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Quasi-linear formulation of MOND

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ABSTRACT

A new formulation of modified Newtonian dynamics (MOND) as a modified-potential theory of gravity is propounded. In effect, the theory dictates that the MOND potential ϕ produced by a mass distribution ρ is a solution of the Poisson equation for the modified source density $\hat{\rho} = -(4\pi G)^{-1}\nabla \cdot \boldsymbol{g}$, where $\boldsymbol{g} = \nu(|\boldsymbol{g}^N|/a_0)\boldsymbol{g}^N$, and $\boldsymbol{g}^N$ is the Newtonian acceleration field of ρ. This makes ϕ simply the scalar potential of the algebraic acceleration field $\boldsymbol{g}$. The theory thus involves solving only linear-differential equations, with one non-linear, algebraic step. It is derivable from an action, satisfies all the usual conservation laws, and gives the correct centre-of-mass acceleration to composite bodies. The theory is akin in some respects to the non-linear Poisson formulation of Bekenstein and Milgrom, but it is different from it, and is obviously easier to apply. The two theories are shown to emerge as natural modifications of a Palatini-type formulation of Newtonian gravity, and are members in a larger class of bi-potential theories.

Key words: galaxies: kinematics and dynamics – cosmology: theory – dark matter.

1 INTRODUCTION

The only known, full-fledged, non-relativistic (NR) formulation of modified Newtonian dynamics (MOND) has been that of Bekenstein & Milgrom (1984). This is a modified gravity theory in which the MOND gravitational potential, ϕ, produced by a mass density ρ, is gotten from the non-linear generalization of the Poisson equation,

$$\nabla \cdot [\mu(|\nabla\phi|/a_0)\nabla\phi] = 4\pi G\rho. \quad (1)$$

Here, $\mu(x)$ is a function characterizing the theory.[1] This field equation is derived from an action, and enjoys all the standard conservation laws resulting from the usual space symmetries of the underlying action.[2]

There is also extensive use in the literature of the pristine formulation of MOND (Milgrom 1983), in which the MOND acceleration, $\boldsymbol{g}$, is calculated from the Newtonian value, $\boldsymbol{g}^N$, via an algebraic relation of the form

$$\boldsymbol{g} = \nu(g_N/a_0)\boldsymbol{g}^N, \quad (2)$$

where $g_N = |\boldsymbol{g}^N|$. For the case of circular motion in an axisymmetric system, appropriate for rotation curve analysis, equation (2) is, in fact, an exact relation in the class of theories dubbed 'modified inertia' formulation of MOND (Milgrom 1994a). Expression (2) is also exact in the non-linear Poisson formulation for systems of one-dimensional symmetry, such as a spherical galaxy. In this case, $\nu(y)$ is related to $\mu(x)$ by $\nu(y) = 1/\mu(x)$, where x and y are related by $x\mu(x) = y$. This algebraic formulation is very easy to use – hence its attraction as a wieldy tool – and does capture the salient MOND effects in many instances. However, it can definitely not be used as a complete[3] theory, especially for non-test-particle motion: in the first place $\boldsymbol{g}$ resulting from equation (2) is, generally, not derivable from a potential, with all the adverse effects of this (e.g. there is no conserved momentum). Also, when applying equation (2) to describe the centre-of-mass motion of a composite system based on the accelerations of its constituents it fails completely. The same is true when applying it, e.g. to the galactic external-field effect in the Solar system (Milgrom 2009a). This had left us with only the non-linear Poisson formulation of MOND as a reliable, complete theory.

Here, I present a new MOND formulation that combines the benefits of the above two formulations: it constitutes a complete theory derivable from an action, and enjoying the standard conservation laws. Yet, the unavoidable non-linearity – a direct corollary of the basic tenets of MOND – enters in an easy to handle, algebraic manner. The resulting field equations then involve only linear-differential equations.

This theory came to light as follows. In a recent paper (Milgrom 2009a), concerning the MOND external-field effect in the Solar system, I applied the non-linear Poisson formulation to a specific

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[1]There are also the obvious generalizations to multipotential theories, whereby the MOND potential is a sum of several potentials, each satisfying an equation such as equation (1).

[2]Blanchet (2007) has given an interpretation to this theory, as resulting from the omnipresence of a gravitationally polarizable medium, and Blanchet & Le Tiec (2008, 2009) extended the idea to an appropriate relativistic version.

[3]'Complete' in the sense of being applicable to an arbitrary problem in the realm for which it is meant; in the present case, to any finite, self-gravitating system.

problem: a point mass in a constant background field representing the sun in the field of the Galaxy. Beside the exact treatment of the problem, I considered a certain approximation for the MOND potential of this configuration. It was defined as the solution of the (linear) Poisson equation with a source density $\hat{\rho} = -(4\pi G)^{-1}\nabla\cdot \boldsymbol{g}$, with $\boldsymbol{g}$ given by equation (2). This approximation was justified on the basis of $\hat{\rho}$ having properties similar to those of the density-like field $(4\pi G)^{-1}\Delta\phi$, where ϕ is the exact solution of equation (1). I have now come to realize that this approximation may, in fact, form the basis for a complete MOND theory, standing on its own.

The non-linear Poisson formulation of MOND has been applied, analytically and numerically, to many problems to which it would be interesting to apply the present theory as well. Among these: Solar system tests (Milgrom 1986a, 2009a; Bekenstein & Magueijo 2006), forces on non-test-mass bodies (Milgrom 1997, 2002a; Dai, Matsuo & Starkman 2008), disc stability and bar formation (Brada & Milgrom 1999; Tiret & Combes 2008), two-body relaxation (Ciotti & Binney 2004), dynamical friction (Nipoti et al. 2008), escape speed from a galaxy (Famaey, Bruneton & Zhao 2007; Wu et al. 2007), galaxy interactions (Nipoti, Londrillo & Ciotti 2007a; Tiret & Combes 2007), and collapse (Nipoti, Londrillo & Ciotti 2007b), triaxial models of galactic systems (Wang, Wu & Zhao 2008; Wu et al. 2009), the external-field effect as applied to dwarf spheroidals and warp induction (Brada & Milgrom 2000a,b; Angus 2008), structure formation (e.g. Llinares, Knebe & Zhao 2008) and quite a few more.

Relativistic extensions of the non-linear Poisson theory where also propounded (see, e.g. Sanders 1997; Bekenstein 2004; Bekenstein & Magueijo 2006; Zlosnik, Ferreira & Starkman 2006, 2007; Skordis 2009). The bi-, or multipotential NR theories described here have also inspired a class of relativistic, bi-, or multi-metric MOND theories (Milgrom 2009c,d, see below).

The new, quasi-linear MOND (QUMOND) theory – in particular its emergence as a modification of a Palatini formulation of Newtonian gravity, and its generalizations – is described in Section 2. In section 3, I derive some of its general properties pertaining to forces on bodies. Section 4 concerns the deep-MOND limit of the theory. Section 5 deals with systems in a constant external field, relevant for the external field effect, and for the question of the centre of mass motion of composite systems. Section 6 is a discussion.

2 FORMALISM

Consider a gravitating system of density $\rho(\boldsymbol{r}, t)$ and velocity field $\boldsymbol{v}(\boldsymbol{r}, t)$. Describe the dynamics of the system by the action $I = \int L\mathrm{d}t$, with $L \equiv \int \mathcal{L}\mathrm{d}^3r$, and

$$\mathcal{L} = \mathcal{L}_{\mathrm{K}} - \mathcal{L}_{\mathrm{P}} = -\frac{1}{8\pi G}\left\{2\nabla\phi\cdot\nabla\phi^N - a_0^2\mathcal{Q}\left[(\nabla\phi^N/a_0)^2\right]\right\} + \rho\left(\frac{1}{2}\boldsymbol{v}^2 - \phi\right) \quad (3)$$

involving the two potentials ϕ and ϕ^N, where $\mathcal{L}_{\mathrm{K}} = \rho\boldsymbol{v}^2/2$ is the kinetic energy density, and $\mathcal{L}_{\mathrm{P}}$ is the potential energy density. The density ρ may be viewed as made up of the masses of the constituents $\rho(\boldsymbol{r}, t) = \sum_i m_i\delta^3[\boldsymbol{r} - \boldsymbol{r}_i(t)]$, with each of the masses, m_i, negligible with respect to the total mass; so each mass can be treated as a test mass in the field of the rest.

Varying the action over the particle degrees of freedom gives

$$\ddot{\boldsymbol{r}}_i = -\nabla\phi(\boldsymbol{r}_i). \quad (4)$$

Varying over ϕ gives

$$\Delta\phi^N = 4\pi G\rho, \quad (5)$$

and varying over ϕ^N gives

$$\Delta\phi = \nabla\cdot\left[\nu(|\nabla\phi^N|/a_0)\nabla\phi^N\right], \quad (6)$$

where $\nu(y) \equiv \mathcal{Q}'(y^2)$. Equation (4) tells us that the masses move according to the standard Newtonian law of inertia in the potential ϕ; thus ϕ is the MOND potential. Equation (5) tells us that ϕ^N solves the Poisson equation with ρ as source; thus ϕ^N is the standard Newtonian potential (when we impose the standard boundary condition $\phi^N \to 0$ at infinity). Equation (6) tells us that the MOND potential satisfies the Poisson equation for the density

$$\hat{\rho} = -\frac{1}{4\pi G}\nabla\cdot\left[\nu(g_N/a_0)\boldsymbol{g}^N\right] = \nu\rho - \frac{1}{4\pi G a_0}\nu'\nabla g_N\cdot\boldsymbol{g}^N, \quad (7)$$

as a source, with $\boldsymbol{g}^N = -\nabla\phi^N$. Thus, $\hat{\rho}$ would be the density that gives the correct MOND potential if we interpret the observations in the framework of Newtonian dynamics. Then, $\rho_{\mathrm{p}} \equiv \hat{\rho} - \rho$ would be the phantom mass density, introduced in Milgrom (1986b), which will be interpreted by a Newtonist as the density of dark matter. The density ρ_{p} is what was called in Milgrom (2009a) 'the surrogate mass density', and was used as an approximation for the phantom density of the non-linear Poisson theory.

