Quasi-periodic solutions of Hamiltonian perturbations of 2D linear Schrödinger equations

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0. Introduction

This paper is a continuation of the author's work $[B_{1,2}]$ on constructing periodic and quasi-periodic solutions of Hamiltonian perturbations of linear PDE's (with periodic boundary conditions). The method used for this purpose was initiated in the work of Craig and Wayne (see $[C\text{-}W_{1,2}]$) for 1D-equations and time periodic solutions. It is an infinite dimensional phase space version of the Liapounov-Schmidt argument for the construction of periodic solutions. The basic idea of the Liapounov-Schmidt scheme (cf. [C-W]) consists in splitting the problem in a resonnant finite dimensional piece given by the Q-equation and the remainder of the problem, the P-equation, which is infinite dimensional and contains the small divisors issues. To achieve frequency variation, we will rely on outer parameters contained in the equation (cf. [Kuk]) rather than amplitude-frequency modulation depending on the nonlinear term (cf. [C-W]). The model equation considered here is the nonlinear Schrödinger equation

$$iu_t - \Delta u + V(x)u + \varepsilon \frac{\partial F}{\partial \overline{u}} (u, \overline{u}, x) = 0 \tag{0.1}$$

or

$$iu_t - \Delta u + (u * V) + \varepsilon \frac{\partial}{\partial \overline{u}} F(u, \overline{u}, x) = 0 \tag{0.2}$$

where u is a space periodic function, i.e. $u = u(x,t)$, $x \in \mathbf{T}^{\mathbf{d}}$. In (0.1) $V = V(x)$ is a real periodic potential and in (0.2) $u * V$ defines a real Fourier multiplier. The advantage of replacing the term $V(x)u$ by a convolution $u * V$ is the fact that the eigenfunction basis of $\Delta u + (u * V)$ are given by exponentials which in dimension $d \geq 2$ avoids certain difficulties which are not really the primary issues here.(**) The parameter $\varepsilon \geq 0$ in (0.1), (0.2) corresponds to a perturbation and F is a real valued function in $u, \overline{u}, x$, periodic in x. For simplicity, we assume F is a trigonometric polynomial in x with polynomial coefficients in $u, \overline{u}$. This hypothesis may be relaxed however. The outer parameters in the equation responsible for the frequency modulation are introduced using the term $V(x) \cdot u$ in (0.1) and $V * u$ in (0.2). Time periodic solutions for the linear equation $iu_t - \Delta u + (u * V) = 0$ are indeed given by

$$u(x,t) = e^{i(\langle m.x\rangle + \lambda t)} \tag{0.3}$$

where $\lambda = |m|^2 + \widehat{V}(m)$. In $[B_2]$, we prove their "persistency" for the perturbed equation (0.2) (λ is considered as a parameter taken outside an exceptional set of measure $\to 0$ for $\varepsilon \to 0$ and of course perturbed according to the Q-equation). The P-equation constitutes the main problem and is solved by a Newton scheme, which has the advantage (on convergent expansions for instance) of converging rapidly. The main difficulty is the control of the inverses of the linearized operators, which has the same flavor as the localization theory for the Anderson model on a lattice. Writing the linearized operator in the form $T = D + \varepsilon T_1$ where D is diagonal

$$D_{m,n} = -\langle \lambda, n\rangle + \mu_m \tag{0.4}$$

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(**) One could also consider (0.1) with V of the form $V(x_1, x_2) = V(x_1) + V(x_2)$.

$$\mu_m = |m|^2 + \hat{V}(m) \tag{0.5}$$

(the lattice settup is obtained by passing to Fourier transform), those difficulties depend on the structure of the "singular sites" (m, n), i.e. such that $D_{m,n}$ is small. In the periodic problem, this structure is rather easy to deal with. Both in 1D and in higher dimension one encounters a pattern of separated islands for the singular sites. Using Pöschel's lemma (cf. [P]), i.e. the localization identity for the inverse operator, it suffices to control the inverses of the restrictions of T to neighborhoods of these islands. This is achieved by simple eigenvalue perturbation, considered as functions of λ (here one relies on self-adjointness properties of T). This singular site structure in the quasi periodic case is more complicated and requires different arguments. The nature of the singular sites is related to the work of Fröhlich and Spencer on localization for $-\varepsilon\Delta + V(n)$ where Δ is the lattice Laplacian and $V(n)$ is a quasi-periodic potential. The study of those requires a multi-scale analysis. In $[\mathrm{B}_1]$, we established for 1D-equations (0.1) or (0.2) the persistency of quasi-periodic solutions of the unperturbed equation, extending some of Kuksin's work [Kuk] for Dirichlet boundary conditions to the case of periodic boundary conditions. Recall that the periodic spectrum of the operator $-\frac{d^2}{dx^2} + V$ appears in pairs $\omega_{2n-1}, \omega_{2n}$ of nearby frequencies

$$|\omega_{2n} - \omega_{2n-1}| \sim \left|\hat{V}(n)\right|. \tag{0.6}$$

This multiplicity or almost multiplicity of normal frequencies is an essential obstacle if one tries to apply the KAM scheme (as Kuksin does). The first step in $[\mathrm{B}_1]$ consists in establishing a Melnikov result in finite dimensional phase space (i.e. the persistency of a low dimensional torus in a high dimensional phase space) using the Liapounov-Schmidt type technique from [C-W]. This argument is essentially different and more flexible than KAM. It avoids restrictive assumptions on the frequencies, such as Melnikov's second condition, which excludes in particular multiplicities in the normal frequencies. Once the finite dimensional result obtained, the analysis of PDE-models involving our infinite dimensional phase space appears as a technical elaboration of the arguments, which we pursued in $[\mathrm{B}_1]$ for 1D NL Schrödinger and wave equations. Our aim is to study here the 2D NLSE (0.2), following the same method. The main difficulty is an unbounded multiplicity for normal frequencies, i.e. the equation (lattice points on a circle)

$$m_1^2 + m_2^2 = R^2 \qquad (m_1, m_2 \in \mathbf{Z}) \tag{0.7}$$

may have a large number of solutions for given R. However, these solutions appear in small clusters (of cardinality ≤ 2) which are well-separated and the total number of solution is at most $\exp \frac{\log R}{\log\log R} \ll R^\varepsilon$. These facts which are special for $2D$ play a role in our analysis. At this stage, there seems to be essential difficulties to extend this work for $D > 2$.

Defining $v = \overline{u}$ and taking complex conjugate of (0.2), replace (0.2) by 2 equations

$$\begin{cases} iu_t - \Delta u + (u * V) + \varepsilon \frac{\partial}{\partial v} F(u, v, x) = 0 \\ -iv_t - \Delta v + (v * \bar{V}) + \varepsilon \frac{\partial}{\partial u} F(u, v, x) = 0. \end{cases} \tag{0.8}$$

Passing to Fourier transform, the linearized operator $T = D + \varepsilon T_1$ is given by

$$D = \begin{pmatrix} -\langle\lambda, n\rangle + |m|^2 + \hat{V}(m) & 0 \\ 0 & \langle\lambda, n\rangle + |m|^2 + \hat{V}(-m) \end{pmatrix} \tag{0.9}$$

$$T_1 = \begin{pmatrix} S_{\frac{\partial^2 F}{\partial u\, \partial v}} & S_{\frac{\partial^2 F}{\partial v^2}} \\ S_{\frac{\partial^2 F}{\partial u^2}} & S_{\frac{\partial^2 F}{\partial u\, \partial v}} \end{pmatrix} \tag{0.10}$$

(S_ϕ = Toeplitz operator with symbol ϕ)

acting on pairs $(\hat{u}, \hat{v})$. Those pairs fulfil the condition $v(x) = \overline{u(-x)}$ and T preserves this property, since $\hat{V}, F$ are real. Consider an unperturbed solution (for $d = 2$ say)

$$u_0(x,t) = a_1\ e^{i((m_1,x)+\lambda_{1,0}t)} + a_2\ e^{i((m_2,x)+\lambda_{2,0}t)} \tag{0.11}$$

where $\lambda_{\alpha,0} = |m_\alpha|^2 + \hat{V}(m_\alpha)$, $\alpha = 1, 2$. The perturbed solution will have the form

$$u_\varepsilon(x,t) = \sum_{m,n} a_{m,n}\ e^{i((n_1\lambda_1+n_2\lambda_2)t+(m,x))} \tag{0.12}$$

where in particular (for $\rho < 1$)

$$\lambda_\alpha = \lambda_{\alpha,0} + 0(\varepsilon)^{(*)} \tag{0.13}$$

$$a_{m_1,(1,0)} = a_1 \quad , \quad a_{m_2,(0,1)} = a_2 \tag{0.14}$$

$$\sum_{(m,n)\notin \mathcal{R}} |a_{mn}|\ e^{(|m|+|n|)^c} = 0(\varepsilon^\rho) \tag{0.15}$$

for some constant $c > 0^{(**)}$ and denoting $\mathcal{R} = \{(m_1,(1,0)),(m_2,(0,1))\}$. We consider here $\lambda_0 = (\lambda_{1,0}, \lambda_{2,0})$ as a parameter. The perturbed solution will be constructed for $\lambda_0 \in \Delta$, where Δ depends on $|a_1|$, $|a_2|$ in particular and has a Contor type structure and mes $(\Delta^c) \to 0$ in the parameter set for $\varepsilon \to 0$. In case there is a (weak) dependence on λ_0 of normal frequencies μ_m, $m \neq m_1, m_2$, we assume moreover the non-resonnance condition (2.9) below.

Denote $\mu_m = |m|^2 + \hat{V}(m)$. The P-equation is obtained by projecting (the Fourier transform of) (0.2) on the complement of $\mathcal{R}$. Thus the linearized operator is obtained by restricting $T = D + \varepsilon\ T_1$ to the complement of the "resonnant set"

$$R = \{(\mp m_1, \pm, (\mp 1, 0))\ ,\ (\mp m_2, \pm, (0, \mp 1))\} \tag{0.16}$$

with sign correspondence.

The remaining 4 (in fact 2 independent) equations form the Q-system and determine $\lambda_\alpha - \lambda_{\alpha,0}$. In order to solve the P-equation, we restrict (λ_0, λ) to a Cantor set $\tilde{\Delta} = \cap\tilde{\Delta}_k$ constructed along the Newton iteration scheme. The restrictions on (λ_0, λ) are "admissible" in the sense that for any function of the form (0.13)

$$\lambda = \lambda_0 + \varepsilon(\lambda_0)$$

the condition $(\lambda_0, \lambda) \in \tilde{\Delta}$ may be achieved for $\lambda_0 \in \Delta$, with Δ as above. We also assume that the solution to the P-equation extends as a smooth function of (λ_0, λ) to the full parameter set (and actually solves the P-equation for $(\lambda_0, \lambda) \in \tilde{\Delta}$). This point is of importance to determine $\lambda = \lambda_0 + \varepsilon(\lambda_0)$ by the Q-equations. These considerations are analogous to [C-W]$^{(***)}$ and [B1].

(*) The $0(\varepsilon)$-term depends on $\lambda_{\alpha,0}$, $|a_\alpha|$ $(\alpha = 1, 2)$.

(**) See the discussion in [B1] in this respect.

(***) Except that here we do not invoke amplitude frequency modulation using the nonlinearity but use directly the frequency vector λ as a parameter, which simplifies certain matters.

The remainder of the paper deals with the linearized P-equation and the control of the inverse of restrictions T_N^λ of T to lattice sets $|n_1|, |n_2|, \ldots < N$. Since this paper is a continuation of [C-W$_{1,2}$] and [B$_1$] the reader may wish to consult them first for a more complete discussion of the Newton scheme and since the cases treated there are simpler. We will present in Appendix a brief argument for construction of time periodic solutions in general dimension.

1. Description of the matrix $T_N^{\lambda,\sigma}$

Matrix with index set $(n, \pm, m)$ where

$$\begin{cases} n \in \mathbf{Z}^d\ ,\ |n_j| \leq N & \text{(time frequencies)} \\ m \in \mathbf{Z}^b & \text{(space frequencies)}. \end{cases}$$

In this discussion, we let $b = 2$.

