Quasi-Orthogonal Space-Time Block Codes with Full Diversity

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Outline

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  - Tirkkonen-Boariu-Hottinen (TBH) Scheme
- Quasi-Orthogonal STBCs with Full Diversity
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Motivation

- Space-time block codes from orthogonal designs (Alamouti, Tarokh–Jafarkhani-Calderbank)
  - They have fast ML decoding, i.e., all information symbols can be separately decoded.
  - They have full diversity.
- However, the rates of STBC from complex orthogonal designs are upper bounded by $3/4$ for two or more transmit antennas.
  - Square codes by amicable designs
  - General codes by Wang-Xia 02
- It is difficult to construct orthogonal designs with rate higher than $1/2$ for more than four transmit antennas.

- To improve the symbol transmission rate, one natural way is to relax the requirement of the orthogonality.
- Recently, Jafarkhani (2001) and Tirkkonen-Boariu-Hottinen (TBH) proposed STBCs from quasi-orthogonal designs, where the orthogonality is relaxed to provide higher symbol transmission rate.
  - The ML decoding becomes the **pairwise** information symbol decoding instead of single information symbol decoding.
  - They do not have the full diversity.

- Jafarkhani Scheme:

  – For 4 transmit antennas, a quasi-orthogonal STBC with symbol transmission rate 1 was constructed from the Alamouti scheme as follows:

$$C = \begin{bmatrix} A & B \\ -\overline{B} & \overline{A} \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ -x_2^* & x_1^* & -x_4^* & x_3^* \\ -x_3^* & -x_4^* & x_1^* & x_2^* \\ x_4 & -x_3 & -x_2 & x_1 \end{bmatrix},$$

  where

$$A = \begin{bmatrix} x_1 & x_2 \\ -x_2^* & x_1^* \end{bmatrix}, \quad B = \begin{bmatrix} x_3 & x_4 \\ -x_4^* & x_3^* \end{bmatrix}.$$

- TBH scheme:
  - For 4 transmit antennas, a similar scheme is given as follows:

$$C = \begin{bmatrix} A & B \\ B & A \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ -x_2^* & x_1^* & -x_4^* & x_3^* \\ x_3 & x_4 & x_1 & x_2 \\ -x_4^* & x_3^* & -x_2^* & x_1^* \end{bmatrix},$$

where

$$A = \begin{bmatrix} x_1 & x_2 \\ -x_2^* & x_1^* \end{bmatrix}, \quad B = \begin{bmatrix} x_3 & x_4 \\ -x_4^* & x_3^* \end{bmatrix}.$$

- Assume the whole system is

$$Y = \sqrt{\rho/n}\, CH + W,$$

then the maximum-likelihood decoding at the receiver is

$$\arg \min_{(s_1, s_2, s_3, s_4) \in \mathcal{A}^4} ||Y - \sqrt{\rho/n}\, CH||_F^2$$

$$= (\arg \min_{(s_1, s_3) \in \mathcal{A}^2} f_1(s_1, s_3), \arg \min_{(s_2, s_4) \in \mathcal{A}^2} f_1(s_2, s_4)).$$

  – In the orthogonal space-time code case,

$$\arg \min_{(s_1, s_2, s_3) \in \mathcal{A}^3} ||Y - \sqrt{\rho/n}\, CH||_F^2$$

$$= (\arg \min_{s_1 \in \in \mathcal{A}} f_1(s_1), \arg \min_{s_2 \in \in \mathcal{A}} f_2(s_2), \arg \min_{s_3 \in \in \mathcal{A}} f_3(s_3)).$$

- The *diversity product* of the TBH scheme can be calculated as follows:

$$\begin{aligned}
\zeta &= \frac{1}{4}\min_{\Delta C \neq 0}\left|\det\left[(\Delta C)^{\mathcal{H}}(\Delta C)\right]\right|^{1/8} \\
&= \frac{1}{4}\min_{S \neq \tilde{S}}\left(\sum_{i=1}^{2}\left|(s_i - \tilde{s}_i) + (s_{i+2} - \tilde{s}_{i+2})\right|^2\right)^{1/4} \\
&\qquad \left(\sum_{i=1}^{2}\left|(s_i - \tilde{s}_i) - (s_{i+2} - \tilde{s}_{i+2})\right|^2\right)^{1/4} \\
&= \frac{1}{4}\min_{1 \leq i \leq 2}\min_{(u,v) \neq (\tilde{u},\tilde{v})}\left|(u-\tilde{u})^2 - (v-\tilde{v})^2\right|^{1/2},
\end{aligned}$$

in which $u, \tilde{u} \in \mathcal{A}_i$ and $v, \tilde{v} \in \mathcal{A}_{i+2}$. Moreover, $S = (s_1, s_2, s_3, s_4)$, $\tilde{S} = (\tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4)$, and $s_i, \tilde{s}_i \in \mathcal{A}_i$.

- If all of the signal constellations $\mathcal{A}_i$ are the same, then $\zeta = 0$, i.e., the TBH scheme does not have full diversity.

2. Quasi-Orthogonal STBCs with full diversity

- In both the Jafarkhani scheme and the TBH scheme, all information symbols are chosen in a single signal constellation. Thus, after modulation, the modulated signals do not have full diversity.
- The main idea of our new scheme is to choose the signal contellations properly to ensure the full diversity.

- For convenience, let us define the *minimum ζ-distance* between two signal constellations $\mathcal{A}$ and $\mathcal{B}$ as follows:

$$d_{\min,\zeta}(\mathcal{A},\mathcal{B}) \triangleq \min_{(u,v)\neq(\tilde{u},\tilde{v})} \left|(u-\tilde{u})^2-(v-\tilde{v})^2\right|^{1/2},$$

where $u,\tilde{u}\in\mathcal{A}$ and $v,\tilde{v}\in\mathcal{B}$.

- Obviously, we have

$$d_{\min,\zeta}(\mathcal{A},\mathcal{B}) \leq \min\left\{d_{\min}(\mathcal{A}), d_{\min}(\mathcal{B})\right\},$$

where $d_{\min}(\mathcal{A})$ and $d_{\min}(\mathcal{B})$ are the minimum Euclidean distances of the signal constellations $\mathcal{A}$ and $\mathcal{B}$, respectively.

- Then the diversity product of the TBH scheme can be rewritten as

$$\begin{aligned} \zeta &= \frac{1}{4} \min\nolimits_{1\le i\le 2}\, d_{\min,\zeta}(\mathcal{A}_i, \mathcal{A}_{i+2}) \\ &\le \frac{1}{4} \min\nolimits_{1\le i\le 2}\, d_{\min}(\mathcal{A}_i). \end{aligned}$$

- The diversity product is determined by the minimum ζ-distance of each pair of signal constellations $\mathcal{A}_i$ and $\mathcal{A}_{i+2}$.