As usual, the symmetries of the action under space and time translations, and rotations, lead to conserved momentum, energy and angular momentum (see also Section 3).

In effect, this theory starts with the acceleration field defined by the algebraic relation equation (2), and remedies its not being derived from a potential by projecting it on the space of gradient vector fields. In other words, write

$$\nu(g_N/a_0)\boldsymbol{g}^N = -\nabla\phi - \nabla\times\boldsymbol{A}, \quad (8)$$

which is a unique decomposition if we require that $\nabla\phi$ vanish at infinity, then ϕ is the MOND potential of the theory.

Restoration of the Newtonian limit for $a_0 \to 0$ requires $\mathcal{Q}(z) \to z + Q_1$ for large z, where Q_1 is a constant. Space–time-scale invariance, which is the defining tenet of the deep-MOND limit $a_0 \to \infty$ (Milgrom 2009b), and the standard normalization of a_0 (defined so that the mass-asymptotic-rotational-speed relation is $V^4 = MGa_0$) for which $\nu(y) \approx y^{-1/2}$, dictate $\mathcal{Q}(z) \approx (4/3)z^{3/4} + Q_0$ for $z \ll 1$. Since Q is defined up to an immaterial additive constant, we choose it so that $Q_0 = 0$.

In one-dimensional cases – e.g. for spherically symmetric systems – equation (6) implies relation (2), and all three formulations give the same acceleration field with $\nu(y) = 1/\mu(x)$, where $y = x\mu(x)$. The QUMOND theory will produce unique solutions if and only if $y\nu(y)$ is monotonic, which we assume; and since this function is $\approx y$ for large values of y, it has to be increasing everywhere, so $y\nu' + \nu > 0$.

For an isolated mass distribution of total mass M, bounded in a finite region, we have asymptotically at infinity $\phi^N \approx -MG/r$. Thus, $\hat{\rho} \approx (4\pi G)^{-1}(MGa_0)^{1/2}r^{-2} = (1/3)\rho_{\mathrm{M}}(r/R_{\mathrm{M}})^{-2}$, where $\rho_{\mathrm{M}} = 3M/4\pi R_{\mathrm{M}}^3$, $R_{\mathrm{M}} = (MG/a_0)^{1/2}$, are, respectively, the MOND density and the MOND radius for the mass M. It thus follows that the MOND potential, relative to its value at a finite point, diverges logarithmically at infinity (as in any MOND theory). Note also that the above asymptotic behaviour of the fields makes L_{P} logarithmically divergent upon space integration. This is the same situation as in the case of the non-linear Poisson formulation. Differences in L_{P} for systems with the same total mass are finite, and only such differences will concern us.

We can write the MOND potential, relative to some arbitrary origin $\boldsymbol{r} = 0$, as

$$\phi(\boldsymbol{r}) = -G \int \mathrm{d}^3 r' \hat{\rho}(\boldsymbol{r}') \left(\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} - \frac{1}{|\boldsymbol{r}'|} \right)$$
$$= \frac{1}{4\pi} \int \mathrm{d}^3 r' \nabla \cdot \left[\nu(g_N/a_0) \boldsymbol{g}^N \right] (\boldsymbol{r}') \left(\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} - \frac{1}{|\boldsymbol{r}'|} \right), \quad (9)$$

which is finite. This is a closed form expression of the MOND field for an arbitrary (bounded) mass distribution, with $\boldsymbol{g}^N$ itself being expressed as usual in terms of ρ.

2.1 Emergence from a Palatini formulation

The proposed modification amounts to replacing $(\nabla\phi)^2$ in the standard Newtonian Lagrangian density,

$$\mathcal{L} = -\frac{1}{8\pi G}(\nabla\phi)^2 + \rho \left(\frac{1}{2} \boldsymbol{v}^2 - \phi \right) \quad (10)$$

by $2\nabla\phi \cdot \nabla\tilde{\phi} - a_0^2 \mathcal{Q}[(\nabla\tilde{\phi}/a_0)^2]$, where $\tilde{\phi}$ is an auxiliary potential, called here ϕ^N because it turns out to equal the Newtonian potential. This modification can be performed in two steps: first add to $\mathcal{L}$ a term $(8\pi G)^{-1}(\nabla\phi - \nabla\tilde{\phi})^2$, which is the same as $-(8\pi G)^{-1}(\phi - \tilde{\phi})(\Delta\phi - \Delta\tilde{\phi})$ up to a divergence. This does not modify the theory since varying over $\tilde{\phi}$ gives $\Delta\tilde{\phi} = \Delta\phi$, which annuls the new addition to the action. The free field part of the new Lagrangian density now has $-(\nabla\tilde{\phi})^2 + 2\nabla\phi \cdot \nabla\tilde{\phi}$ replacing $(\nabla\phi)^2$.

The resulting theory after this step may be viewed as a Palatini-type formulation of Newtonian gravity. In the standard Palatini formulation of general relativity (GR), the connection (representing an auxiliary gravitational acceleration field) is not assumed, a priori, to be the Levi–Civita connection of the metric. The analogous approach here would be to introduce an auxiliary-acceleration degree of freedom, $\tilde{\boldsymbol{g}}$, and not assume, a priori, that it is the gradient of the potential. Write then the Lagrangian density as

$$\mathcal{L} = \frac{1}{8\pi G}(\tilde{\boldsymbol{g}}^2 - 2\phi\nabla \cdot \tilde{\boldsymbol{g}}) + \rho \left(\frac{1}{2} \boldsymbol{v}^2 - \phi \right). \quad (11)$$

Variation over $\tilde{\boldsymbol{g}}$ gives $\tilde{\boldsymbol{g}} = -\nabla\phi$, and over ϕ, $\nabla \cdot \tilde{\boldsymbol{g}} = -4\pi G\rho$, yielding Newtonian gravity.[4] The first part in $\mathcal{L}$ falls from the gravitational action $-c^4(16\pi G)^{-1} \int g^{1/2} g^{\mu\nu} \tilde{R}_{\mu\nu}$ of the relativistic Palatini formulation of GR. Then, $\tilde{\boldsymbol{g}}$ descends from the independent connection degrees of freedom, from which $\tilde{R}_{\mu\nu}$ is constructed, and ϕ from the metric. The second part in $\mathcal{L}$ falls from the matter Lagrangian density, with matter coupled minimally to the metric, and not to the connection. In the relativistic Palatini formulation, extremization over the connection (assumed symmetric) establishes it as the Levi–Civita connection of the metric (here, $\tilde{\boldsymbol{g}}$ is established as the gradient of the gravitational potential), and extremization over the metric (here, ϕ) gives the Einstein equation for the connection with the standard matter energy-momentum as source (here, the Poisson equation for ϕ).

MOND is then introduced in the second step, replacing $\tilde{\boldsymbol{g}}^2$ in $\mathcal{L}$ of equation (11) by $a_0^2 \mathcal{Q}(\tilde{\boldsymbol{g}}^2/a_0^2)$. To get our Lagrangian (3), we now do impose, beyond the standard Palatini formalism, that $\tilde{\boldsymbol{g}}$ is a gradient of some auxiliary potential (not of ϕ itself): $\tilde{\boldsymbol{g}} = -\nabla\tilde{\phi}$; so, $\tilde{\phi}$, not $\tilde{\boldsymbol{g}}$, is the basic degree of freedom beside ϕ. If we do not impose this restriction on $\tilde{\boldsymbol{g}}$ we get instead of equations (5) and (6) a different theory

$$\nu(|\tilde{\boldsymbol{g}}|/a_0)\tilde{\boldsymbol{g}} = -\nabla\phi, \qquad \nabla \cdot \tilde{\boldsymbol{g}} = -4\pi G\rho, \quad (12)$$

which, quite interestingly, is equivalent to the non-linear Poisson formulation, equation (1), in the form used, e.g. in Milgrom (1986a) [with $\tilde{\boldsymbol{g}} = -\mu(|\nabla\phi|/a_0)\nabla\phi$].

To recapitulate, the Lagrangian density (11) underlies a Palatini-like formulation of Newtonian gravity, whether or not we restrict $\tilde{\boldsymbol{g}}$, a priori, to be a gradient. However, the MOND extension, even with a given $\mathcal{Q}(z)$, does depend on whether we make the restriction or not: with the restriction, we get our present QUMOND theory. Without it – varying the action over a larger space of trial acceleration fields $\tilde{\boldsymbol{g}}$ – we get the non-linear Poisson theory. This captures at once the affinity and the disparity between the two theories, and also explains why they coincide in cases of one-dimensional symmetry, where any vector field is a gradient.

