QUASI-INVARIANT MANIFOLDS, STABILITY, AND GENERALIZED HOPF BIFURCATION

by

S. R. Bernfeld
Department of Mathematics
University of Texas at Arlington
Arlington, Texas 76019

P. Negrini
Istituto di Matematica
Universita de Camerino
62032 Camerino, Italy

and

L. Salvadori
Dipartimento di Matematica
Universita di Trento
38050-POVO (Trento)
Italy

Technical Report #153
March, 1981

QUASI-INVARIANT MANIFOLDS, STABILITY, AND GENERALIZED HOPF BIFURCATION

by

S. R. Bernfeld*, P. Negrini**, and L. Salvadori*

1. INTRODUCTION

We are interested in obtaining an analysis of the bifurcating periodic orbits arising in the generalized Hopf bifurcation problems in R^n. The existence of these periodic orbits has often been obtained by using such techniques as the Lyapunov-Schmidt method or topological degree arguments (see Marsden and McCracken [8] and Hale [6] and their references). Our approach, on the other hand, is based upon stability properties of the equilibrium point of the unperturbed system. Andronov et. al. [1] showed the fruitfulness of this approach in studying bifurcation problems in R^2 (for more recent papers see Negrini and Salvadori [9] and Bernfeld and Salvadori [2]). In the case of R^2, in contrast to that of R^n, $n > 2$, the stability arguments can be effectively applied because of the Poincaré-Bendixson theory. Bifurcation problems in R^n can be reduced to that of R^2 when two dimensional invariant manifolds are known to exist. The existence of such manifolds occurs, for example when the unperturbed system contains only two purely imaginary eigenvalues.

In this paper we shall be concerned with the general situation in R^n in which the unperturbed system may have several pairs of purely imaginary

*This research was partially supported by U. S. Army Research Grant DAAG29-80-C-0060.

**Work performed under the auspices of Italian Council of Research (CNR).

eigenvalues. To be more precise, let us consider the differential system

(1.1) $$\dot{p} = f_0(p),$$

where $f_0 \in C^\infty[B^n(a_0),R^n]$, $f_0(0) = 0$, and $B^n(a_0) = \{p \in R^n: \|p\| < a_0\}$. Assume the Jacobian matrix $f_0'(0)$ has two purely imaginary eigenvalues $\pm i$ and that the remaining eigenvalues $\{\lambda_j\}_{j=1}^{n-2}$ satisfy $\lambda_j \neq mi$, $m = 0, \pm 1, \ldots$.

For those $f \in C^\infty[B^n(a_0),R^n]$, $f(0) = 0$, which are close to f_0 (in an appropriate topology) consider the perturbed system

(1.2) $$\dot{p} = f(p).$$

We are interested in determining the number of nontrivial periodic orbits of (1.2) lying near the origin and having period close to 2π for those f close to f_0.

In approaching this problem, we will consider for any positive integer k the following property:

(a) (i) there exists a neighborhood N^* of f_0, an $a_1 > 0$ and a number $\delta_1 > 0$ such that for every $f \in N^*$ there are at most k nontrivial periodic orbits of (1.2) lying in $B^n(a_1)$ whose period is in $[2\pi-\delta_1, 2\pi+\delta_1]$;

(ii) for each integer j, $0 \leq j \leq k$, for each $a_2 \in (0,a_1)$ for each $\delta_2 \in (0,\delta_1)$ and for each neighborhood N of f_0, $N \subseteq N^*$, there exists $f \in N$ such that (1.2) has exactly j nontrivial periodic orbits lying in $B^n(a_2)$ whose period is in $[2\pi-\delta_2, 2\pi+\delta_2]$;

(iii) for any $\bar{a} \in (0,a_1)$, $\bar{\delta} \in (0,\delta_1)$ there exists a neighborhood $\bar{N}$ of f_0, $\bar{N} \subseteq N^*$ such that if $f \in \bar{N}$ and if γ is a periodic orbit of

(1.2) lying in $B^n(a_1)$ whose period is in $[2\pi-\delta_1, 2\pi+\delta_1]$ then γ lies in $B^n(\bar{a})$ with period in $[2\pi-\bar{\delta}, 2\pi+\bar{\delta}]$.

In contrast to (a) another property which we consider in this paper is:

(A) For any neighborhood N of f_0, for any integer $j \geq 0$, for any $a \in (0,a_1]$, and for any $\delta > 0$ there exists $f \in N$ such that (1.2) has j nontrivial periodic orbits lying in $B^n(a)$ whose period is in $[2\pi-\delta_2, 2\pi+\delta_2]$.

In R^2, Andronov et.al. [1] proved that property (a) ((i),(ii)) is a consequence of the origin of (1.1) being h-asymptotically stable or h-completely unstable where h is an odd integer and $k = \frac{h-1}{2}$. The origin of (1.1) in R^n is said to be h-asymptotically stable (h-completely unstable) if h is the smallest positive integer such that the origin of (1.2) is asymptotically stable (completely unstable) for all f for which $f(p) - f_0(p) = o(\|p\|^h)$; that is h is the smallest positive integer such that asymptotic stability (complete instability) of the origin for (1.1) is recognizable by inspecting the terms up to order h in the Taylor expansion of f_0 (see Negrini and Salvadori [9] for further information on h-asymptotic stability). In a recent paper Bernfeld and Salvadori [2] in R^2 extended the results of Andronov et.al. [1] by proving property (a) is equivalent to the h-asymptotic stability (h-complete instability) of the origin of (1.1) (where again $k = \frac{h-1}{2}$). It was also shown that property (A) is equivalent to the case in which the origin of (1.1) is neither h-asymptotically stable nor h-completely unstable for any positive integer h.