- Problem: How to design the signal constellation pair $\mathcal{A}_i$ and $\mathcal{A}_{i+2}$ such that

$$d_{\min,\zeta}(\mathcal{A}_i, \mathcal{A}_{i+2}) > 0,$$

i.e., the quasi-orthogonal STBC has full diversity?

How to design the signal constellations?

- Problem: how to design signal constellations $\mathcal{A}$ and $\mathcal{B}$ such that $d_{\min,\zeta}(\mathcal{A},\mathcal{B}) > 0$?

- We found that for an arbitrary signal constellation $\mathcal{A}$, if

$$\mathcal{B} = e^{j\phi}\mathcal{A} \stackrel{\triangle}{=} \{e^{j\phi}s \,:\, s \in \mathcal{A}\},$$

  then $d_{\min,\zeta}(\mathcal{A}, e^{j\phi}\mathcal{A}) > 0$ for some specific rotation angle ϕ.

- We know that the minimum ζ-distance is upper bounded by

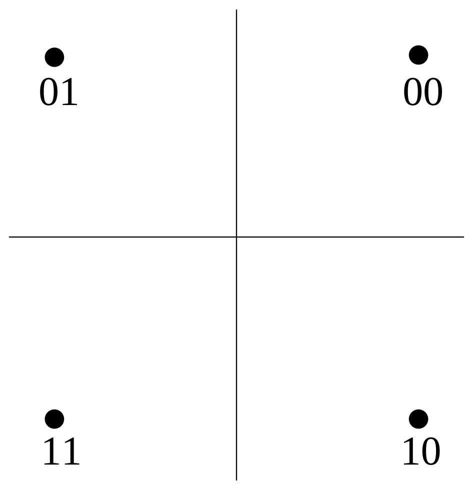
$$d_{\min,\zeta}(\mathcal{A}, e^{j\phi}\mathcal{A}) \leq d_{\min}(\mathcal{A}).$$

  So, the problem remaining is how to choose the rotation angle ϕ?

- For example, if $\mathcal{A}$ is BPSK $\{1, -1\}$, and the rotation angle ϕ is chosen as $\pi/2$, then

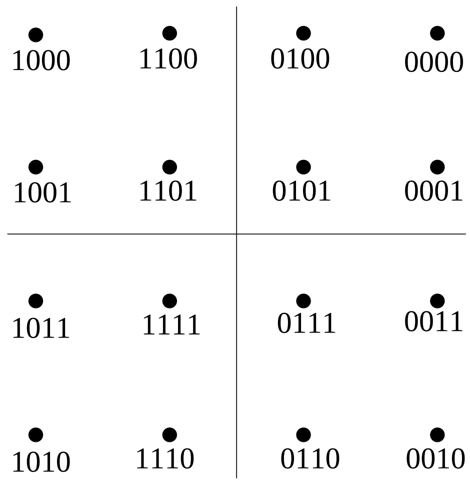
$$d_{\min,\zeta}(\mathcal{A}, e^{j\pi/2}\mathcal{A}) = d_{\min}(\mathcal{A}).$$

- If $\mathcal{A}$ is chosen as 4-QAM,



then the optimum rotation angle ϕ is $\pi/4$.

- If $\mathcal{A}$ is chosen as 16-QAM,



the optimum rotation angle ϕ?.

Theorem 1: Assume $\mathcal{A}$ is a signal constellation drawn from a square lattice, where the side length of the squares in the lattice is equal to $d_{\min}(\mathcal{A})$. Then, the minimum ζ-distance between $\mathcal{A}$ and $e^{j\pi/4}\mathcal{A}$ is $d_{\min}(\mathcal{A})$, i.e.,

$$d_{\min,\zeta}(\mathcal{A}, e^{j\pi/4}\mathcal{A}) = d_{\min}(\mathcal{A}).$$

- Recall that the minimum ζ-distance is upper bounded by $d_{\min}$, i.e.,

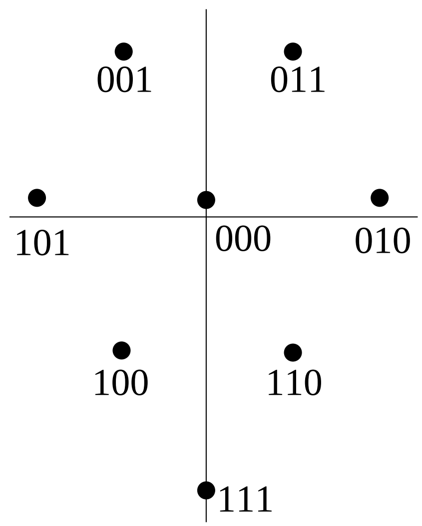
$$d_{\min,\zeta}(\mathcal{A}, e^{j\phi}\mathcal{A}) \leq d_{\min}(\mathcal{A}).$$

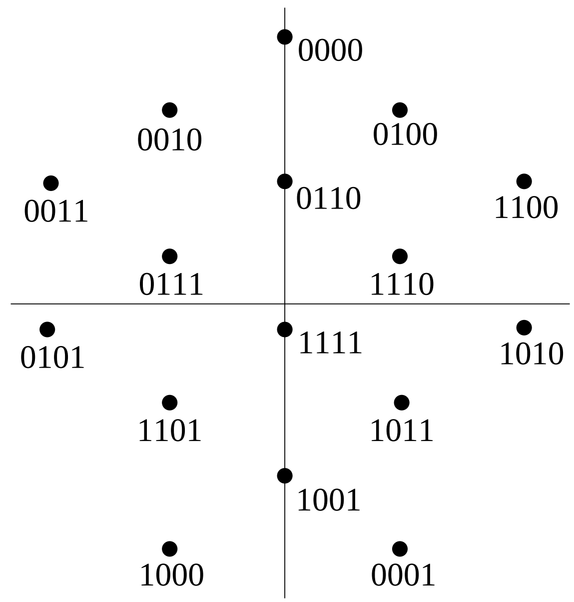
  If $\mathcal{A}$ is r-QAM, then the optimum rotation angle ϕ is $\pi/4$, and

$$d_{\min,\zeta}(\mathcal{A}, e^{j\pi/4}\mathcal{A}) = d_{\min}(\mathcal{A}).$$

- It is desired to have the signal constellation $\mathcal{A}$ with $d_{\min}$ as large as possible, meanwhile the minimum ζ-distance between $\mathcal{A}$ and $e^{j\phi}\mathcal{A}$ achieving $d_{\min}$.