We assume $(n, \pm, m)$ not in the resonnance set, which we identify with

$$R = \{(-e_j, +, -m_j) \mid j = 1, \ldots, d\} \cup \{(e_j, -, m_j) \mid j = 1, \ldots, d\}$$

where $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ and m_j some $\mathbf{Z}^2$-element, since T_N^λ is the linearization of the P-equation.

The diagonal of $T_N^{\lambda,\sigma}$ is given by

$$\pm\langle\lambda, n\rangle \pm \sigma + \mu_{\mp m} \quad \text{(with sign correspondence)}. \tag{1.1}$$

The matrix T is selfadjoint and satisfies the off-diagonal estimate

$$\left|T_{(n,\pm,m),(n',\pm,m')}\right| < \varepsilon\ e^{-|n-n'|^{c_1} - |m-m'|^{c_1}}. \tag{1.2}$$

Moreover, for all sign choices $(+,+), (+,-), (-,+), (-,-)$, $T_{(n,\pm,m),(n',\pm,m')}$ depends only on $n-n'$, $m-m'$ if $n \neq n'$ or $m \neq m'$. (Off-diagonal produced by Toeplitz-type operators).

2. Control of the inverse

$\left(T_N^{\lambda,\sigma}\right)^{-1}$ will be controlled by reciprocals of expressions

$$\{p\left(\pm\langle\lambda, n\rangle \pm \sigma + \mu_{\mp m}\right)\} \tag{2.1}$$

provided not to small, i.e. $> \frac{1}{\Phi_1(N)}$ ($\Phi_1(N)$ growing slightly faster than polynomial).

In (2.1), $|n_j| \leq N$, $(n, \pm, m) \notin \mathcal{R}$ and p is a polynomial taken in a set $\mathcal{K}_N^q$, $1 \leq q < Q \equiv Q(N) \sim \exp \frac{\log\log N}{\log\log\log N} \ll (\log N)^\varepsilon$ of the form

$$p(\sigma_1) = \sigma_1^q + \sum_{j<q} a_j(\lambda)\ \sigma_1^j \tag{2.2}$$

with

$$|\partial_\lambda^\alpha\ a_j| < e^{C_2(q+|\alpha|)^2}. \tag{2.3}$$

The structure of each $\mathcal{K}_N^q$ is that of a countable compact satisfying $\mathcal{K}_N^{(\alpha)} \neq \emptyset$ only for finitely many α-derivations. This property has to be made metrically more precise. We have

$$\mathcal{K}_N^{(\alpha)} = \emptyset \quad \text{for} \quad \alpha > \alpha_0 \ (= \text{some positive integer}) \tag{2.4}$$

$$\# \, \mathcal{K}_N^{(\alpha_0)} < \Phi_2(N). \tag{2.5}$$

For $\alpha \leq \alpha_0$,

$$\# \left[\mathcal{K}_N^{(\alpha-1)} \cap (\mathcal{K}_N^{\alpha} + B(0,\delta)) \right] < \Phi_2(N) \cdot \delta^{-C_3} \tag{2.6}$$

where $B(0,\delta)$ denotes an L^∞-ball $[\|a_j\|_\infty < \delta$ for $j = 0, 1, \ldots, q-1]$, and where again $\Phi_2(N)$ grows slightly faster than polynomial.

In the present situation, we may in fact let $q \leq 2$ except for at most $\Phi_2(N)$ expressions of one of following forms

(I)

$$\prod_{\substack{|n-n'| < C_4 \\ |m| < C_4 \\ (n', \pm, m) \notin R}} (\pm\langle \lambda, n' \rangle \pm \sigma + \mu_{\mp m}) + \sum_{j<q} a_j(\lambda)\, \sigma_1^j \tag{2.7}$$

where the first term of (2.7) is of degree $q < C_5$ in σ, $\sigma_1 = \sigma + \langle n, \lambda \rangle$ and $|\partial_\lambda^\alpha\, a_j| \leq \varepsilon$.

These expressions arise from small m and are also of the form (2.1)

(II)

$p\,(\pm\langle \lambda, n\rangle \pm \sigma + \mu_m)$ where p is as in (2.2), (2.3) of degree $q < Q(N)$ defined above.

We assume moreover that these expressions may be replaced alternatively by

$$\pm\langle \lambda, n \rangle \pm \sigma + \mu_m + a_0(\lambda)$$

$|a_0(\lambda)|, |a_0'(\lambda)| \leq \varepsilon$. Hence $q = 1$ but only first derivative control on coefficients.

We let

$$\mu_m = |m|^2 + 0\left(|m|^{-C_6}\right) \tag{2.8}$$

with possible weak dependence on λ.

Assume further

$$|\pm\langle \lambda, n \rangle + \mu_m| > \varepsilon_0 \gg \varepsilon \quad \text{for} \quad |n| < N_0\ ,\ (n, \pm, m) \notin R \tag{2.9}$$

where N_0 is taken sufficiently large (first Melnikov condition).

Lemma 1. *There is a λ-set of measure at most $\Phi_2(N)^2\, \delta^{1/C_7}$ on which complement*

$$|p\,(\pm\langle \lambda, n\rangle + \mu_m)| > \delta \quad \textit{for} \quad |n_j| \leq N\ ,\ m \in \mathbf{Z}^2.$$

Here $C_7 = C_7(C_3, C_5, \alpha_0)$.

Proof. Since for polynomials $p(\sigma_1)$ with leading coefficient σ_1^q only q derivatives will be involved, we may assume $q < C_5$ by the Comment in (II). Observe that $\pm\langle \lambda, n\rangle + \mu_m$ has at least to be bounded if $|p\,(\pm\langle \lambda, n\rangle + \mu_m)| \leq$

δ. Thus for small n, m is small too and $p(\pm\langle\lambda, n\rangle + \mu_m)$ is given by (2.7) for $\sigma = 0$, which is at least $(\varepsilon_0)^{(10C_4)^{2+d}} + 0(\varepsilon) > \delta$ invoking (2.9).

For larger n, denoting $\sigma_1 = \pm\langle\lambda, n\rangle + \mu_m$, one has $\nabla_\lambda\, \sigma_1 \approx \pm n$ and hence $\left|\left(\frac{\partial}{\partial_\lambda}\right)^q p(\sigma_1(\lambda))\right| \approx |n|^q$. Thus for fixed n, m, p, we have $|p(\sigma_1)| > \delta_1$, except for λ in a set of measure $< |n|^{-1}\, \delta_1^{1/q} < \delta_1^{1/q}$. Since σ_1 has to remain bounded, $|m|^2 < |n| + 1$, bounding $|m|$. It follows that to each p there corresponds a set of measure $< N^{d+1}\, \delta_1^{1/q}$. One next uses the metric structure of $\mathcal{K}_N$. Assume for instance $\alpha_0 = 1$. Then $\mathcal{K}_N^{(1)}$ has at most $\Phi_2(N)$ elements and $\mathcal{K}_N \backslash \left(\mathcal{K}_N^{(1)} + B_{\frac{1}{2}\delta_1}\right)$ has at most $\Phi_2(N)\ \delta_1^{-C_3}$-elements. Requiring $|p| > \delta_1$ for $p \in \mathcal{K}_N^{(1)}$ and $|p| > \delta$ for $p \in \mathcal{K}_N \backslash \left(\mathcal{K}_N^{(1)} + B_{\frac{1}{2}\delta_1}\right)$, addition of the measures yields an exceptional set of size

$$\Phi_2(N)\ N^{d+1}\ \left[\delta_1^{1/q} + \delta_1^{-C_3}\ \delta^{1/q}\right] \tag{2.10}$$

leading to a bound as stated in the lemma, if we minimize (2.10) in δ_1.

For the general case $\mathcal{K}_N^{(\alpha_0)} \neq \phi$, $\mathcal{K}_N^{(\alpha_0+1)} = \phi$, a straightforward iteration of the previous construction α_0 times yields a similar estimate.

Assume $p_1, p_2 \in \mathcal{K}$. There are polynomials $p_\pm = (p_1, p_2)_\pm$

$$p_\pm(\sigma) = \sigma^{q'} + \sum_{j<q'} a_j(\lambda)\ \sigma^j \tag{2.11}$$

$$q' \leq Q(N)^2\ ,\ |\partial_\lambda^\alpha\ a_j| < \Phi_3(N) \ll \exp(\log N)^\varepsilon \quad \text{for} \quad |\alpha| \leq Q(N)^2 \tag{2.12}$$

such that

$$p_\pm(\sigma_1 \pm \sigma_2) \in \ \text{Ideal}\ (p_1(\sigma_1), p_2(\sigma_2))\,. \tag{2.13}$$

The set $\widetilde{\mathcal{K}} = \{p_\pm \mid p_1, p_2 \in \mathcal{K}\}$ will still satisfy (2.4)-(2.6) replacing α_0 by $2\alpha_0$ and C_3 by $2C_3$, $\Phi_2(N)$ by $\Phi_2(N)^2$.

Lemma 2. *Let n_1, n_2 be fixed and $|n_1 - n_2| > N$. Let $\delta < N^{-1}$. Except for λ in a set of measure at most $\Phi_2(N)^3\ \delta^{1/\Phi_4(N)}$, $\Phi_4 = \Phi_4(Q(N), C_3, \alpha_0, c_6) \ll (\log N)^\varepsilon$, we have*

$$|p(\langle n_1 - n_2, \lambda\rangle \pm \mu_{m_1} \pm \mu_{m_2})| > \delta \tag{2.14}$$

for all $m_1, m_2 \in \mathbf{Z}^2$, $p \in \widetilde{\mathcal{K}}$.

Proof. Replace first μ_m by $|m|^2 \in \mathbf{Z}$. Observe that by (2.12) and assumption $|n_1 - n_2| > N$, one has again $\left|\left(\frac{\partial}{\partial_\lambda}\right)^q\ p(\sigma_1(\lambda))\right| \sim |n_1 - n_2|^q$ and using the structure of $\widetilde{\mathcal{K}}$ one may as in Lemma 1 ensure

$$p\left(\langle n_1 - n_2, \lambda\rangle \pm |m_1|^2 \pm |m_2|^2\right) > \delta_1 \tag{2.15}$$

except for λ in a set of measure $< \Phi_2(N)^2\ \delta_1^{1/\Phi}$ where $\Phi = \Phi(Q(N), C_3, \alpha_0) \ll (\log N)^\varepsilon$. By (2.8), replacement of μ_m by $|m|^2$ is allowed for $|m_1|^{-c_6}, |m_2|^{-c_6} < 0\left(\frac{\delta_1}{N}\right) > \delta_1^2$. Also, one only needs to consider pairs (m_1, m_2) such that $|m_1 - m_2| < |n_1 - n_2|$. Thus it remains to verify (2.14) for at most $N^2\ \delta_1^{-\frac{4}{c_6}}$ pairs (m_1, m_2). This yields the estimate

$$\Phi_2(N)^2\ \delta_1^{1/\Phi} + N^2\ \delta_1^{-\frac{4}{c_6}}\ \Phi_2(N)^2\ \delta^{1/\Phi} \tag{2.16}$$

for the exceptional λ-set. Optimizing (2.16) in δ_1 yields the result.

3. Structure of the matrices

For given $m_0 \in \mathbf{Z}^2$, consider the box $Q = \{m \in \mathbf{Z}^2 \mid |(m - m_0)_i| < N^{10}\}$ and let $T_Q = T_N\, |_Q$. We aim to establish conditions on (λ, σ) to ensure that for each Q

$$\|T_Q^{-1}\| < M < \Phi_1(N) \tag{3.1}$$

$$\left|T_Q^{-1}\left((n, \pm, m), (n', \pm, m')\right)\right| < \exp\left(-\frac{1}{2}\, |m - m'|^{c_1}\right) \quad \text{if} \quad |m - m'| > N^4. \tag{3.2}$$

This will essentially be achieved by a pattern of "isolated islands" which will be constructed later in this section. Properties (3.1), (3.2) may then be extended to T_N or any restriction of T_N to a union of Q-cubes.

