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Quasi-linear evolution of tearing modes

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Résumé. — On étudie la croissance d'une instabilité tearing dans le régime non linéaire de Rutherford. L'usage des techniques de perturbations singulières nous permet de retrouver, à l'ordre le plus bas, les résultats de Rutherford. A l'ordre suivant nous montrons que le mode engendre une déformation quasi linéaire du profil d'équilibre du flux magnétique ; la diffusion résistive de cette déformation ralentit la croissance du mode et indique la possibilité d'une saturation de l'instabilité.

Abstract. — The growth of a tearing instability in Rutherford's non-linear regime is investigated. Using a singular perturbations technique, we recover to lowest order Rutherford's result. To the following order we show that the mode generates a quasi-linear deformation of the equilibrium flux profile, whose resistive diffusion slows down the growth and shows the possibility of a saturation of the instability.

1. Tearing instabilities [1] have been shown to play an important rôle in many areas of plasma physics, including magnetic fusion experiments, solar flares and magnetospheric activity. A decisive step in the understanding of their non-linear behaviour was accomplished by Rutherford [2]. He showed that when the width of a magnetic island, due to a high-m ($m \geqslant 2$) tearing mode, exceeds that of the resistive singular layer, the non-linear $\mathbf{j} \times \mathbf{B}$ force becomes more important than plasma inertia to oppose the fluid motion. Then there results a regime where the island size grows linearly — rather than exponentially — with time. This result has been extended to a model with anomalous electron viscosity, modelling the braiding of magnetic field lines in the vicinity of the separatrix [3].

This however does not explain the saturation of the instability, observed both in experiments [4] and in numerical simulations [5, 6]. Saturation was found in a semi-analytical calculation by White *et al.* [5]. They observed that in numerical simulations the perturbed current behaves linearly with the total magnetic flux, inside the magnetic island. Then the ansatz that $J = a + b\psi$, where a and b are constants to be determined by matching with the outer solution, allows them to find the island's growth rate as :

$$\frac{\partial x_T}{\partial t} = A \, \Delta'(x_T) \, [1 - \alpha x_T]$$

where A is a constant, x_T is the island width, and α is due to the cylindrical geometry and to the resulting asymmetry of the island. $\Delta'(x_T)$ is the slope-jump of the outer solution, found numerically at the separatrix. Saturation results when the island is wide enough to cancel the slope-jump, which measures the available magnetic energy.

However this work raises many questions, mainly about the validity of the original ansatz and of the matching method. The authors, as usual in tearing-mode theory [1, 2], define an « outer » solution and an « inner » solution of the MHD equations. Then they match these solutions and their first derivatives at the separatrix of the magnetic island. But this leads them to a confusion between the solution inside the magnetic island, and the « inner » solution of a boundary-layer problem. To be more specific, consider Rutherford's work : its main point is the existence of a non-linear current, which dominates in the region of the magnetic island : this current exists both inside and outside the island, and extends to a few island widths from the resonant surface. Using a boundary-layer technique, Rutherford correctly matches his inner and outer solutions at a distance larger than the island width, but smaller than the plasma radius. On the other hand, White *et al.* extend the « outer » solution (obtained from ideal-MHD equations) to the separatrix, and use the « inner solution » (with resistive effects included approximately) only inside the island : thus their incorrect use of boundary-layer theory leads them to neglect all resistive effects, including Rutherford's current, outside of the island. As will be seen in this paper, these neglected effects turn out to be the main cause of the decrease of the growth rate of the instability. The definition in reference [5] of the slope-jump $\Delta'(x_T)$ is thus invalid, because it is not part of a consistent boundary-layer technique.

We address this problem in a more systematic manner, using Van Dyke's formulation [7] of singular perturbation theory (which is an extension of usual boundary-layer theory). This technique has already been used [8] to regularize Rutherford's solution, which was singular as the separatrix due to the neglect of inertia and viscosity.

The first step is to define a small parameter which is the ratio between the scale-length of the singular layer (in our case the island width x_T), and the scale-length of the outer solution (in our case this scale is given by the inverse of the slope-jump Δ'). Thus our small parameter is $\varepsilon = \Delta' x_T \sim x_T \left(\frac{1}{\psi_1} \frac{\partial \psi_1}{\partial x} \right)$. It was noted by Drake *et al.* [9, 10] that ε is also the ratio τ_R^i/τ_c, where $\tau_R^i = x_T^2/\eta$ is the skin time through the island and $\tau_c = \left(\frac{1}{\psi_1} \frac{\partial \psi_1}{\partial t} \right)^{-1}$ is the growth time of the island [10] : then the usual « constant-ψ » approximation is equivalent to $\varepsilon = 0$. The present work carries the calculation to first order in ε, thus taking into account a weak departure from constant ψ.

Its main result is that, as the tearing instability grows in Rutherford's regime, it generates a quasi-linear deformation $\delta\psi_0$ of the equilibrium magnetic flux ψ_0. Because $\varepsilon = \frac{\tau_R^i}{\tau_c}$ is small, $\delta\psi_0$ can diffuse resistively far from the island $\left(\text{in a time } \tau_c, \text{ it diffuses to a distance } \delta x \sim \frac{x_T}{\varepsilon^{1/2}} \right)$. Then it substantially alters the outer solution, decreasing the « outer » slope-jump. At the same time, the growth of the island self-consistently modifies its shape (through $\delta\psi_0$ and the radial dependence of ψ_1), further reducing the growth-rate of the instability.

Section 2 of this paper outlines the main steps of the determination of the inner and outer solutions for the perturbed magnetic flux. Section 3 gives the final results and the conclusion. Detailed calculations are given in separate appendices.

2. We use the same model as that of Rutherford [2], considering a plane plasma slab with the main magnetic field in the z direction, the periodicity in the y direction, and the equilibrium gradients in the x direction. The plasma is assumed to have a constant resistivity and is submitted to an externally-applied electric field $E_0(x)\,\mathbf{e}_z$, resulting in an equilibrium current $j_0(x) = E_0(x)/\eta$. The magnetic field and velocity are represented by the flux ψ and stream function u with :

$$\mathbf{B} = B_z\,\mathbf{e}_z + \mathbf{e}_z \times \nabla\psi$$

$$\mathbf{V} = \mathbf{e}_z \times \nabla u\,.$$

In the small plasma pressure limit, neglecting inertia and viscosity (as was done by Rutherford) the equations of resistive MHD are written as :

$$\frac{\partial \psi}{\partial t} - \mathbf{B}.\nabla u = \eta(j_z - j_0) \tag{1a}$$

$$j_z = j_z(\psi) \tag{1b}$$

$$\Delta\psi = j_z\,. \tag{1c}$$

At equilibrium we have $\psi = \psi_0(x)$, $B_y = B_0(x) = \frac{\partial \psi_0}{\partial x}$, and in the vicinity of the resonant surface (where $B_0 = 0$) we approximate : $\psi_0(x) \approx B_y' \frac{x^2}{2}$.