- Notice that for any number of points in two dimensions, the best constellations known by now, from a minimum Euclidean distance point of view, are drawn from the lattices of equilateral triangles (Foschini *et al* 1974, Conway and Sloane 1983, Forney *et al* 1984). For example,





$(a)\, 8 - TRI$ $\qquad$ $(b)\, 16 - TRI\ or\ Voronoi\, code$

Theorem 2: Assume $\mathcal{A}$ is a signal constellation drawn from a lattice of equilateral triangles, where the side length of the equilateral triangles is equal to $d_{\min}(\mathcal{A})$. Then, the minimum ζ-distance between $\mathcal{A}$ and $e^{j\pi/6}\mathcal{A}$ is $d_{\min}(\mathcal{A})$, i.e.,

$$d_{\min,\zeta}(\mathcal{A}, e^{j\pi/6}\mathcal{A}) = d_{\min}(\mathcal{A}).$$

3. Simulation Results

- For 4 transmit antennas, the quasi-orthogonal STBC with full diversity is given by

$$C = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ -x_2^* & x_1^* & -x_4^* & x_3^* \\ x_3 & x_4 & x_1 & x_2 \\ -x_4^* & x_3^* & -x_2^* & x_1^* \end{bmatrix},$$

where $x_1, x_2 \in \mathcal{A}$, and $x_3, x_4 \in e^{j\phi}\mathcal{A}$ for some signal constellation $\mathcal{A}$, and the rotation angle ϕ is determined by the signal constellation $\mathcal{A}$, which is specified in Theorem 1 and 2.

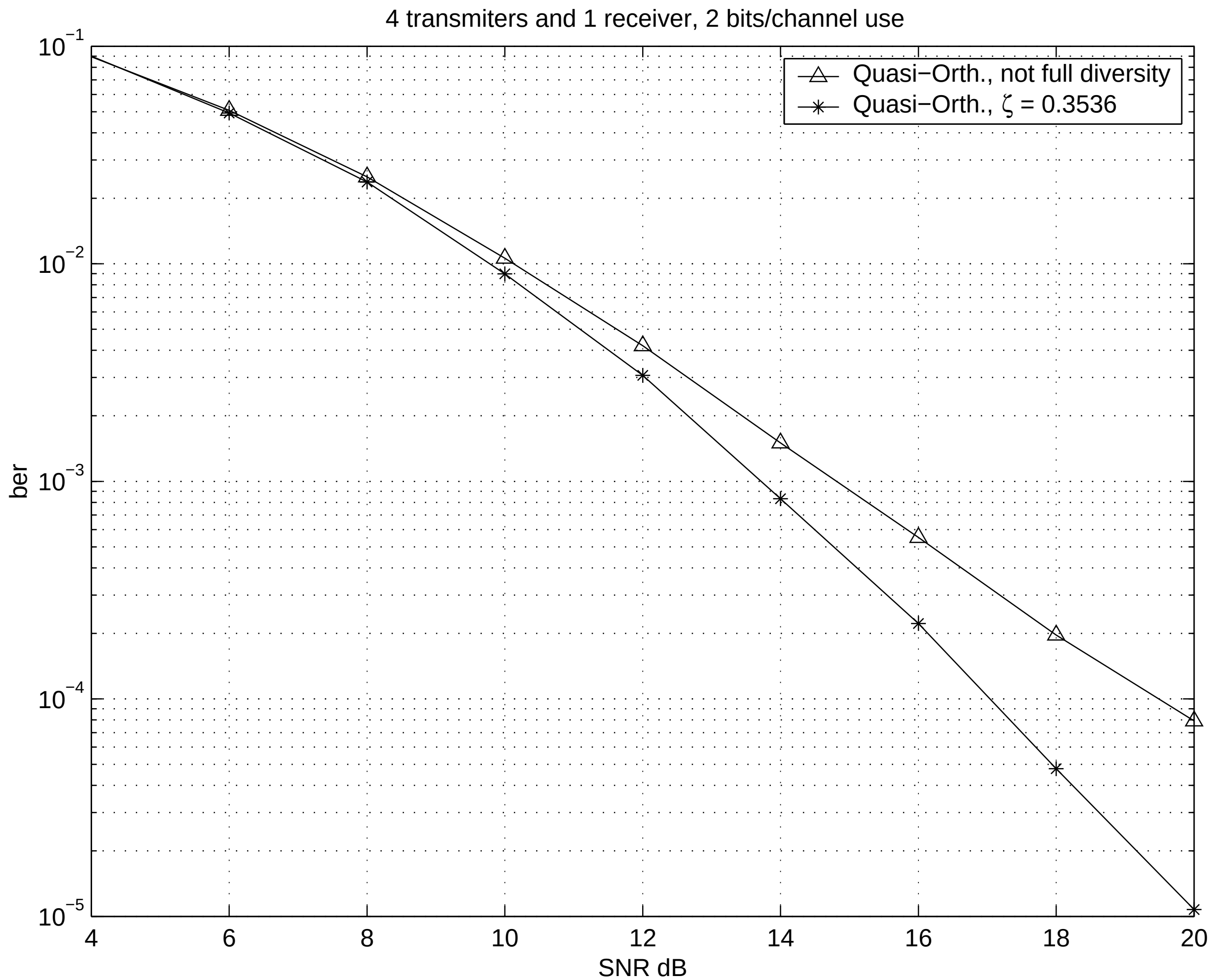


Figure 1: Performances of the new scheme (line with $*$) and the TBH scheme (line with $\triangle$).

