automorphism of $(\otimes_{\mathbb{Z}} M_n(\mathbb{C}))^G$, where the G -action of the UHF algebra $\otimes_{\mathbb{Z}} M_n(\mathbb{C})$ is the product action $\otimes_{\mathbb{Z}} \text{Ad } \pi_\alpha(g)$. This follows from Kishimoto’s result [9, Theorem 2.1] (see [4, Lemma 5.5] for details).

The equivalence of (3) and (4) follows from Lemma 3.3 below.

We will show the equivalence of (4) and (5) in Appendix as it follows from a rather lengthy computation, and we do not really require it in the rest of this note. \square

To show the equivalence of (3) and (4), we first recall the following well-known fact.

LEMMA 3.2. *Let A be a C^* -algebra, and let $\{t_i\}_{i=1}^n \subset M(A)$ be isometries with mutually orthogonal ranges. Let $\{e_{ij}\}_{i,j=1}^n$ be the system of matrix units of the matrix algebra $M_n(\mathbb{C})$. We define two homomorphisms $\rho_1 : A \rightarrow A \otimes M_n(\mathbb{C})$ and $\rho_2 : A \otimes M_n(\mathbb{C}) \rightarrow A$ by $\rho_1(a) = a \otimes e_{11}$, and $\rho_2(a \otimes e_{ij}) = t_i a t_j^*$. Then $K_*(\rho_2)$ is the inverse of $K_*(\rho_1)$.*

PROOF. Since $K_*(\rho_1)$ is an isomorphism, it suffices to show that the homomorphism $\rho_2 \circ \rho_1(x) = t_1 x t_1^*$ induces the identity on $K_*(A)$. This follows from a standard argument. \square

Recall that we regard $K_*(\hat{\beta}_{\pi_\alpha})$ as an element of $\text{End}(K_*(B \rtimes_\beta G))$ by identifying $K_*(B \rtimes_\beta G)$ with $K_*((B \rtimes_\beta G) \otimes B(\mathcal{H}_n))$.

LEMMA 3.3. *With the above notation, we have the equality $K_*(j_\beta) \circ K_*(\Lambda_\varphi) = K_*(\hat{\beta}_{\pi_\alpha}) \circ K_*(j_\beta)$.*

PROOF. Identifying $B(\mathcal{H}_n)$ with the linear span of $\{s_i s_j^*\}_{i,j=1}^n$ acting on \mathcal{H}_n by left multiplication, we have

$$\pi_\alpha(g) = \sum_{i=1}^n \alpha_g(s_i) s_i^*.$$

We define a homomorphism $\rho : (B \rtimes_\beta G) \otimes B(\mathcal{H}_n) \rightarrow B \rtimes_\beta G$ by $\rho(x \otimes s_i s_j^*) = \varphi(s_i) x \varphi(s_j)^*$, which plays the role of ρ_2 in Lemma 3.2 with $A = B \rtimes_\beta G$ and $t_i = \varphi(s_i)$. Then for $x \in B^G$, we have

$$\begin{aligned} \rho \circ \hat{\beta}_{\pi_\alpha} \circ j_\beta(x) &= \frac{1}{\#G} \sum_{g \in G} \rho \circ \hat{\beta}_{\pi_\alpha}(\lambda_g^\beta x) = \frac{1}{\#G} \sum_{g \in G} \rho(\lambda_g^\beta x \otimes \pi_\alpha(g)) \\ &= \frac{1}{\#G} \sum_{g \in G} \sum_{i=1}^n \rho(\lambda_g^\beta x \otimes \alpha_g(s_i) s_i^*) = \frac{1}{\#G} \sum_{g \in G} \sum_{i=1}^n \varphi(\alpha_g(s_i)) \lambda_g^\beta x \varphi(s_i)^* \\ &= \frac{1}{\#G} \sum_{g \in G} \sum_{i=1}^n \lambda_g^\beta \varphi(s_i) x \varphi(s_i)^* = j_\beta \circ \Lambda_\varphi(x), \end{aligned}$$

which proves the statement thanks to Lemma 3.2. \square

4. Equivariant Lin-Phillips’s theorem. The purpose of this section is to show the following theorem, which is an equivariant version of Lin-Phillips’s theorem [10, Theorem 3.3], [18, Proposition 7.2.5].

THEOREM 4.1. *Let G be a finite group, let α be a quasi-free action of G on \mathcal{O}_∞ , and let (B, β) be a unital G - C^* -algebra. We assume that B is purely infinite and simple, and β is outer. Then any two unital G -homomorphisms in $\text{Hom}_G(\mathcal{O}_\infty, B)$ are G -approximately unitarily equivalent.*

Until the end of this section, we assume that G , $(\mathcal{O}_\infty, \alpha)$ and (B, β) are as in Theorem 4.1. To prove Theorem 4.1, we basically follow Lin-Phillips’s strategy based on Theorem 3.1 in place of [17, Theorem 3.6], though we will take a short cut by using a ultraproduct technique.

Let n be a natural number larger than or equal to 2, and let \mathcal{E}_n be the Cuntz-Toeplitz algebra, which is the universal C^* -algebra generated by isometries $\{t_i\}_{i=1}^n$ with mutually orthogonal ranges. Note that $p_n = 1 - \sum_{i=1}^n t_i t_i^*$ is a non-zero projection not as in the case of the Cuntz algebras. We denote by \mathcal{K}_n the linear span of $\{t_i\}_{i=1}^n$. Quasi-free actions on \mathcal{E}_n are defined as in the case of the Cuntz algebras. For a quasi-free action γ of G on \mathcal{E}_n , we denote by $(\pi_\gamma, \mathcal{K}_n)$ the corresponding unitary representation of G in \mathcal{K}_n .

LEMMA 4.2. *Let γ be a quasi-free action of G on \mathcal{E}_n with finite n , and let $\varphi, \psi \in \text{Hom}_G(\mathcal{E}_n, B)$ be injective G -homomorphisms, either both unital or both nonunital. If $[\varphi(1)] = [\psi(1)] = 0$ in $K_0(B^G)$, then φ and ψ are G -approximately unitarily equivalent.*

PROOF. In the same way as in the proof of Lemma 3.3, we can see

$$K_0(j_\beta)\left(\sum_{i=1}^n [\varphi(t_i t_i^*)]\right) = K_0(\hat{\beta}_{\pi_\alpha}) \circ K_0(j_\beta)([\varphi(1)]),$$

and so we have

$$K_0(j_\beta)([\varphi(p_n)]) = K_0(j_\beta)([\varphi(1)]) - K_0(\hat{\beta}_{\pi_\gamma}) \circ K_0(j_\beta)([\varphi(1)]) = 0,$$

in $K_0(B \rtimes_\beta G)$. This implies $[\varphi(p_n)] = 0$ in $K_0(B^G)$, and for the same reason, $[\psi(p_n)] = 0$ in $K_0(B^G)$. Thus the statement follows from essentially the same argument as in the proof of [10, Proposition 1.7] by using Theorem 3.1 in place of [17, Theorem 3.6]. \square

Since every quasi-free G -action on \mathcal{O}_∞ is the inductive limit of a system of quasi-free actions of the form $\{(\mathcal{E}_{n_k}, \gamma^{(k)})\}_{k=1}^\infty$, we get the following corollary.

COROLLARY 4.3. *Let $\varphi, \psi \in \text{Hom}_G(\mathcal{O}_\infty, B)$ be either both unital or both nonunital. If $[\varphi(1)] = [\psi(1)] = 0$ in $K_0(B^G)$, then φ and ψ are G -approximately unitarily equivalent.*

Let $\omega \in \beta N \setminus N$ be a free ultrafilter, and let $\iota_\omega : \mathcal{O}_\infty \rightarrow \mathcal{O}_\infty^\omega$ be the inclusion map. For $\varphi \in \text{Hom}_G(\mathcal{O}_\infty, B)$, we denote by φ^ω the G -homomorphism in $\text{Hom}_G(\mathcal{O}_\infty^\omega, B^\omega)$ induced by φ . Then it is easy to show the following three conditions for $\varphi, \psi \in \text{Hom}_G(\mathcal{O}_\infty, B)$ are equivalent:

- (1) φ and ψ are G -approximately unitarily equivalent,
- (2) $\varphi^\omega \circ \iota_\omega$ and $\psi^\omega \circ \iota_\omega$ are G -approximately unitarily equivalent,
- (3) $\varphi^\omega \circ \iota_\omega$ and $\psi^\omega \circ \iota_\omega$ are G -unitarily equivalent.