Rutherford has also shown that when a high-m ($m \geqslant 2$) tearing mode is growing, its harmonics can be neglected, provided they are sufficiently damped. Accordingly we assume a perturbation given by :

$$\psi = \psi_0(x) - \tilde{\psi}(x, y, t)$$

$$\tilde{\psi}(x, y, t) = \tilde{\psi}_1(x, t) \cos ky + \delta\tilde{\psi}_0(x, t)$$

$\delta\tilde{\psi}_0$ is the quasi-linear perturbation of the equilibrium flux due to the growth of the unstable mode ψ_1.

Averaging equation (1a) over field lines, we get :

$$\tilde{j} = \frac{\left\langle \frac{1}{\eta} \frac{\partial \tilde{\psi}}{\partial t} \Big/ \frac{\partial \psi}{\partial x} \right\rangle}{\left\langle 1 \Big/ \frac{\partial \psi}{\partial x} \right\rangle} + \left[j_0(x) - \frac{\left\langle j_0 \Big/ \frac{\partial \psi}{\partial x} \right\rangle}{\left\langle 1 \Big/ \frac{\partial \psi}{\partial x} \right\rangle} \right] \tag{2}$$

where the brackets mean the average over y at constant ψ. Following Rutherford we solve equation (2) separately in the island region ($x \sim x_T = 2(\tilde{\psi}_1/B_y')^{1/2}$) and outside of it ($x \gg x_T$).

2.1 Inner solution. — In the inner region (not to be confused with the inside of the island, $\psi < \tilde{\psi}_1$) we use Rutherford's result giving $\frac{\partial x_T}{\partial t} \sim \eta\,\Delta'$ to order the different terms in equation (2). As pointed out by Drake *et al.* [9] $\varepsilon = 0$ corresponds to the familiar « constant-ψ » approximation. Assuming $\varepsilon = \Delta' x_T$ small but finite allows us to relax this approximation and to treat explicity the weak variation of $\tilde{\psi}$ in the island region. We write it as :

$$\tilde{\psi}(x, y, t) = \tilde{\psi}_1(x, t) \cos ky + \delta\tilde{\psi}_0(x, t) \tag{3a}$$

$$\tilde{\psi}_1(x, t) = \psi_i(t)\,[1 + \Delta' sx + \varepsilon h_0(x) + \varepsilon^2 h_1(x) + \cdots] \tag{3b}$$

$$\delta\tilde{\psi}_0(x, t) = \varepsilon\delta\psi_0(t)\,[1 + g_0(x) + \varepsilon g_1(x) + \cdots] \tag{3c}$$

where ψ_i and $\delta\psi_0$ are only functions of time, and the radial dependences of $\tilde{\psi}_1$ and $\delta\tilde{\psi}_0$ have been expanded in ε. h and g will appear as integrals of Rutherford's current, which is even in x. Then we require h and g to be even, and to be zero at the resonant surface. s in an integration constant, implying an asymmetry of the island due to that of the outer solution. The fact that $\delta\tilde{\psi}_0$ is of order ε will be justified later.

We write equation (2) as :

$$\Delta\tilde{\psi} = \frac{\left\langle \frac{1}{\eta}\frac{\partial\tilde{\psi}}{\partial t}\Big/\frac{\partial\psi}{\partial x} \right\rangle}{\left\langle 1\Big/\frac{\partial\psi}{\partial x}\right\rangle} + \left[j_0(x) - \frac{\left\langle j_0\Big/\frac{\partial\psi}{\partial x}\right\rangle}{\left\langle 1\Big/\frac{\partial\psi}{\partial x}\right\rangle}\right]. \quad (4)$$

In the vicinity of the island the Laplacian reduces to $\partial^2/\partial x^2$. The term in square brackets (linearized current) writes to lowest order $j_0'\left[x - 1\Big/\left\langle \frac{1}{x}\right\rangle\right]$. With $j_0' \sim B_y'/a$ and $\frac{1}{\eta}\frac{\partial\psi_i}{\partial t} \sim \Delta' B_y' x_T$ the linearized current is thus of order $(a\Delta')^{-1}$ compared to the first term (Rutherford's current) in the right hand side of equation (4). Then we must require $(a\Delta')$ to be large to neglect this term, as was done by Rutherford. Writing :

$$\frac{\partial\psi}{\partial x} = B_y' x^*[1 + \varepsilon M(x, y, t)] + 0(\varepsilon^2) \quad (5a)$$

$$x^{*2} = \frac{2}{B_y'}[\psi + \psi_i(t)\cos Y + \varepsilon\delta\psi_0(t)] \quad (5b)$$

$$M(x, y, t) = \frac{1}{B_y' x^{*2}} \times$$

$$\times\ [(h_0 - xh_0')\,\psi_i(t)\cos Y + (g_0 - xg_0')\,\delta\psi_0] \quad (5c)$$

$$\frac{1}{\eta}\frac{\partial}{\partial t} \sim \frac{\varepsilon}{x_T^2}, \quad Y = ky$$

where primes denote derivatives with respect to x, and solving equation (4) order by order in ε, we get :

$$\varepsilon\frac{\partial^2}{\partial x^2}[\psi_i h_0 \cos Y + g_0\,\delta\psi_0] = \frac{1}{\eta}\frac{\partial\psi_i}{\partial t}\frac{\langle \cos Y/x^*\rangle}{\langle 1/x^*\rangle} \quad (6a)$$

$$\varepsilon^2\frac{\partial^2}{\partial x^2}[\psi_i h_1\cos Y + g_1\,\delta\psi_0] = \varepsilon\frac{\left\langle \frac{1}{\eta}\frac{\partial}{\partial t}(\psi_i h_0\cos Y + g_0\,\delta\psi_0)/x^*\right\rangle}{\langle 1/x^*\rangle} +$$

$$+\ \varepsilon\frac{1}{\eta}\frac{\partial\psi_i}{\partial t}\frac{1}{\langle 1/x^*\rangle^2}\left\{\left\langle\frac{M}{x^*}\right\rangle\left\langle\frac{\cos Y}{x^*}\right\rangle - \left\langle\frac{M\cos Y}{x^*}\right\rangle\left\langle\frac{1}{x^*}\right\rangle\right\}. \quad (6b)$$

Equation (6a) is just that solved by Rutherford. M is the first order modification of the metrics, due to the departure from constant ψ_1. It must be noted again that s represents the asymmetry of the island due to that of the outer solution, coming in practical cases from the cylindrical geometry. It figures only to second order in the metrics, and thus must be neglected in our calculation. This constitutes an important difference with the results of reference [5] where, due to the incorrect ordering, the asymmetry of the island was found to play an important rôle.