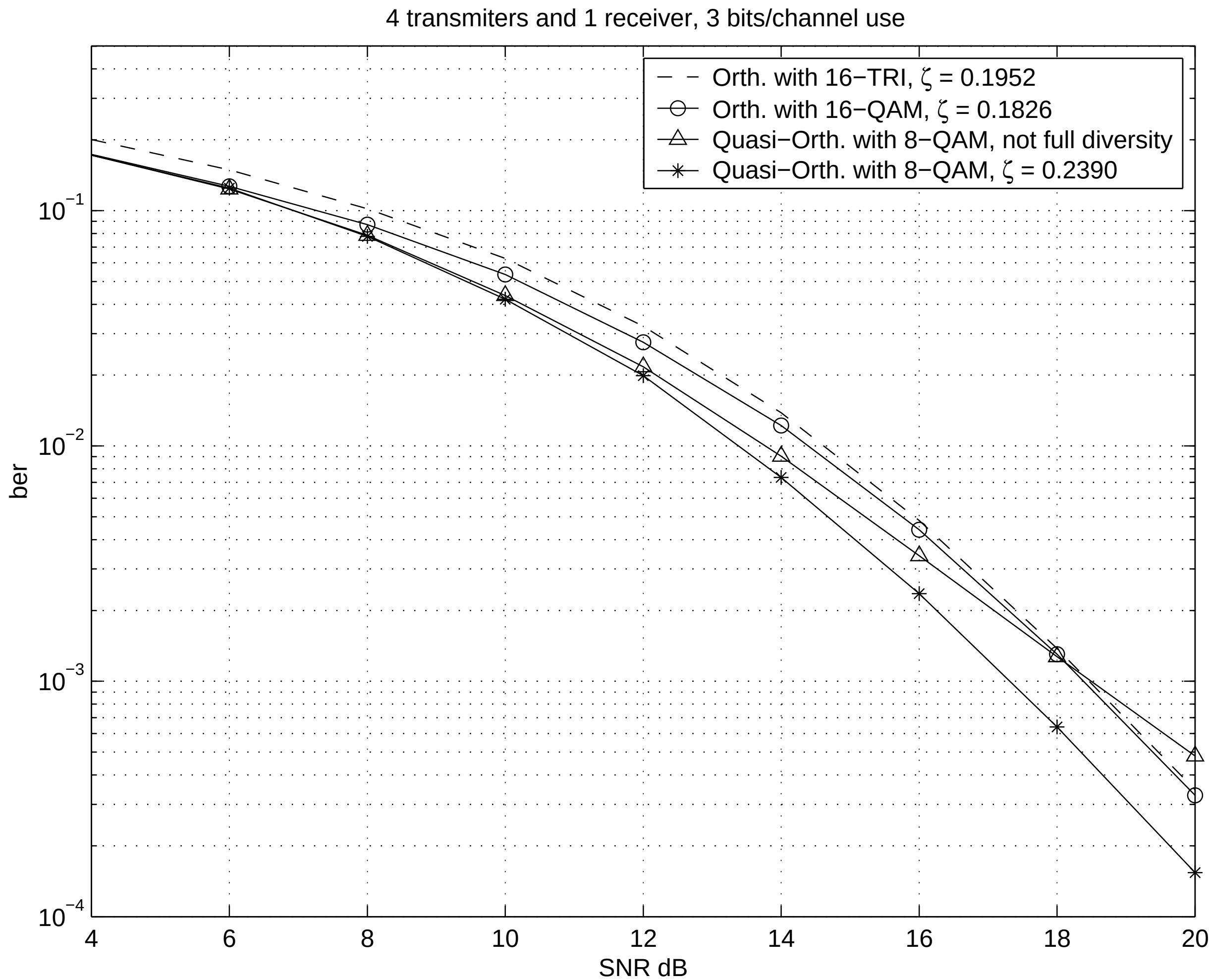


Figure 2: Performances of the new scheme (line with $*$), the TBH scheme (line with $\triangle$), and the orthogonal design (line with $\circ$ for 16-QAM and dash line for 16-TRI).

- For 8 transmit antennas, the quasi-orth. STBC with full diversity is given by

$$C = \sqrt{\frac{4}{3}} \begin{bmatrix} x_1 & x_2 & x_3 & 0 & x_4 & x_5 & x_6 & 0 \\ -x_2^* & x_1^* & 0 & x_3 & -x_5^* & x_4^* & 0 & x_6 \\ -x_3^* & 0 & x_1^* & -x_2 & -x_6^* & 0 & x_4^* & -x_5 \\ 0 & -x_3^* & x_2^* & x_1 & 0 & -x_6^* & x_5^* & x_4 \\ x_4 & x_5 & x_6 & 0 & x_1 & x_2 & x_3 & 0 \\ -x_5^* & x_4^* & 0 & x_6 & -x_2^* & x_1^* & 0 & x_3 \\ -x_6^* & 0 & x_4^* & -x_5 & -x_3^* & 0 & x_1^* & -x_2 \\ 0 & -x_6^* & x_5^* & x_4 & 0 & -x_3^* & x_2^* & x_1 \end{bmatrix},$$

where $x_1, x_2, x_3 \in \mathcal{A}$, and $x_4, x_5, x_6 \in e^{j\phi}\mathcal{A}$ for some signal constellation $\mathcal{A}$, and the rotation angle ϕ is determined by the signal constellation $\mathcal{A}$, which is specified in Theorem 1 and 2.

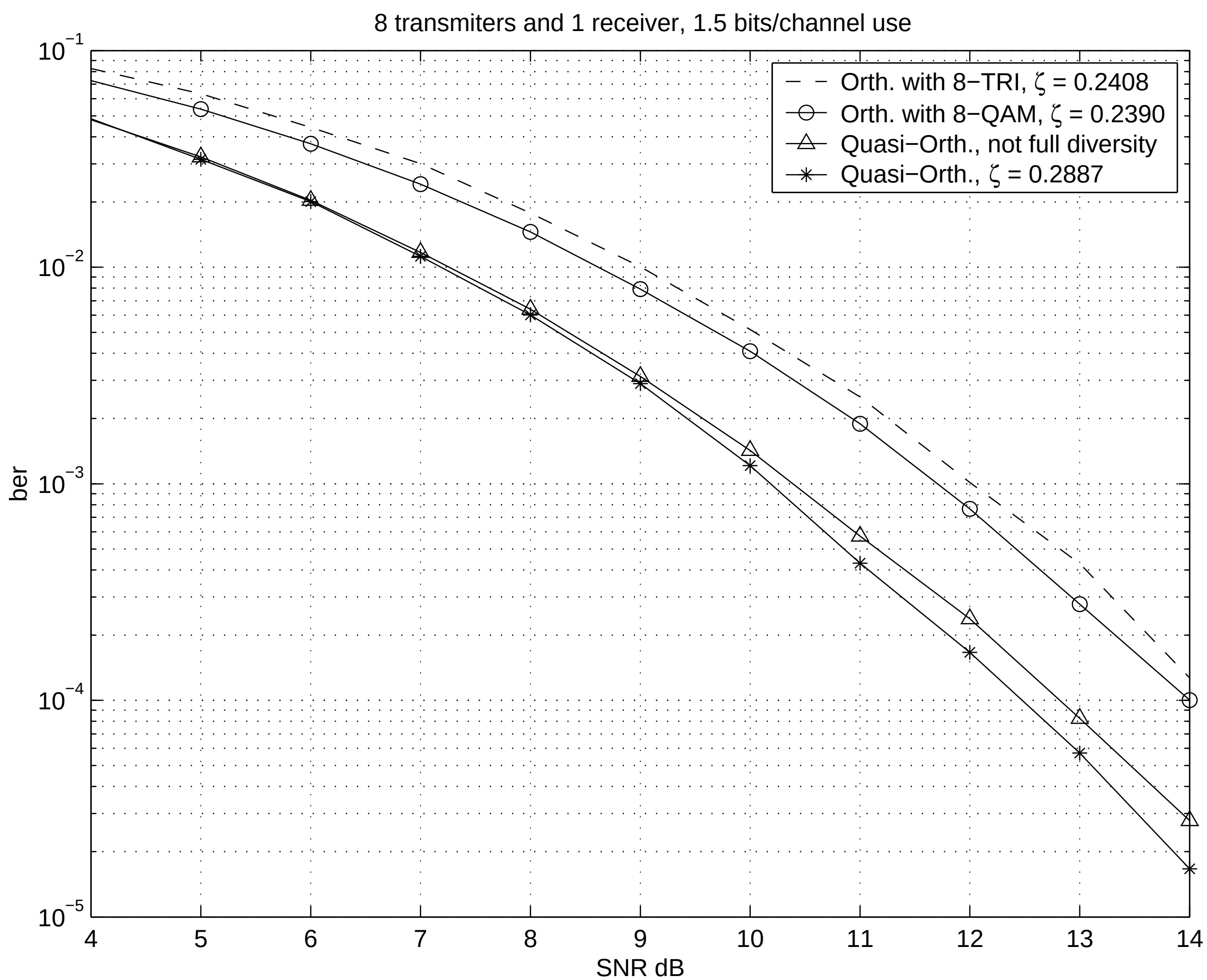


Figure 3: Performances of the new scheme (line with $*$), the Jafarkhani scheme (line with $\triangle$), and the orthogonal design (line with $\circ$ for 8-QAM and dash line for 8-TRI).