Note that since G is a finite group, we have $(\mathcal{O}_{\infty\omega})^G = (\mathcal{O}_\infty^G)^\omega \cap \mathcal{O}'_\infty$ and $(B^\omega)^G = (B^G)^\omega$.

PROOF OF THEOREM 4.1. Let $\varphi, \psi \in \text{Hom}_G(\mathcal{O}_\infty, B)$ be unital. Since \mathcal{O}_∞ is a Kirchberg algebra, the ω -central sequence algebra $\mathcal{O}_{\infty\omega}$ is purely infinite and simple. Let H be the kernel of $\alpha : G \rightarrow \text{Aut}(\mathcal{O}_\infty)$. Then we may regard α as a faithful quasi-free action of

G/H , which is outer. Therefore α_ω is outer as an action of G/H . This implies that $(\mathcal{O}_{\infty\omega})^G$ is purely infinite and simple.

Choosing three nonzero projections $q_1, q_2, q_3 \in (\mathcal{O}_{\infty\omega})^G$ satisfying $q_1 + q_2 + q_3 = 1$ and $[1] = [q_1] = [q_2] = -[q_3]$ in $K_0((\mathcal{O}_{\infty\omega})^G)$, we introduce $\varphi_i, \psi_i \in \text{Hom}_G(\mathcal{O}_\infty, B^\omega)$, $i = 1, 2, 3$, by $\varphi_i(x) = \varphi^\omega(q_i x)$ and $\psi_i(x) = \psi^\omega(q_i x)$ for $x \in \mathcal{O}_\infty$. Then we have

$$\begin{aligned} \varphi(x) &= \varphi_1(x) + \varphi_2(x) + \varphi_3(x), \quad x \in \mathcal{O}_\infty, \\ \psi(x) &= \psi_1(x) + \psi_2(x) + \psi_3(x), \quad x \in \mathcal{O}_\infty, \end{aligned}$$

$$[1] = [\varphi_1(1)] = [\varphi_2(1)] = -[\varphi_3(1)] = [\psi_1(1)] = [\psi_2(1)] = -[\psi_3(1)] \in K_0((B^\omega)^G).$$

Since $[(\varphi_2 + \varphi_3)(1)] = [(\psi_2 + \psi_3)(1)] = 0$ in $K_0((B^\omega)^G)$, Corollary 4.3 implies that there exists a unitary $u \in U((B^\omega)^G)$ satisfying $u(\varphi_2 + \varphi_3)(x)u^* = (\psi_2 + \psi_3)(x)$ for $x \in \mathcal{O}_\infty$. We set $\varphi_1^u(x) = u\varphi_1(x)u^*$. Then φ_1^u is in $\text{Hom}_G(\mathcal{O}_\infty, B^\omega)$ satisfying $\varphi_1^u(1) = \psi_1(1)$, and $\varphi^\omega \circ \iota_\omega$ and $\varphi_1^u + \psi_2 + \psi_3$ are G -approximately unitarily equivalent. Since $(\varphi_1^u + \psi_3)(1) = (\psi_1 + \psi_3)(1)$ whose class in $K_0((B^\omega)^G)$ is 0, Corollary 4.3 again implies that there exists a unitary $v \in U((B^\omega)^G)$ satisfying $v\psi_2(1)v^* = \psi_2(1)$ and $v(\varphi_1^u + \psi_3)(x)v^* = (\psi_1 + \psi_3)(x)$ for $x \in \mathcal{O}_\infty$. This shows that $vu\varphi(x)u^*v^* = \psi(x)$ for $x \in \mathcal{O}_\infty$, and so φ and ψ are G -approximately unitarily equivalent. \square

5. Splitting theorem and Uniqueness theorem. Thanks to Theorem 4.1, we can obtain a G -equivariant version of Kirchberg-Phillips’s \mathcal{O}_∞ theorem [7, Theorem 3.15], [18, Theorem 7.2.6].

THEOREM 5.1. *Let G be a finite group, and let (A, α) be a G - C^* -algebra. We assume that A is a unital Kirchberg algebra and α is outer. Let $\{\gamma^{(i)}\}_{i=1}^\infty$ be any sequence of quasi-free actions of G on \mathcal{O}_∞ . Then (A, α) is conjugate to*

$$\left(A \otimes \bigotimes_{i=1}^\infty \mathcal{O}_\infty, \quad \alpha \otimes \bigotimes_{i=1}^\infty \gamma^{(i)} \right).$$

PROOF. Let H_i be the kernel of the homomorphism $\gamma^{(i)} : G \rightarrow \text{Aut}(\mathcal{O}_\infty)$. Since we may regard $\gamma^{(i)}$ and $\gamma^{(i)} \otimes \gamma^{(i)}$ as outer actions of G/H_i , Theorem 4.1 implies that there exist unitaries $\{u_n^{(i)}\}_{n=1}^\infty$ in $(\mathcal{O}_\infty \otimes \mathcal{O}_\infty)^{\gamma^{(i)} \otimes \gamma^{(i)}}$ satisfying

$$\lim_{n \rightarrow \infty} \|u_n^{(i)}(x \otimes 1)u_n^{(i)*} - 1 \otimes x\| = 0, \quad \text{for all } x \in \mathcal{O}_\infty.$$

Let

$$(B, \beta) = \left(\bigotimes_{i=1}^\infty \mathcal{O}_\infty, \quad \bigotimes_{i=1}^\infty \gamma^{(i)} \right),$$

and let $\rho_l, \rho_r \in \text{Hom}_G(B, B \otimes B)$ be as in Theorem 2.3. Then ρ_r and ρ_l are G -approximately unitarily equivalent.

To prove the statement applying Theorem 2.3, it suffices to construct a unital embedding of (B, β) in $(A_\omega, \alpha_\omega)$. For this, it suffices to construct a unital embedding of $(\mathcal{O}_\infty, \gamma^{(i)})$ into $(A_\omega, \alpha_\omega)$ for each i because the usual trick of taking subsequences can make the embeddings

commute with each other. Let γ be the quasi-free action of G on \mathcal{O}_∞ such that $(\pi_\gamma, \mathcal{H}_\infty)$ is unitarily equivalent to the infinite direct sum of the regular representation. Since there is a unital embedding of $(\mathcal{O}_\infty, \gamma^{(i)})$ into $(\mathcal{O}_\infty, \gamma)$, in order to prove the theorem, it only remains to construct a unital embedding of $(\mathcal{O}_\infty, \gamma)$ into $(A_\omega, \alpha_\omega)$.

Thanks to [13, Lemma 3], we can find a nonzero projection $e \in A_\omega$ satisfying $e\alpha_{\omega g}(e) = 0$ for any $g \in G \setminus \{e\}$. We choose an isometry $v \in A_\omega$ satisfying $vv^* \leq e$, and set $s_{0,g} = \alpha_{\omega g}(v)$. Then $\{s_{0,g}\}_{g \in G}$ are isometries in A_ω with mutually orthogonal ranges satisfying $\alpha_{\omega g}(s_{0,h}) = s_{0,gh}$. Let $p = \sum_{g \in G} s_{0,g}s_{0,g}^*$, which is a projection in $(A_\omega)^G$. Replacing v if necessary, we may assume that $p \neq 1$. Since $(A_\omega)^G$ is purely infinite and simple, we can find a sequence of partial isometries $\{w_i\}_{i=0}^\infty$ in $(A_\omega)^G$ with $w_0 = p$ such that $w_i^*w_i = p$ for all i , and $\{w_iw_i^*\}_{i=0}^\infty$ are mutually orthogonal. Let $s_{i,g} = w_i s_{0,g}$. Then $\{s_{i,g}\}_{(i,g) \in \mathbb{N} \times G}$ is a countable family of isometries in A_ω with mutually orthogonal ranges satisfying $\alpha_{\omega g}(s_{i,h}) = s_{i,gh}$. Thus we get the desirable embedding of $(\mathcal{O}_\infty, \gamma)$ into $(A_\omega, \alpha_\omega)$. \square

Applying Theorem 5.1 to $A = \mathcal{O}_\infty$ with a faithful quasi-free action α , we obtain

COROLLARY 5.2. *Any two faithful quasi-free actions of a finite group on \mathcal{O}_∞ are mutually conjugate.*

6. Asymptotic representability.

DEFINITION 6.1. An action α of a discrete group G on a unital C^* -algebra A is said to be *asymptotically representable* if there exists a continuous family of unitaries $\{u_g(t)\}_{t \geq 0}$ in $U(A)$ for each $g \in G$ satisfying

$$\begin{aligned} \lim_{t \rightarrow \infty} \|u_g(t)xu_g(t)^* - \alpha_g(x)\| &= 0, & \text{for all } x \in A, g \in G, \\ \lim_{t \rightarrow \infty} \|u_g(t)u_h(t) - u_{gh}(t)\| &= 0, & \text{for all } g, h \in G, \\ \lim_{t \rightarrow \infty} \|\alpha_g(u_h(t)) - u_{ghg^{-1}}(t)\| &= 0, & \text{for all } g, h \in G. \end{aligned}$$

An action α is said to be *approximately representable* if α satisfies the above condition with a sequence $\{u_g(n)\}_{n \in \mathbb{N}}$ in place of the continuous family $\{u_g(t)\}_{t \geq 0}$.