Equations (5) and (6) are solved for h_0, g_0 and h_1. Detailed calculations and solutions are given in appendix A. Expanding the solutions for $\tilde{X} = \frac{x}{x_T} \gg 1$ (where they will be matched to the outer ones) we obtain :

$$\psi_i(\varepsilon h_0 + \varepsilon^2 h_1) = \varepsilon a_0 + (\varepsilon a_1 + \varepsilon^2 a_2)\,x_T\,|\tilde{X}| \quad (7a)$$

$$\varepsilon\delta\psi_0\,g_0 = \varepsilon a_3 \,\mathrm{Ln}\,|\tilde{X}| \quad (7b)$$

where

$$\varepsilon a_0 = \frac{-2\psi_i}{\pi B_y'}\frac{1}{\eta}\frac{\partial\psi_i}{\partial t}c_0$$

$$\varepsilon a_1 = \frac{1}{\pi}\left(\frac{2\psi_i}{B_y'}\right)^{1/2}\frac{1}{\eta}\frac{\partial\psi_i}{\partial t}c_1$$

$$\varepsilon^2 a_2 = \frac{1}{\pi B_y'}\left(\frac{2\psi_i}{B_y'}\right)^{1/2}\left\{\left(\frac{1}{\eta}\frac{\partial\psi_i}{\partial t}\right)^2 b_0 + \frac{\psi_i^{1/2}}{\eta^2}\frac{\partial}{\partial t}\left(\psi_i^{1/2}\frac{\partial\psi_i}{\partial t}\right)b_1 + \frac{1}{\eta^2}\frac{\partial}{\partial t}\left(\psi_i\frac{\partial\psi_i}{\partial t}\right)b_2\right\}$$

$$\varepsilon a_3 = \frac{\psi_i}{2B_y'}\frac{1}{\eta}\frac{\partial\psi_i}{\partial t}$$

and :

$$c_0 = \int_{-1}^{+1} \mathrm{d}w \frac{\alpha(w)}{\beta(w)} \sqrt{1-w^2} = 0.67\,, \quad c_1 = \int_{-1}^{+\infty} \mathrm{d}w \frac{\alpha^2(w)}{\beta(w)} = 1.82$$

$$b_0 = \int_{-1}^{+\infty} \mathrm{d}w \frac{\alpha(w)}{\beta^2(w)} \int_{-1}^{w} \mathrm{d}w' \frac{\alpha(w')}{\beta(w')} \left[\alpha(w) \frac{\partial}{\partial w} \beta(w, w') - 2\,\beta(w) \frac{\partial}{\partial w} \alpha(w, w') \right] = 0.41$$

$$b_1 = \int_{-1}^{+\infty} \mathrm{d}w \frac{\alpha(w)}{\beta(w)} \int_{-1}^{w} \mathrm{d}w'\, \alpha(w') = -\,0.10$$

$$b_2 = -\int_{-1}^{+\infty} \mathrm{d}w \frac{\alpha(w)}{\beta(w)} \int_{-1}^{w} \mathrm{d}w' \frac{\alpha(w')}{\beta(w')} \beta(w, w') = -\,0.89$$

$$\alpha(w, w') = \int_0^{Y_M(w')} \frac{\mathrm{d}Y \cos Y}{\sqrt{w + \cos Y}} \qquad \beta(w, w') = \int_0^{Y_M(w')} \frac{\mathrm{d}Y}{\sqrt{w + \cos Y}}$$

$$\alpha(w) = \alpha(w, w) \qquad \beta(w) = \beta(w, w)$$
$$Y_M(w') = \cos^{-1}(-\,w') \quad (w' \leqslant 1)$$
$$Y_M(w') = \pi \qquad (w' \geqslant 1)\,.$$

The functions α and β are expressed in terms of elliptic integrals, and the integrals over w and w' are performed numerically.

2.2 Outer solution. — In the outer region we solve equation (4) by expansion in $\tilde{\psi}/\psi \ll 1$. Detailed calculations are given in appendix B. To second order we obtain :

$$\Delta[\psi_1(x) \cos Y] - \frac{j_0'}{B_0} \psi_1 \cos Y = \left[\frac{j_0'}{B_0^2} \delta\psi_0' - \frac{1}{B_0} \frac{1}{\eta} \frac{\partial}{\partial t} \delta\psi_0' \right] \psi_1 \cos Y \tag{8a}$$

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{\eta} \frac{\partial}{\partial t} \right) \delta\psi_0 = -\frac{\partial}{\partial x} \left(\frac{\psi_1^2 j_0'}{2 B_0^2} \right) + \frac{1}{\eta} \frac{\partial}{\partial t} \frac{\partial}{\partial x} \left(\frac{\psi_1^2}{4 B_0} \right). \tag{8b}$$

The linear terms in equation (8a) give the familiar ideal MHD equation for the outer solution ψ_1, while far from the island $\left(\frac{\partial^2}{\partial x^2} \ll \frac{1}{\eta} \frac{\partial}{\partial t} \right)$ equation (8b) gives the ideal result for $\delta\psi_0$, used by White *et al.* [5] :

$$\delta\psi_0 = -\frac{\partial}{\partial x} \left(\frac{\psi_1^2}{4 B_0} \right) \tag{9}$$

but this is easily shown to contribute only to order ε^3 to the slope-jump.