Conclusion

- We proposed quasi-orthogonal STBCs with full diversity.
- We also studied the optimality of the signal constellations. The optimum rotation angles were given for some commonly used signal constellations.

On Optimal Quasi-Orthogonal Space-Time Block Codes with Minimum Decoding Complexity

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Outline

- Background
- Linear Transformations for Quasi-Orthogonal Space-Time Block Codes (QOSTBC) with Minimum Decording Complexity
- Optimal Linear Transformations for Square QAM
- Optimal Linear Transformations for Retangular QAM
- Simulation Results
- Conclusion

Background

- Orthogonal space-time block codes from complex orthogonal designs (Alamouti, Tarokh-Jarfarkhani-Calderbank)
  - Full diversity
  - Complex symbol-wise (equivalently real symbol pair-wise) ML decoding
  - Rates are upper bounded by 3/4 for more than 2 Tx antennas (Wang-Xia)
- Quasi orthogonal space-time block codes (Jafarkhani, Tirkkonen-Boariu-Hottinen, Papadias-Foschini) without information symbol rotations
  - Not full diversity
  - Rates are 1 for 4 Tx antennas
  - Complex symbol pair-wise ML decoding
- Full diversity QOSTBC with information symbol rotation (Tirkkonen, Sharma-Papadias, Su-Xia)

- Full diversity and diversity products are maximized (Su-Xia) over all possible linear transformations of information symbols arbitrarily located on square or equal-literal triangular lattices
- Rates 1 for 4 Tx antennas
- Complex symbol pair-wise ML decoding

- Faster ML decoding QOSTBC with information symbol rotation (Yuen-Guan-Tjhung'04)
  - Real symbol pair-wise ML decoding
  - Full diversity and diversity products are maximized among orthogonal rotations of information symbols
  - Rates 1 for 4 Tx antennas
- Co-ordinate interleaved orthogonal designs (CIOD) (Khan-Rajan-Lee'03) with information rotations
  - Real symbol pair-wise ML decoding
  - Full diversity and diversity products are maximized among orthogonal

rotations of square QAM information symbols

– Rates 1 for 4 Tx antennas

- **Goals** of this work
  - Use the most general setting of linear transformations of information symbols for both QOSTBC and CIOD (NOT limited to orthogonal rotations as studied by Khan-Rajan-Lee and Yuen-Guan-Tjhung)
  - Present if and only if conditions on the linear transformations for QOSTBC to possess the real symbol pair-wise ML decoding
  - Present the optimal linear transformations for QOSTBC (optimal rotations of rectangular QAM for CIOD) with real symbol pair-wise ML decoding such that the diversity products are maximized (optimized)

Complex Orthogonal Designs

A *complex orthogonal design* (COD) in complex variables $z_1, z_2, \cdots, z_k$ is a $T \times n$ matrix $G(z_1, \cdots, z_k)$ such that

any entry of $G(z_1, \cdots, z_k)$ is a complex linear combinations of $z_1, z_2, \cdots, z_k, z_1^*, z_2^*, \cdots, z_k^*$.

G satisfies the orthogonality

$$G^\dagger G = (|z_1|^2 + |z_2|^2 + \cdots + |z_k|^2) I_n \tag{1}$$

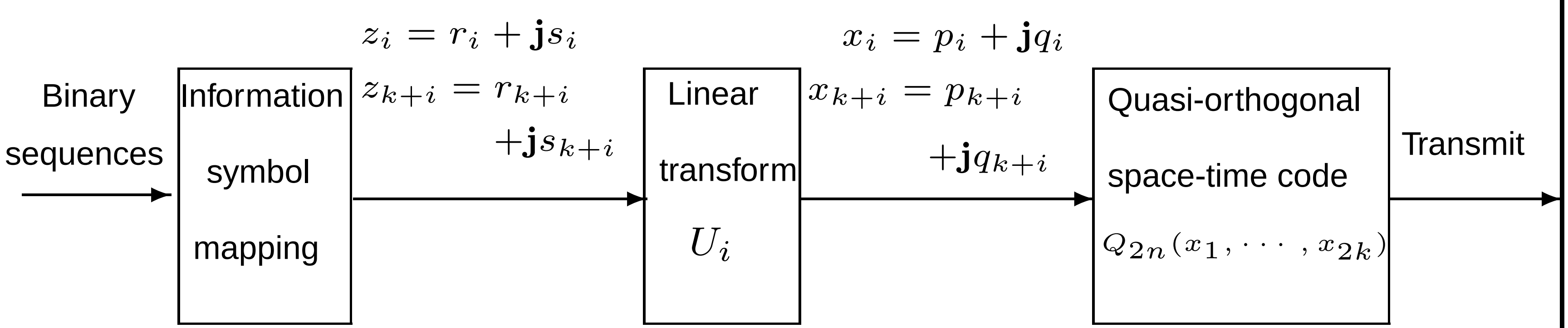
for all complex values $z_1, z_2, \cdots, z_k$.

QOSTBC

- Let $G_n(z_1, \cdots, z_k)$ be a $T \times n$ complex orthogonal design in complex variables $z_1, \cdots, z_k$.
- Let $A = G_n(z_1, \cdots, z_k)$ and $B = G_n(z_{k+1}, \cdots, z_{2k})$.
- We consider the following quasi-orthogonal space-time block code (QOSTC) $Q_{2n}(z_1, \cdots, z_{2k})$:

$$Q_{2n}(z_1, \cdots, z_{2k}) = \begin{pmatrix} A & B \\ B & A \end{pmatrix}.$$

Encoding for QOSTBC with General Linear Transformations of Information Symbols



Encoding for QOSTBC with General Linear Transformations of Information Symbols

- Let $\mathcal{S}$ be a signal constellation, i.e., a set of complex numbers on the complex plane.
- A binary information sequence is mapped to points z_i in $\mathcal{S}$ as $z_i = r_i + \mathbf{j}s_i$ for $1 \leq i \leq 2k$.
- For each i, $1 \leq i \leq k$, take a pre-designed real linear transform U_i and the real vector $(r_i, s_i, r_{k+i}, s_{k+i})^t$ of dimension 4 is transformed to another real vector $(p_i, q_i, p_{k+i}, q_{k+i})^t$ of dimension 4:

$$(p_i, q_i, p_{k+i}, q_{k+i})^t = U_i(r_i, s_i, r_{k+i}, s_{k+i})^t, \quad (2)$$

  where U_i is non-singular.
- Form complex variables $x_i = p_i + \mathbf{j}q_i$ for $1 \leq i \leq 2k$.
- With these complex variables x_i, form a QOSTBC $Q_{2n}(x_1, x_2, \cdots, x_{2k})$ that is used as a space-time block code and transmitted through $2n$ transmit

antennas.