The problem in R^n was first considered by Chafee [5]. Using the Lyapunov-Schmidt method he obtained a determining equation $\psi(\xi,f) = 0$

where ψ is a measure of the amplitude of the bifurcating period orbits of (1.2) and f represents the right hand side of (1.2). By assuming that the multiplicity of the zero root of $\psi(\cdot, f_0)$ is a finite number k, he proved that property (a) holds for this k.

Our goal in this paper is to relate the number k in property (a) with the conditional asymptotic stability properties of the origin for a differential system which is close in some sense to the unperturbed system (1.1). These stability properties are precisely the h-asymptotic stability (h-complete instability) of the origin for a particular differential equation (S_h) in R^2. The construction of (S_h) as well as the recognition of the h-asymptotic stability (h-complete instability) of the origin of (S_h) can be accomplished by solving linear algebraic systems. Indeed, these stability properties can be recognized by applying the classical Poincaré procedure (see [9] or [10]). Thus, the number k, $k = \frac{h-1}{2}$, can be determined using elementary algebraic techniques. The analysis of our problem is completed by observing that when the origin for (S_h) is neither h-asymptotically stable nor h-completely unstable for every $h > 0$ then property (A) holds.

The main ingredients of our analysis are: (i) the construction of a quasi-invariant manifold $\sum_h$ for the unpertubed system (1.1); (ii) the use of the Poincaré map along a particular set of solutions of (1.1) which are initially close to $\sum_h$.

In conclusion, the quantitative problem of determining the number of bifurcating periodic solutions of the perturbed system (1.2) can be reduced to an analysis of the qualitative behavior of the flow near the origin of a two dimensional system appropriately related to the unperturbed system

(1.1). In addition, an algebraic procedure allows for a concrete solution to the problem.

Finally, we remark that an announcement of our results was presented at a conference in Trento, Italy [3].

2. RESULTS

We will endow the space $C^{\infty}[B^n(a_0),R^n]$ with the following topology: define a function $|||\cdot|||$ mapping $C^{\infty}[B^n(a_0),R^n]$ into R as

$$|||f||| = \sum_{\ell=0}^{\infty} \frac{\|f\|^{(\ell)}}{2^{\ell}(1+\|f\|^{(\ell)})}$$

where $\|f\|^{(\ell)}$ denotes the usual $C^{(\ell)}$-supremum norm of f on $B^n(a_0)$. Then $C^{\infty}[B^n(a_0),R^n]$ is a metric linear space under $|||\cdot|||$. For any vector $w \in R^n$ we shall denote by $\|w\|$ the Euclidean norm of w.

By an appropriate change of coordinates depending on f we may write systems (1.1) and (1.2) respectively in the form

$$\begin{aligned} \dot{x} &= -y + X_0(x,y,z) \\ \dot{y} &= x + Y_0(x,y,z) \\ \dot{z} &= A_0 z + Z_0(x,y,z) \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} \dot{x} &= \alpha x - \beta y + X(x,y,z,f) \\ \dot{y} &= \alpha y + \beta x + Y(x,y,z,f) \\ \dot{z} &= Az + Z(x,y,z,f). \end{aligned} \tag{2.2}$$

For each fixed f, α and β are constants satisfying $\alpha(f_0) = 0$, $\beta(f_0) = 1$

and A is an $(n-2) \times (n-2)$ constant matrix satisfying $A(f_0) = A_0$. Moreover, for fixed f, X and Y belong to $C^\infty[B^n(a_0), R]$, Z belongs to $C^\infty[B^n(a_0), R^{n-2}]$, and X, Y, Z are of order greater than one. The eigenvalues of A_0, $\{\lambda_j\}_{j=1}^{n-2}$ satisfy the condition that $\lambda_j \neq mi$, $m = 0, \pm 1 \ldots$.

We now consider an $(n-2)$ dimensional polynomial of some degree h, $h \geq 1$, given by

$$\phi^{(h)}(x,y) = \phi_1(x,y) + \ldots + \phi_h(x,y), \tag{2.3}$$

where $\phi_j(x,y)$ is homogeneous of degree j. We attempt to determine $\phi_1, \ldots, \phi_h$ in order to obtain along the solutions of (2.1)

$$\left[\frac{d}{dt}\left(z - \phi^{(h)}(x,y)\right)\right]_{z=\phi^{(h)}(x,y)} = o(x^2+y^2)^{h/2} \tag{2.4}$$

that is, we have to satisfy

$$\begin{aligned} &\frac{\partial \phi^{(h)}(x,y)}{\partial x}\left[-y + X_0(x,y,\phi^{(h)}(x,y))\right] \\ &+ \frac{\partial \phi^{(h)}(x,y)}{\partial y}\left[x + Y_0(x,y,\phi^{(h)}(x,y))\right] \\ &= A_0\phi^{(h)}(x,y) + Z_0(x,y,\phi^{(h)}(x,y)) + o(x^2+y^2)^{h/2}. \end{aligned} \tag{2.5}$$

This implies for every $j \in \{1,\ldots,h\}$, ϕ_j has to satisfy the partial differential equation

$$\frac{\partial \phi_j}{\partial y}x - \frac{\partial \phi_j}{\partial x}y = A_0\phi_j + U_j, \tag{2.6}$$

where U_j is an $(n-2)$ dimensional homogeneous polynomial of degree j depending on the functions $\phi_1 \ldots \phi_{j-1}$. Under the assumptions on A_0 (2.6) has a unique solution and can be solved recursively by observing that $\phi_1(x,y) \equiv 0$ (see for instance Bibikov [4]).

The two dimensional surface $z = \phi^{(h)}(x,y)$ is tangent at the origin to the eigenspace corresponding to the eigenvalues $\pm i$. This surface will be called a quasi-invariant manifold of order h.

Given any $h > 0$ define the following two dimensional system

$$(S_h) \qquad \begin{aligned} \dot{x} &= -y + X_0(x,y,\phi^{(h)}(x,y)) \\ \dot{y} &= x + Y_0(x,y,\phi^{(h)}(x,y)). \end{aligned}$$

(This is the system referred to in the introduction).