We solve equation (8b) for $\delta\psi_0$ with the conditions that it matches for small x to equation (7b) and for large x to equation (9). For $x \sim x_T\, \varepsilon^{-1/2}$ (which is the resistive depth on a time τ_c) we find :

$$\frac{1}{\eta} \frac{\partial}{\partial t} \delta\psi_0' = -\frac{1}{\eta^2} \left[\frac{\partial^2}{\partial t^2} \frac{\partial^2}{\partial x^2} \frac{\psi_1^2}{4 B_0} \right] \frac{x^3}{2\,\eta^2 t^2} \frac{1}{2\,i\pi} \int_{p_0 - i\infty}^{p_0 + i\infty} \mathrm{d}p\, \mathrm{e}^{p\eta t} p^{-5/2} H(x\sqrt{p})$$

with :

$$H(z) = \mathrm{e}^{-z} E_i(z) + \mathrm{e}^{z} E_1(z) - \frac{2}{z}\,.$$

Substituting this result into equation (8a) we get : $\psi = \psi_{1L} + \delta\psi_1$, where ψ_{1L} is the linear solution and :

$$\frac{\partial}{\partial x} \delta\psi_1 = \frac{-\,\psi_{1L}}{4\,B_y'^2} \left[\frac{1}{\eta^2} \frac{\partial^2}{\partial t^2} \psi_{1L}^2 \right] \frac{1}{\eta^2 t^2} \frac{1}{2\,i\pi} \int_{p_0 - i\infty}^{p_0 + i\infty} \mathrm{d}p\, \mathrm{e}^{p\eta t} p^{-5/2} \int_{x\sqrt{p}}^{\infty} \mathrm{d}z \frac{H(z)}{z}\,. \tag{10}$$

For $x \ll x_T \varepsilon^{-1/2}$ this gives :

$$\psi_1(x, t) = \psi_1(t) \left[1 + b_\pm X + \frac{j_0'}{B_y' \Delta'} \left(b_\pm \frac{X^2}{2} + X \operatorname{Ln} |X| \right) + \varepsilon^{3/2} d |X| \right] \tag{11}$$

where $X = \Delta' x$ and $b_+ - b_- = 1$, giving the linear outer slope-jump. The quadratic and logarithmic terms in parentheses are of order $\frac{j_0'}{B_y' \Delta'} \sim \frac{1}{a\Delta'}$, and can be neglected (they match to the « linearized current » term neglected in the inner solution), and :

$$\varepsilon^{3/2} d = \frac{-3\gamma}{2^{5/2} \Delta' B_y'^2} \psi_1^{-1/2} \left(\frac{1}{\eta} \frac{\partial \psi_1}{\partial t} \right)^{5/2}$$

with :

$$\gamma = \frac{1}{\Gamma\left(\frac{5}{2}\right)} \int_0^\infty \frac{dz}{z} \left(H(z) + \frac{2}{z} \right) = 3.63 .$$

The ordering in $\varepsilon^{3/2}$ comes from the resistive diffusion on a scale length $x_T \varepsilon^{-1/2}$.

3. Matching the inner solution (Eq. 7a) to the outer one (Eq. 11), as shown in appendix C, gives our final result :

$$\psi_i = \psi_1 - \varepsilon a_0 , \quad s = \left(\frac{b_+ + b_-}{2} \right) \frac{\psi_1}{\psi_i} , \quad \varepsilon = \Delta' x_T = 2 \left(\frac{\psi_i}{B_y'} \right)^{1/2} \Delta$$

$$\frac{1}{\eta} \frac{\partial \psi_1}{\partial t} = \frac{0.61}{4\pi} \Delta' (B_y' \psi_1)^{1/2} [1 - 0.16\, \varepsilon - 0.40\, \varepsilon^{3/2}] \tag{12}$$

where we have re-introduced the factor 4π neglected for simplicity in equation (4c), and where Δ' is still the linear slope-jump.

To lowest order we recover Rutherford's result, although in reference [2] the numerical value of c_1 was incorrect and s was not given explicitly. To the following orders, although the use of a boundary-layer technique restricts us to small values of ε, our result shows that saturation can occur for $\varepsilon \sim 1$. This is indeed the order of magnitude observed in numerical simulations, with $(a\Delta') \sim$ 10-20 and $\frac{x_T}{a} \sim 0.1$ in most cases. Because of the differences in the techniques used and in the physics involved in the calculations, it is difficult to compare our result (Eq. (12)) with that of White *et al.* In equation (12), $\Delta'(1 - 0.40\, \varepsilon^{3/2})$ is the non-linear « outer » slope-jump. It plays the same rôle as $\Delta'(x_T)$ in the result of White *et al.* However in the later case the evolution of $\Delta'(x_T)$ is due to the ideal-MHD quasi-linear flux $\delta\psi_0$; our work shows that this contribution is smaller than the one we take into account, which is due to the resistive diffusion of $\delta\psi_0$, away from the island region.

On the other hand the term $(1 - 0.16\, \varepsilon)$ in equation (12) represents a correction to the « inner » slope-jump, and plays the same rôle as the term $(1 - \alpha x_T)$ in the results of reference [5].

However, although these terms seem to have the same order of magnitude, it is impossible to know whether they have the same origin, because the result of reference [5] relies on an ansatz on the current profile, and is limited to a contribution form inside the island, and not from the whole island region.

Finally we note that our « inner » correction, proportional to ε, comes only from terms proportional to $\frac{\partial \psi_1}{\partial t}$. Thus this term should not play an important rôle at saturation. The latter should thus essentially be due to the outer correction (in $\varepsilon^{3/2}$), as $\delta\psi_0$ would keep diffusing, from the level reached during the growth of the island, on a scale-length $x_T(t/t_c)^{1/2}$. This slow evolution of ψ_0 (and of the saturated island width) was indeed observed, but not explained, in the numerical simulations of reference [5].

In conclusion, we have shown that improper use of boundary-layer techniques has led White *et al.* to neglect the most important effect causing the decrease of the slope-jump Δ'. A more correct technique, based on singular-perturbation theory, allows us to give the ordering of the saturated-island width

($\Delta' x_T \sim 1$) and to identify the mechanism leading to this saturation : the resistive diffusion, away from the island, of the quasi-linear magnetic flux $\delta\psi_0$ which flattens the current gradient responsible for the instability. It should be possible to measure $\delta\psi_0$ in numerical calculations, and to check the validity of this mechanism.

Work is in progress to study a similar effect, due to the evolution of the temperature profile in Rutherford's regime.

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Appendix A.

INNER EQUATIONS AND SOLUTIONS.

We start from equation (4) :

$$\Delta\tilde{\psi} = \frac{\left\langle \frac{1}{\eta} \frac{\partial\tilde{\psi}}{\partial t} \Big/ \frac{\partial\psi}{\partial x} \right\rangle}{\left\langle 1 \Big/ \frac{\partial\psi}{\partial x} \right\rangle} \tag{A-1}$$

where

$$\psi = \psi_0 - \tilde{\psi}$$

$$\tilde{\psi} = \psi_i(t)\,[1 + \Delta' sx + \varepsilon h_0(x) + \varepsilon^2 h_1(x)] \cos Y + \varepsilon\, \delta\psi_0(t)\,[1 + g_0(x) + \varepsilon g_1(x)]$$

and we have neglected the linearized current term.