We will also conc[illegible]ing off-diagonal estimates in the n-variable for previous N-scales and the decay (1.2) for T that

$$\left|T_N^{-1}\left((n, \pm, m), (n', \pm, m')\right)\right| < \exp\left(-\frac{1}{2}\, |n - n'|^{c_1}\right) \quad \text{if} \quad |n - n'| > \Phi_0(N) = \exp(\log N)^{1/2} \tag{3.3}$$

and similarly for restrictions of T_N to unions of Q-cubes.

Uniform off-diagonal estimates for the inverses f[illegible] the successive N-scales result from applications of the localization identity for inverses and only the specific exponential decay rate of T matters here (taking $c_1 < 1$).

These off-diagonal estimates will be used in particular when restricting to finite matrices having certain bounds on their inverses, in the process of the Newton scheme.

Considering for given Q the matrix $T_Q^{\lambda, \sigma}$, we aim to ensure (3.1), (3.2) by conditions on λ, σ of the form described in section (2). For a given Q, the number of polynomials will be at most $\Phi_2(N)$. Moreover one has to analyze the dependence on Q and show that the resulting polynomials for varying Q will belong to a compact $\mathcal{K}$ of the form (2.4)-(2.6). This fact will result from the structure of the matrices $T_Q^{\lambda, \sigma}$. The relevant feature for this issue is the behaviour of the diagonal

$$\pm\langle \lambda, n\rangle + |m|^2 \pm \sigma.$$

Assume Q contains a point m_0 such that

$$\left||m_0|^2 \pm \sigma\right| < N^2 \tag{3.4}$$

(otherwise $|\pm\langle \lambda, n\rangle + \mu_m \pm \sigma| > N^2$ for all $m \in Q$).

Put $\tilde{\sigma} = \sigma \pm |m_0|^2$ and write the diagonal

$$\pm\langle \lambda, n\rangle + |m|^2 \pm \sigma = \pm\langle \lambda, n\rangle + \left(|m|^2 - |m_0|^2\right) \pm \tilde{\sigma}. \tag{3.5}$$

The only relevant part is $\left(|m|^2 - |m_0|^2\right)_{m \in Q}$. Write $m = m_0 + \Delta m$. Assume $m', m'' \in Q$ such that $\Delta m'$, $\Delta m''$ are linearly independent and

$$\begin{cases} \left||m'|^2 - |m_0|^2\right| < |m_0|^{1/2} \\ \left||m''|^2 - |m_0|^2\right| < |m_0|^{1/2}. \end{cases} \tag{3.6}$$

Hence (assuming $|m_0| > N^{40}$)

$$\begin{cases} |\langle m_0, \Delta m'\rangle| < 2\ |m_0|^{1/2} \\ |\langle m_0, \Delta m''\rangle| < 2\ |m_0|^{1/2} \end{cases} \tag{3.7}$$

implying

$$|m_0| < N^{10}\ |m_0|^{1/2}. \tag{3.8}$$

From the assumption on m_0, it follows that

$$\dim \left\{ m - m_0 \mid m \in Q \quad \text{and} \quad \left||m|^2 - |m_0|^2\right| < |m_0|^{1/2} \right\} \leq 1. \tag{3.9}$$

Consequently, there is a vector $v \neq 0$ in $\mathbf{Z}^2$, $|v| < N^{10}$ such that if $m \in Q$, $m - m_0 \notin \mathbf{Z}v$, then

$$\left||m|^2 - |m_0|^2\right| > |m_0|^{1/2}$$

and hence

$$|\pm\langle\lambda, n\rangle + \mu_m \pm \sigma| > \frac{1}{2}|m_0|^{1/2}. \tag{3.10}$$

Define

$$Q_1 = \{m \in Q \mid m - m_0 \in \mathbf{Z}v\}. \tag{3.11}$$

Thus

$$\left\| T^{-1}_{Q\setminus Q_1} \right\| < |m_0|^{-1/2}.$$

Writing

$$T_Q = \begin{pmatrix} T_{Q\setminus Q_1} & P^* \\ P & T_{Q_1} \end{pmatrix} \tag{3.12}$$

one has for the inverse

$$T_Q^{-1} = \begin{pmatrix} T^{-1}_{Q\setminus Q_1} + T^{-1}_{Q\setminus Q_1}\ P^*(T_{Q_1} - P\ T^{-1}_{Q\setminus Q_1}\ P^*)^{-1}\ P\ T^{-1}_{Q\setminus Q_1} & -T^{-1}_{Q\setminus Q_1}\ P^*(T_{Q_1} - P\ T^{-1}_{Q\setminus Q_1}\ P^*)^{-1} \\ -(T_{Q_1} - P\ T^{-1}_{Q\setminus Q_1}\ P^*)^{-1}\ P\ T^{-1}_{Q\setminus Q_1} & (T_{Q_1} - P\ T^{-1}_{Q\setminus Q_1}\ P^*)^{-1} \end{pmatrix} \tag{3.13}$$

which is controlled by

$$(T_{Q_1} - P\ T^{-1}_{Q\setminus Q_1}\ P^*)^{-1}. \tag{3.14}$$

The matrix $T_{Q_1} - P\ T^{-1}_{Q\setminus Q_1}\ P^*$ is a perturbation of T_{Q_1} and the conditions on $(\lambda, \tilde\sigma)$ to control (3.14) will in the limit (here for $|m_0| \to \infty$) amount to those controlling $T^{-1}_{Q_1}$. For $m = m_0 + s.v$, one has

$$\pm\langle\lambda, n\rangle + |m|^2 \pm \sigma = \pm\langle\lambda, n\rangle + 2\ \langle m_0, v\rangle\ s + s^2|v|^2 \pm \tilde\sigma \quad (|\tilde\sigma| < N^2 \text{ from (3.4)}). \tag{3.15}$$

Clearly there are at most N^{20} possibilities for v and the only remaining parameter is the integer $\langle m_0, v\rangle$ which is not necessarily bounded. We distinguish following cases

(a) $|\langle m_0, v\rangle| > N^{20}$

Then $|\pm\langle\lambda, n\rangle + \mu_m \pm \sigma| > N^{20}$ for all $(n, \pm, m) \in Q$ with $m \neq m_0$.

(b) $|\langle m_0, v\rangle| \leq N^{20}$.

Case (a)

In this case, there are possibly only small sites for $m = m_0$ and it will suffice for (λ, σ) to ensure the properties

$$\|T_{Q_0}^{-1}\| < M \tag{3.16}$$

where

$$Q_0 = \{ |(m - m_0)_1| < N^2 \} . \tag{3.17}$$

The effect of the other $m \neq m_0$ on the conditions is to generate an $\alpha_0 = 2$ compact satisfying (2.6) for some absolute c_3.

Case (b)

In this case, also (m_0, v) is determined up to N^{20} choices and the elements outside Q_1 grow at least as $|m_0|^{1/2}$. Conditions on (λ, σ) will clearly generate on $\alpha_0 = 1$ compact satisfying again (2.6) for a specific c_3.

Coming to conditions (3.1), (3.2). From the structure of the Q_1-diagonal

$$\pm\langle\lambda, n\rangle + 2\,\langle m_0, v\rangle\; s + |v|^2\; s^2 \pm \tilde{\sigma} \tag{3.18}$$

considered as a function of s, it follows that one may cover the set

$$\{m \in Q \mid |\pm\langle\lambda, n\rangle + \mu_m \pm \sigma| < 1 \quad \text{for some} \quad |n| < N\} \subset \{m \in Q \mid \big||m|^2 \pm \sigma\big| < N^2\}$$

by sets Q_α such that

$$\begin{cases} \text{each } Q_\alpha \text{ is a union of } N\text{-squares} & \\ \operatorname{diam} Q_\alpha < N^3 & (3.19) \\ \operatorname{dist}(Q_\alpha, Q_\beta) > N \quad \text{for} \quad \alpha \neq \beta. & (3.20) \end{cases}$$

Again from the localization identify for inverses, one needs to satisfy for each α

$$\|T_{Q_\alpha}^{-1}\| < M. \tag{3.21}$$

4. Inductive step

Let $N = N_r$ and fix a region Q in the m-variable which is a union of N-size boxes. We assume $N_r > N_{r-1}^{10}$, so that Q is admissible for all stages $r' \leq r - 1$. In fact this Q-restriction will play little role in what follows and will be ignored therefore.

We first consider the range of the n-variable $(|n_j| \leq N)$. The matrix $T = T_N^{\lambda,\sigma}\,|_Q$ appears as a (2×2) block matrix

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{12}^* & T_{22} \end{pmatrix}.$$

For a subset $A \subset [-N, N]^d$, denote then T_A the restriction

$$\begin{pmatrix} T_{11}\,|_A & T_{12}\,|_A \\ T_{21}^*\,|_A & T_{22}\,|_A \end{pmatrix}$$

(unless otherwise specified).

The first purpose is to write

$$[-N,N]^d = \Omega_1 \cup \Omega_2$$

where $T_{\Omega_1}^{-1}$ will be well-controlled and Ω_2 at most of bounded size.

Let $s \leq r-1$ and $A = n_0 + [-N_s, N_s]^d$ a box contained in $[-N,N]^d$. Observe that by the structure of T described in (1)

$$T_A = T^{\lambda,\sigma+\langle\lambda,n_0\rangle}_{[-N_s,N_s]^d}. \tag{4.1}$$

Choose $s = r-1$. We claim that with $A(n_0)$ as above, we may ensure

$$\|T^{-1}_{A(n_0)}\| < \Phi_1(N_{r-1}) \tag{4.2}$$

except for n_0 in a cube $\Lambda_{r-1} \subset [-N_r, N_r]^d$ of size $< 10N_{r-1}$.

To achieve (4.2), we need to ensure

$$|p(\pm\langle\lambda,n\rangle \pm \sigma + \mu_m)| > \frac{1}{\Phi_1(N_{r-1})} \tag{4.3}$$

for $n \in A(n_0)$, $m \in Q$ and p a polynomial in $\mathcal{K}_{N_{r-1}}$ as described in section 2.

Assume $n_1 \in A(n_0)$, $n_1' \in A(n_0')$ such that

$$|n_0 - n_0'| > 10N_{r-1} \tag{4.4}$$

$$|p(\nu\langle\lambda,n_1\rangle + \nu\sigma + \mu_m)| < \frac{1}{\Phi_1(N_{r-1})} \tag{4.5}$$

$$|p'(\nu'\langle\lambda,n_1'\rangle + \nu'\sigma + \mu_{m'})| < \frac{1}{\Phi_1(N_{r-1})} \tag{4.6}$$

where $p,p' \in \mathcal{K}_{N_{r-1}}$ and $\nu,\nu' = \pm 1$.

Distinguishing the cases $\nu = \nu'$ (resp. $\nu = -\nu'$) and denoting $P_- = (p,p')_-$ (resp. $P_+ = (p,p')_+$) the polynomial (2.13), it follows from (4.5), (4.6) that

$$|P_\mp(\langle\lambda, n_1 - n_1'\rangle + \mu_m \mp \mu_{m'})| \ll \frac{\exp(\log N_{r-1})^\varepsilon}{\Phi_1(N_{r-1})} < \frac{1}{\Phi_1(N_{r-1})^{1/2}}. \tag{4.7}$$

From (4.4), $|n_1 - n_1'| > 8N_{r-1}$. By Lemma 2, if one wants to avoid (4.5), (4.6), an exceptional λ-set of measure

$$N_r^{2d}\,\Phi_2(N_{r-1})^3\,\Phi_1(N_{r-1})^{-\frac{1}{2\Phi_4(N_{r-1})}} \tag{4.8}$$

has to be excluded.

To ensure say

$$(4.8) < N_{r-1}^{-1} \tag{4.9}$$

we assume

$$\log N_r \sim (\log N_{r-1})^{10} \quad \text{and} \quad \Phi_2(N) \leq \exp(\log N)^{10} \tag{4.10}$$

and hence

$$\log\Phi_1(N_{r-1}) > (4d\,\log N_r + 6\log\Phi_2(N_{r-1}))\;\;\Phi_4(N_{r-1}) < (\log N_{r-1})^{11}. \tag{4.11}$$

Thus by excluding a λ-set of measure at most N_{r-1}^{-1}, (4.2) will hold except for n_0 in a box Λ_{r-1} of size $\sim N_{r-1}$. Observe that the λ-set to be removed may be taken to be a union of intervals of length $\Phi_1(N_{r-1})^{-2}$

say of total measure $< N_{r-1}^{-1}$. This last point is of importance in extending the approximate solutions (for "good" λ) to the entire λ-range.