We distinguish the two possible cases:

I. There exists $h > 1$ (and then h must be odd) such that $x \equiv y \equiv 0$ is either h-asymptotically stable or h-completely unstable for (S_h).

II. Case I does not hold.

We are now able to state our main result.

<u>Theorem 1</u>. In Case I property (a) holds with $k = \frac{h-1}{2}$. In Case II, property (A) holds.

If all the eigenvalues of A_0 have real part not equal to zero, then for every $h > 1$ there exists a C^{h+1} two dimensional center manifold which will be denoted by H_h. We notice that if $z = \phi(x,y)$ is the equation of this center manifold, we can write

$$(2.7) \qquad \phi(x,y) = \phi^{(h)}(x,y) + o(x^2+y^2)^{h/2}.$$

As a consequence of Theorem 1 the following result holds.

<u>Corollary 1</u>. Suppose that all the eigenvalues of A_0 have real part different than zero. Then: (i) if there exists an h (and h must be odd) such that the origin of the unperturbed system (2.1) is either h-asymptotically

stable or h-completely unstable on H_h (that is, with respect to initial points on H_h) then (a) holds with $k = \frac{h-1}{2}$; (ii) if for every $h > 1$ the origin for the unperturbed system (2.1) is neither h-asymptotically stable nor h-completely unstable on H_h then (A) holds.

Under some more particular hypotheses on the eigenvalues of $f_0'(0)$ the stability properties in Theorem 2 can be expressed in terms of the unperturbed system (2.1) without any explicit involvement of H_h. This can be proved by the extension of the Poincaré procedure [10] given by Lyapunov [7]. Precisely the following result holds.

Corollary 2. Suppose all the eigenvalues of A_0 have negative real part. Then (i) if the origin of the unperturbed system (2.1) is either h-asymptotically stable or h-unstable (in the whole) then (a) holds with $k = \frac{h-1}{2}$; (ii) if for every $h > 1$ the origin for the unperturbed system (2.1) is neither h-asymptotically stable nor h-unstable, then (A) holds.

Notice that we are using the concept of h-unstable whose definition is analogous to that of h-complete instability. A similar theorem can be stated when $f_0'(0)$ has two purely imaginary eigenvalues $\pm i$ and the remaining eigenvalues have positive real part.

3. PRELIMINARY PROPOSITIONS

Using the transformation

$$\zeta = z - \phi^{(h)}(x,y),$$

we can rewrite the unperturbed system (2.1) as

$$
(3.1)\qquad
\begin{aligned}
\dot{x} &= -y + X_0^{(h)}(x,y,\zeta)\\
\dot{y} &= x + Y_0^{(h)}(x,y,\zeta)\\
\dot{\zeta} &= A_0\zeta + W_0^{(h)}(x,y,\zeta),
\end{aligned}
$$

where $X_0^{(h)}(x,y,0) = X_0(x,y,\phi^{(h)}(x,y))$, $Y_0^{(h)}(x,y,0) = Y_0(x,y,\phi^{(h)}(x,y))$. From (2.4) we observe that $W_0^{(h)}(x,y,0)$ is of order greater than h. Analogously, we can rewrite the perturbed system (2.2) as

$$
(3.2)\qquad
\begin{aligned}
\dot{x} &= \alpha x - \beta y + X^{(h)}(x,y,\zeta,f)\\
\dot{y} &= \alpha y + \beta x + Y^{(h)}(x,y,\zeta,f)\\
\dot{\zeta} &= A\zeta + W^{(h)}(x,y,\zeta,f),
\end{aligned}
$$

where $X^{(h)}(x,y,0,f) = X(x,y,\phi^{(h)}(x,y),f)$, $Y^{(h)}(x,y,0,f) = Y(x,y,\phi^{(h)}(x,y),f)$ and for fixed f, $X^{(h)}$, $Y^{(h)}$, $W^{(h)}$ are of order ≥ 2. Let us set $I_\delta = [2\pi - \delta, 2\delta + \delta]$ for any $\delta > 0$. We now give the following proposition.

__Proposition 1.__ There exist $\varepsilon, \delta, L > 0$ and a neighborhood N_0 of f_0 such that whenever $f \in N_0$ and $(x,y,\zeta) \in \gamma$, where γ is any T-periodic orbit of (3.2) lying in $B^n(\varepsilon)$ with $T \in I_\delta$, then $\|\zeta\| \leq L(x^2+y^2)$.

__Proof.__ Choose $\varepsilon_1 \in (0,a_0)$, $\delta_1 > 0$ and a neighborhood N_1 of f_0 such that: (i) $\det(I-e^{AT}) \neq 0$ for $f \in N_1$ and $T \in I_{\delta_1}$; (ii) the solutions of (3.2) through the initial point $(0,x_0,y_0,\zeta_0)$, $(x(t,x_0,y_0,\zeta_0,f)$, $y(t,x_0,y_0,\zeta_0,f)$, $\zeta(t,x_0,y_0,\delta_0,f))$ exist and belong to $B^n(a_0)$ for all $(x_0,y_0,\zeta_0) \in B^n(\varepsilon_1)$, $f \in N_1$ and $t \in [0,2\pi + \delta_1]$. Here I is the $(n-2) \times (n-2)$ identity matrix. Condition (i) can be satisfied for f close to f_0 and δ_1 small because our assumptions on the eigenvalues of A_0 implies that $\det(I-e^{2\pi A_0}) \neq 0$.