Every asymptotically representable action is approximately representable, but the converse may not be true in general. When G is a finite abelian group, an action α is approximately representable if and only if its dual action has the Rohlin property. When G is a cyclic group of prime power order, approximately representable quasi-free actions on \mathcal{O}_n with finite n are completely characterized in [5], and there exist quasi-free actions that are not approximately representable.

The purpose of this section is to show the following theorem:

THEOREM 6.2. *Every quasi-free action of a finite group G on \mathcal{O}_∞ is asymptotically representable.*

It is unlikely that one could show Theorem 6.2 directly from the definition of quasi-free actions. Our proof uses the intertwining argument, Theorem 2.2, between two model actions;

one is obviously quasi-free, and the other is an infinite tensor product action, that can be shown to be asymptotically representable.

We first introduce the notion of K -trivial embeddings of the group C^* -algebra. We denote by $\{\lambda_g\}_{g \in G}$ the left regular representation of a finite group G . The group C^* -algebra $C^*(G)$ is the linear span of $\{\lambda_g\}_{g \in G}$.

DEFINITION 6.3. Let G be a finite group, and let A be a unital C^* -algebra. An unital injective homomorphism $\rho : C^*(G) \rightarrow A$ is said to be a K -trivial embedding if $KK(\rho) = KK(C^*(G) \ni \lambda_g \mapsto 1 \in A)$.

For each irreducible representation (π, H_π) of G , we choose an orthonormal basis $\{\xi(\pi)_i\}_{i=1}^{n_\pi}$ of H_π , where $n_\pi = \dim \pi$. We set $\pi(g)_{ij} = \langle \pi(g)\xi(\pi)_j, \xi(\pi)_i \rangle$, and

$$e(\pi)_{ij} = \frac{n_\pi}{\#G} \sum_{g \in G} \overline{\pi(g)_{ij}} \lambda_g.$$

Then $\{e(\pi)_{ij}\}_{1 \leq i, j \leq n_\pi}$ is a system of matrix units, and we have

$$\lambda_g = \sum_{\pi \in \hat{G}} \sum_{i, j=1}^{n_\pi} \pi(g)_{ij} e(\pi)_{ij}.$$

Let $C^*(G)_\pi$ be the linear span of $\{e(\pi)_{ij}\}_{i, j=1}^{n_\pi}$. Then $C^*(G)_\pi$ is isomorphic to the matrix algebra $M_{n_\pi}(C)$, and $C^*(G)$ has the direct sum decomposition

$$C^*(G) = \bigoplus_{\pi \in \hat{G}} C^*(G)_\pi.$$

Let $\chi_\pi(g) = \text{Tr}(\pi(g))$ be the character of π . Then

$$z(\pi) = \frac{n_\pi}{\#G} \sum_{g \in G} \overline{\chi_\pi(g)} \lambda_g = \sum_{i=1}^{n_\pi} e(\pi)_{ii}$$

is the unit of $C^*(G)_\pi$.

It is easy to show the following lemma.

LEMMA 6.4. Let G be a finite group, and let A, B be unital simple purely infinite C^* -algebras.

(1) A unital injective homomorphism $\rho : C^*(G) \rightarrow A$ is a K -trivial embedding if and only if $[\rho(e(\pi)_{11})] = 0$ in $K_0(A)$ for any nontrivial irreducible representation π . When $K_0(A)$ is torsion free, it is further equivalent to the condition that $[\rho(z(\pi))] = 0$ in $K_0(A)$ for any nontrivial irreducible representation π .

(2) Any two K -trivial unital embeddings of $C^*(G)$ into A are unitarily equivalent.

(3) If $\rho : C^*(G) \rightarrow A$ and $\sigma : C^*(G) \rightarrow B$ are K -trivial embeddings, so is the tensor product embedding $C^*(G) \ni \lambda_g \mapsto \rho(\lambda_g) \otimes \sigma(\lambda_g) \in A \otimes B$.

We now construct a K -trivial embedding of $C^*(G)$ into \mathcal{O}_∞ . We fix a nonzero projection $p \in \mathcal{O}_\infty$ with $[p] = 0$ in $K_0(\mathcal{O}_\infty)$, and fix unital embeddings

$$B(\ell^2(G)) \subset \mathcal{O}_2 \subset p\mathcal{O}_\infty p.$$

We denote by $\sigma_0 : C^*(G) \rightarrow p\mathcal{O}_\infty p$ the resulting embedding, and set $u_g = \sigma_0(\lambda_g) + 1 - p$. Then $\sigma : C^*(G) \ni \lambda_g \mapsto u_g \in \mathcal{O}_\infty$ is a K -trivial embedding of $C^*(G)$ into \mathcal{O}_∞ .

Using $\{u_g\}_{g \in G}$, we introduce a G - C^* -algebra (A, α) by

$$(A, \alpha_g) = \bigotimes_{k=1}^\infty (\mathcal{O}_\infty, \text{Ad } u_g).$$

More precisely, we set

$$A_n = \bigotimes_{k=1}^n \mathcal{O}_\infty, \quad u_g^{(n)} = \bigotimes_{k=1}^n u_g,$$

and $\alpha_g^{(n)} = \text{Ad } u_g^{(n)}$. Then (A, α) is the inductive limit of the system $\{(A_n, \alpha^{(n)})\}_{n=1}^\infty$ with the embedding $\iota_n : A_n \ni x \mapsto x \otimes 1 \in A_{n+1}$. The C^* -algebra A is isomorphic to \mathcal{O}_∞ , and the action α is outer.

LEMMA 6.5. *Let the notation be as above.*

- (1) *The action α is asymptotically representable.*
- (2) *The embedding $\iota_\alpha : C^*(G) \ni \lambda_g \mapsto \lambda_g^\alpha \in A \rtimes_\alpha G$ gives KK -equivalence.*

PROOF. (1) It suffices to construct a homotopy $\{v_g(t)\}_{t \in [0,1]}$ of unitary representations of G in A_3 satisfying $v_g(0) = u_g \otimes 1 \otimes 1$, $v_g(1) = u_g^{(2)} \otimes 1$, and $\alpha_g^{(3)}(v_h(t)) = v_{ghg^{-1}}(t)$. Since $\{u_g \otimes 1\}_{g \in G}$, $\{u_g^{(2)}\}_{g \in G}$, and $\{1 \otimes u_g\}_{g \in G}$ give K -trivial embeddings of $C^*(G)$ into A_2 , there exist unitaries $w_1, w_2 \in U(A_2)$ satisfying $w_1(u_g \otimes 1)w_1^* = w_2(1 \otimes u_g)w_2^* = u_g^{(2)}$. Let $w = (w_1 \otimes 1)(1 \otimes w_2^*)$, which is a unitary in $A_3^G = A_3 \cap \{u_g^{(3)}\}'_{g \in G}$ satisfying $w(u_g \otimes 1 \otimes 1)w^* = u_g^{(2)} \otimes 1$. Since A_3^G is isomorphic to a finite direct sum of C^* -algebras Morita equivalent to \mathcal{O}_∞ , there exists a homotopy $\{w(t)\}_{t \in [0,1]}$ in $U(A_3^G)$ with $w(0) = 1$ and $w(1) = w$. Thus $v_g(t) = w(t)(u_g \otimes 1 \otimes 1)w(t)^*$ gives the desired homotopy.

(2) We identify $B_n = A_n \rtimes_{\alpha^{(n)}} G$ with the C^* -subalgebra of $A \rtimes_\alpha G$ generated by A_n and $\{\lambda_g^\alpha\}_{g \in G}$, and we denote by $\iota'_n : B_n \rightarrow B_{n+1}$ the embedding map. Then $A \rtimes_\alpha G$ is the inductive limit of the system $\{B_n\}_{n=1}^\infty$. Let $\iota_\alpha^{(n)} : C^*(G) \ni \lambda_g \mapsto \lambda_g^\alpha \in B_n$. Since we have $\iota'_n \circ \iota_\alpha^{(n)} = \iota_\alpha^{(n+1)}$, in order to prove the statement it suffices to show that $\iota_\alpha^{(n)}$ induces isomorphisms of the K -groups for every n .