2.2 Generalizations

Since the Palatini formulation introduces a second degree of freedom, we can further generalize the Lagrangian density[5] (11) by taking

$$\mathcal{L}_{\mathrm{P}} = -\frac{a_0^2}{8\pi G}\mathcal{F}(\zeta, \kappa, \xi) - \rho\phi, \quad \text{with } \zeta = (\tilde{\boldsymbol{g}}/a_0)^2,$$
$$\kappa = (\nabla\phi/a_0)^2, \ \xi = 2\tilde{\boldsymbol{g}} \cdot \nabla\phi/a_0^2. \quad (13)$$

Extremization over $\tilde{\boldsymbol{g}}$ and over ϕ gives, respectively

$$\mathcal{F}_{\zeta}\tilde{\boldsymbol{g}} = -\mathcal{F}_{\xi}\nabla\phi, \quad (14)$$

$$\nabla \cdot (\mathcal{F}_{\kappa}\nabla\phi + \mathcal{F}_{\xi}\tilde{\boldsymbol{g}}) = 4\pi G\rho. \quad (15)$$

Since by equation (14) $\tilde{\boldsymbol{g}}$ and $\nabla\phi$ must be aligned, one of the variables, e.g. κ, determines the other two, e.g. by solving the equations $\mathcal{F}_{\zeta}^2\zeta = \mathcal{F}_{\xi}^2\kappa = -\mathcal{F}_{\zeta}\mathcal{F}_{\xi}\xi/2$, which follow from equation (14). Substituting the resulting expressions in equation (15) then gives $\nabla \cdot [\mu(\kappa^{1/2})\nabla\phi] = 4\pi G\rho$. So even in this general case we get the non-linear Poisson theory equation (1), and $\mathcal{F}(\zeta, \kappa, \xi)$ enters the dynamics of the system only through the function μ of a single variable, extracted from it.

If, however, we assume a priori that $\tilde{\boldsymbol{g}}$ is a gradient field $\tilde{\boldsymbol{g}} = -\nabla\tilde{\phi}$, the variety of choices of $\mathcal{F}$ in the above Lagrangian yields a richer family of theories,

$$\nabla \cdot (\mathcal{F}_{\zeta}\nabla\tilde{\phi} - \mathcal{F}_{\xi}\nabla\phi) = 0, \quad \nabla \cdot (\mathcal{F}_{\kappa}\nabla\phi - \mathcal{F}_{\xi}\nabla\tilde{\phi}) = 4\pi G\rho. \quad (16)$$

It includes our QUMOND theory with $\mathcal{F}_{\zeta} = -\nu(\zeta^{1/2})$, $\mathcal{F}_{\kappa} = 0$, $\mathcal{F}_{\xi} = -1$, and also the non-linear Poisson theory with $\mathcal{F}_{\zeta} = \mathcal{F}_{\xi} = 0$, $\mathcal{F}_{\kappa} = \mu(\kappa^{1/2})$.

Another interesting subclass is

$$\mathcal{F} = \mathcal{O}(\zeta) + \mathcal{N}(\kappa) - \mathcal{M}(\zeta + \kappa + \xi) \quad (17)$$

[note that $\zeta + \kappa + \xi = (\nabla\phi - \nabla\tilde{\phi})^2/a_0^2$]. With $\mathcal{O} = 0$ and $\mathcal{N} = \kappa$ we get Newtonian dynamics. For $\mathcal{N} = 0$, we have $\mathcal{F}_{\kappa} = \mathcal{F}_{\xi} = -\mathcal{M}'$, $\mathcal{F}_{\zeta} = \mathcal{O}' - \mathcal{M}'$; so, defining $\phi^* = \phi - \tilde{\phi}$, we get from equation (16),

$$\nabla \cdot \left\{ \mathcal{O}' \left[(\nabla\tilde{\phi}/a_0)^2 \right] \nabla\tilde{\phi} \right\} = -\nabla \cdot \left\{ \mathcal{M}' \left[(\nabla\phi^*/a_0)^2 \right] \nabla\phi^* \right\}$$
$$= 4\pi G\rho. \quad (18)$$

[4] The second term in the first part of $\mathcal{L}$ can be replaced by $+2\nabla\phi \cdot \tilde{\boldsymbol{g}}$ from which it differs by a divergence.

[5] We require invariance under $\phi \to \phi + \text{const}$. Also we limit ourselves here to actions where only $\tilde{\boldsymbol{g}}$ appears, and not its derivative, because we want to compare with theories where $\tilde{\boldsymbol{g}}$ is a gradient of some potential, and we want to avoid higher derivatives of that potential.

Thus, $\phi = \tilde{\phi} + \phi^*$, with $\tilde{\phi}$ and ϕ^* satisfying the non-linear Poisson equation. This covers the NR limit of Tensor–Vector–Scalar (TeVeS) for $\mathcal{O} \propto \zeta$, so $\tilde{\phi}$ satisfies the linear Poisson equation.

An interesting subfamily is $\mathcal{O} = \alpha\zeta$ ($\alpha \neq 0$), $\mathcal{N} = \beta\kappa$, for which the Lagrangian density is

$$\mathcal{L} = -\frac{1}{8\pi G}\left\{\alpha(\nabla\tilde{\phi})^2 + \beta(\nabla\phi)^2 - a_0^2\mathcal{M}\left[(\nabla\phi - \nabla\tilde{\phi})^2/a_0^2\right]\right\} + \rho\left(\frac{1}{2}\boldsymbol{v}^2 - \phi\right), \tag{19}$$

or, in terms of ϕ and $\phi^* = \phi - \tilde{\phi}$:

$$\mathcal{L} = -\frac{1}{8\pi G}\left\{(\alpha+\beta)(\nabla\phi)^2 - 2\alpha\nabla\phi\cdot\nabla\phi^* + \alpha(\nabla\phi^*)^2 - a_0^2\mathcal{M}\left[(\nabla\phi^*)^2/a_0^2\right]\right\} + \rho\left(\tfrac{1}{2}\boldsymbol{v}^2 - \phi\right), \tag{20}$$

where I used the freedom to normalize α, β and $\mathcal{M}$ so that G is the Newton constant. The field equations are then

$$\nabla\cdot[\mu^*(|\nabla\phi^*|/a_0)\nabla\phi^*] = 4\pi G\rho,$$
$$\Delta\phi = 4\pi G\beta^{-1}\rho + \beta^{-1}\nabla\cdot(\mathcal{M}'\nabla\phi^*) = \nabla\cdot[(1-\alpha^{-1}\mathcal{M}')\nabla\phi^*], \tag{21}$$

with

$$\mu^* = \beta - \frac{\alpha+\beta}{\alpha}\mathcal{M}'[(\nabla\phi^*/a_0)^2]. \tag{22}$$

One has to solve the non-linear Poisson equation for ϕ^*, and then the linear version for ϕ. In fact, for $\alpha + \beta \neq 0$, the second equation (21) can be written as

$$\Delta\phi = \frac{1}{\alpha+\beta}\Delta(\alpha\phi^* + \phi^N), \tag{23}$$

so the MOND potential is simply a linear combination of the Newtonian potential and a solution of the first of equation (21), which is of the type of equation (1).

The case $\alpha + \beta = 0$ ($\mu^* = \beta$) is an interesting special case as it gives our QUMOND theory (see below).

Consider now the Newtonian and MOND limits of the theory underlaid by the Lagrangian density (20). It is enough to consider the spherical case so that applying Gauss theorem to the field equations (21) we get algebraic relations between the (minus) radial accelerations $g = \mathrm{d}\phi/\mathrm{d}r \geq 0$, $g^* = \mathrm{d}\phi^*/\mathrm{d}r$, and the Newtonian acceleration $g_N = \mathrm{d}\phi_N/\mathrm{d}r \geq 0$:

$$\mu^* g^* = g_N, \quad g = (1-\alpha^{-1}\mathcal{M}')g^* = \frac{\alpha - \mathcal{M}'}{\alpha\beta - (\alpha+\beta)\mathcal{M}'}g_N. \tag{24}$$

In the Newtonian limit $g/g_N \to 1$ so

$$\mathcal{M}' \to \mathcal{M}'_\infty = \frac{\alpha(\beta-1)}{\alpha+\beta-1}. \tag{25}$$

This relation between α, β and $\mathcal{M}'_\infty$ does not require any tuning between different contributions to the action. It reflects our taking G to be the phenomenological Newton constant. Starting with a coupling G', we would end up with a relation $\mathcal{M}'_\infty = \alpha(\beta - G'/G)/(\alpha + \beta - G'/G)$, which is only used to define Newton's constant in terms of α, β, G' and $\mathcal{M}'_\infty$.

If $\mathcal{M}$ is such that $\mathcal{M}'_\infty = 0$, we must have $\beta = 1$ to get the correct Newtonian limit (or rather $G = G'/\beta$, which allows us to normalize the coefficients so that $G' = G, \beta = 1$), α is not constrained. For $\mathcal{M}'_\infty$ infinite we have to have $\alpha + \beta = 1$. When $\mathcal{M}'_\infty$ is finite we have neither $\beta = 1$ nor $\alpha + \beta = 1$.