To first order the metrics writes :

$$\frac{\partial\psi}{\partial x} = B'_y\, x^*[1 + \varepsilon M(x, Y, t)]$$

where

$$x^{*2} = \frac{2}{B'_y}\,[\psi + \varepsilon\, \delta\psi_0(t) + \psi_i(t) \cos Y]$$

and

$$M(x, Y, t) = \frac{1}{B'_y\, x^{*2}}\,[(h_0 - x h'_0)\, \psi_i \cos Y + (g_0 - x g'_0)\, \delta\psi_0]\,. \tag{A-2}$$

We solve equation (A-1) order by order in $\varepsilon = \Delta' x_T$, using Rutherford's result $\left(\frac{1}{\eta} \frac{\partial}{\partial t} \sim \frac{\Delta'}{x_T}\right)$ to get :

$$\varepsilon \frac{\partial^2}{\partial x^2}(\psi_i\, h_0 \cos Y + g_0\, \delta\psi_0) = \frac{1}{\eta} \frac{\partial\psi_i}{\partial t} \frac{\left\langle \frac{\cos Y}{x^*} \right\rangle}{\left\langle \frac{1}{x^*} \right\rangle} \tag{A-3}$$

$$\varepsilon \frac{\partial^2}{\partial x^2}(\psi_i\, h_1 \cos Y + g_1\, \delta\psi_0) = \frac{\left\langle \frac{\varepsilon}{\eta} \frac{\partial}{\partial t} (\psi_i\, h_0 \cos Y + g_0\, \delta\psi_0)/x^* \right\rangle}{\langle 1/x^* \rangle} + $$

$$+ \frac{\varepsilon}{\eta} \frac{\partial\psi_i}{\partial t} \frac{1}{\langle 1/x^* \rangle^2} \left\{ \left\langle \frac{M}{x^*} \right\rangle \left\langle \frac{\cos Y}{x^*} \right\rangle - \left\langle \frac{M \cos Y}{x^*} \right\rangle \left\langle \frac{1}{x^*} \right\rangle \right\}. \tag{A-4}$$

From equations (A-2) and (A-3) we get :

$$\frac{\partial}{\partial x}(\varepsilon M x^{*2}) = -\frac{x^*}{B'_y} \frac{1}{\eta} \frac{\partial\psi_i}{\partial t} \frac{\langle \cos Y/x^* \rangle}{\langle 1/x^* \rangle}$$

M, like h and g, is even and is zero at $x = 0$. This gives :

$$\varepsilon M(x, Y, t) = -\frac{1}{B_y'^2 x^{*2}} \psi_i \frac{1}{\eta} \frac{\partial \psi_i}{\partial t} \int_{-\cos Y}^{w} \mathrm{d}w' \frac{\alpha(w')}{\beta(w')} \tag{A-5}$$

where $w = \dfrac{\psi + \varepsilon \, \delta\psi_0}{\psi_i}$, and :

$$\alpha(w, w') = \int_0^{Y_M(w')} \frac{\mathrm{d}Y \cos Y}{\sqrt{w + \cos Y}}, \qquad \beta(w, w') = \int_0^{Y_M(w')} \frac{\mathrm{d}Y}{\sqrt{w + \cos Y}}$$

$$\alpha(w) = \alpha(w, w), \qquad \beta(w) = \beta(w, w)$$

$$Y_M(w') = \cos^{-1}(-w') \qquad (w' \leqslant 1)$$

$$Y_M(w') = \pi \qquad (w' \geqslant 1).$$

After some algebra we find :

$$\left\langle \frac{\varepsilon M}{x^*} \right\rangle = \frac{1}{(2\,\psi_i B_y')^{1/2}} \frac{1}{\eta} \frac{\partial \psi_i}{\partial t} \int_{-1}^{w} \mathrm{d}w' \frac{\alpha(w')}{\beta(w')} \frac{\partial}{\partial w} \beta(w, w') \tag{A-6}$$

$$\left\langle \frac{\varepsilon M \cos Y}{x^*} \right\rangle = \frac{1}{(2\,\psi_i B_y')^{1/2}} \frac{1}{\eta} \frac{\partial \psi_i}{\partial t} \int_{-1}^{w} \mathrm{d}w' \frac{\alpha(w')}{\beta(w')} \frac{\partial}{\partial w} \alpha(w, w') \tag{A-7}$$

$$\left\langle \frac{1}{\eta} \frac{\partial}{\partial t} (\varepsilon \tilde{\phi})/x^* \right\rangle = \frac{1}{(2\,B_y')^{1/2}} \frac{1}{\eta^2} \frac{\partial}{\partial t} \psi_i^{1/2} \frac{\partial \psi_i}{\partial t} \int_{-1}^{w} \mathrm{d}w' \, \alpha(w') - \\ - \frac{1}{(2\,\psi_i B_y')^{1/2}} \frac{1}{\eta^2} \frac{\partial}{\partial t} \left(\psi_i \frac{\partial \psi_i}{\partial t} \right) \int_{-1}^{w} \mathrm{d}w' \frac{\alpha(w')}{\beta(w')} \beta(w, w') \tag{A-8}$$

where

$$\tilde{\phi} = \psi_i h_0 \cos Y + g_0 \, \delta\psi_0 \,.$$

Using these results, we integrate and Fourier transform equation (A-4) to obtain, for $w \gg 1$ (where the inner and outer solutions will be matched) :

$$\left[\varepsilon^2 \frac{\partial}{\partial x} (\psi_i h_1) \right]_{w \to \infty} = \frac{\sqrt{2}}{\pi} \frac{\psi_i}{B_y'^{3/2}} \frac{1}{\eta^2} \frac{\partial}{\partial t} \left(\psi_i^{1/2} \frac{\partial \psi_i}{\partial t} \right) \int_{-1}^{\infty} \mathrm{d}w \frac{\alpha(w)}{\beta(w)} \int_{-1}^{w} \mathrm{d}w' \, \alpha(w') - \\ - \frac{\sqrt{2}}{\pi} \frac{\psi_i^{1/2}}{B_y'^{3/2}} \frac{1}{\eta^2} \frac{\partial}{\partial t} \left(\psi_i \frac{\partial \psi_i}{\partial t} \right) \int_{-1}^{\infty} \mathrm{d}w \frac{\alpha(w)}{\beta(w)} \int_{-1}^{w} \mathrm{d}w' \frac{\alpha(w')}{\beta(w')} \beta(w, w') + \\ + \frac{\sqrt{2}}{\pi} \frac{\psi_i^{1/2}}{B_y'^{3/2}} \left(\frac{1}{\eta} \frac{\partial \psi_i}{\partial t} \right)^2 \int_{-1}^{\infty} \mathrm{d}w \frac{\alpha(w)}{\beta^2(w)} \int_{-1}^{w} \mathrm{d}w' \frac{\alpha(w')}{\beta(w')} \left\{ \alpha(w) \frac{\partial}{\partial w} \beta(w, w') - \beta(w) \frac{\partial}{\partial w} \alpha(w, w') \right\}. \tag{A.9}$$