Question: *how to design a real linear transformations* U_i *of size* 4×4 *for a QOSTBC such that it possess a real symbol pair-wise ML decoding and to have full diversity (or optimal diversity product).*

Real Symbol Pair-wise ML Decoding

- Let

$$\begin{aligned} g_i(p_i, q_i, p_{k+i}, q_{k+i}) &\triangleq p_i^2 + q_i^2 + p_{k+i}^2 + q_{k+i}^2, \\ f_i(p_i, q_i, p_{k+i}, q_{k+i}) &\triangleq 2(p_i p_{k+i} + q_i q_{k+i}). \end{aligned}$$

- Then the quadratic term in the ML decoding objective function is

$$Q_{2n}^{\dagger} Q_{2n} = \begin{pmatrix} aI_n & bI_n \\ bI_n & aI_n \end{pmatrix},$$

where

$$\begin{aligned} a &= \sum_{i=1}^{2k} |x_i|^2 = \sum_{i=1}^{k} g_i \\ b &= \sum_{i=1}^{k} (x_i x_{k+i}^* + x_{k+i} x_i^*) = \sum_{i=1}^{k} f_i. \end{aligned}$$

- To possess a real symbol pair-wise ML decoding, the linear transformation U_i needs to be chosen such that one of the following three cases holds

Case 1. Functions g_i and f_i can be separated as

$$g_i(p_i, q_i, p_{k+i}, q_{k+i}) = g_{i1}(r_i, s_i) + g_{i2}(r_{k+i}, s_{k+i}),$$
$$f_i(p_i, q_i, p_{k+i}, q_{k+i}) = f_{i1}(r_i, s_i) + f_{i2}(r_{k+i}, s_{k+i}).$$

Case 2. Functions g_i and f_i can be separated as

$$g_i(p_i, q_i, p_{k+i}, q_{k+i}) = g_{i1}(r_i, r_{k+i}) + g_{i2}(s_i, s_{k+i}),$$
$$f_i(p_i, q_i, p_{k+i}, q_{k+i}) = f_{i1}(r_i, r_{k+i}) + f_{i2}(s_i, s_{k+i}).$$

Case 3. Functions g_i and f_i can be separated as

$$g_i(p_i, q_i, p_{k+i}, q_{k+i}) = g_{i1}(r_i, s_{k+i}) + g_{i2}(s_i, r_{k+i}),$$
$$f_i(p_i, q_i, p_{k+i}, q_{k+i}) = f_{i1}(r_i, s_{k+i}) + f_{i2}(s_i, r_{k+i}).$$

Linear Transformation Characterization for Real Symbol Pair-wise ML Decoding

Theorem 1: Let U_i be a 4×4 non-singular matrix with all real entries. Then, we have the following results.

- Case 1 holds if and only if U_i can be written as

$$U_i = \begin{pmatrix} U_{i1} & U_{i2} \\ U_{i1}R_{i1} & U_{i2}R_{i2} \end{pmatrix},$$

  where $U_{i1}, U_{i2}, R_{i1}, R_{i2}$ are 2×2 matrices of real entries, $R_{i1}^2 = I_2$ and $R_{i2}^2 = I_2$, and

$$R_{i1}^t U_{i1}^t U_{i2} + U_{i1}^t U_{i2} R_{i2} = 0.$$

- Case 2 holds if and only if U_i can be written as

$$U_i = \begin{pmatrix} U_{i1} & U_{i2} \\ U_{i1}R_{i1} & U_{i2}R_{i2} \end{pmatrix} P_1,$$

where $U_{i1}, U_{i2}, R_{i1}, R_{i2}$ are the same as in (i) for Case 1 and

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- Case 3 holds if and only if U_i can be written as

$$U_i = \begin{pmatrix} U_{i1} & U_{i2} \\ U_{i1}R_{i1} & U_{i2}R_{i2} \end{pmatrix} P_2,$$

where $U_{i1}, U_{i2}, R_{i1}, R_{i2}$ are the same as in (i) for Case 1 and

$$P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Optimal Linear Transformation for Square QAM

- We want to determine the optimal linear transformation that maximize the following normalized diversity product of the QOSTBC Q_{2n} in terms of the mean signal power

$$\bar{\zeta}(Q_{2n}) \triangleq \frac{\zeta(Q_{2n})}{\left(\prod_{i=1}^{k} |\det(U_i)|\right)^{1/(4k)}},$$

where $\zeta(Q_{2n})$ is the diversity product of QOSTBC $Q_{2n}(x_1, \cdots, x_{2k})$.

Optimal Linear transformation for Square QAM

Theorem 2: Let

$$\alpha = \arctan(2) \text{ and } R = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ \sin(\alpha) & -\cos(\alpha) \end{pmatrix},$$

and P_1 and P_2 be the 4×4 matrices defined in (ii) and (iii) in Theorem 1, respectively. For the three cases, we have the following results,respectively.

- For Case 1, let

$$U_i = \frac{1}{\sqrt{2}} \begin{pmatrix} I_2 & I_2 \\ R & -R \end{pmatrix}.$$

  Then, the above orthogonal matrices U_i satisfy (i) for Case 1 in Theorem 1, i.e., the quadratic forms f_i and g_i of 4 variables can be separated as Case 1, and furthermore, U_i are optimal in the sense that the normalized diversity product $\bar{\zeta}(Q_{2n})$ is maximized among all other non-singular linear transformations U_i that satisfy (i) in Theorem 1 and

$$\max_{U_i \text{ in (i) Theorem 1}} \bar{\zeta}(Q_{2n}) = \frac{1}{2\sqrt{2T}}(\frac{4}{5})^{1/4}.$$

- For Case 2, let

$$U_i = \frac{1}{\sqrt{2}} \begin{pmatrix} I_2 & I_2 \\ R & -R \end{pmatrix} P_1.$$

Then, the above orthogonal matrices U_i satisfy (ii) for Case 2 in Theorem 1 and they are optimal, and the same maximum normalized diversity product in above case is achieved.

- For Case 3, let

$$U_i = \frac{1}{\sqrt{2}} \begin{pmatrix} I_2 & I_2 \\ R & -R \end{pmatrix} P_2.$$

Then, the above orthogonal matrices U_i satisfy (iii) for Case 3 in Theorem 1 and they are optimal, and the same maximum normalized diversity product in above is achieved.

Remark: For the above square QAM case, the above optimal normalized diversity product coincides with the one obtained by Yuen-Guan-Tjhung from optimal orthogonal rotation.