We now want to determine $(x_0,y_0,\zeta_0) \in B^n(\varepsilon_1)$, $f \in N_1$ and $T \in I_{\delta_1}$ which satisfy the condition

$$(3.3) \qquad \zeta(T,x_0,y_0,\zeta_0,f) = \zeta.$$

From the third equation in (3.2) it follows that (3.3) is equivalent to the equation $F(x_0,y_0,\zeta_0,T,f) = 0$, where

$$F(x_0,y_0,\zeta_0,T,f) = (I-e^{AT})\zeta_0 - \int_0^T e^{A(T-s)}[W^{(h)}(x(s,x_0,y_0,\zeta_0,f),\ y(s,x_0,y_0,\zeta_0,f),$$

$$\zeta(s,x_0,y_0,\zeta_0,f),f)]ds.$$

Since $W^{(h)}$ is of order ≥ 2 in (x,y,ζ) for each f we have $F(0,0,0,2\pi,f_0) = 0$ and let $D_{\zeta_0}F(0,0,0,2\pi,f_0) = \det(I-e^{2\pi A_0}) \neq 0$. Then, by the implicit function theorem, there exist $\varepsilon \in (0,\varepsilon_1]$, $\delta \in (0,\delta_1]$, $N_0 \subseteq N_1$ and $\sigma \in C[B^2(\varepsilon) \times I_\delta \times N_0,\ R^{n-2}]$, $\sigma(0,0,2\pi,f_0) = 0$ such that

(α) For every $(x_0,y_0,\zeta_0) \in B^n(\varepsilon)$, $T \in I_\delta$, and $f \in N_0$ (3.3) holds if and only if $\zeta_0 = \sigma(x_0,y_0,T,f)$.

(β) $\|\sigma(x_0,y_0,T,f)\| \leq L(x_0^2+y_0^2)$ for some constant $L > 0$ and for all $(x_0,y_0) \in B^2(\varepsilon)$, $T \in I_\delta$, $f \in N_0$.

The function σ is C^∞ in (x_0,y_0) and its derivatives are continuous in all variables x_0,y_0,T,f. For any $T \in I_\delta$ and $f \in N_0$ we have $F(0,0,0,T,f) = 0$ and then $\sigma(0,0,T,f) = 0$. Moreover, $\det D_{\zeta_0}F(0,0,0,T,f) = \det(I-e^{AT}) \neq 0$ (because of (i)) and $D_{x_0}F(0,0,0,T,f) = D_{y_0}F(0,0,0,T,f) = 0$, which implies $D_{x_0}\sigma(0,0,T,f) = D_{y_0}\sigma(0,0,T,f) = 0$. In particular, consider any T-periodic solution of (3.2) lying in $B^n(\varepsilon)$, with $T \in I_\delta$ and $f \in N_0$ and denote its orbit by γ. Since (3.2) is autonomous condition (3.3) is satisfied for any point $(x,y,\zeta) \in \gamma$. Thus, Proposition 1 immediately follows from (α), (β).

The substitution

$$x = r\cos\theta,\ y = r\sin\theta,\ \zeta = rv, \tag{3.4}$$

into (3.2) gives a system which we write as

$$\frac{dr}{d\theta} = R^{(h)}(\theta,r,v,f)$$
$$\frac{dv}{d\theta} = \frac{A}{\beta}v + \eta^{(h)}(\theta,r,v,f), \tag{3.5}$$

where $R^{(h)}$, $\eta^{(h)} \in C^\infty$. The solutions of (3.5) for which $r(\theta) \neq 0$ for all θ are the orbits of corresponding solutions of (3.2). Moreover, the origin is a solution of both (3.2) and (3.5). We denote by $(r(\theta,c,v_0,f),\ v(\theta,c,v_0,f))$ the solution of (3.5) passing through $(0,c,v_0)$. When the solutions $(r(\theta),v(\theta))$ of (3.5) are known, the corresponding solutions of (3.2) can be completely determined by solving the equation

$$\frac{d\theta}{dt} = \Theta(\theta,r(\theta),v(\theta),f),$$

where Θ is greater than some positive number in a neighborhood of the origin in R^{n-1} and for f close to f_0. Every 2π-periodic solution of (3.5), $(r(\theta),v(\theta))$ represents a periodic orbit of (3.2) whose period T is given by

$$T = \int_0^{2\pi} \frac{d\theta}{\Theta(\theta,r(\theta),v(\theta),f)} \tag{3.6}$$

For any $\varepsilon > 0$ denote by $P(\varepsilon) = \{(r,v) \in R^{n-1},\ r \geq 0,\ 2^2 + \|v\|^2 < \varepsilon^2\}$.

We now introduce for system (3.5) property (a') which corresponds to property (a) for system (3.2).

(a')(i) There exists a neighborhood N^* of f_0 and an $\varepsilon_1 > 0$ such that for every $f \in N^*$ there are at most k nontrivial 2π-periodic orbits of (3.5) lying in $P(\varepsilon_1)$.

(ii) For each integer j, $0 \leq j \leq k$, for each neighborhood N of f_0, $N \subseteq N^*$, and for each $\varepsilon_2 > 0$ there exists $f \in N$ such that (3.5) has exactly j nontrivial 2π-periodic solutions lying in $P(\varepsilon_2)$.

(iii) for any $\bar{\varepsilon} \in (0,\varepsilon_1)$ there exists a neighborhood $N_{\bar{\varepsilon}}$ of f_0, $N_{\bar{\varepsilon}} \subseteq N^*$ such that if $f \in N_{\bar{\varepsilon}}$ and if γ is a 2π-periodic solution of (3.5) lying in $P(\varepsilon_1)$ then γ lies in $P(\bar{\varepsilon})$.

The solutions of (3.5) are the representation in polar coordinates of the orbits of the solutions of (3.2). It is not clear apriori that (a') implies (a) because these properties involve neighborhoods of the origin in (r,v) space and of the origin in (x,y,ζ) space respectively while the substitution (3.4) is singular at $r = 0$. Nonetheless, we can prove the following proposition.