From equation (A.3) we also get :

$$\left(\varepsilon \, \delta\psi_0 \frac{\partial g_0}{\partial x} \right)_{w \to \infty} = \left(\frac{\psi_i}{2\,B_y'} \right)^{1/2} \frac{1}{\eta} \frac{\partial \psi_i}{\partial t} \int_{-1}^{w} \mathrm{d}w' \, \alpha(w') \\ = -\left(\frac{\psi_i}{2\,B_y'} \right)^{1/2} \frac{1}{\eta} \frac{\partial \psi_i}{\partial t} \int_{w}^{\infty} \mathrm{d}w' \, \alpha(w') \tag{A.10}$$

$$\approx \frac{\psi_i}{2\,B_y'} \frac{1}{\eta} \frac{\partial \psi_i}{\partial t} \frac{1}{x}$$

where we have used the facts that :

$$\int_0^\infty \mathrm{d}x \int_0^\pi \mathrm{d}Y\, \tilde{j}_z(x, Y) \sim \int_{-1}^\infty \mathrm{d}w\, \alpha(w) = 0$$

and that, for $w \gg 1$, $w \approx \dfrac{B'_y x^2}{\psi_i}$.

For h_0 we obtain :

$$\left(\varepsilon\psi_i \frac{\partial h_0}{\partial x}\right)_{w\to\infty} = \frac{1}{\pi}\left(\frac{2\,\psi_i}{B'_y}\right)^{1/2} \frac{1}{\eta}\frac{\partial \psi_i}{\partial t}\int_{-1}^\infty \mathrm{d}w \int_0^{Y_M(w)} \mathrm{d}Y \frac{\cos Y}{\sqrt{w + \cos Y}} \frac{\alpha(w)}{\beta(w)}(1 - \varepsilon M)$$

$$= \frac{1}{\pi}\left(\frac{2\,\psi_i}{B'_y}\right)^{1/2} \frac{1}{\eta}\frac{\partial \psi_i}{\partial t}\int_{-1}^{+\infty} \mathrm{d}w \frac{\alpha^2(w)}{\beta(w)} -$$

$$- \frac{1}{\pi B'_y}\left(\frac{2\,\psi_i}{B'_y}\right)^{1/2}\left(\frac{1}{\eta}\frac{\partial \psi_i}{\partial t}\right)^2 \int_{-1}^{+\infty} \mathrm{d}w \frac{\alpha(w)}{\beta(w)}\int_{-1}^{w} \mathrm{d}w' \frac{\alpha(w')}{\beta(w')}\frac{\partial}{\partial w}\alpha(w, w') .$$

Finally we get $h_0 - xh'_0$ by equations (A-2) and (A-5), giving :

$$[\varepsilon\psi_i(h_0 - xh'_0)]_{w\to\infty} = -\frac{2}{\pi}\frac{\psi_i}{B'_y}\frac{1}{\eta}\frac{\partial \psi_i}{\partial t}\int_{-1}^{+1} \mathrm{d}w \sqrt{1 - w^2}\,\frac{\alpha(w)}{\beta(w)} .$$

Collecting the results, we have for $w \gg 1$:

$$\psi_i(\varepsilon h_0 + \varepsilon^2 h_1) = \varepsilon a_0 + (\varepsilon a_1 + \varepsilon^2 a_2)\, x_T \,|\, \tilde{X} \,|$$

$$\varepsilon g_0\, \delta\psi_0 = \varepsilon a_3 \,\mathrm{Ln}\, |\, \tilde{X} \,|$$

with

$$\tilde{X} = \frac{x}{x_T}$$

$$\varepsilon a_0 = -\frac{2\,\psi_i}{\pi B'_y}\frac{1}{\eta}\frac{\partial \psi_i}{\partial t}\, c_0$$

$$\varepsilon a_1 = \frac{1}{\pi}\left(\frac{2\,\psi_i}{B'_y}\right)^{1/2}\frac{1}{\eta}\frac{\partial \psi_i}{\partial t}\, c_1$$

$$\varepsilon^2 a_2 = \frac{1}{\pi B'_y}\left(\frac{2\,\psi_i}{B'_y}\right)^{1/2}\left\{\left(\frac{1}{\eta}\frac{\partial \psi_i}{\partial t}\right)^2 b_0 + \frac{\psi_i^{1/2}}{\eta^2}\frac{\partial}{\partial t}\left(\psi_i^{1/2}\frac{\partial \psi_i}{\partial t}\right) b_1 + \frac{1}{\eta^2}\frac{\partial}{\partial t}\left(\psi_i \frac{\partial \psi_i}{\partial t}\right) b_2\right\}$$

$$\varepsilon a_3 = \frac{\psi_i}{2\,B'_y}\,\frac{1}{\eta}\frac{\partial \psi_i}{\partial t}$$

and :

$$c_0 = \int_{-1}^{+1} \mathrm{d}w \frac{\alpha(w)}{\beta(w)}\sqrt{1 - w^2} = 0.67$$

$$c_1 = \int_{-1}^{+\infty} \mathrm{d}w \frac{\alpha^2(w)}{\beta(w)} = 1.82$$

$$b_0 = \int_{-1}^{\infty} \mathrm{d}w \frac{\alpha(w)}{\beta^2(w)}\int_{-1}^{w} \mathrm{d}w' \frac{\alpha(w')}{\beta(w')}\left\{\alpha(w)\frac{\partial}{\partial w}\beta(w, w') - 2\,\beta(w)\frac{\partial}{\partial w}\alpha(w, w')\right\} = 0.41$$

$$b_1 = \int_{-1}^{+\infty} \mathrm{d}w \frac{\alpha(w)}{\beta(w)}\int_{-1}^{w} \mathrm{d}w'\, \alpha(w') = -\,0.10$$

$$b_2 = -\int_{-1}^{\infty} \mathrm{d}w \frac{\alpha(w)}{\beta(w)}\int_{-1}^{w} \mathrm{d}w' \frac{\alpha(w')}{\beta(w')}\beta(w, w') = -\,0.89$$

Appendix B.