We now repeat the preceding letting $n_0 \in \Lambda_{r-1}$. Remark that the dependence of (4.7) on n_1, n_1' only appears through the difference $n_1 - n_1'$ where $|n_1 - n_1'| < 10N_{r-1}$. Consequently the conditions (4.7) lead again to a measure estimate

$$(10N_{r-1})^{2d}\ \Phi_2(N_{r-2})^2\ \Phi_1(N_{r-2})^{-\frac{1}{2\Phi_4(N_{r-2})}} < N_{r-2}^{-1} \tag{4.12}$$

for the λ-set to be removed. One may then obtain a subbox Λ_{r-2} of Λ_{r-1} of size $< 10N_{r-2}$ such that for $n_0 \in \Lambda_{r-1}\backslash\Lambda_{r-2}$, $A = n_0 + [-N_{r-2}, N_{r-2}]^d$, one has

$$\|T_A^{-1}\| < \Phi_1(N_{r-2}). \tag{4.13}$$

Iterating this construction permits to obtain a set $\Lambda_0 = \Omega_2$ of bounded size such that

$$\|T_A^{-1}\| < \Phi_1(N_s) \tag{4.14}$$

if $A = n_0 + [-N_s, N_s]^d$ and

$$20N_{s+1} > \text{dist}(n_0, \Omega_2) > 20N_s. \tag{4.15}$$

Besides (4.14), one will also satisfy for such A the off-diagonal estimates

$$\left|T_A^{-1}\left((n,m),(n',m')\right)\right| < \begin{cases} \exp\left(-\frac{1}{2}\,|n-n'|^{c_1}\right) & \text{if } \ |n-n'| > \Phi_0(N_s) \ll N_s^{\varepsilon} \\ \exp\left(-\frac{1}{2}\,|m-m'|^{c_1}\right) & \text{if } \ |m-m'| > N_s^5 \end{cases} \tag{4.16}$$

by (3.3).

Denote

$$T_A^{-1}(n,n') = \left(T_A^{-1}\left((n,m),(n',m')\right)\right)_{m,m'\in Q} \tag{4.17}$$

which is a matrix with index set Q. Clearly by (4.14), (4.16) and Shur's estimate, we have

$$\|T_A^{-1}(n,n')\| < \Phi_1(N_s) \tag{4.18}$$

and for

$$|n-n'| > \Phi_0(N_s) \tag{4.19}$$

$$\begin{aligned} \|T_A^{-1}(n,n')\| &\le \max_m \sum_{m'} \left|T_A^{-1}\left((n,m),(n',m')\right)\right| \\ &< \max_m \left(\sum_{|m-m'|<N_s^5} + \sum_{|m-m'|>N_s^5} \right) \left|T_A^{-1}\left((n,m),(n',m')\right)\right| \\ &< N_s^{10}\ \exp\left(-\frac{1}{2}\,|n-n'|^{c_1}\right) + \sum_{|m'-m|>N_s^5} \exp\left(-\frac{1}{2}\,|m-m'|^{c_1}\right) \\ &< \exp\left(-\frac{1}{3}\,|n-n'|^{c_1}\right). \end{aligned} \tag{4.20}$$

Fix $n_1, n_2 \in \Omega_2$, $\text{dist}(n_1,\Omega_2) \ge \text{dist}(n_2,\Omega_2)$ and let for $s \le r-1$

$$20N_{s+1} > \text{dist}(n_1,\Omega_2) > 20N_s \tag{4.21}$$

$A = n_1 + [-N_s, N_s]^d$. From the localization identity, one has

$$T_{\Omega_1}^{-1}(n_1, n_2) = T_A^{-1}(n_1, n_2) + \sum_{n' \in A, n'' \in \Omega_1 \setminus A} T_A^{-1}(n_1, n')\ T(n', n'')\ T_{\Omega_1}^{-1}(n'', n_2). \tag{4.22}$$

Hence from (4.18), (4.20), (4.10)

$$\begin{aligned} \|T_{\Omega_1}^{-1}(n_1, n_2)\| < \\ \Phi_1(N_s) + \sum_{\substack{|n'-n_1| < \Phi_0(N_s) \\ |n''-n_1| > N_s}} \Phi_1(N_s)\ e^{-\frac{1}{3}|n'-n''|^{c_1}}\ \|T_{\Omega_1}^{-1}(n'', n_2)\| \\ + \sum_{\substack{|n'-n_1| > \Phi_0(N_s) \\ |n''-n_1| > N_s}} e^{-\frac{1}{3}|n_1-n'|^{c_1} - \frac{1}{3}|n'-n''|^{c_1}}\ \|T_{\Omega_1}^{-1}(n'', n_2)\| \\ < \Phi_1(\operatorname{dist}(n_1, \Omega_2)) + \sup_{|n-n_1| > N_s} e^{-\frac{1}{4}|n-n_1|^{c_1}}\ \|T_{\Omega_1}^{-1}(n, n_2)\|. \end{aligned} \tag{4.23}$$

Take

$$\Phi_1(N) < \exp(\log N)^{12} \tag{4.24}$$

(compatible with (4.11)).

From iterating (4.23), one deduces that

$$\|T_{\Omega_1}^{-1}(n_1, n_2)\| < \Phi_1\left(\operatorname{dist}'(n_1, \Omega_2) + \operatorname{dist}'(n_2, \Omega_2)\right) \tag{4.25}$$

with

$$\operatorname{dist}'(n, \Omega_2) = \min(\operatorname{dist}(n, \Omega_2), N_{r-1}) \tag{4.26}$$

and

$$\|T_{\Omega_1}^{-1}(n_1, n_2)\| < e^{-\frac{1}{5}|n_1-n_2|^{c_1}} \tag{4.27}$$

if

$$|n_1 - n_2| > 10\Phi_0\left(\operatorname{dist}'(n_1, \Omega_2) + \operatorname{dist}'(n_2, \Omega_2)\right) \tag{4.28}$$

and in particular if

$$|n_1 - n_2| > 10\Phi(N_{r-1}).$$

Coming back to Ω_2, assume $n_1, n_2 \in \Omega_2$ and

$$\begin{aligned} |\nu_1\langle\lambda, n_1\rangle + \nu_1\sigma + \mu_{m_1}| < \varepsilon_0 \\ |\nu_2\langle\lambda, n_2\rangle + \nu_2\sigma + \mu_{m_2}| < \varepsilon_0 \end{aligned} \tag{4.29}$$

for some $\nu_1, \nu_2 = \pm 1$, $m_1, m_2 \in \mathbf{Z}^2$. Here $\varepsilon_0 \gg \varepsilon$ is a small number.

It follows from (4.29) that

$$|\langle\lambda, n_1 - n_2\rangle + \nu_1\ \mu_{m_1} - \nu_2\ \mu_{m_2}| < 2\ \varepsilon_0. \tag{4.30}$$

Let $|n_1 - n_2| < C_{12}$. We distinguish two cases

Case 1. $|m_1|$, $|m_2|$ are small (bounded by C_{12} say).

Case 2. Either $|m_1| > C_{12}$ or $|m_2| > C_{12}$.

If (4.30) holds, necessarily $\nu_1 = \nu_2$ and both $|m_1|, |m_2| > \frac{1}{2} C_{12}$. In particular $\mu_{m_i} \notin \{\lambda_1, \lambda_2\}$, $\mu_{m_i} \approx |m_i|^2$. Hence, by approximate choice of λ, we may ensure $n_1 = n_2$ as only possibility. Thus Ω_2 is restricted to a single site, $\Omega_2 = \{n_0\}$, with either + or − sign and

$$|\langle \lambda, n\rangle + \sigma \pm \mu_m| < \varepsilon_0 \tag{4.31}$$

only appears for $m \in \mathbf{Z}^2$ such that $|m|^2$ takes a given value.

In either case, fix $n_0 \in \Omega_2$ and redefine

$$\sigma_1 = \sigma + \langle n_0, \lambda\rangle.$$

Thus the diagonal elements of T are $\pm\langle \lambda, n - n_0\rangle \pm \sigma_1 + \mu_{\mp m}$.

For simplicity we will put $n_0 = 0$.

We next show that

$$\left|T_{\Omega_1}^{-1}((n_1, m_1), (n_2, m_2))\right| < \exp\left(-\frac{1}{10}|m_1 - m_2|^{c_1/20}\right) \quad \text{if} \quad |m_1 - m_2| > (|n_1| + |n_2|)^{10} \tag{4.32}$$

establishing an off-diagonal estimate with respect to the m-variable.

Assume $|n_1| \geq |n_2|$. Let A be as above (i.e. a neighborhood of n_1 of size $\sim |n_1|$).

Applying again the localization identity for the inverse

$$T_{\Omega_1}^{-1}((n_1, m_1), (n_2, m_2)) = T_A^{-1}((n_1, m_1), (n_2, m_2)) \tag{4.33}$$

$$+ \sum_{n\in A, n'\notin A} T_A^{-1}((n_1, m_1), (n, m))\ T((n, m), (n', m'))\ T_{\Omega_1}^{-1}((n', m'), (n_2, m_2)). \tag{4.34}$$

From the induction hypothesis (4.16)

$$|(4.33)| < e^{-\frac{1}{2}|m_1 - m_2|^{c_1}} \tag{4.35}$$

since

$$|m_1 - m_2| > (\text{size } A)^5. \tag{4.36}$$

Examine the second term (4.34). Using the first index, estimate by

$$\max_{|n' - n_1| > \frac{1}{2}|n_1|} \varepsilon\, e^{-\frac{1}{4}|n_1 - n'|^{c_1}} \left((4.37) + (4.38) + (4.39)\right) \tag{4.40}$$

where

$$(4.37) = \sum_{\substack{|m - m_1| < |n_1|^5 \\ |m - m'| < |n_1|^5}} \Phi_1(|n_1|)\ e^{-|m - m'|^{c_1}}\ \left|T_{\Omega_1}^{-1}((n', m'), (n_2, m_2))\right|$$

$$(4.38) = \sum_{\substack{|m - m_1| < |n_1|^5 \\ |m - m'| > |n_1|^5}} \Phi_1(|n_1|)\ e^{-|m - m'|^{c_1}}\ \left|T_{\Omega_1}^{-1}((n', m'), (n_2, m_2))\right|$$

$$(4.39) = \sum_{\substack{|m - m_1| > |n_1|^5 \\ m'}} e^{-\frac{1}{2}|m_1 - m|^{c_1} - |m - m'|^{c_1}}\ \left|T_{\Omega_1}^{-1}((n', m'), (n_2, m_2))\right|.$$

Thus

$$(4.37) < \max_{\substack{|n' - n_1| > \frac{1}{2}|n_1| \\ |m' - m_1| < 2|n_1|^5}} \varepsilon\, e^{-\frac{1}{4}|n_1 - n'|^{c_1}}\ \left|T_{\Omega_1}^{-1}((n', m'), (n_2, m_2))\right| \tag{4.41}$$

$$(4.38) + (4.39) < \max_{\substack{|n'-n_1| > \frac{1}{2}|n_1| \\ m'}} \varepsilon\, e^{-\frac{1}{4}|n_1-n'|^{c_1} - \frac{1}{2}|m_1-m'|^{c_1}} \left|T_{\Omega_1}^{-1}\left((n', m'), (n_2, m_2)\right)\right| . \tag{4.42}$$

If $|n_1 - n'| > |m_1 - m_2|^{1/20}$, one gets the bound

$$e^{-\frac{1}{8}|n_1-n'|^{c_1} - \frac{1}{8}|m_1-m_2|^{c_1/20}} \Phi_1\left(|n_1| + |n' - n_1|\right) < e^{-\frac{1}{9}|m_1-m_2|^{c_1/20}} \tag{4.43}$$

since $|m_1 - m_2| > |n_1|^{10}$.