Since $\alpha^{(n)}$ is inner, there exists an isomorphism $\theta_n : B_n \rightarrow A_n \otimes C^*(G)$ given by $\theta_n(a) = a \otimes 1$ for $a \in A_n$ and $\theta_n(\lambda_g^\alpha) = u_g^{(n)} \otimes \lambda_g$. Thus all we have to show is that the map $\theta_n \circ \iota_\alpha^{(n)} : C^*(G) \ni \lambda_g \mapsto u_g^{(n)} \otimes \lambda_g \in A_n \otimes C^*(G)$ induces isomorphisms of the K -groups. This follows from the facts that A_n is isomorphic to \mathcal{O}_∞ and $\{u_g^{(n)}\}_{g \in G}$ gives a K -trivial embedding of $C^*(G)$ into A_n . □

LEMMA 6.6. *For the G - C^* -algebra (A, α) as constructed above, any unital $\varphi \in \text{Hom}_G(A, A)$ is G -asymptotically unitarily equivalent to id .*

PROOF. Let $B = A \rtimes_\alpha G$, and let $\hat{\alpha} : B \rightarrow B \otimes C^*(G)$ be the dual coaction of α . Then φ extends to a unital endomorphism $\tilde{\varphi}$ in $\text{End}(B)$ with $\tilde{\varphi}(\lambda_g^\alpha) = \lambda_g^\alpha$, which satisfies $\hat{\alpha} \circ \tilde{\varphi} = (\tilde{\varphi} \otimes \text{id}_{C^*(G)}) \circ \hat{\alpha}$. By Lemma 6.5,(2), we have $KK(\tilde{\varphi}) = KK(\text{id}_B)$. Thus Lemma 6.5,(1) and [6, Theorem 4.8] imply that there exists a continuous family of unitaries $\{u(t)\}_{t \geq 0}$ in A satisfying

$$\lim_{t \rightarrow \infty} \|u(t)xu(t)^* - \tilde{\varphi}(x)\| = 0 \text{ for all } x \in B.$$

Setting $x = \lambda_g^\alpha$, we know that $\{\alpha_g(u(t)) - u(t)\}_{t \geq 0}$ converges to 0. Since G is a finite group, there exists a conditional expectation from A onto A^G , and we can construct a continuous family of unitaries $\{\tilde{u}(t)\}_{t \geq 0}$ in A^G such that $\{u(t) - \tilde{u}(t)\}_{t \geq 0}$ converges to 0 by a standard perturbation argument. Therefore φ and id are G -asymptotically unitarily equivalent. \square

PROOF OF THEOREM 6.2. Let γ be a faithful quasi-free G -action on \mathcal{O}_∞ . Thanks to Corollary 5.2, we may assume that \mathcal{O}_∞ has the canonical generators $\{s_i\}_{i \in J}$ with $G \subset J$ satisfying $\gamma_g(s_h) = s_{gh}$. Since α is asymptotically representable, it suffices to show that α and γ are conjugate. Thanks to Theorem 5.1, the action α is conjugate to $\alpha \otimes \gamma$, and so there exists a unital embedding of $(\mathcal{O}_\infty, \gamma)$ into (A, α) . Thus if there exists a unital embedding of (A, α) into $(\mathcal{O}_\infty, \gamma)$, Theorem 2.2, Theorem 4.1, and Lemma 6.6 imply that α and γ are conjugate. Since γ is conjugate to the infinite tensor product of its copies thanks to Theorem 5.1 again, all we have to show is that there exists a unital embedding of $(\mathcal{O}_\infty, \text{Ad } u.)$ into $(\mathcal{O}_\infty, \gamma)$.

We denote by $\mathcal{O}_\infty^\gamma$ the fixed point subalgebra of \mathcal{O}_∞ under the G -action γ . Since $\mathcal{O}_\infty^\gamma$ is purely infinite and simple, we can choose a nonzero projection $q_0 \in \mathcal{O}_\infty^\gamma$ with $[q_0] = 0$ in $K_0(\mathcal{O}_\infty^\gamma)$. We set $q_1 = \sum_{g \in G} s_g q_0 s_g^*$. A similar argument as in the proof of Lemma 3.3 implies that $[q_1] = 0$ in $K_0(\mathcal{O}_\infty^\gamma)$. We set

$$v_g = \sum_{h \in G} s_{gh} q_0 s_h^* + 1 - q_1.$$

Then $\{v_g\}_{g \in G}$ is a unitary representation of G in \mathcal{O}_∞ satisfying $\gamma_g(v_h) = v_{ghg^{-1}}$, and so $\{v_g^*\}_{g \in G}$ is a γ -cocycle. We show that this is a coboundary by using [4, Remark 2.6]. Indeed, we have

$$\begin{aligned} (6.1) \quad \frac{1}{\#G} \sum_{g \in G} v_g^* \lambda_g^\gamma &= (1 - q_1)e_\gamma + \frac{1}{\#G} \sum_{g \in G} \sum_{h \in G} s_h q_0 s_{gh}^* \lambda_g^\gamma \\ &= (1 - q_1)e_\gamma + \frac{1}{\#G} \sum_{g \in G} \sum_{h \in G} s_h q_0 \lambda_g^\gamma s_h^* = (1 - q_1)e_\gamma + \sum_{h \in G} s_h q_0 e_\gamma s_h^*. \end{aligned}$$

This means that the class of this projection in $K_0(\mathcal{O}_\infty \rtimes_\gamma G)$ is

$$[(1 - q_1)e_\gamma] + \#G[q_0 e_\gamma] = [e_\gamma],$$

which implies that $\{v_g^*\}_{g \in G}$ is a coboundary. Thus there exists a unitary $v \in \mathcal{O}_\infty$ satisfying $v_g^* = v\gamma_g(v^*)$.

We set $w_g = v^*v_gv$, and claim that $\{w_g\}_{g \in G}$ gives a K -trivial embedding of $C^*(G)$ into $\mathcal{O}_\infty^\gamma$. Indeed,

$$\gamma_g(w_h) = \gamma_g(v^*)\gamma_g(v_h)\gamma_g(v) = v^*v_g^*v_{ghg^{-1}}v_gv = w_h,$$

which shows $w_g \in \mathcal{O}_\infty^\gamma$. Let $\rho : C^*(G) \ni \lambda_g \mapsto w_g \in \mathcal{O}_\infty^\gamma$. Thanks to Lemma 6.4,(1), in order to prove the claim it suffices to show that $[\rho(e(\pi)_{11})] = 0$ in $K_0(\mathcal{O}_\infty^\gamma)$ for any nontrivial irreducible representation (π, H_π) of G . Indeed, we have

$$\begin{aligned} K_0(j_\gamma)([\rho(e(\pi)_{11})]) &= \left[\frac{n_\pi}{\#G^2} \sum_{g,h \in G} \overline{\pi(h)_{11}} \lambda_g^\gamma w_h \right] = \left[\frac{n_\pi}{\#G^2} \sum_{g,h \in G} \overline{\pi(h)_{11}} \lambda_g^\gamma v^* v_h v \right] \\ &= \left[\frac{n_\pi}{\#G^2} \sum_{g,h \in G} \overline{\pi(h)_{11}} \gamma_g(v^*) v_{ghg^{-1}} \lambda_g^\gamma v \right] = \left[\frac{n_\pi}{\#G^2} \sum_{g,h \in G} \overline{\pi(h)_{11}} v^* v_g^* v_{ghg^{-1}} \lambda_g^\gamma v \right] \\ &= \left[\frac{n_\pi}{\#G^2} \sum_{g,h \in G} \overline{\pi(h)_{11}} v_{hg^{-1}} \lambda_g^\gamma \right] = \left[\frac{n_\pi}{\#G^2} \sum_{g,h \in G} \overline{\pi(h)_{11}} v_h v_g^* \lambda_g^\gamma \right]. \end{aligned}$$

Equation (6.1) implies that this is equal to

$$\left[\frac{n_\pi}{\#G} \sum_{h \in G} \overline{\pi(h)_{11}} v_h \left\{ (1 - q_1) e_\gamma + \sum_{k \in G} s_k q_0 e_\gamma s_k^* \right\} \right].$$

Let $\rho_0 : C^*(G) \ni \lambda_g \mapsto v_g \in \mathcal{O}_\infty$. Since $v_h(1 - q_1) = 1 - q_1$ and π is nontrivial, we see that this is equal to

$$\left[\rho_0(e(\pi)_{11}) \sum_{k \in G} s_k q_0 e_\gamma s_k^* \right] = n_\pi [q_0 e_\gamma] = 0.$$

Thus the claim is shown.