OUTER EQUATIONS AND SOLUTIONS.

In the outer region, we solve equation (4) :

$$\Delta\tilde{\psi} = \frac{\left\langle \frac{1}{\eta}\frac{\partial\tilde{\psi}}{\partial t}\Big/\frac{\partial\psi}{\partial x}\right\rangle}{\left\langle 1\Big/\frac{\partial\psi}{\partial x}\right\rangle} + \left[j_0(x) - \frac{\left\langle j_0\Big/\frac{\partial\psi}{\partial x}\right\rangle}{\left\langle 1\Big/\frac{\partial\psi}{\partial x}\right\rangle}\right] \tag{B-1}$$

by expansion to second order in $\tilde{\psi}/\psi$, with

$$\tilde{\psi} = \psi_1(x, t)\cos Y + \delta\psi_0(x, t), \quad \psi = \psi_0 - \tilde{\psi}.$$

To do this, we must express the average at constant ψ of x-dependent quantities. Writing :

$$\begin{aligned} A(x) &= \mathcal{A}(\psi)\,|_{\psi=\psi_0(x)} \\ &= \mathcal{A}(\psi) + \tilde{\psi}\frac{\partial\mathcal{A}}{\partial\psi} + \frac{\tilde{\psi}^2}{2}\frac{\partial^2\mathcal{A}}{\partial\psi^2} \end{aligned}$$

we obtain after some algebra :

$$\langle A(x)\rangle = A(x) + (\langle\tilde{\psi}\rangle - \tilde{\psi})\frac{1}{B_0}\frac{\partial A}{\partial x} + \frac{1}{2}(\tilde{\psi}^2 + \langle\tilde{\psi}\rangle^2 - 2\tilde{\psi}\langle\tilde{\psi}\rangle)\frac{1}{B_0}\frac{\partial}{\partial x}\frac{1}{B_0}\frac{\partial A}{\partial x} \tag{B-2}$$

$$\langle A(x)\cos Y\rangle = \frac{\psi_1}{2B_0}\frac{\partial A}{\partial x}, \quad \langle\tilde{\psi}^2\rangle = \delta\psi_0^2 + \frac{\psi_1^2}{2}$$

$$\langle\tilde{\psi}\rangle = \delta\psi_0(x) + \frac{1}{2B_0}\psi_1\psi_1' - \frac{1}{B_0}\delta\psi_0'\psi_1\cos Y$$

and finally :

$$\langle A(x)\rangle = A(x) - \frac{1}{B_0}\frac{\partial A}{\partial x}\psi_1\cos Y + \frac{1}{B_0^2}\frac{\partial A}{\partial x}\left[\frac{\psi_1\psi_1'}{2} - \delta\psi_0'\psi_1\cos Y\right] + \frac{\psi_1^2}{2B_0}\frac{\partial}{\partial x}\left[\frac{1}{B_0}\frac{\partial A}{\partial x}\right]. \tag{B-3}$$

This procedure avoids us to assume, as in reference [5], the conservation of the current between flux lines. Then equation (B-1) gives :

$$\left[\Delta - \frac{j_0'}{B_0}\right]\psi_1\cos Y = \left[\frac{j_0'}{B_0^2}\delta\psi_0' - \frac{1}{B_0}\frac{1}{\eta}\frac{\partial}{\partial t}\delta\psi_0'\right]\psi_1\cos Y \tag{B-4}$$

$$\left[\frac{\partial^2}{\partial x^2} - \frac{1}{\eta}\frac{\partial}{\partial t}\right]\delta\psi_0 = -\frac{\partial}{\partial x}\left[\frac{\psi_1^2 j_0'}{2B_0^2}\right] + \frac{1}{\eta}\frac{\partial}{\partial t}\frac{\partial}{\partial x}\frac{\psi_1^2}{4B_0}. \tag{B-5}$$

The linear terms of equation (B-4) give the usual linear equation for the outer solution, while far from the island $\left(\frac{\partial^2}{\partial x^2} \ll \frac{1}{\eta}\frac{\partial}{\partial t}\right)$ equation (B-5) gives the ideal-MHD result :

$$\delta\psi_0 = -\frac{\partial}{\partial x}\left(\frac{\psi_1^2}{4B_0}\right). \tag{B-6}$$

This solution is easily shown to contribute only, for $x \gtrsim \Delta'^{-1}$, to order ε^3 to the slope-jump of ψ_1.

Then we solve equations (B-4) and (B-5) for $x \ll \Delta'^{-1}$, where we can estimate the non-linear terms :

$$\left[\frac{\partial}{\partial x}\left(\frac{\psi_1^2 j_0'}{2B_0^2}\right)\Big/\frac{1}{\eta}\frac{\partial}{\partial t}\frac{\partial}{\partial x}\left(\frac{\psi_1^2}{4B_0}\right)\right] \sim \left[\frac{j_0'\,\delta\psi_0'\,\psi_1}{B_0^2}\Big/\frac{\psi_1}{B_0}\frac{1}{\eta}\frac{\partial}{\partial t}\delta\psi_0'\right] \sim \frac{x_T}{xa\Delta'} \ll 1 .$$

The solution of equation (B-5) must match for large x ($x \gg x_T \varepsilon^{-1/2}$) to equation (B-6) and for small x ($x \sim x_T$) to the inner solution, equation (7b). By Laplace transform in time we find :

$$\frac{1}{\eta}\frac{\partial}{\partial t}\frac{\partial}{\partial x}\delta\psi_0 = \frac{c}{2\,i\pi}\int_{p_0 - i\infty}^{p_0 + i\infty} \mathrm{d}p\, \mathrm{e}^{p\eta t}\, p^{-5/2}\, H(x\sqrt{p}) \tag{B-7}$$

where

$$H(z) = \mathrm{e}^{z}\,E_1(z) + \mathrm{e}^{-z}\,Ei(z) - \frac{2}{z}$$

$$c = -\frac{x^3}{2\,\eta^2\,t^2}\,\frac{1}{\eta^2}\frac{\partial^2}{\partial t^2}\frac{\partial^2}{\partial x^2}\left(\frac{\psi_1^2}{4\,B_0}\right)$$

and to lowest order c is a constant. Integrating equation (B-7) would give the value of $\varepsilon\delta\psi_0(t)$ in the inner solution, but this result is not needed here.