We can write g as a function of g_N alone, distinguishing between two cases. If $\alpha + \beta \neq 0$,

$$g^* = \nu^*(|g_N|/a_0)g_N,$$
$$g = g_N + \frac{1-\alpha-\beta}{\alpha+\beta}g_N + \frac{\alpha}{\alpha+\beta}\nu^*(|g_N|/a_0)g_N, \tag{26}$$

where $\nu^*(y)$, as before, is such that if $y = x\mu^*(x)$ then $x = y\nu^*(y)$. If $\alpha + \beta = 0$, we have

$$g^* = \beta^{-1}g_N, \quad g = g_N + \beta^{-2}\left\{\mathcal{M}'\left[(g_N/\beta a_0)^2\right] - \mathcal{M}'_\infty\right\}g_N, \tag{27}$$

where here, according to equation (25), $\mathcal{M}'_\infty = \beta(\beta - 1)$. From these we can read the requirements for the Newtonian and MOND limits. Consider first the case $\alpha + \beta \neq 0$. In the MOND limit, $g_N/a_0 \to 0$ we have to have $g/g_N \to (g_N/a_0)^{-1/2}$ diverging. The last term in expression (26) dominates, and we must have $[\alpha/(\alpha+\beta)]\nu^*(y)y \to y^{1/2}$.

In considering the Newtonian limit for the $\alpha + \beta \neq 0$ case, we have to fork again: If $\alpha + \beta = 1$ (which, as we saw, applies when $\mathcal{M}'_\infty$ is infinite), the second term in the second of equations (26) vanishes, and so $\nu(y)y$ has to vanish for $y \to \infty$. However, if the first of equation (21) is to be elliptic, as it must, $y\nu^*(y)$ must be monotonic function. We saw that in the MOND regime $[\alpha/(\alpha+\beta)]\nu^*(y)y \approx y^{1/2}$ is an increasing function, and so it must be increasing everywhere. This means that $g/g_N - 1$ must vanish in the Newtonian limit slower than $(g_N/a_0)^{-1}$. This, however, is in conflict with Solar system constraints (Milgrom 1983, 2009a; Sereno & Jetzer 2006); this would thus rule out the case $\alpha + \beta = 1$.[6]

When $\alpha + \beta \neq 1$, ν^* has to tend to a constant value for large arguments: $\nu^*(y) \to (\alpha + \beta - 1)/\alpha$, and this it can do with arbitrary speed without violating the ellipticity condition. However, comparing the MOND and Newtonian limits in this case we see that if $0 < \alpha + \beta < 1$, ν^* has to vanish for some finite value of its argument, and this means that $\mu^*(x) = 1/\nu^*(y)$ as to blow up at a finite argument value. This is undesirable, and excludes the parameter range $0 < \alpha + \beta < 1$, leaving us with $\alpha + \beta > 1$ or $\alpha + \beta \leq 0$. There may be other constraints on α, β to be investigated (e.g. positivity conditions).

Now return to the interesting boundary case $\alpha + \beta = 0$. There is no matter-of-principle constraint on how fast $\mathcal{M}' - \mathcal{M}'_\infty$ can vanish for large arguments; so this theory can be made to approach Newtonian dynamics arbitrarily fast. The MOND limit phenomenology dictates $\mathcal{M}'(z) \approx \beta^{3/2}z^{-1/4}$ for $z \ll 1$. In fact, we can, when $\alpha + \beta = 0$, absorb β in the definition of ϕ^*, so that $\phi^* \to \beta^{-1}\phi^*$, giving an equivalent theory with $\mathcal{M}(z/\beta^2) - (1-\beta^{-1})z$ as the new $\mathcal{M}$. The new $\mathcal{M}'$ then vanishes for high arguments. Without loss of generality we can thus put in this case $\beta = 1$ and $\mathcal{M}'_\infty = 0$, with the equations of motion,

$$\Delta\phi^* = 4\pi G\rho,$$
$$\Delta\phi = \nabla\cdot[(1+\mathcal{M}')\nabla\phi^*] = 4\pi G\rho + \nabla\cdot(\mathcal{M}'\nabla\phi^*). \tag{28}$$

This theory is equivalent to the QUMOND theory I started with, with $\mathcal{Q}(z) = z + \mathcal{M}(z)$, and so $\nu(y) = 1 + \mathcal{M}'(y^2)$. $\nabla\phi^*$ is the Newtonian acceleration, and $(4\pi G)^{-1}\nabla\cdot(\mathcal{M}'\nabla\phi^*)$ is the density of the 'phantom matter' representing DM. In the MOND limit $z \to 0$ we have $\mathcal{M}'(z) \to z^{-1/4}$.

[6] This finding is of the same kind arrived at by Zhao & Famaey (2006) in regard to certain versions of TeVeS.

In all the above examples, the theory can be cast as two equations (for two potentials) that can be solved separately [as for the system (18)], or sequentially [as for system (21)]. For a general $\mathcal{F}$, this is presumably not possible and solving the resulting theory is rather more challenging.

Generalizing even further, we could include several auxiliary acceleration fields, $\boldsymbol{g}_i$, in addition to ϕ, and have $\mathcal{L}_{\rm P}$ a function of all the scalars $\boldsymbol{g}_i^2$, $(\nabla\phi)^2$, $\boldsymbol{g}_i\cdot\boldsymbol{g}_j$, $\boldsymbol{g}_i\cdot\nabla\phi$. Without constraining $\boldsymbol{g}_i$ to be gradients we still get the non-linear Poisson theory; with the constraints we seem to get a yet richer family. All the above theories are equivalent for a spherical system.

2.3 Extensions

Here, I describe succinctly two interesting extensions of the class of theories described above.

These bi-potential theories have inspired a class of relativistic formulations of bi-metric MOND (BIMOND) theories (Milgrom 2009c,d). The BIMOND theories not only were constructed in analogy with the bi-potential theories discussed here, but, in fact, reduce to them in the NR limit. The Lagrangian density of the BIMOND theories is constructed after the fashion of the Lagrangian density (19). Instead of two potentials we now have two metrics. One, $g_{\mu\nu}$, is the MOND metric, descending to the MOND potential in the NR limit, and the other, $\hat{g}_{\mu\nu}$, is an auxiliary one giving $\tilde{\phi}$ in the limit. Matter couples only to the MOND metric in the standard way, echoing the fact that here matter couples only to ϕ. As in many bimetric formulations discussed in the literature, the free Lagrangian densities $(\nabla\phi)^2$, $(\nabla\tilde{\phi})^2$ are replaced by the corresponding Ricci scalars of the two metrics. The novelty enters in designing the interaction term between the two metrics. The difference between the Levi–Civita connections of the two metrics,

$$C^{\alpha}_{\beta\gamma}=\Gamma^{\alpha}_{\beta\gamma}-\hat{\Gamma}^{\alpha}_{\beta\gamma} \tag{29}$$

is a tensor, which, moreover, plays the role of gravitational accelerations. Such acceleration-like tensors are obviously crucial in the context of MOND, as they permit us to construct dimensionless scalars from $a_0^{-1}C^{\alpha}_{\beta\gamma}$ and then take functions of these to serve as interpolating functions between the GR and the MOND regime.[7]

I have thus considered actions of the form:

$$I=-\frac{1}{16\pi G}\int\left[\beta g^{1/2}R+\alpha\hat{g}^{1/2}\hat{R}-(g\hat{g})^{1/4}f(\kappa)a_0^2\mathcal{M}(\Upsilon/a_0^2)\right]\mathrm{d}^4x + I_{\rm M}(g_{\mu\nu},\psi_i)+\hat{I}_{\rm M}(\hat{g}_{\mu\nu},\chi_i), \tag{30}$$

where in the argument of $\mathcal{M}$, Υ is a scalar quadratic in the tensor $C^{\alpha}_{\beta\gamma}$. As in our NR case $\mathcal{M}'$ plays the role of an interpolating function between the MOND and GR regimes. $I_{\rm M}$ is the matter action, with matter degrees of freedom represented by ψ_i, coupling only to $g_{\mu\nu}$ [g and $\hat{g}$ are minus the determinants of the two metrics, and $\kappa=(g/\hat{g})^{1/4}$]. I also allow for 'twin' matter described by degrees of freedom χ_i, which couples only to $\hat{g}_{\mu\nu}$. Since the scalar Υ contains only first derivatives of the metrics this action leads to second-order field equations.

The other interesting extension, inspired, in return, by the BIMOND theories, is the possible inclusion of 'twin' matter in our NR theories. This can be effected by adding to our Lagrangian density (19) a term $\gamma\tilde{\rho}[(1/2)\tilde{\boldsymbol{v}}^2-\tilde{\phi}]$ (which would be the NR limit of $\hat{I}_{\rm M}$). We then have two types of matter, each accelerated directly, in the standard manner, only by its own potential, but interacting indirectly, 'gravitationally', through the coupling between their potentials, rather unlike standard gravity. The field equations (21) now read

$$\nabla\cdot[\mu^*(|\nabla\phi^*|/a_0)\nabla\phi^*]=4\pi G(\rho-\beta\gamma\alpha^{-1}\tilde{\rho}),$$
$$\Delta\phi=4\pi G\beta^{-1}\rho+\beta^{-1}\nabla\cdot(\mathcal{M}'\nabla\phi^*)=\nabla\cdot[(1-\alpha^{-1}\mathcal{M}')\nabla\phi^*]+4\pi G\gamma\alpha^{-1}\tilde{\rho}. \tag{31}$$

I will discuss the interesting implications of such theories with twin matter in a separate paper (Milgrom, in preparation), taking in the rest of this paper $\tilde{\rho}=0$.

3 FORCES AND VIRIAL RELATIONS

I concentrate hereafter on the simpler, QUMOND special case. Most results are easily carried to the more general case. Because of the similarities between the theories, many of the properties of the non-linear Poisson formulation derived in the past (see e.g. Bekenstein & Milgrom 1984; Milgrom 1986a,b, 1994b, 1997, 1998, 2002a) invite derivation of analogous properties of the QUMOND formulation. I derive some of these here and in the following sections.