We choose a unital embedding $\mu_0 : \mathcal{O}_\infty \rightarrow \mathcal{O}_\infty^\gamma$. Since both $\{\mu_0(u_g)\}_{g \in G}$ and $\{w_g\}_{g \in G}$ give K -trivial embeddings of $C^*(G)$ into $\mathcal{O}_\infty^\gamma$, Lemma 6.4,(2) shows that we may assume $\mu_0(u_g) = w_g$ by replacing μ_0 if necessary. Let $\mu(x) = v\mu_0(x)v^*$. Then

$$\begin{aligned} \gamma_g \circ \mu(x) &= \gamma_g(v)\mu_0(x)\gamma_g(v^*) = v_gv\mu_0(x)v^*v_g^* = vw_g\mu_0(x)w_g^*v^* \\ &= v\mu_0(u_gxu_g^*)v^* = \mu \circ \text{Ad } u_g(x). \end{aligned}$$

Thus μ is the desired embedding of $(\mathcal{O}_\infty, \text{Ad } u.)$ into $(\mathcal{O}_\infty, \gamma)$. □

From Theorem 6.2 and Lemma 6.6, we get

COROLLARY 6.7. *Let G be a finite group, and let γ be a quasi-free action of G on \mathcal{O}_∞ . Then any unital $\varphi \in \text{Hom}_G(\mathcal{O}_\infty, \mathcal{O}_\infty)$ is G -asymptotically unitarily equivalent to id.*

7. Equivariant Rørdam group. Let A and B be simple C^* -algebras. For simplicity we assume that A and B are unital. Following Rørdam [18, p. 40], we denote by $H(A, B)$ the set of the approximately unitary equivalence classes of nonzero homomorphisms from A into $B \otimes \mathbf{K}$. Choosing two isometries s_1 and s_2 satisfying the \mathcal{O}_2 relation in $M(B \otimes \mathbf{K})$, we can define the direct sum $[\varphi] \oplus [\psi]$ of two classes $[\varphi]$ and $[\psi]$ in $H(A, B)$ to be the class of the homomorphism

$$A \ni x \mapsto s_1\varphi(x)s_1^* + s_2\psi(x)s_2^* \in B \otimes \mathbf{K}.$$

This makes $H(A, B)$ a semigroup. When A is a separable simple nuclear C^* -algebra and B is a Kirchberg algebra, the Rørdam semigroup $H(A, B)$ is in fact a group. Moreover, if A satisfies the universal coefficient theorem, it is isomorphic to $KK(A, B)$, a certain quotient of $KK(A, B)$.

Let G be a finite group, and let α and β be outer G -actions on A and B respectively. We equip $B \otimes \mathbf{K}$ with a G - C^* -algebra structure by the diagonal action $\beta_g^s = \beta_g \otimes \text{Ad } u_g$, where $\{u_g\}$ is a countable infinite direct sum of the regular representation of G . Then we can introduce an equivariant version $H_G(A, B)$ as the set of the G -approximately equivalence classes of nonzero G -homomorphisms in $\text{Hom}_G(A, B \otimes \mathbf{K})$.

THEOREM 7.1. *Let (A, α) and (B, β) be unital G - C^* -algebras with outer actions α and β . We assume that A is separable, simple, and nuclear, and B is a Kirchberg algebra. Then $H_G(A, B)$ is a group.*

Let (A, α) and (B, β) be as above. We say that $\varphi \in \text{Hom}_G(A, B)$ is \mathcal{O}_2 -absorbing if there exists $\varphi' \in \text{Hom}_G(A \otimes \mathcal{O}_2, B)$ with $\varphi = \varphi' \circ \iota_A$, where $A \otimes \mathcal{O}_2$ is equipped with the G -action $\alpha \otimes \text{id}_{\mathcal{O}_2}$, and $\iota_A : A \ni x \mapsto x \otimes 1 \in A \otimes \mathcal{O}_2$ is the inclusion map. We say that $\varphi \in \text{Hom}_G(A, B)$ is \mathcal{O}_∞ -absorbing if there exists a unital embedding of \mathcal{O}_∞ in $(\varphi(1)B^G\varphi(1)) \cap \varphi(A)'$.

The proof of Theorem 7.1 follows from essentially the same argument as in [18, Lemma 8.2.5] with the following lemma.

LEMMA 7.2. *Let the notation be as above.*

- (1) *Let $\varphi, \psi \in \text{Hom}_G(A, B)$ be \mathcal{O}_2 -absorbing G -homomorphisms, either both unital or both nonunital. Then φ and ψ are G -approximately unitarily equivalent.*
- (2) *Any element in $\text{Hom}_G(A, B)$ is G -asymptotically unitarily equivalent to an \mathcal{O}_∞ -absorbing one in $\text{Hom}_G(A, B)$.*

PROOF. (1) When φ and ψ are nonunital, the two projections $\varphi(1)$ and $\psi(1)$ are equivalent in B^G , and we may assume $\varphi(1) = \psi(1)$. Replacing B with $\varphi(1)B\varphi(1)$, we may assume that φ and ψ are unital.

Let γ be a faithful quasi-free action of G on \mathcal{O}_∞ . Since $(A \otimes \mathcal{O}_2, \alpha \otimes \text{id}_{\mathcal{O}_2})$ is conjugate to $(\mathcal{O}_\infty \otimes \mathcal{O}_2, \gamma \otimes \text{id}_{\mathcal{O}_2})$ thanks to [4, Corollary 4.3], it suffices to show that any unital $\varphi, \psi \in \text{Hom}_G(\mathcal{O}_\infty \otimes \mathcal{O}_2, B)$ are G -approximately unitarily equivalent. Theorem 4.1 implies that there exists $u \in U((B^\omega)^G)$ satisfying $u\varphi(x \otimes 1)u^* = \psi(x \otimes 1)$ for any $x \in \mathcal{O}_\infty$, where $\omega \in \beta N \setminus N$ is a free ultrafilter. Let $D = (B^\omega)^G \cap \psi(\mathcal{O}_\infty \otimes 1)'$. Then it suffices to show

that the two unital homomorphisms $\rho, \sigma \in \text{Hom}(\mathcal{O}_2, D)$ defined by $\rho(y) = u\varphi(1 \otimes y)u^*$, $\sigma(y) = \psi(1 \otimes y)$ for $y \in \mathcal{O}_2$, are approximately unitarily equivalent. Indeed, since $(B^\omega)^G \cap B' = (B_\omega)^G$ is purely infinite and simple, for any separable C^* -subalgebra C of D there exists a unital embedding of \mathcal{O}_∞ in $D \cap C'$. Thus essentially the same proof of [15, Lemma 2.1.7] shows that $\text{cel}(D)$ is finite (see [15, Lemma 2.1.1] for the definition). Therefore ρ and σ are approximately unitarily equivalent thanks to [17, Theorem 3.6].

(2) Since (B, β) is conjugate to $(B \otimes \mathcal{O}_\infty, \beta \otimes \text{id}_{\mathcal{O}_\infty})$ thanks to [4, Corollary 2.10], the statement follows from the same argument as in the proof of [18, Lemma 8.2.5,(i)]. \square

REMARK 7.3. There are two natural homomorphisms

$$\mu : H_G(A, B) \rightarrow H(A, B),$$

$$\nu : H_G(A, B) \rightarrow H(A \rtimes_\alpha G, B \rtimes_\beta G).$$

The first one is the forgetful functor. Every $\varphi \in \text{Hom}_G(A, B)$ extends to $\tilde{\varphi} \in \text{Hom}(A \rtimes_\alpha G, B \rtimes_\beta G)$ by $\tilde{\varphi}(\lambda_g^\alpha) = \lambda_g^\beta$, and the second one is given by associating $[\tilde{\varphi}] \in H(A \rtimes_\alpha G, B \rtimes_\beta G)$ with $[\varphi] \in H_G(A, B)$. The following hold for the two maps (see [6, Section 4] for more general treatment):

(1) If β has the Rohlin property, then μ is injective, and the image of μ is

$$\{[\rho] \in H(A, B); [\beta_g^s \circ \rho] = [\rho \circ \alpha_g], \text{ for all } g \in G\}.$$

(2) If β is approximately representable, then ν is injective, and the image of ν is

$$\{[\rho] \in H(A \rtimes_\alpha G, B \rtimes_\beta G); [\hat{\beta}^s \circ \rho] = [(\rho \otimes \text{id}_{C^*(G)}) \circ \hat{\alpha}]\}.$$

REMARK 7.4. Let $\hat{H}_G(A, B)$ be the set of the G -asymptotically equivalence classes of nonzero G -homomorphisms in $\text{Hom}_G(A, B \otimes \mathbf{K})$. It is tempting to conjecture that the natural map from $\hat{H}_G(A, B)$ to the equivariant KK -group $KK_G(A, B)$ is an isomorphism, as it is the case for trivial G (see [15]).