Substituting equation (B-7) into equation (B-4) we get :

$$\psi_1(x, t) = \psi_{1\mathrm{L}} + \delta\psi_1$$

where $\psi_{1\mathrm{L}}$ is the linear outer solution, and :

$$\frac{\partial}{\partial x}\delta\psi_1 = \left(-\frac{\psi_1}{4\,B_y'^2}\,\frac{1}{\eta^2}\,\frac{\partial^2}{\partial t^2}\psi_1^2\right)\cdot\frac{1}{\eta^2\,t^2}\cdot\frac{1}{2\,i\pi}\int_{p_0 - i\infty}^{p_0 + i\infty}\mathrm{d}p\,\mathrm{e}^{p\eta t}\,p^{-5/2}\int_{x\sqrt{p}}^{\infty}\mathrm{d}z\,\frac{H(z)}{z}. \tag{B-8}$$

For $X = \Delta' x \sim \varepsilon$ we obtain :

$$\psi_{1\mathrm{L}}(x, t) = \psi_1(t)\left[1 + b_{\pm}\,X + \frac{j_0'}{B_y'\,\Delta'}\left(b_{\pm}\,\frac{X^2}{2} + X\,\mathrm{Ln}\,|\,X\,|\right)\right] \tag{B-9}$$

where $b_+ - b_- = 1$ (giving the linear slope-jump) and the terms in parentheses can be neglected, as they are of order $(a\Delta')^{-1}$ and match to the linearized current term neglected in the inner solution. For $\delta\psi_1$ we get to lowest order :

$$\frac{\partial}{\partial x}\delta\psi_1 = \frac{\psi_1}{4\,B_y'^2}\,\frac{1}{\eta^2}\,\frac{\partial^2}{\partial t^2}\,\psi_1^2\left[\frac{1}{x} - \frac{\gamma}{\sqrt{\eta t}}\right] \tag{B-10}$$

where

$$\gamma = \frac{1}{\Gamma(5/2)}\int_0^{\infty}\frac{\mathrm{d}z}{z}\left[H(z) + \frac{2}{z}\right] = 3.63\,.$$

The term in $1/x$ matches to a similar term in the inner solution, contributing only to order $\varepsilon^2\,\mathrm{Ln}\,\varepsilon$ to the growth rate, and we neglect it. Using $\eta t = 2\,\psi_1\left(\frac{1}{\eta}\,\frac{\partial\psi_1}{\partial t}\right)^{-1}$ we obtain from Eqs. (B-9) and (B-10) :

$$\psi_1(x, t) = \psi_1(t)\,[1 + b_{\pm}\,X + \varepsilon^{3/2}\,d\,|\,X\,|] \tag{B-11}$$

with

$$\varepsilon^{3/2}\,d = \frac{-\,3\,\gamma}{2^{5/2}\,\Delta'\,B_y'^2}\,\psi_1^{-1/2}\left(\frac{1}{\eta}\,\frac{\partial\psi_1}{\partial t}\right)^{5/2}.$$

Appendix C.

MATCHING AND FINAL RESULT.

We match the inner solution taken at $x \gg x_T$ (Eq. (7a)) to the outer solution taken at $x \ll x_T\,\varepsilon^{-1/2}$ (Eq. (11)). Details of the matching technique can be found in reference [7]. Then we have :

$$\psi_i(x, t) = \psi_i(t)\,[1 + \varepsilon s\tilde{X}] + \varepsilon a_0 + (\varepsilon a_1 + \varepsilon^2\,a_2)\,x_T\,|\,\tilde{X}\,| \tag{C-1}$$

$$\psi_1(x, t) = \psi_1(t)\,[1 + b_{\pm}\,X + \varepsilon^{3/2}\,d\,|\,X\,|] \tag{C-2}$$

$$\tilde{X} = \frac{x}{x_T},\quad X = \varepsilon\,\frac{x}{x_T},\quad b_+ - b_- = 1\,.$$

Matching these solutions gives :

$$\psi_i = \psi_1 - \varepsilon a_0 \tag{C-3}$$

$$b_\pm \psi_1 \pm \varepsilon^{3/2} d\psi_1 = s\psi_i \pm (\varepsilon a_1 + \varepsilon^2 a_2) x_T \tag{C-4}$$

hence :

$$s = \frac{b_+ + b_-}{2} \frac{\psi_1}{\psi_i} \tag{C-5}$$

and :

$$\varepsilon a_1 = \frac{\Delta'}{2} \psi_1 + \Delta' \varepsilon^{3/2} d\psi_1 - \varepsilon^2 a_2 . \tag{C-6}$$

Solving this equation by iteration we get to lowest order :

$$\frac{1}{\eta}\frac{\partial \psi_i}{\partial t} = \frac{1}{\eta}\frac{\partial \psi_1}{\partial t} = \frac{\pi \Delta'}{2^{3/2} c_1} (B_y' \psi_1)^{1/2} \tag{C-7}$$

which is just Rutherford's result, and to order $\varepsilon^{3/2}$

$$\frac{1}{\eta}\frac{\partial \psi_i}{\partial t} = \frac{\pi}{2^{3/2} c_1} \Delta'(B_y' \psi_1)^{1/2} \left\{ 1 - \frac{c_0}{\sqrt{2}\, c_1} \Delta' \left(\frac{\psi_1}{B_y'}\right)^{1/2} - \right.$$
$$\left. - \frac{\pi}{2^{3/2} c_1^2} \Delta' \left(\frac{\psi_1}{B_y'}\right)^{1/2} \left[b_0 + b_1 + \frac{3}{2} b_2 \right] - \frac{3\pi^{5/2}}{2^{21/4} c_1^{5/2}} \gamma \Delta'^{3/2} \left(\frac{\psi_1}{B_y'}\right)^{3/4} \right\} .$$

Using equation (C-3) we finally get :

$$\frac{1}{\eta}\frac{\partial \psi_1}{\partial t} = 0.61\ \Delta'(B_y' \psi_1)^{1/2} [1 - 0.16\, \varepsilon - 0.40\, \varepsilon^{3/2}] \tag{C-8}$$

with

$$\varepsilon = \Delta' x_T = 2\, \Delta' \left(\frac{\psi_1}{B_y'}\right)^{1/2} .$$

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