Assume $|n_1 - n'| \leq |m_1 - m_2|^{1/20}$. Clearly, in any case (4.41) + (4.42) is at most

$$\varepsilon \max_{\substack{|n'-n_1| < |m_1-m_2|^{1/20} \\ |m'-m_2| > |m_1-m_2| - 2|n_1|^5}} \left|T_{\Omega_1}^{-1}\left((n', m'), (n_2, m_2)\right)\right| \tag{4.44}$$

$$+ \varepsilon \max_{\substack{|n'-n_1| < |m_1-m_2|^{1/20} \\ m'}} e^{-\frac{1}{2}|m_1-m'|} \left|T_{\Omega_1}^{-1}\left((n', m'), (n_2, m_2)\right)\right| . \tag{4.45}$$

The worse contribution comes from (4.44). If one performs an iteration, the original pairs (n_1, m_1), (n_2, m_2) get replaced by pairs $(\overline{n}_1, \overline{m}_1)$, $(\overline{n}_2, \overline{m}_2)$ and we need to preserve the property

$$|\overline{m}_1 - \overline{m}_2| > \left(|\overline{n}_1| + |\overline{n}_2|\right)^5 \tag{4.46}$$

to exploit (4.16).

If we iterate (4.44) k times, the pair $(\overline{n}_1, \overline{m}_1)$, $(\overline{n}_2, \overline{m}_2)$ will clearly satisfy

$$|\overline{n}_1 - n_1| < k|m_1 - m_2|^{1/20} \quad , \quad |\overline{n}_2 - n_2| < k|m_1 - m_2|^{1/20}$$

$$|\overline{m}_1 - \overline{m}_2| > |m_1 - m_2| - 2k\, |n_1|^5 - 2k^6|m_1 - m_2|^{1/4}.$$

Hence

$$|\overline{n}_1| + |\overline{n}_2| < |n_1| + |n_2| + 2k\, |m_1 - m_2|^{1/20} < |m_1 - m_2|^{1/10} + 2k\, |m_1 - m_2|^{1/20} < 2\, |m_1 - m_2|^{1/10} \tag{4.47}$$

and

$$|\overline{m}_1 - \overline{m}_2| > |m_1 - m_2| - 2k\, |m_1 - m_2|^{1/2} - 2k^6|m_1 - m_2|^{1/4} > \frac{1}{2}\, |m_1 - m_2| \tag{4.48}$$

provided

$$k < |m_1 - m_2|^{1/20}. \tag{4.49}$$

At this stage however, such iteration factor yields already a factor $\varepsilon^{|m_1-m_2|^{1/20}} < e^{-|m_1-m_2|^{c_1}}$ again. This yields (4.32).

Considering the decomposition $\Omega = \Omega_1 \cup \Omega_2$, write

$$T = \begin{pmatrix} T_{\Omega_1} & P^* \\ P & T_{\Omega_2} \end{pmatrix} . \tag{4.50}$$

We distinguish the cases (1), (2) above. Thus in case (2), Ω_2 consists of a single element for either the + or − sign and (in the case of + sign)

$$T_{\Omega_1} = \begin{pmatrix} T_{11}|_{\Omega_1} & T_{12}\,|_{\Omega_1} \\ (T_{12}\,|_{\Omega_1})^* & T_{22} \end{pmatrix} \qquad T_{\Omega_2} = \begin{pmatrix} T_{11}|_{\Omega_2} & 0 \\ 0 & 0 \end{pmatrix} .$$

In case (1)

$$T_{\Omega_1} = \begin{pmatrix} T_{11}|_{\Omega_1} & T_{12}|_{\Omega_1} \\ T_{12}^*|_{\Omega_1} & T_{22}|_{\Omega_1} \end{pmatrix}.$$

Write

$$T^{-1} = \begin{pmatrix} T_{\Omega_1}^{-1} + T_{\Omega_1}^{-1} P^* \left(T_{\Omega_2} - P\, T_{\Omega_1}^{-1} P^*\right)^{-1} P\, T_{\Omega_1}^{-1} & -T_{\Omega_1}^{-1} P^* \left(T_{\Omega_2} - P\, T_{\Omega_1}^{-1} P^*\right)^{-1} \\ -\left(T_{\Omega_2} - P\, T_{\Omega_1}^{-1} P^*\right)^{-1} P\, T_{\Omega_1}^{-1} & \left(T_{\Omega_2} - P\, T_{\Omega_1}^{-1} P^*\right)^{-1} \end{pmatrix}. \tag{4.51}$$

Recall (4.25), thus

$$\|T_{\Omega_1}^{-1}\| < \Phi_1(N_{r-1}). \tag{4.52}$$

Hence it remains to control

$$\left(T_{\Omega_2} - P\, T_{\Omega_1}^{-1} P^*\right)^{-1}. \tag{4.53}$$

Estimate for $n_1, n_2 \in \Omega_2$

$$\left\|(P\, T_{\Omega_1}^{-1} P^*)(n_1, n_2)\right\| \le \sum_{n', n''} \|P(n_1, n')\|\, \|T_{\Omega_1}^{-1}(n', n'')\|\, \|P^*(n'', n_2)\|$$

and by (4.25)

$$\begin{aligned} &< \varepsilon^2 \sum_{n', n''} e^{-|n_1 - n'|^{c_1} - |n_2 - n''|^{c_1}}\ \Phi_1\ (|n'| + |n''|) \\ &< \varepsilon^2 \sum_{n', n''} e^{-\frac{1}{2}|n'|^{c_1} - \frac{1}{2}|n''|^{c_1}}\ \Phi_1\ (|n'| + |n''|) \\ &\le \varepsilon^2. \end{aligned}$$

Recall that Ω_2 is a neighborhood of 0 of bounded size. Hence also

$$\|P\, T_{\Omega_1}^{-1} P^*\| \le \varepsilon^2. \tag{4.54}$$

Also, by (4.32)

$$\left|(P\, T_{\Omega_1}^{-1} P^*)\left((n_1, m_1), (n_2, m_2)\right)\right| \le$$

$$\sum_{n', n'', m', m''} \left|P\left((n_1, m_1), (n', m')\right)\right|\, \left|(T_{\Omega_1}^{-1})\left((n', m'), (n'', m'')\right)\right|\, \left|P^*\left((n'', m''), (n_2, m_2)\right)\right| \le$$

$$\varepsilon^2 \sum_{\substack{n', n'' \\ |m' - m''| < (|n'| + |n''|)^{10}}} e^{-\frac{1}{2}|n'|^{c_1} - |m_1 - m'|^{c_1} - \frac{1}{2}|n''|^{c_1} - |m_2 - m''|^{c_1}}\ \Phi_1\ (|n'| + |n''|) \tag{4.55}$$

$$+\, \varepsilon^2 \sum_{\substack{n', n'' \\ |m' - m''| \ge (|n'| + |n''|)^{10}}} e^{-\frac{1}{2}|n'|^{c_1} - |m_1 - m'|^{c_1} - \frac{1}{2}|n''|^{c_1} - |m_2 - m''|^{c_1} - \frac{1}{10}|m' - m''|^{c_1/20}}. \tag{4.56}$$

Clearly

$$(4.55) < \varepsilon^2\ e^{-\frac{1}{4}|m_1 - m_2|^{c_1/10}}$$

$$(4.56) < \varepsilon^2\ e^{-\frac{1}{10}|m_1 - m_2|^{c_1/20}}$$

so that

$$\left|(P\, T_{\Omega_1}^{-1} P^*)\ \left((n_1, m_1), (n_2, m_2)\right)\right| < \varepsilon^2\ e^{-\frac{1}{10}|m_1 - m_2|^{c_1/20}}. \tag{4.57}$$

Next, consider following decomposition according to the m-variable

$$\Omega_2 = \Omega_{2,1} \cup \Omega_{2,2}.$$

In case (1), Ω_{22} is a bounded set, such that

$$|\pm\langle n, \lambda\rangle \pm \sigma + \mu_{\mp m}| > 1 \quad \text{for} \quad m \in \Omega_{2,1}. \tag{4.58}$$

In case (2), Ω_{22} will consist of the m-values for which $|m|^2$ takes a certain given value and

$$|\pm\langle n_0, \lambda\rangle \pm \sigma + \mu_{\mp m}| > \frac{1}{2} \quad \text{for} \quad \Omega_2 = \{n_0\}\ ,\ m \in \Omega_{2,1}. \tag{4.59}$$

Decompose $P\ T_{\Omega_1}^{-1}\ P^*$ accordingly

$$P\ T_{\Omega_1}^{-1}\ P^* = \begin{pmatrix} U_{11} & U_{12}^* \\ U_{12} & U_{22} \end{pmatrix} \tag{4.60}$$

so that

$$T_{\Omega_2} - P\ T_{\Omega_1}^{-1}\ P^* = \begin{pmatrix} T_{\Omega_{2,1}} - U_{11} & Q^* - U_{12}^* \\ Q - U_{12} & T_{\Omega_{22}} - U_{22} \end{pmatrix}. \tag{4.61}$$

Observe that by (4.54), (4.58), (4.59), the diagonal elements of $T_{\Omega_{2,1}} - U_{11}$ are at least $\frac{1}{2} - \varepsilon^2 > \frac{1}{3}$. It remains thus to control

$$S = T_{\Omega_{22}} - U_{22} - (Q - U_{12})\ (T_{\Omega_{21}} - U_{11})^{-1}\ (Q^* - U_{12}^*). \tag{4.62}$$

Recall that by (4.54), (4.57)

$$\|U_{ij}\| \leq \varepsilon^2 \tag{4.63}$$

$$|U_{ij}(m_1, m_2)| \leq \varepsilon^2\ e^{-\frac{1}{10}|m_1 - m_2|^{c_1/20}}. \tag{4.64}$$

From the preceding it follows in particular that

$$\|(T_{\Omega_{2,1}} - U_{11})^{-1}\| \leq 3. \tag{4.65}$$

Since Q appears as an off-diagonal part of T

$$\|Q\| < \varepsilon. \tag{4.66}$$

Consequently, from (4.63)-(4.66)

$$S = T_{\Omega_{22}} + 0(\varepsilon^2). \tag{4.67}$$

Also

$$|T_{\Omega_{2,1}}(m_1, m_2)| < \varepsilon\ e^{-|m_1 - m_2|^{c_1}}\quad,\quad |T_{\Omega_{22}}(m_1, m_2)| < \varepsilon\ e^{-|m_1 - m_2|^{c_1}} \quad \text{for} \quad m_1 \neq m_2 \tag{4.68}$$

$$|Q(m_1, m_2)| < \varepsilon\ e^{-|m_1 - m_2|^{c_1}} \tag{4.69}$$

from which by (4.64) one easily derives that

$$|S(m_1, m_2)| < \varepsilon\ e^{-\frac{1}{11}|m_1 - m_2|^{c_1/20}}. \tag{4.70}$$

Case (1)

The diagonal of $T_{\Omega_{22}}$ is given by

$$\pm\langle \lambda, n - n_0\rangle \pm \sigma_1 + \mu_{\mp m}$$

where $\sigma_1 = \langle \lambda, n_0\rangle + \sigma$, $|n - n_0|$ and $|m|$ are bounded by C_{12}. Thus S is a $(d \times d)$-matrix for some bounded d. To control the inverse $(S^{\lambda,\sigma_1})^{-1}$, we consider the determinant $\det S^{\lambda,\sigma_1}$ which is thus of the form

$$\det S^{\lambda,\sigma_1} = \prod_{\substack{|n-n_0|<C_{12} \\ |m|<C_{12}}} (\pm\langle \lambda, n - n_0\rangle \pm \sigma_1 + \mu_{\mp m}) + \varphi(\lambda, \sigma_1) \tag{4.71}$$

where (cf. (4.67))

$$\varphi(\lambda, \sigma_1) = 0(\varepsilon). \tag{4.72}$$

Case (2)

The diagonal of $T_{\Omega_{22}}$ is given by

$$\pm\sigma_1 + \mu_{\mp m}$$

where $\sigma_1 = \langle \lambda, n_0 \rangle + \sigma$, $|m|^2 = k$ (specific value) and the sign is well specified.