3.1 Forces

The force $\boldsymbol{F}$ on the collection of masses in a volume V is defined as the generator of translations, in the sense that under a rigid translation of the masses in V alone by a small increment $\delta\boldsymbol{r}$ the potential energy changes by

$$\delta L_{\rm P}=-\delta\boldsymbol{r}\cdot\boldsymbol{F}. \tag{32}$$

Such a translation causes a change in the density $\delta\rho=-\delta\boldsymbol{r}\cdot\nabla\rho$, inside V and zero outside. This generates corresponding changes in the potentials; but taking into account the stationarity of $L_{\rm P}$ under such changes (since the total mass is fixed, the variation in the potentials vanishes at infinity) we have $\delta L_{\rm P}=\int_V\phi\delta\rho=-\delta\boldsymbol{r}\cdot\int_V\phi\nabla\rho$. Integrating by parts then gives

$$\boldsymbol{F}=-\int_V\mathrm{d}^3r\,\rho\nabla\phi. \tag{33}$$

It is useful to consider the stress tensor of the gravitational field, P. One way to drive it is to write $L_{\rm P}$ as a coordinate scalar in curved space, and consider its variation under a change δg_{ij} in the metric, to get P from

$$\delta L_{\rm P}=\frac{1}{2}\int g^{1/2}\mathrm{d}^3r\,\mathsf{P}_{ij}\delta g^{ij}, \tag{34}$$

where summation over repeated indices is understood. One finds from equation (3), going back to the Euclidean case,

$$4\pi G\mathsf{P}=\left(\boldsymbol{g}\cdot\boldsymbol{g}^N-\frac{a_0^2}{2}\mathcal{Q}\right)\mathsf{I}+\nu\boldsymbol{g}^N\otimes\boldsymbol{g}^N-\boldsymbol{g}\otimes\boldsymbol{g}^N-\boldsymbol{g}^N\otimes\boldsymbol{g}, \tag{35}$$

where I is the unit tensor.[8] For solutions of the equations of motion, the divergence of P is found to be

$$\nabla\cdot\mathsf{P}=\rho\boldsymbol{g}. \tag{36}$$

[7]Here $c=1$, otherwise we use the MOND scalelength $\ell=c^2/a_0$ in the dimensionless tensors $\ell C^{\alpha}_{\beta\gamma}$.

[8]Since P is derived from the action without reference to whether $\boldsymbol{g}_N$ is a gradient, this expression for P, and the subsequent expressions that use it, remains valid in the non-linear Poisson formulation if we replace $\boldsymbol{g}^N$ by $-\mu\nabla\phi$, and $\boldsymbol{g}$ by $-\nabla\phi$. This gives $4\pi G\mathsf{P}=[\mu(\nabla\phi)^2-(a_0^2/2)\mathcal{Q}]\mathsf{I}-\mu\nabla\phi\otimes\nabla\phi$, which agrees with the expression given in Milgrom (2002a).

We can thus write the force on V as an integral over its surface Σ,

$$\boldsymbol{F} = \int_V \mathrm{d}^3r\,\rho\boldsymbol{g} = \int_\Sigma \mathsf{P}\cdot\mathrm{d}\boldsymbol{\sigma}, \tag{37}$$

or

$$\boldsymbol{F} = \frac{1}{4\pi G}\int_\Sigma \;(\nu\boldsymbol{g}^N - \boldsymbol{g})\boldsymbol{g}^N\cdot\mathrm{d}\boldsymbol{\sigma} - \boldsymbol{g}^N\boldsymbol{g}\cdot\mathrm{d}\boldsymbol{\sigma} + \left(\boldsymbol{g}\cdot\boldsymbol{g}^N - \frac{a_0^2}{2}\mathcal{Q}\right)\mathrm{d}\boldsymbol{\sigma}. \tag{38}$$

The torque on the volume V – the generator of rotations in the above sense – is

$$\boldsymbol{T} = \int_V \mathrm{d}^3r\,\rho\boldsymbol{r}\times\boldsymbol{g} = \int_\Sigma \boldsymbol{r}\times\mathsf{P}\cdot\mathrm{d}\boldsymbol{\sigma}, \tag{39}$$

where I integrated by parts making use of the symmetry of P or

$$\boldsymbol{T} = \frac{1}{4\pi G}\int_\Sigma \;[\boldsymbol{r}\times(\nu\boldsymbol{g}^N - \boldsymbol{g})]\boldsymbol{g}^N\cdot\mathrm{d}\boldsymbol{\sigma} - \boldsymbol{r}\times\boldsymbol{g}^N\boldsymbol{g}\cdot\mathrm{d}\boldsymbol{\sigma} + \left(\boldsymbol{g}\cdot\boldsymbol{g}^N - \frac{a_0^2}{2}\mathcal{Q}\right)\boldsymbol{r}\times\mathrm{d}\boldsymbol{\sigma}. \tag{40}$$

In Appendix A, I give other expressions for $\boldsymbol{F}$ and $\boldsymbol{T}$ as surface integrals, which might also be useful.

For an isolated bounded system, we get vanishing total force and torque, as can be seen by taking the integration surface in equations (38) and (40) as the sphere at infinity, and using the asymptotic behaviour of the fields. This is tantamount to the conservation of the total momentum and angular momentum in an isolated system.

3.2 Virial relations

Multiplying equation (6) by ϕ^N and integrating over all space gives an integral relation satisfied by solutions of the field equations,[9]

$$\int\left[\boldsymbol{g}^N\cdot\boldsymbol{g} - \nu(g_N/a_0)g_N^2\right]\mathrm{d}^3r = 0. \tag{41}$$

This relation, together with the useful inequality

$$zQ'(z) > \frac{1}{2}Q(z), \qquad \text{for} \quad z > 0, \tag{42}$$

implies that the free-field energy $(8\pi G)^{-1}\int\mathrm{d}^3r[2\nabla\phi\cdot\nabla\phi^N - a_0^2\mathcal{Q}]$ is always positive for solutions of the field equations.[10] To derive inequality (42) look at $H(y)\equiv zQ'(z) - Q(z)/2 = y^2\nu(y) - Q(y^2)/2$, with $z = y^2$. We have $H(0) = 0$ because $\mathcal{Q}(0) = 0$, and $H'(y) = y(y\nu' + \nu) > 0$ for $y > 0$, from the uniqueness condition; so $H(y) > 0$ for $y > 0$.

Define the virial as

$$\mathcal{V} \equiv -\int\mathrm{d}^3r\,\rho\boldsymbol{r}\cdot\boldsymbol{g}, \tag{43}$$

which is useful in several contexts. For example, since the force density on a system can be written as $\boldsymbol{f}(\boldsymbol{r}) = \rho(\boldsymbol{r})\,\boldsymbol{g}(\boldsymbol{r})$, we can write $\mathcal{V} = -\int\mathrm{d}^3r\,\boldsymbol{r}\cdot\boldsymbol{f}$. The virial is the generator of space scaling in the sense that under $\boldsymbol{r}\to(1+\epsilon)\boldsymbol{r}$ for small ϵ, $L_\mathrm{P}\to L_\mathrm{P} + \epsilon\mathcal{V}$ (see detailed discussion in Milgrom 1997). Using equation (36), we write

$$\mathcal{V} = -\int\mathrm{d}^3r(\nabla\cdot\mathsf{P})\cdot\boldsymbol{r} = -\int\mathrm{d}^3r\nabla\cdot(\mathsf{P}\cdot\boldsymbol{r}) + \int\mathrm{d}^3r\mathcal{P}, \tag{44}$$

where the trace

$$\mathcal{P}\equiv\mathrm{Tr}(\mathsf{P}) = \frac{1}{4\pi G}\left[\nu g_N^2 + \boldsymbol{g}\cdot\boldsymbol{g}^N - \frac{3}{2}a_0^2\mathcal{Q}\right]. \tag{45}$$

Write the first term in equation (44) as a surface integral to get

$$4\pi G\mathcal{V} = \int\mathrm{d}^3r\left[2\nu g_N^2 - \frac{3}{2}a_0^2\mathcal{Q}\right] - \int_\Sigma\left(\boldsymbol{g}\cdot\boldsymbol{g}^N - \frac{a_0^2}{2}\mathcal{Q}\right)\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{\sigma} + \nu\boldsymbol{r}\cdot\boldsymbol{g}^N\boldsymbol{g}^N\cdot\mathrm{d}\boldsymbol{\sigma} - \boldsymbol{r}\cdot\boldsymbol{g}\boldsymbol{g}^N\cdot\mathrm{d}\boldsymbol{\sigma} - \boldsymbol{r}\cdot\boldsymbol{g}^N\boldsymbol{g}\cdot\mathrm{d}\boldsymbol{\sigma}, \tag{46}$$

where I made use of the integral relation (41).[11,12] In Appendix A, I give another useful expression for the virial.