<u>Proposition 2</u>. Property (a') implies (a).

<u>Proof</u>. Without loss of generality we may assume $\varepsilon_1 < 1$. Define the mapping ψ: $(x,y,\zeta) \to (r,v)$, $(x,y) \neq (0,0)$, given by the substitution (3.4), that is $r = \sqrt{x^2 + y^2}$ and $v = \zeta/\sqrt{x^2 + y^2}$. Then $\psi^{-1}(P(\varepsilon_1)) \subset B^n(\varepsilon_1)$ since $r^2 + \|v\|^2 < \varepsilon_1^2$ implies $r^2 + \|\zeta\|^2 = r^2 + r^2\|v\|^2 \leq r^2 + \|v\|^2 < \varepsilon_1^2$. Each 2π-periodic solution of (3.5) lying in $P(\varepsilon_1)$ corresponds in polar coordinates

to the orbit of a periodic solution of (3.2) lying in $B^n(\varepsilon_1)$ whose period is included in some interval I_{δ_1}. Let ε, δ, L, and N_0 be the constants and neighborhood of f_0 defined in Proposition 1 and assume $\varepsilon_1 < \varepsilon$. In view of (3.6) and the observation $\Theta(\theta,0,0,f_0) \equiv 1$ we may choose δ_1 and N^* such that $\delta_1 < \delta$ and $N^* \subset N_0$. Let $a_1 = \varepsilon_1/\sqrt{1+L^2}$. If $f \subseteq N^*$ and γ is any periodic orbit of (3.2) lying in $B^n(a_1)$ whose period is in I_{δ_1} then by Proposition 1 $\psi(\gamma) \subset P(\varepsilon_1)$. Indeed, if $(x,y,\zeta) \subset \gamma$ and $r = \sqrt{x^2+y^2}$, then $(r,v) = \psi(x,y,\zeta)$ satisfies $\|v\|^2 + r^2 = \frac{\|\zeta\|^2}{r^2} + r^2 < L^2r^2 + r^2 = r^2(1+L^2) < a_1{}^2(1+L^2) < \varepsilon_1{}^2$. Thus, property (a)(i) follows from (a')(i).

<u>Proof of (a)(ii)</u>. Pick any $\bar{a}_2 < a_2 < \varepsilon_1/\sqrt{1+L^2}$ and $\tilde{N} \subset N^*$ such that for every $f \in \tilde{N}$ we have: (1) the periodic orbits of (3.2) lying in $B^n(\bar{a}_2)$, which correspond to the 2π-periodic orbits of (3.5) lying in $P(\bar{a}_2)$, have period in I_{δ_2}. This follows by using (3.6) and the fact that $\Theta(\theta,0,0,f_0) \equiv 1$. (2) $f \in \tilde{N}$ implies that all the 2π-periodic orbits of (3.5) lying in $P(\varepsilon_1)$ lie in $P(\bar{a}_2)$. This can be done in view of (a')(iii). In view of (a')(ii) pick $f \in \tilde{N}$ such that (3.5) has exactly j 2π-periodic orbits lying in $P(\bar{a}_2)$. If γ is any periodic orbit of (3.2) lying in $B^n(a_2)$ whose period is in I_{δ_2} then $\psi(\gamma) \subseteq P(\varepsilon_1)$. In view of (2) $\psi(\gamma) \subseteq P(\bar{a}_2)$. This completes the proof of (a)(ii).

<u>Proof of (a)(iii)</u>. Assume $\bar{a}_1 < \bar{a} < \varepsilon_1/\sqrt{1+L^2}$ and $\bar{N} \subset N^*$ such that conditions (1) and (2) above hold with $\tilde{N}$ replaced by $\bar{N}$ and $\delta_2, \bar{a}_2$, and a_2 replaced by $\bar{\delta}, \bar{a}_1$, and $\bar{a}$ respectively. Then, if γ is a periodic orbit of (3.2) lying in $B^n(a_1)$ with period in I_{δ_1}, then $\psi(\gamma) \subseteq P(\varepsilon_1)$ which implies $\psi(\gamma) \subseteq P(\bar{a}_1)$. Then, in view of (1) and (2),

$v \subseteq B^n(\bar{a}_1) \subseteq B^n(\bar{\varepsilon})$ with period in I_δ. This completes the proof of Proposition 2.

A solution $(r(\theta), v(\theta))$ of (3.5) that exists on $[0,2\pi]$ will be called a $(2\pi, v)$ solution if $v(2\pi) = v(0)$. Every 2π-periodic solution is obviously a $(2\pi, v)$ solutions; thus, in order to find the 2π-periodic sol[illegible] only need to inspect the set of $(2\pi, v)$ solutions. This leads us to [illegible] the following proposition.

Proposition 3. There exists an $\varepsilon > 0$ and a neighborhood N of f_0, and a function $\tau \in C[[0,\varepsilon) \times N, R^{n-2}]$, $\tau(\cdot, f) \in C^\infty$, $\tau(0,f) = 0$ for $f \in N$, [illegible] for every $(c, v_0) \in P(\varepsilon)$ and $f \in N$ the solution of (3.5) through $(0, c, [illegible]$ is a $(2\pi, v)$ solution if and only if $v_0 = \tau(c, f)$.