8. Appendix. In this appendix, we show the equivalence of (4) and (5) in Theorem 3.1. Since our argument works for a compact group G , we assume that G is compact in what follows. Our proof is new even for trivial G . Let α be a quasi-free action of G on \mathcal{O}_n with finite n , and let (B, β) be a unital G - C^* -algebra. Now the definition of the projection $e_\beta \in B \rtimes_\beta G$ should be modified to $e_\beta = \int_G \lambda_g^\beta dg$, where dg is the normalized Haar measure of G . For two unital $\varphi, \psi \in \text{Hom}_G(\mathcal{O}_n, B)$, we define $u_{\psi, \varphi} \in U(B^G)$ as in Theorem 3.1.

Let \mathcal{E}_n be the Cuntz-Toeplitz algebra with the canonical generators $\{t_i\}_{i=1}^n$. We denote by q_n the surjection $q_n : \mathcal{E}_n \rightarrow \mathcal{O}_n$ sending t_i to s_i for $i = 1, 2, \dots, n$. Then the kernel J_n of q_n is the ideal generated by $p_n = 1 - \sum_{i=1}^n t_i t_i^*$, and is isomorphic to the compact operators \mathbf{K} . We denote by $i_n : J_n \rightarrow \mathcal{E}_n$ the inclusion map. Since \mathcal{O}_n is nuclear, the exact sequence

$$(8.1) \quad 0 \longrightarrow J_n \xrightarrow{i_n} \mathcal{E}_n \xrightarrow{q_n} \mathcal{O}_n \longrightarrow 0,$$

is semisplit, that is, there exists a unital completely positive lifting $l_n : \mathcal{O}_n \rightarrow \mathcal{E}_n$ of q_n . We denote by $\tilde{\alpha}$ the quasi-free action of G on \mathcal{E}_n that is a lift of α . By replacing l_n with l_n^G given

by

$$l_n^G(x) = \int_G \tilde{\alpha}_g \circ l_n \circ \alpha_{g^{-1}}(x) dg, \quad x \in \mathcal{O}_n,$$

we see that (8.1) is a semisplit exact sequence of G - C^* -algebras. Thus it induces the following 6-term exact sequence of KK_G -groups (see [1, p. 208]):

$$\begin{array}{ccccc} KK_G^0(J_n, B) & \xleftarrow{i_n^*} & KK_G^0(\mathcal{E}_n, B) & \xleftarrow{q_n^*} & KK_G^0(\mathcal{O}_n, B) \\ \delta \downarrow & & & & \uparrow \delta \\ KK_G^1(\mathcal{O}_n, B) & \xrightarrow{q_n^*} & KK_G^1(\mathcal{E}_n, B) & \xrightarrow{i_n^*} & KK_G^1(J_n, B) \end{array}.$$

Let H_n be the n -dimensional Hilbert space \mathbf{C}^n with the canonical orthonormal basis $\{e_i\}_{i=1}^n$. We regard H_n as a \mathbf{C} - \mathbf{C} bimodule with a G -action given by π_α . We denote by \mathcal{F}_n the full Fock space

$$\mathcal{F}_n = \bigoplus_{m=0}^{\infty} H_n^{\otimes m},$$

with a unitary representation $\pi_{\mathcal{F}_n}$ of G coming from π_α . Identifying t_i with the creation operator of e_i acting on \mathcal{F}_n , we regard \mathcal{E}_n as a C^* -subalgebra of $\mathbf{B}(\mathcal{F}_n)$. With this identification, we have $J_n = \mathbf{K}(\mathcal{F}_n)$, and p_n is the projection onto $H_n^{\otimes 0}$. We regard \mathcal{F}_n as J_n - \mathbf{C} bimodule, which gives the KK_G -equivalence of J_n and \mathbf{C} . Pimsner's arguments [16, Theorem 4.4, Theorem 4.9] yield the following 6-term exact sequence:

$$\begin{array}{ccccc} KK_G^0(\mathbf{C}, B) & \xleftarrow{1-[H_n]\hat{\otimes}} & KK_G^0(\mathbf{C}, B) & \xleftarrow{\quad} & KK_G^0(\mathcal{O}_n, B) \\ \delta' \downarrow & & & & \uparrow \delta' \\ KK_G^1(\mathcal{O}_n, B) & \longrightarrow & KK_G^1(\mathbf{C}, B) & \xrightarrow{1-[H_n]\hat{\otimes}} & KK_G^1(\mathbf{C}, B) \end{array},$$

where $[H_n]\hat{\otimes}$ denotes the left multiplication of the class $[H_n] \in KK_G(\mathbf{C}, \mathbf{C})$. Note that the identification of $KK_G^*(J_n, B)$ and $KK_G^*(\mathbf{C}, B)$ is given by $[\mathcal{F}_n] \in KK_G(J_n, \mathbf{C})$, and so $\delta' = \delta \circ (\mathcal{F}_n \hat{\otimes})$.

With the Green-Julg isomorphism $h_* : KK_G^*(\mathbf{C}, B) \rightarrow K_*(B \rtimes_\beta G)$ ([1, Theorem 11.7.1]), we have the commutative diagram

$$\begin{array}{ccc} KK_G^*(\mathbf{C}, B) & \xrightarrow{[\mathcal{H}_n]\hat{\otimes}} & KK_G^*(\mathbf{C}, B) \\ h_* \downarrow & & \downarrow h_* \\ K_*(B \rtimes_\beta G) & \xrightarrow{K_*(\hat{\beta}_{\pi_\alpha})} & K_*(B \rtimes_\beta G) \end{array},$$

and so we get the following 6-term exact sequence

$$\begin{array}{ccccc}
 K_0(B \rtimes_{\beta} G) & \xleftarrow{1-K_0(\hat{\beta}_{\pi_{\alpha}})} & K_0(B \rtimes_{\beta} G) & \longleftarrow & KK_G^0(\mathcal{O}_n, B) \\
 \delta'' \downarrow & & & & \uparrow \delta'' \\
 KK_G^1(\mathcal{O}_n, B) & \longrightarrow & K_1(B \rtimes_{\beta} G) & \xrightarrow{1-K_1(\hat{\beta}_{\pi_{\alpha}})} & K_1(B \rtimes_{\beta} G)
 \end{array}$$

with $\delta'' = \delta \circ ([\mathcal{F}_n] \hat{\otimes}) \circ h_*^{-1}$. Now the proof of the equivalence of (4) and (5) in Theorem 3.1 follows from the next theorem.

THEOREM 8.1. *With the above notation, we have*

$$\delta''(K_1(j_{\beta})([u_{\psi, \varphi}])) = KK_G(\psi) - KK_G(\varphi).$$

The proof of Theorem 8.1 follows from a standard and rather tedious computation below. In what follows, we freely use the notation in Blackadar’s book [1] for KK -theory. We regard $C_1, C = C_0[0, 1]$, and $S = C_0(0, 1)$ as G - C^* -algebras with trivial G -actions.

[1, Theorem 19.5.7] shows that δ' is given by the left multiplication of the class δ_{q_n} of the extension (8.1) in $KK_G^1(\mathcal{O}_n, C) = KK_G(\mathcal{O}_n, C_1)$, whose Kasparov module $(E_1, \phi_1, F_1) \in E_G(\mathcal{O}_n, C_1)$ is given as follows. By the Stinespring dilation of the G -equivariant lifting $l_n^G : \mathcal{O}_n \rightarrow \mathcal{E}_n \subset \mathbf{B}(\mathcal{F}_n)$, we get a Hilbert space H including \mathcal{F}_n , with a unitary representation π_H of G extending $\pi_{\mathcal{F}_n}$, satisfying the following condition: there is a unital G -homomorphism $\Phi : \mathcal{O}_n \rightarrow \mathbf{B}(H)$ such that if P is the projection from H onto \mathcal{F}_n , then $l_n^G(x) = P\Phi(x)P$ for any $x \in \mathcal{O}_n$. Now we have

$$(E_1, \phi_1, F_1) = (H \hat{\otimes} C_1, \Phi \hat{\otimes} 1, (2P - 1) \hat{\otimes} \varepsilon),$$

where $\varepsilon = 1 \oplus -1$ is the generator of $C_1 \cong C^*(\mathbf{Z}_2)$.