In a theory in which $\mathcal{Q}(z)$ vanishes at small z faster than the MOND behaviour $\mathcal{Q}\propto z^{3/4}$, such as in Newtonian dynamics, the surface integrals vanish at infinity, and what remains of equation (46) forms the basis of a virial relation upon its integration over time. In the case of MOND both the volume integral and the surface integrals are finite. Integrating over a sphere at infinity, all the vectors appearing in the surface integral in equation (46) are radial. It follows then that all cancel except $(a_0^2/2)\int_\Sigma\mathcal{Q}\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{\sigma}$, which contributes $(8\pi G/3)M(MGa_0)^{1/2}$, where M is the total mass of the system.[13] So,

$$\mathcal{V} = \frac{1}{4\pi G}\int\mathrm{d}^3r\left[2\nu g_N^2 - \frac{3}{2}a_0^2\mathcal{Q}\right] + \frac{2}{3}M(MGa_0)^{1/2}. \tag{47}$$

If we view ρ as made up of many discrete small point masses, so that each can be considered a test particle in the field of the rest, then the force, $\boldsymbol{F}_i$, on particle i at position $\boldsymbol{r}_i$ is $-\,m_i\nabla\phi(\boldsymbol{r}_i)$ (this is not true for a non-test particle). We can then write the virial as[14]

$$\mathcal{V} = -\sum_i\boldsymbol{r}_i\cdot\boldsymbol{F}_i = -\sum_i m_i\boldsymbol{r}_i\cdot\ddot{\boldsymbol{r}}_i = \sum_i m_i\boldsymbol{v}_i^2 - \frac{\mathrm{d}}{\mathrm{d}t}\sum_i m_i\boldsymbol{r}_i\cdot\boldsymbol{v}_i. \tag{48}$$

Use equation (47) and, as usual, average over a long time, dropping the last term in equation (48) for systems that remain bound over long times, we get for such systems the virial relation

$$\langle\boldsymbol{v}^2\rangle\equiv M^{-1}\left\langle\sum_i m_i\boldsymbol{v}_i^2\right\rangle_t = \frac{2}{3}(MGa_0)^{1/2} + \frac{1}{4\pi MG}\left\langle\int\mathrm{d}^3r\left[2\nu g_N^2 - \frac{3}{2}a_0^2\mathcal{Q}\right]\right\rangle_t, \tag{49}$$

where $\langle\rangle_t$ denotes the long-time average.

There is an analogous relation that holds in the non-linear Poisson formulation (Milgrom 1994b), but the integral there involves the MOND potential, for which we have to solve before we can use this virial relation. In the present theory, the integral involves only the Newtonian acceleration field and is readily calculated from the mass distribution – an example of the added amenability of the present formulation.

[9]With the MOND behaviour of ν and the asymptotic fields, each of the two terms in the integrand gives rise to an integral that diverges logarithmically; the integral of the difference not only converges, but vanishes.

[10]The energy integral actually diverges logarithmically for an isolated system.

[11]Each of the terms in expression (45) for $\mathcal{P}$ behaves as r^{-3} at infinity, but $\mathcal{P}$ itself vanishes faster, so it gives a finite integral over space.

[12]The above expressions for $\boldsymbol{F}, \boldsymbol{T}$ and $\mathcal{V}$ are invariant to an addition of a constant to $\mathcal{Q}$, as they should be.

[13]This applies for the particular choice of the additive constant in the definition of $\mathcal{Q}$ for which $\mathcal{Q}(0) = 0$.

[14]In the fluid approximation, use of the Euler and continuity equations leads directly to $\mathcal{V} = \int\mathrm{d}^3r\rho\boldsymbol{v}^2 - (\mathrm{d}/\mathrm{d}t)\int\mathrm{d}^3r\rho\boldsymbol{r}\cdot\boldsymbol{v}$.

4 THE DEEP-MOND LIMIT

In the deep-MOND limit, the Lagrangian density becomes

$$\mathcal{L} = \frac{a_0^2}{6\pi G}\left[(\nabla\phi^N/a_0)^2\right]^{3/4} - \frac{1}{4\pi G}\nabla\phi\cdot\nabla\phi^N + \rho\left(\frac{1}{2}\boldsymbol{v}^2 - \phi\right). \tag{50}$$

Multiplying by a_0G and absorbing $a_0\phi^N \rightarrow \phi^N$ gives an action with only a_0Gm_i appearing, as required by the basic MOND tenets. Seen differently, under space–time-scaling $(\boldsymbol{r}, t) \rightarrow \lambda(\boldsymbol{r}, t)$, with ϕ having zero internal dimension so that $\phi(\boldsymbol{r}) \rightarrow \phi(\boldsymbol{r}/\lambda)$, and ϕ^N having dimension -1, so $\phi^N(\boldsymbol{r}) \rightarrow \lambda^{-1}\phi^{\mathrm{N}}(\boldsymbol{r}/\lambda)$, L is invariant. The action is thus multiplied by λ; so the equations of motion of the masses are invariant. This is the defining requirement of the deep-MOND limit of any theory (Milgrom 2009b).

For the present theory, there exists another symmetry of the potential Lagrangian, L_{P}, alone. Under space scaling $\boldsymbol{r} \rightarrow \lambda\boldsymbol{r}$, for a constant λ, where, again, $\phi(\boldsymbol{r}) \rightarrow \phi(\boldsymbol{r}/\lambda)$, $\phi^N(\boldsymbol{r}) \rightarrow \lambda^{-1}\phi^N(\boldsymbol{r}/\lambda)$, the potential Lagrangian density $\mathcal{L}_{\mathrm{P}}(\boldsymbol{r}) \rightarrow \lambda^{-3}\mathcal{L}_{\mathrm{P}}(\boldsymbol{r}/\lambda)$, and so L_{P} and the potential action are invariant. This means that the field equations in the deep-MOND limit of the theory are invariant to space scaling: if $\phi(\boldsymbol{r})$ is a solution for $\rho(\boldsymbol{r})$, then $\phi(\boldsymbol{r}/\lambda)$ is a solution for $\lambda^{-3}\rho(\boldsymbol{r}/\lambda)$. Recall that the deep-MOND limit of the non-linear Poisson equation is invariant to the whole group of conformal transformations in space, of which scaling is one (Milgrom 1997).

4.1 Virial relation

If the system is everywhere deep in the MOND regime, for which $a_0^2\mathcal{Q} = (4/3)\nu g_N^2$ with our choice of the additive normalization of $\mathcal{Q}$, the volume integral in equation (49) vanishes, and we have

$$\langle \boldsymbol{v}^2\rangle = \frac{2}{3}(MGa_0)^{1/2}. \tag{51}$$

This is an even more useful virial relation than equation (49), since it relates the 3D root-mean square velocity, of any system of test particles deep in the MOND regime, to the total mass, irrespective of the mass distribution.

At many instances, we want to view the system as made up of several finite, non-test masses, ignoring the internal goings on in each of them and using only the centre of mass velocities of these masses (such as galaxies in a small group, or a binary). When the accelerations of these masses are still deep in the MOND regime we can write the above relation (following e.g. the argumentation in Milgrom 1997) as

$$\langle \boldsymbol{v}^2\rangle = \frac{2}{3}(MGa_0)^{1/2}\left(1 - \sum_i q_i^{3/2}\right), \tag{52}$$

where $q_i = m_i/M$. This is based on the relation

$$\sum_i \boldsymbol{r}_i\cdot\boldsymbol{F}_i = -\frac{2}{3}M(MGa_0)^{1/2}\left(1 - \sum_i q_i^{3/2}\right), \tag{53}$$

which applies in this case. All these results are identical to those for the deep-MOND limit of the non-linear Poisson formulation. The general two-body force in the deep-MOND limit follows straightforwardly from this relation, and is thus the same as in the non-linear Poisson formulation (see Milgrom 1997).

Also discussed there is the relation between the above results and the space scale invariance of the non-linear Poisson formulation. This is also relevant to the present theory. I only note here that despite the formal invariance of L_{P} under scaling, the fact that L_{P} itself is infinite leaves room for a finite change in L_{P} (which can be viewed as the potential energy of the system) under scaling of the density distribution $\rho(\boldsymbol{r}) \rightarrow \rho_\lambda(\boldsymbol{r}) = \lambda^{-3}\rho(\boldsymbol{r}/\lambda)$. So, under such a change in the mass distribution we have

$$L_{\mathrm{P}}[\rho_\lambda] = L_{\mathrm{P}}[\rho] + \ln(\lambda^{\mathcal{V}}) = L_{\mathrm{P}}[\rho] + \frac{2}{3}M(MGa_0)^{1/2}\ln(\lambda). \tag{54}$$

This violation of the symmetry by the transformation properties of L_{P} is analogous to so-called 'anomalies' that appear in scale- or conformally invariant field theories, with the virial playing the role of anomalous dimension. In a system made of finite, point masses at positions $\boldsymbol{r}_i$ we have[15]

$$L_{\mathrm{P}}(\lambda\boldsymbol{r}_i) = L_{\mathrm{P}}(\boldsymbol{r}_i) + \frac{2}{3}M(MGa_0)^{1/2}\left(1 - \sum_i q_i^{3/2}\right)\ln(\lambda), \tag{55}$$

giving rise to relation (53) by taking its λ derivative at $\lambda = 1$.

5 A SYSTEM IN A CONSTANT EXTERNAL FIELD

Many times one is dealing with the dynamics in and about a small system that is itself subject to the gravitational field of a large, mother system. Examples are the field around a star, a globular cluster or a dwarf companion in a field of a mother galaxy. Instead of solving the full MOND subsystem-plus-mother-system problem, we may want to treat the subsystem alone, taking the presence of the mother system into account through the constant-acceleration boundary condition that it dictates in the vicinity of the subsystem. This approximation is good if we can erect around the subsystem a volume V, whose surface is Σ, such that (i) V is small enough that the acceleration due to the mother system alone is nearly constant in V, (ii) V is large enough so that on Σ the perturbation due to the subsystem is negligible compared with the acceleration due to the mother system. Assume then that these conditions are satisfied. Solve for the field of the mother system, which gives values $\boldsymbol{g}_0^N$ and $\boldsymbol{g}_0$ for the Newtonian and MOND accelerations at the position of the subsystem. We do not (and need not) specify any relation between these two accelerations. For a spherical mother system, they are related by the algebraic relation (2), but in general they are not so related, and are not even parallel.