Optimal Linear Transformation for Retangular QAM

- Consider a finite-point retangular QAM (RQAM) signal constellations:

$$\mathcal{S} = \left\{ z_i = \frac{n_1 d}{2} + \mathbf{j}\frac{n_2 d}{2} : \; n_i \in \mathcal{N}_i \text{ for } i = 1, 2 \right\},$$

  where

$$\mathcal{N}_i \triangleq \{-(2N_i - 1), \cdots, -1, 1, \cdots, 2N_i - 1\},$$

  where N_1 and N_2 are two positive integers and d is a real positive constant that is used to adjust the total signal energy.

- Let us only consider Case 1 and the other two cases are similar.

Optimal Linear Transformation for RQAM

- **Theorem 3:** For Case 1 and an RQAM with total energy 1. Let $\varepsilon_1 = \frac{4N_1^2-1}{2(2N_1^2+2N_2^2-1)}$, $\varepsilon_2 = \frac{4N_2^2-1}{2(2N_1^2+2N_2^2-1)}$, $\alpha = \arctan(2)$, $\rho = \sqrt{\frac{5}{12(1+\varepsilon_1\varepsilon_2)}}$, and

$$R_1 = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ \sin(\alpha) & -\cos(\alpha) \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\Sigma = \begin{pmatrix} 1+\varepsilon_1 & 1-2\varepsilon_1 \\ 1-2\varepsilon_1 & 2-\varepsilon_1 \end{pmatrix}.$$

Denote a diagonalization of symmetric matrix Σ as $\Sigma = V^t D V$, where $D = \mathrm{diag}(\lambda_1, \lambda_2)$, λ_1, λ_2 are the eigenvalues of Σ and V is an orthogonal matrix.

Let

$$U_{i1} = \rho V^t \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix} V;$$

$$U_{i2} = \rho V^t \begin{pmatrix} \sqrt{\lambda_2} & 0 \\ 0 & \sqrt{\lambda_1} \end{pmatrix} VP; \quad R_2 = -PR_1P.$$

Then,

$$U_i = \begin{pmatrix} U_{i1} & U_{i2} \\ U_{i1}R_1 & U_{i2}R_2 \end{pmatrix} \quad i = 1, 2, \cdots, k,$$

satisfy (i) for Case 1 in Theorem 1, and are optimal in the sense that the diversity product ζ of the QOSTBC is maximized among all U_i under (i) in

Theorem 1 and the optimal diversity product is

$$\zeta_{\rm optimal} = \frac{1}{2\sqrt{2T}}\sqrt{\frac{3}{N_1 N_2}}\times$$
$$\frac{1}{(16N_1^4+16N_2^4+48N_1^2N_2^2-20N_1^2-20N_2^2+5)^{1/4}}.$$

- Optimal normalized diversity product comparison:

$$\zeta_{\text{optimal,SX}} = \frac{1}{2\sqrt{2T}}d > \frac{1}{2\sqrt{2T}}\left(\frac{1}{1+\varepsilon_1\varepsilon_2}\right)^{1/4} d = \zeta_{\text{optimal,WWX}}$$

$$\geq \frac{1}{2\sqrt{2T}}\left(\frac{4}{5}\right)^{1/4} d = \zeta_{\text{optimal,YGT}},$$

where the equality "$=$" holds if and only if $\varepsilon_1 = \varepsilon_2 = 1/2$, i.e., $N_1 = N_2$ or square QAM.

Optimal Rotations for Co-ordinate Interleaved Orthogonal Designs (CIOD)

- Khan-Rajan-Lee have proposed CIOD that has the same rate as QOSTBC for 4 Tx antrennas.
- Khan-Rajan-Lee found the optimal rotations for CIOD to possess the real symbol pair-wise ML decoding and maximized diversity product for square QAM signal constellations.
- We also obtained the optimal rotations for CIOD to possess the real symbol pair-wise ML decoding and maximized diversity product for rectangular QAM (RQAM) signal constellations
- As a consequence of our results, it is shown that the optimal normalized diversity products of QOSTBC and CIOD with the real symbol pair-wise ML decoding are the same in both square and rectangular QAM cases.

Some Examples

Table 1: Diversity product comparison

Constellations	4-QAM	8-QAM	32-QAM
Su-Xia	0.1768	0.0801	0.0198
Case 1	0.1672	0.0757	0.0187
CIOD	0.1672	0.0757	0.0187
RQAM with new optimal transform	0.1672	0.0699	0.0167
RQAM with optimal rotation	0.1672	0.0683	0.0164

Some Examples

Table 2: ML decoding complexity comparison: Number of trials

Constellations	4-QAM	8-QAM	32-QAM
Su-Xia (complex symbol pair-wise)	32	128	2048
Case 1 (real symbol pair-wise)	16	32	128
CIOD (real symbol pair-wise)	16	32	128
RQAM (real symbol pair-wise)	16	32	128