Proof. As in the proof of Proposition 1 we choose $\varepsilon_1 > 0$ and a [illegible] N_1 of f_0 such that (i) $\det(I - e^{2\pi A}) \neq 0$ for $f \in N_1$; and (ii) the [illegible]tions $(r(\theta, c, v_0, f), v(\theta, c, v_0, f))$ of (3.5) exist and belong to $P(a_0)$ [illegible] all $(c, v_0) \in P(\varepsilon_1)$, $f \in N_1$, and $\theta \in [0,2\pi]$. From the second equation [illegible] (3.5) it follows that the condition $v(2\pi, c, v_0, f) = v_0$ is equivalent [illegible] requirement that $F(c, v_0, f) = 0$, where

$$F(c, v_0, f) = (I - e^{2\pi A})v_0 - \int_0^{2\pi} e^{A(2\pi - s)}[n^{(h)}(s, r(s, c, v_0, f), v(s, c, v_0, f), f) [illegible]$$

We recognize that $F(0,0,f) = 0$. Now we prove $D_{v_0} F(0,0,f_0) = \det$ [illegible]

To see this, observe that since $n^{(h)}$ is of order ≥ 2 in v we only [illegible] to prove that $\frac{\partial r(\theta, 0, 0, f_0)}{\partial v_0} \equiv 0$ for $\theta \in [0,2\pi]$ (recall that $n^{(h)}$ may [illegible] linear terms in r). Since $R^{(h)}(\theta, 0, v, f_0) \equiv 0$ then for any v_0 [illegible] $(r(\theta, 0, v_0, f_0), v(\theta, 0, v_0, f_0))$ of (3.5) satisfies $r(\theta, 0, v_0, f_0) \equiv 0$ for $\theta \in [0,2\pi]$. Thus, $\frac{\partial r}{\partial v_0}(\theta, 0, 0, f_0) \equiv 0$ and consequently, $\det D_{v_0} F(0, [illegible]$

$\det(I-e^{2\pi A_0}) \neq 0$. Therefore, the conclusion of Proposition 3 follows from the implicit function theorem and the fact that $F(0,0,f) = 0$.

Denote by $(r(\theta,c,f),\ v(\theta,c,f))$ the $(2\pi,v)$ solution of (3.5) passing through $(0,c,\tau(c,f))$. Because of Proposition 3 we can write

$$
\begin{aligned}
r(\theta,c,f) &= u_1(\theta,f)c + \dots + u_h(\theta,f)c^h + o(c^h) \\
v(\theta,c,f) &= \nu_1(\theta,f)c + \dots + \nu_{h-1}(\theta,f)c^{h-1} + o(c^{h-1}),
\end{aligned}
\tag{3.7}
$$

where $u_1(0,f) = 1$, $u_i(0,f) = 0$ for $i > 1$ and

$$\nu_i(0,f) = \nu_i(2\pi,f) \quad \text{for} \quad i \geq 1. \tag{3.8}$$

Consider now the displacement function relative to the $(2\pi,v)$ solutions which is defined in a right interval of $c = 0$ and in a neighborhood of f_0

$$V(c,f) = r(2\pi,c,f) - c. \tag{3.9}$$

Then the 2π-periodic solutions of (3.5) correspond to the zeros of $V(c,f)$.

We now prove the following result.

Proposition 4. Assume the origin of (S_h) is either h-asymptotically stable or h-completely unstable. Then h is odd and

$$\frac{\partial^i V}{\partial c^i}(0,f_0) = 0,\ i = 1\ldots h-1 \quad \text{and} \quad \frac{\partial^h V}{\partial c^h}(0,f_0) \neq 0. \tag{3.10}$$

Proof. Substitute (3.7) into the second equation in (3.5) for $f = f_0$ and equate coefficients of c. Since $\eta^{(h)}(\theta,r,0,f_0)$ is of order $> h-1$ in r we find that $\nu_i(\theta,f_0) \equiv 0$ for $i = 1, \dots h-1$. Indeed,

$$\frac{\partial \nu_1}{\partial \theta}(\theta,f_0) = A_0\nu_1,$$

implying $\nu_1(\theta,f_0) = \nu_1(0,f_0)e^{A_0\theta}$.

Condition (3.8) and the fact that $\det(I-e^{2\pi A_0}) \neq 0$ implies that $\nu_1(o,f_0) = 0$ and consequently, $\nu_1(\theta,f_0) \equiv 0$. Since $\nu_1(\theta,f_0) \equiv 0$ we then have

$$\frac{\partial \nu_2}{\partial \theta}(\theta,f_0) = A_0 \nu_2$$

and as before, $\nu_2(\theta,f_0) \equiv 0$. Continuing in this manner we obtain $\nu_i(\theta,f_0) \equiv 0$, $i = 1 \ldots h-1$. Thus, in order to compute the functions $u_i(\theta,f_0)$ we may put $v = 0$ into the first equation in (3.5) for $f = f_0$. We then obtain the equation

$$\frac{dr}{d\theta} = R^{(h)}(\theta,r,0,f_0),$$

which is precisely the equation in polar coordinates of the orbits of (S_h). Since $x = y = 0$ is either h-asymptotically stable or h-completely unstable for (S_h) we have that h is odd and

$$u_1(\theta,f_0) \equiv 1,\ u_i(2\pi,f_0) = 0,\ i = 2 \ldots h-1,\ u_h(2\pi,f_0) \neq 0,$$

(see [9] for more details), thus implying (3.10) holds.

Finally, we have the following result concerning the roots of $V(c,f)$ for f close to f_0.

Proposition 5. Assume the origin of (S_n) is either h-asymptotically stable or h-completely unstable. Then there exist $\bar{c} > 0$ and a neighborhood $\bar{N}$ of f_0 such that: (1) $V(c,f)$ is defined for any $c \in [0,\bar{c}]$ and $f \in \bar{N}$; (2) for every $c_1 \in (0,\bar{c})$ there exists a neighborhood $N_{c_1} \subseteq \bar{N}$ such that for $f \in N_{c_1}$ all roots of $V(c,f)$ lying in $[0,\bar{c}]$ lie in $[0,c_1)$.