Let $z(t) = e^{2\pi it}$, and let θ be the element in $\text{Hom}_G(S, B)$ determined by $\theta(z - 1) = u_{\psi, \varphi} - 1$. Then $h_1^{-1} \circ K_1(j_{\beta})([u_{\psi, \varphi}])$ is given by

$$KK_G(\theta) \in KK_G(S, B) \cong KK_G(C_1, B).$$

In order to compute the Kasparov product of $\delta_{q_n} \in KK_G(\mathcal{O}_n, C_1)$ and $KK_G(\theta) \in KK_G(S, B)$, we need to identify $KK_G(S, B)$ with $KK_G(C_1, B)$ explicitly, and we need the invertible element $x \in KK_G(C_1, S)$ defined in [1, Section 19.2]. By the extension

$$0 \longrightarrow S \longrightarrow C \longrightarrow C \longrightarrow 0,$$

we get an invertible element in $KK_G(C, S \hat{\otimes} C_1)$. Then x is the image of this element by the isomorphism

$$\begin{aligned}
 \tau_{C_1} : KK_G(C, S \hat{\otimes} C_1) &\rightarrow KK_G(C \hat{\otimes} C_1, S \hat{\otimes} C_1 \hat{\otimes} C_1) \\
 &= KK_G(C_1, S \hat{\otimes} M_2(C)) = KK_G(C_1, S).
 \end{aligned}$$

For the identification of $C_1 \hat{\otimes} C_1$ and $M_2(C)$ with standard even grading, we follow the convention in the proof of [1, Theorem 18.10.12] (our computation really depends on it). A

direct computation shows that \mathbf{x} is given by the Kasparov module $(E_2, \phi_2, F_2) \in E_G(C_1, S)$ with $E_2 = C^2 \hat{\otimes} (S \oplus S)$,

$$F_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\phi_2(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes Q, \quad \phi_2(\varepsilon) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes Q,$$

where the projection $Q \in M_2(M(S))$ is given by

$$Q(t) = \begin{pmatrix} 1-t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & t \end{pmatrix},$$

and the grading of E_2 is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

With this \mathbf{x} , we have

$$\delta''(K_1(j_\beta)([u_{\psi,\varphi}])) = \delta_{q_n} \hat{\otimes}_{C_1} \mathbf{x} \hat{\otimes}_S K K_G(\theta) = \theta_*(\delta_{q_n} \hat{\otimes}_{C_1} \mathbf{x}),$$

and so our task now is to compute $\delta_{q_n} \hat{\otimes}_{C_1} \mathbf{x}$ explicitly.

LEMMA 8.2. *The class $\delta_{q_n} \hat{\otimes}_{C_1} \mathbf{x} \in K K_G(\mathcal{O}_n, S)$ is given by the quasi-homomorphism $\rho = (\rho^{(0)}, \rho^{(1)})$ from \mathcal{O}_n to S such that $\rho^{(0)}$ and $\rho^{(1)}$ are unital homomorphisms from \mathcal{O}_n to $\mathbf{B}(H \hat{\otimes} S)$ with $\rho^{(0)}(x) = \Phi(x) \hat{\otimes} 1$ and*

$$\rho^{(1)}(x) = (P \hat{\otimes} 1 + (1 - P) \hat{\otimes} z)(\Phi(x) \hat{\otimes} 1)(P \hat{\otimes} 1 + (1 - P) \hat{\otimes} z)^*.$$

PROOF. We regard $H \hat{\otimes} S$ as an \mathcal{O}_n - S bimodule with trivial grading, and we set $E = (H \hat{\otimes} S) \oplus (H \hat{\otimes} S)^{\text{op}}$. We denote by $\Psi : S \rightarrow Q(S \oplus S)$ a Hilbert S -module isomorphism given by

$$\Psi(f)(t) = (\sqrt{1-t}f(t), \sqrt{t}f(t)).$$

Then $E_1 \hat{\otimes}_{C_1} E_2$ is identified with E via the identification of $(\xi_1 \hat{\otimes} f_1, \xi_2 \hat{\otimes} f_2) \in E$ and

$$\xi_1 \hat{\otimes} 1 \hat{\otimes}_{C_1} (1, 0) \hat{\otimes} \Psi(f_1) + \xi_2 \hat{\otimes} 1 \hat{\otimes}_{C_1} (0, 1) \hat{\otimes} \Psi(f_2) \in H \hat{\otimes} C_1 \hat{\otimes}_{C_1} C^2 \hat{\otimes} (S \oplus S).$$

We claim that $\delta_{q_n} \hat{\otimes}_{C_1} \mathbf{x}$ is given by the Kasparov module $(E, \phi, F) \in E_G(\mathcal{O}_n, S)$ with

$$\phi(x) = \text{diag}(\Phi(x) \otimes 1, \Phi(x) \otimes 1),$$

$$F = \begin{pmatrix} 0 & 1 \hat{\otimes} c \\ 1 \hat{\otimes} c & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i(2P - 1) \hat{\otimes} s \\ i(2P - 1) \hat{\otimes} s & 0 \end{pmatrix},$$

where $c(t) = \cos(\pi t)$, $s(t) = \sin(\pi t)$. Indeed, it is easy to show that (E, ϕ, F) is a Kasparov module, and the graded commutator $[F_1 \hat{\otimes} 1_{E_2}, F]$ is positive. We show that F is a F_2 -connection (see [1, Definition 18.3.1] for the definition). Let $\xi \in H$, $x = (x_1, x_2) \in C^2$, and $f = (f_1, f_2) \in S \oplus S$. Then we have

$$T_{\xi \hat{\otimes} 1}(x \hat{\otimes} f) = (x_1 \xi \hat{\otimes} (\sqrt{1-t}f_1 + \sqrt{t}f_2), x_2 \xi \hat{\otimes} (\sqrt{1-t}f_1 + \sqrt{t}f_2)) \in E,$$

$$T_{\xi \hat{\otimes} \varepsilon}(x \hat{\otimes} f) = (-ix_2 \xi \hat{\otimes} (\sqrt{1-t}f_1 + \sqrt{t}f_2), ix_1 \xi \hat{\otimes} (\sqrt{1-t}f_1 + \sqrt{t}f_2)) \in E.$$

A direct computation shows that $T_{\xi \hat{\otimes} 1} \circ F_2 - F \circ T_{\xi \hat{\otimes} 1}$ and $T_{\xi \hat{\otimes} \varepsilon} \circ F_2 + F \circ T_{\xi \hat{\otimes} \varepsilon}$ are in $\mathbf{K}(E_2, E)$. Since F_2 and F are self-adjoint, we see that F is an F_2 -connection. Therefore (E, ϕ, F) gives the Kasparov product $\delta_{q_n} \hat{\otimes}_{C_1} x$.

Note that F satisfies $F = F^*$, $F^2 = 1$. Let

$$U = \begin{pmatrix} 1 \otimes 1 & 0 \\ 0 & 1 \hat{\otimes} c + i(2P - 1) \hat{\otimes} s \end{pmatrix},$$

which is a unitary in $\mathbf{B}(E)$. Then we have

$$U^* F U = \begin{pmatrix} 0 & 1 \otimes 1 \\ 1 \otimes 1 & 0 \end{pmatrix},$$

$$U^* \phi(x) U = \begin{pmatrix} \rho^{(0)}(x) & 0 \\ 0 & \rho^{(1)}(x) \end{pmatrix},$$

which finish the proof. □

To continue the proof, we need more detailed information on the homomorphism Φ .

LEMMA 8.3. *Let the notation be as above.*

(1) *We can choose Φ so that it has the following form with respect to the orthogonal decomposition $H = \mathcal{F}_n \oplus \mathcal{F}_n^\perp$:*

$$\Phi(s_i) = \begin{pmatrix} t_i & r_i \\ 0 & v_i \end{pmatrix}.$$

(2) *For Φ as in (1), the quasi-homomorphism $\rho = (\rho^{(0)}, \rho^{(1)})$ in Lemma 8.2 is expressed as*

$$\rho^{(0)}(s_i) = \begin{pmatrix} t_i \hat{\otimes} 1 & r_i \hat{\otimes} 1 \\ 0 & v_i \hat{\otimes} 1 \end{pmatrix}, \quad \rho^{(1)}(s_i) = \begin{pmatrix} t_i \hat{\otimes} 1 & r_i \hat{\otimes} z^* \\ 0 & v_i \hat{\otimes} 1 \end{pmatrix}.$$

In particular, we have

$$\sum_{i=1}^n \rho^{(1)}(s_i) \rho^{(0)}(s_i)^* = (1_H - p_n) \hat{\otimes} 1 + p_n \hat{\otimes} z^*.$$

PROOF. (1) We first construct $l_n^G : \mathcal{O}_n \rightarrow \mathcal{E}_n$ explicitly. Ignoring the G -actions, we can find a representation Φ' of \mathcal{O}_n on $\mathcal{F}_n \oplus \mathcal{F}_n$ of the form

$$\Phi'(s_1) = \begin{pmatrix} t_1 & p_n \\ 0 & w_1 \end{pmatrix},$$

$$\Phi'(s_i) = \begin{pmatrix} t_i & 0 \\ 0 & w_i \end{pmatrix}, \quad 2 \leq i \leq n.$$

Using Φ' , we define l_n by

$$\begin{pmatrix} l_n(x) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Phi'(x) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and l_n^G by $l_n^G(x) = \int_G \tilde{\alpha}_{g^{-1}} \circ l_n \circ \alpha_g(x) dg$. We have $l_n^G(s_i) = t_i$ for all $1 \leq i \leq n$ by construction.