Some Simulation Results

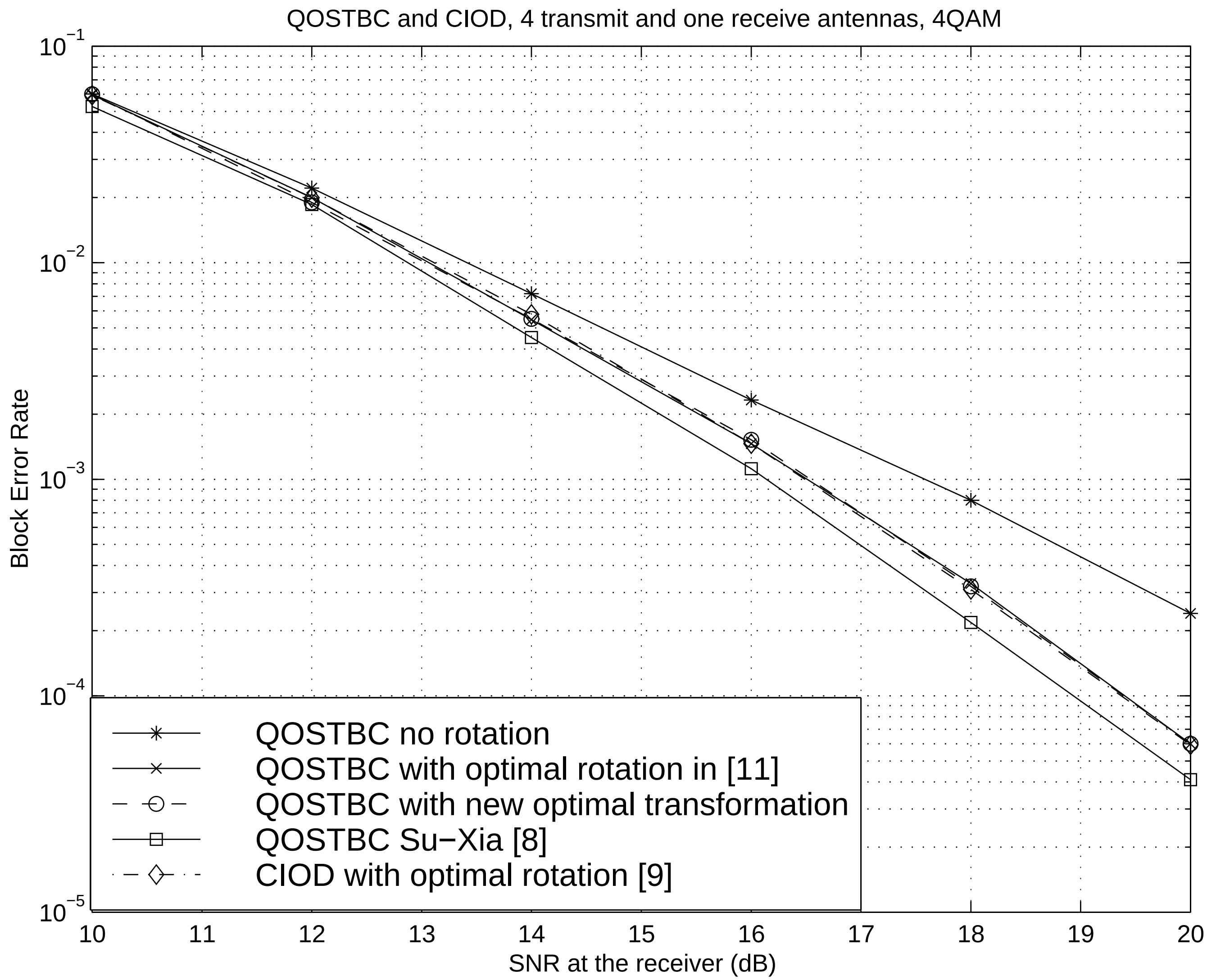


Figure 1: 2 bits/channel use.

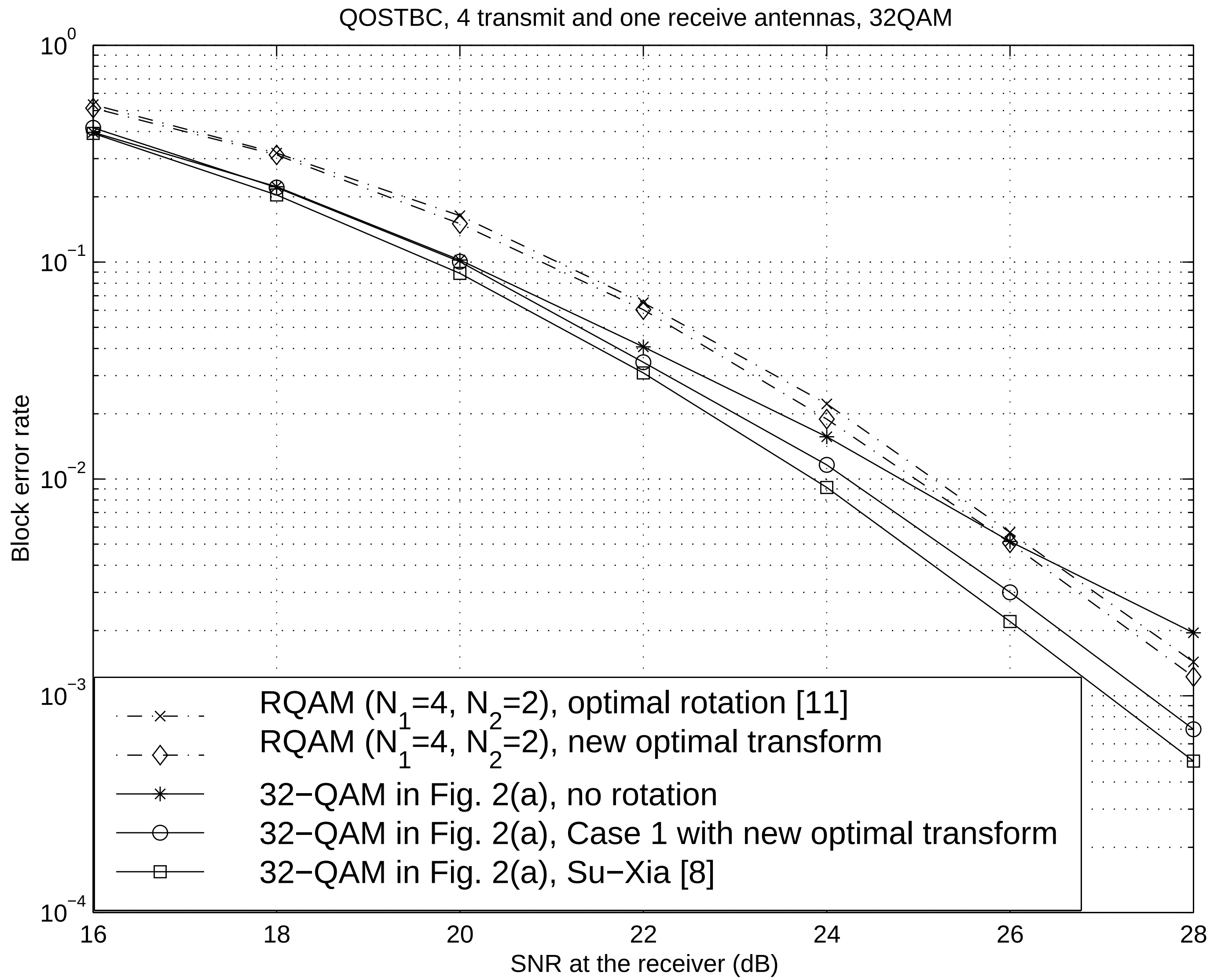


Figure 2: 5 bits/channel use.

Conclusion

- Presented if and only if conditions for linear transformations of information symbols for QOSTBC to possess real symbol pair-wise ML decoding.
- Presented optimal linear transformations of information symbols for QOSTBC to possess real symbol pair-wise ML decoding such that the normalized diversity products are maximized for both square QAM and rectangular QAM cases.
- It turns out that the optimal normalized diversity products of QOSTBC among all linear transformations we obtained coincide with the ones among only orthogonal rotations obtained by Yuen-Guan-Tjhung for square QAM constellations.
- The optimal normalized diversity products of QOSTBC among all linear transformations we obtained are greater than the ones among only orthogonal rotations obtained by Yuen-Guan-Tjhung for non-square rectangular QAM constellations.

- We also obtained optimal rotations for CIOD for non-square rectangular QAM constellations.

Some Papers to Read

- W. Su and X.-G. Xia, Signal Constellations for Quasi-Orthogonal Space-Time Block Codes with Full Diversity, *IEEE Trans. on Information Theory*, Oct. 2004.
- H. Wang, D. Wang, and X.-G. Xia, Optimal Quasi-Orthogonal Space-Time Block Codes with Minimum Decoding Complexity, *IEEE Trans. on Information Theory*, March 2009.
- D. Wang and X.-G. Xia, Optimal Diversity Product Rotations for Quasi-Orthogonal STBC with MPSK, *IEEE Communications Letters*, May, 2005.