The proof of Proposition 5 utilizes Proposition 4 in order to show that for $\bar{c} > 0$ sufficiently small and $c \in [0,\bar{c}]$ we have $|V(c,f_0)| \geq \mu c^h$ for an appropriate constant $\mu > 0$. The continuity of $V(c,f)$ in c and $f = f_0$ allows us to conclude that for every $c_1 \in [0,\bar{c}]$ there exists $N_{c_1} \subset N$ such that for $f \in N_{c_1}$, $|V(c,f)| \geq \frac{1}{2}\mu c_1^h$ for $c \in [c_1,\bar{c}]$. We leave the details to the reader.

4. PROOF OF THEOREM 1

If in the transformation (3.4) we assume $r < 0$ instead of $r > 0$ we obtain a new displacement function $\tilde{V}(c,f)$ defined for $c < 0$. We will extend the domain of V by setting $V(c,f) = \tilde{V}(c,f)$ for $c < 0$. It is easy to recognize that this extended function is continuous and for fixed f is C^∞ in c. In addition, we observe that for any 2π-periodic solution of (3.5) there exist $c_1 > 0$, $c_2 < 0$ such that $V(c_1,f) = V(c_2,f) = 0$.

Assume Case I. We shall prove property (a') holds and in view of Propositio- 3 the proof of the first part of Theorem 1 will be complete.

Proof of property (a')(i). Since the origin is a solution of (3.5) for any f, an application of Rolle's Theorem, in view of (3.10), implies that there exists an $\varepsilon_1 > 0$ and a neighborhood N^* of f_0 such that for any $f \in N^*$, $V(c,f)$ has at most $h-1$ nonzero roots counting multiplicity in $[-\varepsilon_1,\varepsilon_1]$. For each positive root of $V(c,f)$ there is a negative root of $V(c,f)$ corresponding to the same periodic orbit. Thus, there are at most $k = \frac{h-1}{2}$ nontrivial 2π-periodic solutions of (3.5) lying in $P(\varepsilon_1)$ for $f \in N^*$. Thus, property (a')(i) holds.

Proof of property (a')(ii). We essentially adapt to our problem a proceedure used in [1]. Suppose the origin of S_h is h-asymptotically stable (the case in which the origin of (S_h) is h-completely unstable has a similar proof).

Consider a perturbed system of (3.1) of the form

$$\text{(4.1)}\qquad \begin{aligned} \dot{x} &= -y + X_0^{(h)}(x,y,\zeta) + \sum_{i=1}^{j} a_i x(x^2+y^2)^{k-i} \\ \dot{y} &= x + Y_0^{(h)}(x,y,\zeta) + \sum_{i=1}^{j} a_i y(x^2+y^2)^{k-i} \\ \dot{\zeta} &= A_0\zeta + W_0^{(h)}(x,y,\zeta), \end{aligned}$$

where $k = \frac{h-1}{2}$, j is any integer, $1 \leq j \leq k$, and a_i are constants to be determined (the case $j = 0$ follows by letting $f = f_0$). We will denote by $V(c,a_1,\ldots,a_j)$ the displacement function relative to the $(2\pi,v)$ solutions of (3.5) which correspond to (4.1). We will denote by $S(a_1,\ldots,a_j)$ the first two equations in (4.1) for $\zeta = 0$. Since the origin is h-asymptotically stable for $S(0,\ldots,0)$ then from Proposition 4

$$V(c,0,0,\ldots,0) = g_0 c^h + o(c^h),\ g_0 < 0.$$

Thus, for $c_0 > 0$ and sufficiently small we have $V(c_0,0,\ldots,0) < 0$. There exists an $\eta_i > 0$ such that $V(c_0,a_1,\ldots,a_j) < 0$ for $|a_i| < \eta_i$, $i = 1,\ldots,j$. Fix now a_1, $0 < a_1 < \eta_1$. Then

$$V(c,a_1,0,\ldots,0) = g_1 c^{h-2} + o(c^{h-2})$$

where $g_1 = 2\pi a_1 > 0$. This can be recognized by replacing the expression for r in (3.7) into $S(a_1,0,\ldots,0)$ and taking into account that $u_i(2\pi,f) = 0$ $i = 1,\ldots,h-3$ (see [9] for more details). There exists c_1, $0 < c_1 < c_0$ such that $V(c_1,a_1,0,\ldots,0) > 0$ and thus, we can find $\eta_2 > 0$, $\eta_2 < \eta_1$, such that for $|a_i| < \eta_2$, $i = 2,\ldots,j$, $V(c_1,a_1,a_2,\ldots,a_j) > 0$. Fix now $a_2 < 0$, $|a_2| < \eta_2$. Then

$$V(c,a_1,a_2,0,\ldots,0) = g_2 c^{h-3} + O(c^{h-3})$$

where $g_2 = 2\pi a_2 < 0$. Then there exists c_2, $0 < c_2 < c_1$ such that $V(c_2,a_1,a_2,0,\ldots,0) < 0$ and thus, we can find $\eta_3 > 0$, $\eta_3 < \eta_2$ such that for $|a_i| < \eta_3$, $i = 3,\ldots,j$, $V(c_2,a_1,\ldots,a_j) < 0$. Continuing this process we can find a set of numbers $\bar{c}_1,\bar{c}_2,\ldots,\bar{c}_j$, $c_i < \bar{c}_i < c_{i-1}$ $i = 1,\ldots,j$ (and thus $0 < \bar{c}_{i+1} < \bar{c}_i$ $i = 1,\ldots,j-1$) such that $V(\bar{c}_i,a_1,\ldots,a_j) = 0$, $i = 1,\ldots,j$. Since $c = 0$ is a root of $V(c,a_1,\ldots,a_j)$ of order $h - 2j$ (recall $u_i(2\pi,f) = 0$ $i = 1,\ldots,h-2j-1$) and for each positive root of $V(c,a_1,\ldots,a_j)$ we have a negative root then we immediately have that the $\bar{c}_i$ $i = 1,\ldots,j$ are the only positive roots of $V(c,a_1,\ldots,a_j)$. Moreover, we can obtain that the $\bar{c}_i$ can be made close to $c = 0$ by picking c_0 sufficiently small. This completes the proof of (a')(ii).