Assume $|m_0|^2 = k = |m_1|^2 = |m_2|^2$, $|m_1 - m_0| < |m_0|^{1/4}$, $|m_2 - m_0| < |m_0|^{1/4}$. Then

$$2\ |\langle m_0, m_i - m_0 \rangle| < 2\ |m_0|^{1/2} \qquad (i = 1, 2). \tag{4.73}$$

Since $|\det(m_1 - m_0, m_2 - m_0)| \geq 1$, it follows

$$|m_0|^2 \leq |m_0|^{1/2}\ |m_0|^{1/2}\ |m_0|^{1/2}$$

which is contradiction.

This shows that in an $k^{1/8}$-neighborhood of any solution of $|m|^2 = k$, at most one other point may appear. Partition $\Omega_{22} = U\Omega_{22}^\alpha$ where $\#\Omega_{2,2}^\alpha \leq 2$ and

$$\text{dist}\left(\Omega_{22}^\alpha, \Omega_{22}^\beta\right) > k^{1/8} \quad \text{for} \quad \alpha \neq \beta. \tag{4.74}$$

Case (2.1)

$$k < (\log N_r)^{C_{13}}.$$

In this case S is a $(d \times d)$-matrix, with (from the divisor function)

$$d < \exp \frac{\log k}{\log\log k} < \exp\ C_{13}\ \frac{\log\log N_r}{\log\log\log N_r} = \Phi_5(N_r) \ll (\log N_r)^\varepsilon. \tag{4.75}$$

Writing

$$S = \begin{pmatrix} S_{\Omega_{22}^1} & & R^* \\ & S_{\Omega_{22}^2} & \\ R & & \end{pmatrix} \tag{4.76}$$

one has

$$\|R\| < e^{-\frac{1}{11}k^{c_1/200}} \tag{4.77}$$

from (4.70), while the off-diagonal of each $S_{\Omega_{22}^\alpha}$ is at most $0(\varepsilon)$ by (4.67).

We estimate again $(S^{\lambda,\sigma})^{-1}$ by $(\det S^{\lambda,\sigma})^{-1}$, $\det S^{\lambda,\sigma}$ appearing in the form

$$\begin{aligned} \det S^{\lambda,\sigma} &= \prod_{\alpha \leq d} \left(\det S^{\lambda,\sigma}_{\Omega_{22}^\alpha}\right) + \varphi(\lambda, \sigma_1) \\ &= \prod_{\alpha \leq d} ((\pm\sigma_1 + k)^{\pi_\alpha} + \psi_\alpha(\lambda, \sigma_1)) + \varphi(\lambda, \sigma_1) \end{aligned} \tag{4.78}$$

where

$$|\varphi(\lambda, \sigma_1)| = 0\left(d^d\ e^{-\frac{1}{11}k^{c_1/200}}\right) = 0\left(e^{-\frac{1}{12}k^{c_1/200}}\right) \tag{4.79}$$

$$|\psi_\alpha(\lambda, \sigma_1)| = 0(\varepsilon) \tag{4.80}$$

and $\pi_\alpha = 1$ or 2.

Case (2.2)

$$k > (\log N_r)^{C_{13}}.$$

In this case, letting

$$c_1\, C_{13} > 10000 \tag{4.81}$$

we have

$$\|R\| \leq \exp\left(-\frac{1}{11}\,(\log N_r)^{\frac{c_1 C_{13}}{200}}\right) < \exp-(\log N_r)^{20}. \tag{4.82}$$

On the other hand, we only consider (λ, σ)-values for which the inverse is bounded by $\Phi_1(N_r) < \exp(\log N_r)^{12}$. Hence, we may restrict ourself to controlling the $(S_{\Omega_{22}^\alpha})^{-1}$ provided we insure that for all α

$$\left\| S^{-1}_{\Omega_{22}^\alpha} \right\| < \exp(\log N_r)^{12} \tag{4.83}$$

and the off-diagonal part R may thus be ignored in this case.

Estimate again $S^{-1}_{\Omega_{22}^\alpha}$ by the reciprocal of the determinant, of the form

$$(\pm\sigma_1 + k)^{\pi_\alpha} + \psi_\alpha(\lambda, \sigma_1) \tag{4.84}$$

where $\psi(\lambda, \sigma_1) = 0(\varepsilon)$, $\pi_\alpha = 1$ or 2.

In order to apply Malgrange's preparation theorem and replace the error terms $\varphi(\lambda, \sigma_1)$ and $\psi_\alpha(\lambda, \sigma_1)$ by polynomials in σ_1 (of lower degree than the leading σ_1-power) with smooth coefficients in λ, estimates on $\partial_\lambda^\alpha\, \partial_{\sigma_1}^\beta$-derivatives are necessary.

Restrict λ and σ (hence σ_1) to a sufficiently small neighborhood of the initial pair to ensure a perturbation of the operator T small enough to preserve the previous construction reducing the problem to the invertibility of S given by (4.62). In particular, one needs to preserve the bounds on the $(T_A^{\lambda,\sigma})^{-1}$ for the blocks A of size $\leq N_{r-1}$ considered above. Hence one may allow a perturbation of the (λ, σ)-parameter by an amount

$$N_r^{-1}\ \Phi_1(N_{r-1})^{-1} > \exp-(\log N_r)^2$$

(taking (4.10), (4.24) into account).

We will apply Malgrange's theorem w.r.t. σ_1 restricted to a complex disc D centered at the initial $\overline{\sigma}_1 \in \mathbf{R}$ of size $\exp-(\log N_r)^2$ and analytic (in fact the restriction of rational) functions on D satisfying these further derivative estimates (see (4.99)-(4.101) below). This will in particular lead to at most $\exp(\log N_r)^3$ polynomials (restricting the m-variable to a given domain Q). Thus in (2.4)-(2.6), the function $\Phi_2(N_r)$ may be taken $\exp(\log N_r)^3$ say, which is certainly compatible with (4.10).

Assume

$$|\partial_\lambda^\alpha\ T'\,((n_1, m_1), (n_2, m_2))| < \exp\left(-|n_1 - n_2|^{c_1} - |m_1 - m_2|^{c_1}\right)\ e^{|\alpha|^2} \tag{4.85}$$

where T' refers to the off-diagonal of T.

Of course the operator T' results from the approximative solutions $\{y_k\}$ constructed along the Newton iteration scheme and in verifying (4.85) one uses the fast decay of the consecutive increments $\Delta_k\ y = y_k - y_{k-1}$ as well as the way they are obtained. Recall that essentially one has

$$\Delta_k\ y = -\left[F'(y_{k-1})\right]^{-1}\ F(y_{k-1}) \tag{4.86}$$

and T at stage k is constructed from $F'(y_{k-1})$. One needs to control $\partial_\lambda^\alpha(\Delta_k\ y)$ which from (4.86) and induction hypothesis may be bounded by (cf. [B_1])

$$\|T^{-1}\|^{|\alpha|+1} \max_{|\alpha'|\le|\alpha|} \left|\partial_\lambda^{\alpha'} F(y_{k-1})\right| < B^{k(|\alpha|+1)}\, e^{|\alpha|^2}\, e^{-\beta\rho^{k-1}} < e^{|\alpha|^2+\frac{1}{k^2}|\alpha|^2}\, e^{-\rho^k} \tag{4.87}$$

where $1<\rho<\beta<B$ (considering the possibility of large $|\alpha|$). This remark permits to keep essentially the $e^{|\alpha|^2}$-bound over the iteration.

Recall that we defined (4.31)

$$\sigma_1 = \sigma + \langle n_0, \lambda\rangle$$

so that the diagonal of T writes

$$\pm\langle n-n_0, \lambda\rangle \pm \sigma_1 + \mu_{\mp m}. \tag{4.88}$$

For A defined as earlier in this section, size $A\sim N_s$, we have thus

$$\left\|\partial_\lambda^\alpha\, \partial_{\sigma_1}^\beta\, \left[T_A^{\lambda,\sigma_1}\right]^{-1}\right\| < \|T_A^{-1}\|^{|\alpha|+|\beta|+1}\cdot e^{|\alpha|^2}\cdot N_s^{|\alpha|} < e^{|\alpha|^2}\, \Phi_1(N_s)^{2|\alpha|+|\beta|+1}. \tag{4.89}$$

Also, from (4.25)

$$\left\|\partial_\lambda^\alpha\, \partial_{\sigma_1}^\beta\, (T_{\Omega_1}^{-1})\right\| < \|T_{\Omega_1}^{-1}\|^{|\alpha|+|\beta|+1}\cdot e^{|\alpha|^2}\cdot N_r^{|\alpha|} < e^{|\alpha|^2}\, \Phi_1(N_{r-1})^{2|\alpha|+|\beta|+1}. \tag{4.90}$$

As consequence of an induction hypothesis for the derivative estimates of $(T_N^{\lambda,\sigma})^{-1}$ when $N\le N_{r-1}$, we get

$$\left|\partial_\lambda^\alpha\, \partial_{\sigma_1}^\beta\, \left(T_A^{\lambda,\sigma_1}\right)^{-1}((n_1,m_1),(n_2,m_2))\right| <$$
$$\begin{cases} e^{(|\alpha|+|\beta|)^2}\exp\left(-\frac{1}{3}\,|n_1-n_2|^{c_1}\right) & \text{if } |n_1-n_2|>\Phi_0(N_s) \\ e^{(|\alpha|+|\beta|)^2}\exp\left(-\frac{1}{3}\,|m_1-m_2|^{c_1}\right) & \text{if } |m_1-m_2|>N_s^5. \end{cases} \tag{4.91}$$

Next we establish more precise derivative estimates on $(T_{\Omega_1}^{-1})^{\lambda,\sigma_1}$. Our aim is to show (cf. (4.25), (4.27), (4.32) for $\alpha=\beta=0$) that

$$\left|\partial_\lambda^\alpha\, \partial_{\sigma_1}^\beta\, T_{\Omega_1}^{-1}((n_1,m_1),(n_2,m_2))\right| <$$
$$\begin{cases} e^{2(|\alpha|+|\beta|)^2}\,\Phi_1\left(|n_1|+|n_2|\right)^{|\alpha|+|\beta|+1} & (i) \\ e^{2(|\alpha|+|\beta|)^2}\exp\left(-\frac{1}{5}\,|n_1-n_2|^{c_1}\right) \quad \text{if } |n_1-n_2|>\Phi_0\left(|n_1|+|n_2|\right) & (ii) \\ e^{2(|\alpha|+|\beta|)^2}\exp\left(-\frac{1}{10}\,|m_1-m_2|^{c_1/20}\right) \quad \text{if } |m_1-m_2|>\left(|n_1|+|n_2|\right)^{10}. & (iii) \end{cases} \tag{4.92}$$

For (i), (ii), we differentiate (4.22) and for (iii) differential (4.33)-(4.34).