We show that the Stinespring dilation (Φ, H) of this l_n^G has the desired property. Recall that H is the closure of the algebraic tensor product $\mathcal{O}_n \odot \mathcal{F}_n$ with respect to the inner product

$$\langle x \odot \xi, y \odot \eta \rangle = \langle l_n^G(y^*x)\xi, \eta \rangle,$$

and Φ is given by the left multiplication of \mathcal{O}_n . The space \mathcal{F}_n is identified with $1 \odot \mathcal{F}_n$, and the unitary representation π_H is given by $\pi_H(g)(x \odot \xi) = \alpha_g(x) \odot \pi_{\mathcal{F}_n}(g)\xi$. To show that Φ has the desired property, it suffices to show $\|s_i \odot \xi - 1 \odot t_i \xi\| = 0$ for all $\xi \in \mathcal{F}_n$. Indeed,

$$\begin{aligned} & \|s_i \odot \xi - 1 \odot t_i \xi\|^2 \\ &= \langle l_n^G(s_i^*s_i)\xi, \xi \rangle - \langle l_n^G(s_i)\xi, t_i \xi \rangle - \langle l_n^G(s_i^*)t_i \xi, \xi \rangle + \langle t_i \xi, t_i \xi \rangle = 0, \end{aligned}$$

and we get the statement.

(2) The first statement follows from (1) and Lemma 8.2. The Cuntz algebra relation implies

$$\begin{aligned} p_n r_i &= r_i, & r_j^* r_i + v_j^* v_i &= \delta_{i,j}, \\ \sum_{i=1}^n r_i r_i^* &= p_n, & \sum_{i=1}^n r_i v_i^* &= 0, & \sum_{i=1}^n v_i v_i^* &= 1. \end{aligned}$$

These relations and the first statement imply the second statement. □

PROOF OF THEOREM 8.1. Thanks to the previous lemma, we may assume that the class $\theta_*(\delta_{q_n} \hat{\otimes}_{C_1} \mathbf{x}) \in KK_G(\mathcal{O}_n, B)$ is given by a quasi-homomorphism $\sigma = (\sigma^{(0)}, \sigma^{(1)})$ from \mathcal{O}_n to B of the form

$$\sigma^{(0)}(s_i) = \begin{pmatrix} t_i \hat{\otimes} 1 & r_i \hat{\otimes} 1 \\ 0 & v_i \hat{\otimes} 1 \end{pmatrix}, \quad \sigma^{(1)}(s_i) = \begin{pmatrix} t_i \hat{\otimes} 1 & r_i \hat{\otimes} u_{\psi, \varphi}^* \\ 0 & v_i \hat{\otimes} 1 \end{pmatrix},$$

and they satisfy

$$\sum_{i=1}^n \sigma^{(1)}(s_i) \sigma^{(0)}(s_i)^* = (1_H - p_n) \hat{\otimes} 1 + p_n \hat{\otimes} u_{\psi, \varphi}^*.$$

We set $\tilde{\sigma}^{(0)} = \sigma^{(0)} \oplus \varphi$, $\tilde{\sigma}^{(1)} = \sigma^{(1)} \oplus \psi$, which are unital homomorphisms from \mathcal{O}_n to $B((H \oplus C) \hat{\otimes} B)$. Then $\tilde{\sigma} = (\tilde{\sigma}^{(0)}, \tilde{\sigma}^{(1)})$ is a quasi-homomorphism with

$$\sum_{i=1}^n \tilde{\sigma}^{(1)}(s_i) \tilde{\sigma}^{(0)}(s_i)^* = ((1_H - p_n) \hat{\otimes} 1 + p_n \hat{\otimes} u_{\psi, \varphi}^*) \oplus (1_C \hat{\otimes} u_{\psi, \varphi}),$$

which is denoted by u . Then we can construct a norm continuous path $\{u_t\}_{t \in [0,1]}$ of unitaries in $C1 + K(H \oplus C)^G \otimes B^G$ satisfying $u(0) = u$ and $u(1) = 1$. Let $\tilde{\sigma}_t^{(0)} = \tilde{\sigma}^{(0)}$, and let $\tilde{\sigma}_t^{(1)}$ be the homomorphism from \mathcal{O}_n to $B((H \oplus C) \hat{\otimes} B)$ determined by $\tilde{\sigma}_t^{(1)}(s_i) = u(t) \tilde{\sigma}_t^{(0)}(s_i)$. Then $\tilde{\sigma}_t = (\tilde{\sigma}_t^{(0)}, \tilde{\sigma}_t^{(1)})$ gives a homotopy of quasi-homomorphisms connecting $\tilde{\sigma}$ and $\tilde{\sigma}_1 = (\tilde{\sigma}^{(0)}, \tilde{\sigma}^{(0)})$. This shows $[\tilde{\sigma}] = 0$ in $KK_G(\mathcal{O}_n, B)$, and so $\theta_*(\delta_{q_n} \hat{\otimes}_{C_1} \mathbf{x}) = KK_G(\psi) - KK_G(\varphi)$. □

REMARK 8.4. The above argument shows that there exists a short exact sequence

$$0 \rightarrow \text{Coker}(1 - K_{1-*}(\hat{\beta}_{\pi_\alpha})) \rightarrow KK_G^*(\mathcal{O}_n, B) \rightarrow \text{Ker}(1 - K_*(\hat{\beta}_{\pi_\alpha})) \rightarrow 0.$$

REMARK 8.5. From (8.1), we obtain the 6-term exact sequence (see [16, Theorem 4.9]),

$$\begin{array}{ccccc} KK_G^0(B, C) & \xrightarrow{1-\hat{\otimes}[H_n]} & KK_G^0(B, C) & \longrightarrow & KK_G^0(B, \mathcal{O}_n) \\ \uparrow & & & & \downarrow \\ KK_G^1(B, \mathcal{O}_n) & \longleftarrow & KK_G^1(B, C) & \xleftarrow{1-\hat{\otimes}[H_n]} & KK_G^1(B, C) \end{array}$$

In particular, we have the following exact sequence by setting $B = C$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_1(\mathcal{O}_n \rtimes_\alpha G) & \longrightarrow & & & \\ K_0^G(C) & \xrightarrow{1-\hat{\otimes}[H_n]} & K_0^G(C) & \longrightarrow & K_0(\mathcal{O}_n \rtimes_\alpha G) & \longrightarrow & 0 \end{array}$$

Let $\iota_\alpha : C^*(G) \rightarrow \mathcal{O}_n \rtimes_\alpha G$ be the embedding map, let (π, H_π) be an irreducible representation of G , and let

$$e(\pi)_{ij} = \dim \pi \int_G \overline{\pi(g)_{ij}} \lambda_g dg \in C^*(G).$$

Then the canonical isomorphism from $K_0^G(C)$ onto $K_0(C^*(G))$ sends the class of (π, H_π) in $K_0^G(C)$ to $[e(\overline{\pi})_{11}] \in K_0(C^*(G))$. Thus we have the exact sequence

$$0 \longrightarrow K_1(\mathcal{O}_n \rtimes_\alpha G) \longrightarrow Z\hat{G} \xrightarrow{1-[\pi_\alpha]} Z\hat{G} \longrightarrow K_0(\mathcal{O}_n \rtimes_\alpha G) \longrightarrow 0,$$

where $[\pi] \in Z\hat{G}$ is sent to $K_0(\iota_\alpha)([e(\pi)_{11}]) \in K_0(\mathcal{O}_n \rtimes_\alpha G)$. With the identification of $K_*(\mathcal{O}_n \rtimes_\alpha G)$ and $K_*(\mathcal{O}_n^G)$, this recovers the formula of $K_*(\mathcal{O}_n^G)$ obtained in [11], [14].

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