The proof of (a')(iii) is an immediate consequence of Proposition 5. Thus, property (a') holds and so the first part of Theorem 1 is proved.

We now show Case II follows from Case I. For any positive integer j we assume in (3.1) $h = 2j + 1$ and consider perturbed systems of the form

$$\begin{aligned} \dot{x} &= -y + X_0^{(h)}(x,y,\zeta) + bx(x^2+y^2)^{\frac{h-1}{2}} \\ \dot{y} &= x + Y_0^{(h)}(x,y,\zeta) + by(x^2+y^2)^{\frac{h-1}{2}} \qquad (4.2) \\ \dot{\zeta} &= A_0\zeta + W_0^{(h)}(x,y,\zeta), \end{aligned}$$

where b is a constant. We then have that for the corresponding reduced system (S_h) the origin is either h-asymptotically stable or h-completely unstable if $b < 0$ or $b > 0$ respectively. Thus, we have reduced the problem to Case I.

Since j and b are arbitrary, property (A) holds, thus concluding the proof of Theorem 1.

5. PROOF OF COROLLARIES.

The proof of Corollary 1 follows from (2.7) and Theorem 1 by observing that there is an equivalence between the h-asymptotic stability (h-complete instability) of the origin of (3.1) on H_h and the h-asymptotic stability (h-complete instability) of the origin of (S_h).

Proof of Corollary 2.

We observe that the origin of the system (3.1) is h-asymptotically stable in the whole if and only if the origin of (S_h) is h-asymptotically stable. Indeed, if the origin of (S_h) is h-asymptotically stable, there exists a constant $G < 0$ and a polynomial

$$F(x,y,\zeta) = x^2 + y^2 + F_3(x,y,\zeta) + \ldots, F_h(s,y,\zeta)$$

(F_i is a homogeneous polynomial of degree i) such that the derivative of F along the solutions of (3.1) can be written as

$$\dot{F}(x,y,\zeta) = G(x^2+y^2)^{\frac{h+1}{2}} + o(x^2+y^2+\|\zeta\|^2)^{\frac{h+1}{2}}$$

(note that $\dot{F}(x,y,0)$ differs from the derivative of F along the solutions of (S_h) by terms of order greater than $h + 2$). Moreover, there exists a quadratic form in ζ, $Q(\zeta)$, such that along the solutions of system (3.1)

$$\dot{Q}(x,y,\zeta) = -\|\zeta\|^2 + \chi_1(x,y,\zeta) + \chi_2(x,y,\zeta)$$

where χ_1 is of order ≥ 3 and of order ≥ 2 in ζ; and χ_2 is of order $> h + 1$ (see Lyapunov [7] for a detailed analysis of the above statements). Setting $\eta = F + Q$ we have that along the solutions of system (3.1), for x, y, ζ small,

$$\dot{\eta}(x,y,\zeta) = -\|\zeta\|^2 + G(x^2+y^2)^{\frac{h+1}{2}} + \chi_1(x,y,\zeta) + \sigma(x,y,\zeta)$$

$$= \frac{-\|\zeta\|^2}{2} + G(x^2+y^2)^{\frac{h+1}{2}} + \sigma(x,y,\zeta),$$

where σ is of order $> h + 1$ and includes $\chi_2(x,y,\zeta)$. Then η is positive definite and its derivative along solutions of (3.1) is negative definite. This property holds if we perturb (2.1) with terms of order greater than h. On the other hand, this property will not hold by appropriately choosing perturbations of order $\leq h$. Thus, the origin of (2.1) is h-asymptotically stable.

Using similar arguments we can show that if the origin is h-asymptotically stable in the whole then the origin is h-asymptotically stable on H_h.

Analogously, we can prove the origin is h-unstable in the whole if and only if the origin of (S_h) is h-completely unstable. In view of Corollary 1 this completes the proof of Corollary 2.

BIBLIOGRAPHY

[1] Andronov, A., Leontovich, E., Gordon, I., and Maier, A., Theory of bifurcations of dynamical systems in the plane, Israel Program of Scientific Translations, Jerusalem, (1971).

[2] Bernfeld, S. and Salvadori, L., Generalized Hopf bifurcation and h-asymptotic stability, J. Nonlinear Analysis, T.M.A. 4(1980), 1091-1107.

[3] Bernfeld, S., Negrini, P. & Salvadori, L., Stability and generalized Hopf bifurcation through a reduction principle, Proc. of the Conf. "Nonlinear Differential Equations: Invariance, Stability, and Bifurcation" Trento, May 25-30, 1980, Academic Press (to appear).

[4] Bibikov, Y., Local theory of nonlinear analytic ordinary differential equations, Lecture Notes in Mathematics 702, Springer-Verlag, New York (1979).

[5] Chafee, N., Generalized Hopf bifurcation and perturbation in a full neighborhood of a given vector field, Indiana Unir. Math. J. 27(1978), 173-194.

[6] Hale, J. K., Topics in dynamic bifurcation theory, NSF-CBMS Conference, Arlington, Texas, June 16-20, 1980.

[7] Lyapunov, A. M., Probleme général de la stabilité du mouvement, Ann. of Math Studies, 17, Princeton Univ. Press, Princeton, New Jersey, (1947).

[8] Marsden, G. and McCracken, M., The Hopf bifurcation and its applications, Notes in Applied Math. Sci. 19, Springer Verlag, New York, 1976.

[9] Negrini, P. and Salvadori, L., Attractivity and Hopf bifurcation, J. Nonlinear Anal. T.M.A. 3 (1979), 87-100.

[10] Sansone, G. and Conti, R., Nonlinear differential equations, MacMillan, New York, (1964).