Our approximation – becoming increasingly better as the mass of the subsystem is small compared with that of the mother system – is to solve the MOND field equations for the density ρ of the subsystem alone, dictating as boundary conditions at infinity $\nabla\phi \rightarrow -\boldsymbol{g}_0$, $\nabla\phi^N \rightarrow -\boldsymbol{g}_0^N$, instead of the conditions appropriate for an isolated system. We write

$$\nabla\phi^N = -\boldsymbol{g}_0^N + \nabla\chi, \qquad \nabla\phi = -\boldsymbol{g}_0 + \nabla\psi, \tag{56}$$

and determine ψ from equation (6) with the boundary condition $\nabla\psi \rightarrow 0$ at infinity, given that χ is the standard Newtonian potential of the subsystem. Note, importantly, that $\boldsymbol{g}_0$ drops from the equation for the internal potential ψ; so, this potential does not depend on $\boldsymbol{g}_0$, only on $\boldsymbol{g}_0^N$. Take the direction of the latter to define the positive z axis, and its value in units of a_0: $\eta_N = |\boldsymbol{g}_0^N|/a_0$.

Unlike the isolated case, where $\hat{M} = \int\hat{\rho}\,\mathrm{d}^3r$ is infinite, for the case of a constant background field $\hat{M}$ is finite, and is given by (Milgrom 2009a)

$$\hat{M} = -\frac{1}{4\pi G}\int_\infty \nu\boldsymbol{g}^N\cdot\mathrm{d}\boldsymbol{\sigma} = \nu_0(1 + \hat{\nu}_0/3)M, \tag{57}$$

[15] Here, we only move the point masses to new positions $\lambda\boldsymbol{r}_i$, without dilating them intrinsically by a factor of λ, as would be required for equation (54) to apply.

where ν_0 and $\hat{\nu}_0$ are, respectively, the values of ν and its logarithmic derivative at $\eta_N(-1/2 \le \hat{\nu}_0 \le 0)$, and M is the mass of the subsystem.

5.1 Asymptotic behaviour of the field

The asymptotic behaviour of $\hat{\rho}$ is obtained from that of the Newtonian field, and is given by

$$\hat{\rho} \approx \nu_0\hat{\nu}_0 \frac{M}{4\pi}\frac{1}{r^3}\left(1 - \frac{3z^2}{r^2}\right). \tag{58}$$

The asymptotic internal MOND field ψ satisfies the Poisson equation for the above asymptotic form of $\hat{\rho}$. This determines ψ up to a harmonic function (a solution of the Laplace equation). Since ψ is required to vanish at infinity, only harmonic functions that decrease with r have to be considered. Among these we are interested in the leading term, which obviously leaves us with a freedom to add to ψ a multiple of r^{-1}. The coefficient of this term is then determined by imposing $\int_\Sigma \nabla\psi \cdot \mathrm{d}\sigma = 4\pi G\hat{M}$, which follows by applying Gauss's theorem to equation (6). Because $\hat{\rho} \propto r^{-3}P_2(z/r)$ asymptotically, where P_2 is the second Legendre polynomial, it follows that the sought after solution is of the form $r^{-1}P_2(z/r)$. Matching coefficients, we get that

$$\psi^* \approx -\frac{1}{6}\nu_0\hat{\nu}_0\frac{MG}{r}\left(1 - \frac{3z^2}{r^2}\right) \tag{59}$$

solves the above Poisson equation. As can be readily checked, $\int_\Sigma \nabla\psi^* \cdot \mathrm{d}\sigma = 0$ on a sphere. We thus have to add to ψ^* the potential $-\hat{M}G/r$ to satisfy Gauss's theorem. This finally gives for the asymptotic form of ψ,

$$\psi \approx -\frac{MG}{r}\nu_0\left(1 + \frac{\hat{\nu}_0}{2}\right)\left[1 - \frac{\hat{\nu}_0}{2+\hat{\nu}_0}\frac{z^2}{r^2}\right]. \tag{60}$$

To compare our results with the analogue ones for the non-linear Poisson formulation, which depend on the value of the MOND background acceleration, we have to decide which value of $\boldsymbol{g}_0$ to use when comparing with the results for a given $\boldsymbol{g}_0^N$. I pick heuristically, the value $\boldsymbol{g}_0 = \nu(|\boldsymbol{g}_0^N|/a_0)\boldsymbol{g}_0^N$. We can then write equation (58) as

$$\hat{\rho} \approx -\frac{M}{4\pi\mu_0}\frac{L_0}{1+L_0}\frac{1}{r^3}\left(1 - \frac{3z^2}{r^2}\right), \tag{61}$$

where μ_0 and L_0 are, respectively, the values of μ, and of its logarithmic derivative, at $\eta = |\boldsymbol{g}_0|/a_0$ [$\mu(x)$ is related to $\nu(y)$ as described above]. For the same $\boldsymbol{g}_0$, we can write equation (57) as

$$\hat{M} = \frac{3+2L_0}{3\mu_0(1+L_0)}M, \tag{62}$$

and the asymptotic potential as

$$\psi \approx -\frac{MG}{\mu_0 r}\frac{1+L_0/2}{1+L_0}\left(1 + \frac{L_0}{2+L_0}\frac{z^2}{r^2}\right). \tag{63}$$

These can now be compared with the analogue quantities for the non-linear Poisson theory for a MOND external field $\boldsymbol{g}_0$ (Milgrom 2009a),

$$\bar{\rho} \approx -\frac{M}{4\pi\mu_0}\frac{L_0}{(1+L_0)^{3/2}}\frac{1}{\hat{r}^3}\left(1 - \frac{3\hat{z}^2}{\hat{r}^2}\right), \tag{64}$$

where $\hat{z} = (1+L_0)^{-1/2}z,\ \hat{r} = (x^2+y^2+\hat{z}^2)^{1/2}$. We see a similar normalization and radial dependence as in $\hat{\rho}$, but a somewhat different angular dependence. For the total 'dynamical' mass, we have there

$$\bar{M} = \frac{1}{\mu_0 L_0^{1/2}}\arcsin\left(\frac{L_0}{1+L_0}\right)^{1/2}M, \tag{65}$$

which, numerically, is not very different from expression (62), differing by at most 6 per cent in the possible range $0 \le L_0 \le 1$. For the asymptotic, internal potential we have

$$\bar{\psi} \approx -\frac{MG}{\mu_0 r}(1+L_0)^{-1/2}\left(1 - \frac{L_0}{1+L_0}\frac{z^2}{r^2}\right)^{-1/2}. \tag{66}$$

An interesting difference between the two theories in the present context is that the symmetry axis for the QUMOND theory is the direction of the Newtonian background field, while that in the non-linear Poisson theory is the direction of the MOND background field. The two directions are in general different. This shows that the QUMOND theory cannot be equivalent to some form of the non-linear Poisson theory.[16] This difference can also help distinguish observationally between the two types of theories.

5.2 A system dominated everywhere by an external field

If $|\nabla\chi| \ll |\boldsymbol{g}_0^N|$ everywhere in the subsystem, we can write $\boldsymbol{g} = \boldsymbol{g}_0 - \nabla\psi$, with $\boldsymbol{g}_0$ dictated by the mother system, and ψ satisfying to lowest order in χ

$$\Delta\psi = \nu_0[\chi_{,xx} + \chi_{,yy} + (1+\hat{\nu}_0)\chi_{,zz}] = 4\pi G\nu_0\rho + \nu_0\hat{\nu}_0\chi_{,zz}, \tag{67}$$

or, in terms of the above, heuristically chosen, $\boldsymbol{g}_0$

$$\Delta\psi = \mu_0^{-1}\left[\chi_{,xx} + \chi_{,yy} + (1+L_0)^{-1}\chi_{,zz}\right]. \tag{68}$$

The analogue equation for the internal potential in a dominant external field, in the non-linear Poisson formulation is (Milgrom 1986a)

$$[\psi_{,xx} + \psi_{,yy} + (1+L_0)\psi_{,zz}] = \frac{4\pi G}{\mu_0}\rho = \mu_0^{-1}\Delta\chi. \tag{69}$$

In both theories, the main effect is to enhance gravity by a factor of $1/\mu_0$ over Newtonian gravity. The secondary effect, that of stretching the internal potential in the z direction by a factor of about $(1+L_0)^{1/2}$ is different in the two theories, as is the direction of the z-axis itself.

5.3 Centre-of-mass acceleration of composite systems

Consider a small body – such as an atom, a gas cloud or a star – freely falling in a mother system, such as a galaxy. Such subsystems are made of constituents that are, sometimes, subject to high internal accelerations, hence to high total accelerations. Is it possible then that the centre-of-mass motion of the composite subsystem is subject to the MOND acceleration of the galaxy, as it should for MOND phenomenology to work? This was shown to be the case for the non-linear Poisson formulation (Bekenstein & Milgrom 1984), and I now show that it is also the case for the present theory.