The contributions of the second term of (4.22) for instance are

$$\sum_{n\in A, n'\notin A} \left(\partial_\lambda^{\alpha_1}\, \partial_{\sigma_1}^{\beta_1}\, T_A^{-1}\right)(n_1,n)\left(\partial_\lambda^{\alpha_2}\, \partial_{\sigma_1}^{\beta_2}\, T'\right)(n,n')\left(\partial_\lambda^{\alpha_3}\, \partial_{\sigma_1}^{\beta_3}\, T_{\Omega_1}^{-1}\right)(n',n_2) \tag{4.93}$$

where $|\alpha|=|\alpha_1|+|\alpha_2|+|\alpha_3|$, $|\beta|=|\beta_1|+|\beta_2|+|\beta_3|$. Observe that the presence of a square $(|\alpha|+|\beta|)^2$ avoids in particular problems of extra factors due to applying the product rule (since the contributions of $|\alpha_i|+|\beta_i|=|\alpha|+|\beta|$ are dominant in this respect). To estimate (4.93), apply (4.89), (4.90), (4.91), (4.85) and the same argument as earlier for $\alpha=\beta=0$. The first two factors in (4.93) yield the estimate

$$e^{(|\alpha_1|+|\beta_1|)^2}\cdot\Phi_1(N_s)^{2|\alpha_1|+|\beta_1|+1}\cdot e^{|\alpha_2|^2}\cdot e^{-|n-n'|^{c_1}} \quad \text{if } |n_1-n|\le\Phi_0(N_s) \tag{4.94}$$

$$e^{(|\alpha_1|+|\beta_1|)^2}\cdot\exp\left(-\frac{1}{3}\,|n_1-n|^{c_1}\right)\cdot e^{|\alpha_2|^2}\cdot e^{-|n-n'|^{c_1}}\quad\text{if}\quad |n_1-n|>\Phi_0(N_*). \tag{4.95}$$

If (4.94), one gets thus

$$e^{(|\alpha_1|+|\alpha_2|+|\beta_1|+|\beta_2|)^2}\ e^{-\frac{1}{3}|n_1-n'|^{c_1}}\ e^{-\frac{1}{3}\,N_*^{c_1}}\ e^{(2|\alpha_1|+|\beta_1|)(\log N_*)^{12}}. \tag{4.96}$$

If we assume $(2|\alpha_1|+|\beta_1|)(\log N_*)^{12}>\frac{1}{3}N_*^{c_1}$, thus $|\alpha_1|+|\beta_1|>N_*^{c_1/2}$, the last factor in (4.96) may be bounded by $e^{(|\alpha_1|+|\beta_1|)^2}$. Hence, in both cases (4.94), (4.95), there is the estimate

$$e^{2(|\alpha_1|+|\alpha_1|+|\beta_1|+|\beta_2|)^2}\ e^{-\frac{1}{3}|n_1-n'|^{c_1}}. \tag{4.97}$$

This observation permits to carry out essentially the same argument as for $\alpha=\beta=0$.

Observe finally that if $|n_1|+|n_2|$ is large, hence N_*, the last factor of (4.96) may essentially be absorbed in the first. This point is of importance to preserve essentially the $e^{(|\alpha|+|\beta|)^2}$-factor when estimating $\partial_\lambda^\alpha\ \partial_\sigma^\beta\ (T_N^{\lambda,\sigma})^{-1}\ (n_1,n_2)$ for $|n_1-n_2|>\Phi_0(N)$ according to the inductive hypothesis.

Once the derivative bounds for $\left(T_{\Omega_1}^{\lambda,\sigma_1}\right)^{-1}$ gotten, we may estimate derivatives of $P\ T_{\Omega_1}^{-1}\ P^*$ and from the arguments leading to (4.57) we get

$$\left|\partial_\lambda^\alpha\ \partial_{\sigma_1}^\beta\ (P\ T_{\Omega_1}^{-1}\ P^*)\ (m_1,m_2)\right|<\varepsilon^2\ e^{2(|\alpha|+|\beta|)^2}\ e^{-\frac{1}{10}|m_1-m_2|^{c_1/20}}. \tag{4.98}$$

Hence one gets the estimate (4.98) for the matrices U_{ij} given by (4.60) and also for the matrix S defined by (4.62) (case $\alpha=\beta=0$ given by (4.70)).

Case (1)
Besides (4.72), one gets the derivative estimate from (4.98)

$$\left|\partial_\lambda^\alpha\ \partial_{\sigma_1}^\beta\ \varphi(\lambda,\sigma_1)\right|<0\left(\varepsilon\ e^{2(|\alpha|+|\beta|)^2}\right). \tag{4.99}$$

(In this case, α,β will only take bounded values (independent of N).)

Case (2)
In case (2.1), the functions $\psi_\alpha(\lambda,\sigma_1)$ and $\varphi(\lambda,\sigma_1)$ satisfy again from (4.98)

$$\left|\partial_\lambda^\alpha\ \partial_{\sigma_1}^\beta\ \psi_\alpha(\lambda,\sigma_1)\right|=0\left(\varepsilon\ e^{2(|\alpha|+|\beta|)^2}\right) \tag{4.100}$$

while

$$\left|\partial_\lambda^\alpha\ \partial_{\sigma_1}^\beta\ \varphi(\lambda,\sigma_1)\right|=0\left(e^{-k^{c_1/200}}\right)\ e^{2(|\alpha|+|\beta|)^2}. \tag{4.101}$$

Recall that we need to consider the (reciprocal of) the expression (4.78)

$$\prod_{\alpha\le d}\left[(\pm\sigma_1+k)^{\pi_\alpha}+\psi_\alpha(\lambda,\sigma_1)\right]+\varphi(\lambda,\sigma_1) \tag{4.102}$$

where $\pi_\alpha=1,2,\ d\ll k^\varepsilon$.

In case (2.2), we need to consider reciprocals of

$$(\pm\sigma_1+k)^{\pi_\alpha}+\psi_\alpha(\lambda,\sigma_1) \tag{4.103}$$

where $\pi_\alpha=0,1$, ψ_α as in (4.100).

In both (4.102), (4.103) the sign $\pm$ of σ_1 is well specified. Define

$$\sigma_2 = \sigma_1 \pm k = \sigma + \langle n_0, \lambda \rangle \pm k. \tag{4.104}$$

Thus (4.102), (4.103) may be rewritten as

$$\prod_{\alpha \leq d} [\sigma_2^{\mp \alpha} + \psi_\alpha(\lambda, \sigma_2)] + \varphi(\lambda, \sigma_2) \tag{4.105}$$

$$\sigma_2^{\mp} + \psi(\lambda, \sigma_2) \tag{4.106}$$

with ψ_α, ψ satisfying (4.100) and φ (4.101).

Applying in case (1) Malgrange's preparation theorem, (4.71) may be replaced by polynomials

$$\underbrace{\prod_{\substack{|n-n_0|<C_{12} \\ |m|<C_{12}}} (\pm\langle \lambda, n - n_0 \rangle + \sigma_1 + \mu_{\mp m}) + \sum_{j<d} a_j(\lambda)\ \sigma_1^j}_{\text{degree } d} \tag{4.107}$$

where

$$|\partial^\alpha\ a_j| < 0(\varepsilon) \tag{4.108}$$

($|\alpha|$ and d are bounded here).

Similarly in (4.105), (4.106), one may replace $\psi_\alpha(\lambda, \sigma_2)$, $\psi(\lambda, \sigma_2)$ be a lower degree polynomial in σ_2 (of degree 1 or 0) and smooth λ-coefficients. Thus (4.105) gets replaced by

$$\prod_{\alpha \leq d} p_\alpha(\sigma_2, \lambda) + \varphi(\lambda, \sigma_2) \tag{4.109}$$

where

$$p_\alpha(\sigma_2, \lambda) = \sigma_2^d + \sum_{j<d} a_j(\lambda)\ \sigma_2^j \qquad (d \leq 2) \tag{4.110}$$

$$|\partial_\lambda^\alpha\ a_j| < e^{2|\alpha|^2}\ \varepsilon \tag{4.111}$$

and φ as in (4.101) and (4.106) is also replaced by a σ_2-polynomial of the form (4.110)-(4.111).

In order to replace also $\varphi(\lambda, \sigma_2)$ by a polynomial of degree $< \sum_{\alpha \leq d}$ degree p_α, again Malgrange's theorem is applied. In this case however the degree d is large (possibly $\exp \frac{\log\log N}{\log\log\log N}$) and the assumption $d \ll k^\varepsilon$ together with (4.101) become essential. A more quantitative analysis of the preparation theorem taking into account large degrees and specific derivative bounds for large derivatives is thus necessary and will be presented separately.

Finally, observe that in case (2), one has for the self-adjoint S defined by (4.62)

$$S = \sigma_2\ \mathbb{I}\ + 0(\varepsilon). \tag{4.112}$$

Hence, from first eigenvalue variation, S^{-1} may be controlled by reciprocals of expressions of the form

$$\sigma_2 + a(\lambda) \tag{4.113}$$

with $|a|, |a'| = 0(\varepsilon)$. See the comments in (2.II).

Finally, in order to establish the off-diagonal estimates on $T^{-1}(n_1, n_2)$ and $(\partial_\lambda^\alpha\ \partial_\sigma^\beta\ T^{-1})(n_1, n_2)$ for $\|T^{-1}\| < \Phi_1(N_r)$, $|n_1 - n_2| > \Phi_0(N_r)$, we rely again on the localization identity and the estimates (4.27), (4.92) related to $T_{\Omega_1}^{-1}$ (the $\frac{1}{5}$-factor may be improved to any number > 1, from the assumptions on T and derivatives: keep also the observation in mind about the $e^{(|\alpha|+|\beta|)^2}$-factors). In particular one verifies the inductive assumptions in this respect (cf. (4.16), (4.91)) at stage N_r.

5. APPENDIX 1 (Preparation theorem)

We give a proof here of the analytic case with estimates in terms of the data, which make the result applicable in the context of case (2.1) described above. Thus we consider an analytic function

$$f(\lambda, z) = \sum_{j<d} a_j(\lambda)\ z^j + z^d + \varphi(\lambda, z) \tag{1}$$

$$\varphi(\lambda, z) = \sum_{j>d} a_j(\lambda)\ z^j \tag{2}$$

($z \to \sigma_2$) on a neighborhood D of 0 of size δ ($= \exp -(\log N_r)^2$) such that

$$\begin{cases} \left|a_j^{(\alpha)}\right| < e^{2(d+|\alpha|)^2} & \text{for } j < d & (3) \\ \left|a_j^{(\alpha)}\right| < \gamma\ e^{2(j+|\alpha|)^2} & \text{for } j > d & (4) \\ \left|a_j^{(\alpha)}\right| < \gamma\ e^{2|\alpha|^2}\delta^{-j} & \text{for } j > d. & (5) \end{cases}$$

Here

$$\gamma < e^{-B} \quad \text{with} \quad \log d < \frac{\log B}{\log\log B}. \tag{6}$$

Conditions (3), (4), (6) result from (4.101), (4.109)-(4.111) and (5) from the persistency of the $\partial_\lambda^\alpha \varphi$-bound (4.101) for $\sigma_2 \in D$.

Our aim is to replace f by a polynomial

$$P(\lambda, z) = \sum_{j<d} b_j(\lambda)\ z^j + z^d \tag{7}$$

with

$$\left|b_j^{(\alpha)}\right| < e^{C(d+|\alpha|)^2} \quad \text{for} \quad j < d\ ,\ |\alpha| < 2d^2 \tag{8}$$

in the sense that

$$|P(\lambda, z)| \sim |f(\lambda, z)| \quad \text{for} \quad z \in D \tag{9}$$

and λ restricted to a suitable neighborhood of a given parameter choice λ_0.

Assume

$$|\lambda - \lambda_0| < \delta_1 = \delta^{3d}. \tag{10}$$

Denoting $(b_0, \dots, b_{d-1}) = b$ a parameter, define

$$p_b(z) = z^d + \sum_{j<d} b_j\ z^j. \tag{11}$$

Write

$$p_b(z) = p_b(s) + \left[\sum_{j<d} q_j(b, s)\ z^j\right] (z - s) \tag{12}$$

$$\frac{p_b(z)}{p_b(s)(z-s)} = \frac{1}{z-s} + \sum_{j<d} \frac{q_j(b,s)}{p_b(s)}\ z^j. \tag{13}$$

Denote $\Gamma_\tau = \Gamma(0, \tau)$ a τ-circle and assume $|z| < \frac{\tau}{2}$ and $p_b(s) \neq 0$ on Γ_τ.

