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# QUASI-HEREDITY OF ALGEBRAS AND THEIR FACTOR ALGEBRAS

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*Dedicated to Professor Tiande Lei on the occasion of his 65th birthday*

ABSTRACT. Let $A$ be a finite-dimensional algebra over an algebraically closed field and denote by $N$ the Jacobson radical of $A$. If there is an integer $i \geq 2$ such that $A/N^i$ is quasi-hereditary, then $A$ is quasi-hereditary.

Let $A$ be a finite-dimensional algebra over an algebraically closed field $k$. By $N$ we denote the Jacobson radical of $A$. An ideal of $A$ is called a heredity ideal of $A$ if it satisfies (1) $J^2 = J$, (2) $JNJ = 0$, and (3) $J$ is a projective left $A$-module. We recall that the algebra $A$ is said to be quasi-hereditary provided there is a chain

$$0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$$

of ideals of $A$ such that $J_i/J_{i-1}$ is a heredity ideal of $A/J_{i-1}$ for all $i = 1, \ldots, n$. Some basic properties on quasi-hereditary algebras may be found in [DR]. The aim of this note is to show the following: If the algebra $A$ is not quasi-hereditary, then, for any $i \geq 2$, the factor algebra $A/N^i$ never becomes a quasi-hereditary algebra.

Throughout this note all algebras are finite-dimensional $k$-algebras with 1, module means finitely generated left module. By $\bar{a}$ (or $\overline{J}$) we denote the image of $a \in A$ (or $J \subseteq A$) under the canonical map $A \to A/I$, where $I$ is an ideal of $A$.

The above-mentioned result may be reformulated as the following theorem.

**Theorem 1.** *Let $A$ be a basic connected algebra with Jacobson radical $N$. If $A/N^i$ is quasi-hereditary for some $i \geq 2$, then $A$ is quasi-hereditary.*

To prove this result we need some preparations.

**Lemma 2.** *Let $A$ be a basic algebra and $e$ be a primitive idempotent such that $J = AeA$ is a heredity ideal of $A$. Then $eAe \cong k$.*

*Proof.* Since the field $k$ is algebraically closed and the Jacobson radical of $eAe$ is $eNe$, it follows from the definition of a heredity ideal that $eNe = 0$ and $eAe \cong k$.

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**Lemma 3.** *Let $A$ be a basic connected algebra with radical $N$ such that $\overline{A} = A/N^i$ is quasi-hereditary for some $i \geq 2$. Let $e$ be a primitive idempotent of $A$ such that $\overline{A}\bar{e}\overline{A}$ is a heredity ideal of $\overline{A}$. Then $J = AeA$ is a heredity ideal of $A$.*

*Proof.* By [DR] an idempotent ideal $AeA$ of $A$ with $eNe = 0$ is a heredity ideal of $A$ if and only if the multiplication map

$$Ae \otimes_{eAe} eA \to AeA$$

is bijective. Since $\overline{A}\bar{e}\overline{A}$ is a heredity ideal of $\overline{A}$, the multiplication map

$$\overline{N^{i-1}e} \otimes_k \overline{eN} \to \overline{N^{i-1}eN} = 0$$

is bijective. This implies that $\overline{N}^{i-1}\bar{e} = 0$ or $\bar{e}\overline{N} = 0$. Similarly, we consider the multiplication map

$$\overline{Ne} \otimes_k \overline{eN^{i-1}} \to \overline{NeN^{i-1}} = 0.$$

This gives us that either $\overline{Ne} = 0$ or $\overline{eN^{i-1}} = 0$. If $\overline{Ne} = 0$ or $\overline{eN} = 0$, then we get $Ne = 0$ or $eN = 0$. Thus $J = AeA$ is obviously a heredity ideal of $A$. Now let us assume $\overline{Ne} \neq 0$ and $\overline{eN} \neq 0$. Then $\overline{N}^{i-1}\bar{e} = 0$ and $\bar{e}\overline{N}^{i-1} = 0$. It follows from $\overline{N^{i-1}e} = 0$ that $N^{i-1}e = 0$, since $N^{i-1}e \neq 0$ yields that $NN^{i-1}e$ is a proper submodule of $N^{i-1}e$. Similarly, there holds $eN^{i-1} = 0$. In particular, we have $N^ie = 0$ and $eN^i = 0$, and therefore the canonical maps $Ae \to \overline{A}\bar{e}$, $eA \to \bar{e}\overline{A}$ are bijective. On the one hand, it follows from $eNe \subseteq N^i$ that $eNe = 0$. On the other hand, the canonical commutative diagram

$$\begin{array}{ccc} Ae \otimes_k eA & \xrightarrow{\mu} & AeA \\ \downarrow & & \downarrow \\ \overline{A}\bar{e} \otimes_k \bar{e}\overline{A} & \xrightarrow{\bar{\mu}} & \overline{A}\bar{e}\overline{A} \end{array}$$

shows that with $\bar{\mu}$ also $\mu$ is injective. Hence $J = AeA$ is a heredity ideal of $A$.

The following lemma is an easy observation.

**Lemma 4.** *Let $A$ be an artin algebra, $N = \operatorname{rad} A$, $\overline{A} = A/N^i$. Let $e$ be a primitive idempotent in $A$. Then $(A/AeA)/\operatorname{rad}^i(A/AeA) \cong \overline{A}/\overline{AeA}$.*

*Proof.* Let $J = AeA$. Note that $\overline{J} \cong (J + N^i)/N^i$. From

$$\begin{aligned} \overline{A}/\overline{J} &\cong (A/N^i)/((J + N^i)/N^i) \\ &\cong A/(J + N^i) \cong (A/J)/((J + N^i)/J) \\ &\cong (A/J)/((J + N)/J)^i \end{aligned}$$

the lemma follows.

*Proof of the theorem.* We choose a complete set of pairwise nonisomorphic orthogonal primitive idempotents, say $e_1, \dots, e_n$, such that for $J_j = A(e_1 + \cdots + e_j)A$, the chain

$$0 \subset \overline{J}_1 \subset \overline{J}_2 \subset \cdots \subset \overline{J}_n = \overline{A}$$

of ideals of $\overline{A}$ is a heredity chain for $\overline{A}$. Using Lemmas 3 and 4 repeatedly, we then get a heredity chain

$$0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$$

of ideals of $A$. And therefore $A$ is quasi-hereditary.

*Remarks* 5. (1) The converse of the theorem is false. The following simple example is a desired one. Let $A$ be given by the quiver with the relation

$$1\circ \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \circ 2, \qquad \alpha\beta = 0.$$

Then it is easy to verify that $A$ is quasi-hereditary but $A/N^2$ is not.

(2) If one only assumes in the theorem that there exists an ideal $J \subset N^2$ such that $A/J$ is quasi-hereditary then $A$ may not be quasi-hereditary. Let $A$ be given by the above quiver with relations $\alpha\beta\alpha = 0$ and $\beta\alpha\beta = 0$. Then $A$ is not quasi-hereditary, but if one takes $J$ to be the socle of the projective module corresponding to the vertex 1 then $J \subset N^2$ and $A/J$ is isomorphic to the algebra displayed in (1), in particular, it is quasi-hereditary. Further examples may be found in [X].

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