Take an arbitrary, bounded mass distribution ρ, representing the subsystem, placed in a background MOND field $\boldsymbol{g}_0$, such as a

[16] There is other evidence pointing to the same effect: consider, as an example, two unequal point masses. As explained in Milgrom (1986b), a region where $\bar{\rho}$ takes up both signs appears around the critical point where $\nabla\phi = 0$ between the two masses. Using the same arguments, we now see that such a region for $\hat{\rho}$ appears where $\nabla\phi^N = 0$. The critical point for $\nabla\phi$ in a non-linear Poisson theory is never at the Newtonian critical point (unless symmetry dictates it). So $\hat{\rho}$ cannot be a $\bar{\rho}$ for some non-linear Poisson theory. We will see more arguments to this effect below. If the present theory is equivalent to some non-linear Poisson formulation with some interpolating function μ, we would have to have $\mu(x) = 1/\nu(y)$, where $x\mu(x) = y$ in order for the two to coincide for spherical systems. But with this identification of ν the two theories are definitely not equivalent for aspherical systems.

galactic field in the example above. As explained above, provided the mass of the subsystem and its extent are much smaller than the corresponding attributes of the galaxy, we can describe the MOND field of the system by the solution of the MOND equations with a boundary condition of constant acceleration at infinity.

Use expression (38) to calculate the force on the system as an integral over the surface Σ. Take Σ large enough so that on it $\boldsymbol{g}_0$ strongly dominates the internal acceleration, so we can use the asymptotic form of the MOND field $\boldsymbol{g} = \boldsymbol{g}_0 - \nabla\psi$, where ψ is given by equation (60). Substituting this in the integrand term $-\boldsymbol{g}\boldsymbol{g}^N \cdot \mathrm{d}\sigma$ in equation (38), the term with $\boldsymbol{g}_0$ gives $-\boldsymbol{g}_0 \int_\Sigma \boldsymbol{g}^N \cdot \mathrm{d}\sigma = 4\pi G M \boldsymbol{g}_0$, where I used the Newtonian Gauss theorem. The term with $\nabla\psi$ combined with all the other terms in equation (38) can be shown to give a vanishing contribution in the limit where Σ goes to infinity. We are then left with $\boldsymbol{F} = M\boldsymbol{g}_0$, independent of the details of ρ, giving a centre-of-mass acceleration $\boldsymbol{g}_0$.

6 DISCUSSION

I have presented a new NR formulation of MOND as a modified gravitational potential theory. It is derivable from an action and enjoys the standard conservation laws. In a sense, it is an upgrade of the pristine formulation of MOND in which the MOND acceleration field is an algebraic function of the Newtonian acceleration field.

The theory is a member in a class of bi-potential theories in which only one – the MOND potential – couples to matter directly, whereas the other is an auxiliary potential. The MOND departure from Newtonian physics enters not through a modification of the free action of the potential (as happens in the original formulation of the Bekenstein Milgrom theory), but through the interaction between the two potential fields. In Milgrom (2002b), I described a membrane model for MOND in the spirit of the well-known membrane model of gravity. In this model, MOND departure from Newtonian gravity is introduced through modified 'elasticity' of the membrane (compared with that corresponding to Newtonian gravity). In the same vein, the present class of theories gravity can be attributed to the existence of two membranes, one on which matter live and with which alone it interacts directly, the other membrane interacting with the first. MOND effects are then introduced not as modified elasticity of the membranes – which is normal, albeit with the auxiliary membrane having negative elasticity – but through the interaction between the membranes.

These theories are now added to the non-linear Poisson formulation of MOND propounded a quarter century ago by Bekenstein & Milgrom (1984), to which it is similar in many ways. The two theories are shown, in fact, to fork out of the same MOND modification of a Palatini-like formulation of Newtonian gravity.

This addition carries with it the usual benefits of diversity. For example, it brings home the possibility that even more formulations await discovering, and encourages us to look for them.

Also, comparing the predictions of different formulations helps pinpoint results that are not generic to the MOND paradigm, not even to its NR formulations. It may also point to new directions for constructing relativistic formulations of MOND.

In itself the new formulation is a complete theory, on a par with the non-linear Poisson formulation, but rather easier to work with, involving, as it does, only linear differential equations.

Because of the similarities between the two formulations, this QUMOND formulation can also double as a well-motivated approximation for the less wieldy non-linear Poisson formulation (and so also, for the NR limit of theories of the class of TeVeS whose NR limit is a double-potential theory of the non-linear Poisson type). In fact, it is exactly in this role that such a formulation was used in Milgrom (2009a) – without recognizing its completeness as a theory – to calculate the external-field effect in the inner Solar system in the non-linear Poisson formulation. This yielded a closed expression for the desired effect, whereas the application of the non-linear theory required solving the field equation first. Those 'approximate' results are now recognized as exact in the new theory.[17]

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[17] There were also instances in the past, such as in Milgrom (1986b), and in Milgrom & Sanders (2008) where the present formalism was used to approximate the 'phantom' density of the non-linear Poisson theory; these too are exact in the present theory.

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APPENDIX A: ALTERNATIVE EXPRESSIONS FOR THE FORCE, THE TORQUE AND THE VIRIAL

The force on a subsystem made of the masses within the volume V of surface Σ, given in equation (33) can also be written as

$$\boldsymbol{F} = -\frac{1}{4\pi G}\int_V \mathrm{d}^3 r\, \Delta\phi^N \nabla\phi. \tag{A1}$$

The torque from equation (39) can be written as

$$\boldsymbol{T} = -\int_V \mathrm{d}^3 r\, \rho \boldsymbol{r}\times\nabla\phi = -\frac{1}{4\pi G}\int_V \mathrm{d}^3 r\, \Delta\phi^N \boldsymbol{r}\times\nabla\phi, \tag{A2}$$

and similarly for the virial. Integrating by parts several times in a certain order, using the field equation (6), and expressing volume integrals of divergences as surface integrals, we get expressions (38), (40) and (46) for these three quantities. Performing the integration by parts in a different order we get other expressions for these quantities, which are also useful,

$$\boldsymbol{F} = -\frac{1}{4\pi G}\int_\Sigma \Bigg\{ \boldsymbol{g}\boldsymbol{g}^N\cdot\mathrm{d}\boldsymbol{\sigma} + \left(\frac{a_0^2}{2}\mathcal{Q} - \nu g_N^2\right)\mathrm{d}\boldsymbol{\sigma} + \phi^N\left[(\mathrm{d}\boldsymbol{\sigma}\cdot\nabla)\boldsymbol{g} - \nabla(\nu\boldsymbol{g}^N\cdot\mathrm{d}\boldsymbol{\sigma})\right]\Bigg\}, \tag{A3}$$

or, writing, as in equation (8), $\boldsymbol{g} = \nu\boldsymbol{g}^N + \nabla\times\boldsymbol{A}$ we get

$$\boldsymbol{F} = -\frac{1}{4\pi G}\int_\Sigma \Bigg\{ \nu\boldsymbol{g}^N\boldsymbol{g}^N\cdot\mathrm{d}\boldsymbol{\sigma} + \left(\frac{a_0^2}{2}\mathcal{Q} - \nu g_N^2\right)\mathrm{d}\boldsymbol{\sigma} + (\nabla\times\boldsymbol{A})\boldsymbol{g}^N\cdot\mathrm{d}\boldsymbol{\sigma} + \phi^N\nabla[(\nabla\times\boldsymbol{A})\cdot\mathrm{d}\boldsymbol{\sigma}]\Bigg\}. \tag{A4}$$

$$\boldsymbol{T} = \frac{1}{4\pi G}\int_\Sigma \Bigg\{ -\boldsymbol{r}\times\boldsymbol{g}(\boldsymbol{g}^N\cdot\mathrm{d}\boldsymbol{\sigma}) - \frac{a_0^2}{2}\mathcal{Q}\boldsymbol{r}\times\mathrm{d}\boldsymbol{\sigma} + \nu\boldsymbol{r}\times\boldsymbol{g}^N(\boldsymbol{g}^N\cdot\mathrm{d}\boldsymbol{\sigma}) + \phi^N\left[\nabla\cdot(\nu\boldsymbol{g}^N)(\boldsymbol{r}\times\mathrm{d}\boldsymbol{\sigma}) + \boldsymbol{g}\times\mathrm{d}\boldsymbol{\sigma} - (\boldsymbol{r}\times\nabla)(\boldsymbol{g}\cdot\mathrm{d}\boldsymbol{\sigma})\right]\Bigg\}. \tag{A5}$$

$$4\pi G\mathcal{V} = \int \mathrm{d}^3 r\left[2\nu g_N^2 - \frac{3}{2}a_0^2\mathcal{Q}\right] + \int \boldsymbol{r}\cdot\boldsymbol{g}(\boldsymbol{g}^N\cdot\mathrm{d}\boldsymbol{\sigma}) + \frac{1}{2}\int\left[a_0^2\mathcal{Q} - 2\nu g_N^2\right]\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{\sigma} + \int\phi^N[(\boldsymbol{g}-\nu\boldsymbol{g}^N)\cdot\mathrm{d}\boldsymbol{\sigma} + (\boldsymbol{r}\cdot\nabla)(\boldsymbol{g}-\nu\boldsymbol{g}^N)\cdot\mathrm{d}\boldsymbol{\sigma}]. \tag{A6}$$

In all the above expressions ∇ in the last term does not act on the surface element $\mathrm{d}\sigma$.

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