It follows then from Cauchy's theorem and (13) that for all k

$$z^k = \left[\frac{1}{2\pi i}\int_{\Gamma_\tau} \frac{s^k\, ds}{p_b(s)(z-s)}\right] p_b(z) - \sum_{j<d}\left[\frac{1}{2\pi i}\int_{\Gamma_\tau}\frac{s^k\, q_j(b,s)ds}{p_b(s)}\right] z^j. \tag{14}$$

We will make a choice of τ depending on k and according to an iteration process along which b will be more and more specified. Define $k_1 = d^{10}$ say and

$$a_k^* = \gamma^{1/2} \quad \text{for} \quad d < k \le k_1 \quad \text{and} \quad a_k^* = \min\left(\epsilon^{2k^2}, \delta^{-k}\right) \quad \text{for} \quad k > k_1 \tag{15}$$

and let

$$\tau_k \sim (1 + a_k^*)^{-2/k} > \delta^2. \tag{16}$$

Step 1. Let $d < k \le k_1$
Chose $\tau_k = \tau = 0(1) < 1$ such that

$$\left| s^d + \sum_{j<d} a_j(\lambda_0)\, s^j \right| > d^{-d} \equiv \kappa_1 \quad \text{for} \quad s \in \Gamma_\tau. \tag{17}$$

Assume for $j = 0, \ldots, d-1$

$$|b_j - a_j(\lambda_0)| < \frac{\kappa_1}{10d}. \tag{18}$$

Put

$$Q_1(b,\lambda,z) = \sum_{k=d+1}^{k_1} a_k(\lambda)\left[\frac{1}{2\pi i}\int_{\Gamma_{\tau_k}} \frac{s^k\, ds}{p_b(s)(z-s)}\right]$$

and for $j = 0, \ldots, d-1$

$$R_{1,j}(b,\lambda) = \sum_{k=d+1}^{k_1} a_k(\lambda)\left[\frac{1}{2\pi i}\int_{\Gamma_{\tau_k}} \frac{s^k\, q_j(b,s)ds}{p_b(s)}\right]. \tag{19}$$

Hence, from (14)

$$\begin{aligned} f(\lambda, z) &= \sum_{j<d} a_j(\lambda)\, z^j + z^d + Q_1(b,\lambda,z)\, p_b(z) - \sum_{j<d} R_{1,j}(b,\lambda)\, z^j + \sum_{k>k_1} a_k(\lambda)\, z^k \\ &= [1 + Q_1(b,\lambda,z)]\, p_b(z) + \sum_{j<d}(a_j(\lambda) - b_j - R_{1,j}(b,\lambda))\, z^j + \sum_{k>k_1} a_k(\lambda)\, z^k. \end{aligned} \tag{20}$$

From (17), (18)

$$\int_{\Gamma_{\tau_k}} \frac{|s|^k}{|p_b(s)||z-s|} < \frac{\tau_k^k}{\min_{s\in\Gamma_{\tau_k}} |p_b(s)|} < d^d. \tag{21}$$

Hence from (4)

$$|Q_1| \le \sum_{k=d+1}^{k_1} |a_k(\lambda)|\, d^d \le d^d \sum_{k=d+1}^{k_1} a_k^* < d^d\, d^{10}\, \gamma^{1/2} < \gamma^{1/3} \tag{22}$$

by (15), (6). Similarly

$$|R_1| < \gamma^{1/3} \quad \text{and} \quad |\partial_b R_1| < \gamma^{1/3}. \tag{23}$$

Solve for $j = 0, 1, \ldots, d-1$

$$b_j + R_{1,j}(b,\lambda) = a_j(\lambda) \tag{24}$$

which may be done by the implicit function theorem, in view of (23).

This yields

$$a^1(\lambda) = \left(a_j^1(\lambda)\right)_{j=0,\dots,d-1}$$

satisfying

$$a_j^1(\lambda) = a_j(\lambda) + 0(\gamma^{1/3}). \tag{25}$$

Writing $\left|a_j^1(\lambda) - a_j(\lambda_0)\right| \leq \left|a_j^1(\lambda) - a_j(\lambda)\right| + \left|a_j(\lambda) - a_j(\lambda_0)\right| < \gamma^{1/3} + e^{2(d+1)^2}\ \delta_1 < 2\ \gamma^{1/3}$, it follows that (18) is satisfied. We restrict further b to satisfy in particular

$$\left|b_j - a_j^1(\lambda_0)\right| < \frac{\kappa_1}{20d}. \tag{26}$$

Step 2. Let $k_1 < k \leq k_2$.

Choose τ_k satisfying (16) and moreover for $s \in \Gamma_{\tau_k}$

$$\left|s^d + \sum_{j<d} a_j^1(\lambda_0)\ s^j\right| > \left(\frac{\tau_k}{d}\right)^d \geq \left(\frac{\tau_{k_2}}{d}\right)^d \equiv \kappa_2. \tag{27}$$

Assume for $j = 0, 1, \dots, d-1$

$$\left|b_j - a_j^1(\lambda_0)\right| < \frac{\kappa_2}{10d}. \tag{28}$$

Put

$$Q_2(b, \lambda, z) = \sum_{k=d+1}^{k_2} a_k(\lambda)\ \left[\frac{1}{2\pi i}\int_{\Gamma_{\tau_k}} \frac{s^k\ ds}{p_b(s)(z-s)}\right]$$

$$R_{2,j}(b, \lambda) = \sum_{k=d+1}^{k_2} a_k(\lambda)\ \left[\frac{1}{2\pi i}\int_{\Gamma_{\tau_k}} \frac{s^k\ q_j(b,s)}{p_b(s)}\ ds\right]. \tag{29}$$

Hence

$$\begin{aligned} f(\lambda, z) &= \sum_{j<d} a_j(\lambda)\ z^j + z^d + Q_2(b,\lambda,z)\ p_b(z) - \sum_{j<d} R_{2,j}(b,\lambda)\ z^j + \sum_{k>k_2} a_k(\lambda)\ z^k \\ &= [1 + Q_2(b,\lambda,z)]\ p_b(z) + \sum_{j<d}(a_j(\lambda) - b_j - R_{2,j}(b,\lambda))\ z^j + \sum_{k>k_2} a_k(\lambda)\ z^k. \end{aligned} \tag{30}$$

Estimate

$$Q_2 - Q_1 = \sum_{k=k_1+1}^{k_2} a_k(\lambda)\ \left(\frac{1}{2\pi i}\int_{\Gamma_{\tau_k}} \frac{s^k\ ds}{p_b(s)(z-s)}\right]$$

$$\begin{aligned} |Q_2 - Q_1| &< \sum_{k=k_1+1}^{k_2} a_k^*\ \tau_k^k\ \left(\min_{\Gamma_{\tau_k}}\ |p_b(s)|\right)^{-1} \\ &< 2d^d \sum_{k=k_1+1}^{k_2} a_k^*\ \tau_k^{k-d} \quad \text{(by (27), (28))} \\ &< 2d^d \sum_{k=k_1+1}^{k_2} (a_k^*)^{1-2\frac{k-d}{k}} \quad \text{(by (16))} \\ &< 3d^d\ (a_{k_1+1}^*)^{-\frac{2}{3}} \\ &< (a_{k_1+1}^*)^{-1/2} \quad \text{(by (15))}. \end{aligned} \tag{31}$$

Similarly

$$|R_2 - R_1| < (a^*_{k_1+1})^{-1/2} \quad \text{and} \quad |\partial_b(R_2 - R_1)| < (a^*_{k_1+1})^{-1/2}. \tag{32}$$

Solve for $j = 0, 1, \ldots, d-1$

$$b_j + R_{2,j}(b, \lambda) = a_j(\lambda) \tag{33}$$

yielding

$$a^2(\lambda) = (a_j^2(\lambda))_{j=0,\ldots,d-1}.$$

Since, by construction

$$a_j^1(\lambda) + R_{1,j}(a^1(\lambda), \lambda) = a_j(\lambda) \tag{34}$$

it follows by subtraction of (33), (34)

$$(a_j^2 - a_j^1) + R_{1,j}(a^2, \lambda) - R_{1,j}(a^1, \lambda) + (R_{2,j} - R_{1,j})(a^2, \lambda) = 0. \tag{35}$$

Hence, from (23), (32)

$$|a^2 - a^1| < 2(a^*_{k_1+1})^{-1/2} \tag{36}$$

and

$$\left|a^2(\lambda) - a^1(\lambda_0)\right| < 2(a^*_{k_1+1})^{-1/2} + |a_1(\lambda) - a_1(\lambda_0)| < 2(a^*_{k_1+1})^{-1/2} + e^{2(d+1)^2}\,\delta_1. \tag{37}$$

Consider condition (28). From (10), (16), only the first term in (37) has to be considered. This yields

$$(a^*_{k_1+1})^{-1/2} < \frac{1}{20}\, d^{-d-1}\, (a^*_{k_2})^{-\frac{2d}{k_2}} \tag{38}$$

which is easily seen from (15) to hold for

$$k_2 \sim k_1^{3/2}. \tag{39}$$

We again restrict b further to satisfy in particular

$$\left|b_j - a_j^2(\lambda_0)\right| < \frac{\kappa_2}{20d} \tag{40}$$

and perform the next step.

The continuation of this process is clear. Take $k_{s+1} = k_s^{3/2}$. The sequences $\{a^s\}$, $\{Q_s\}$, $\{R_s\}$ are obviously convergent. Denoting b, Q, R their respective limits, we have by construction of a^s for $|z| < \delta^3$, $|\lambda - \lambda_0| < \delta^{3d}$

$$f(\lambda, z) = [1 + Q(b(\lambda), \lambda, z)]\left[z^d + \sum_{j<d} b_j(\lambda)\, z^j\right] \tag{41}$$

where in particular $Q = 0(1)$. Hence (9) holds.

It remains to check (8). We have $(j = 0, \ldots, d-1)$

$$b_j(\lambda) + R_j(b(\lambda), \lambda) = a_j(\lambda) \tag{42}$$

where

$$R_j(b, \lambda) = \sum_{k=d+1}^{\infty} a_k(\lambda)\left[\frac{1}{2\pi i}\int_{\Gamma_{r_k}} \frac{s^k\, q_j(b, s)}{p_b(s)}\, ds\right]. \tag{43}$$

Differentiating (42) yields by (3)

$$|b^{(\alpha)}| \le e^{2(d+|\alpha|)^2} + +(1+|\alpha|)^{|\alpha|} \sup_{|\beta|\le\alpha} |D^\beta R| \sum_{\Sigma|\beta_s|\le|\alpha|} |b^{(\beta)}| \cdot |b^{(\beta_{|\alpha|})}|. \tag{44}$$

Here $D^\beta R$ refers to both b and λ derivatives.

From (43), one has

$$|D^\beta R| < \sum_{k=d+1}^{\infty} \max_{|\beta'|\le|\beta|} |a_k^{(\beta')}| \cdot \tau_k^k \left(\frac{\tau_k}{d}\right)^{-d(1+|\beta|)}. \tag{45}$$

Recall that $|\beta| \le |\alpha| < 2d^2$ by (8).

We distinguish the cases $k \le k_1$ and $k > k_1$

(a) For $k \le k_1$, $\tau_k = 0(1)$ and, by (4), $|a_k^{(\beta')}| \le \gamma\, e^{2(k+|\alpha|)^2} < \gamma\, e^{2(d^{10}+2d^2)^2} < \gamma^{1/2}$.

Hence we get the bound

$$d^{d(1+|\alpha|)}\, \gamma^{1/2} < \gamma^{1/3}. \tag{46}$$

(b) For $k > k_1$, write by (4)

$$|a_k^{(\beta')}| < e^{2(k+|\alpha|)^2} < e^{\frac{11}{5}k^2} \tag{47}$$

since $|\alpha| \le 2d^2 < 2k^{1/5}$. Hence, since also $|a_k^{\beta'}| < e^{2|\alpha|^2}\, \delta^{-k}$, it follows from (15)

$$|a_k^{(\beta')}| < e^{2|\alpha|^2}\, (a_k^*)^{\frac{11}{10}}. \tag{48}$$

Substituting in (45) gives

$$e^{2|\alpha|^2}\, d^{d(1+|\alpha|)} \sum_{k>k_1} (a_k^*)^{\frac{11}{10}-2+\frac{2d(1+|\alpha|)}{k}} < e^{5d^4}\, (a_{k_1}^*)^{-1/2} < e^{-\frac{1}{2}d^{10}}. \tag{49}$$

Consequently, from (46), (49)

$$|D^\beta R| < e^{-\frac{1}{2}d^{10}} \quad \text{for} \quad |\beta| \le |\alpha| \tag{50}$$

and substituting in (44) yields for $|\alpha| < 2d^2$

$$|b_j^{(\alpha)}| < e^{2(d+|\alpha|)^2} + e^{-\frac{1}{3}d^{10}} \sum_{\Sigma|\beta_s|\le|\alpha|} |b^{\beta_1}| \cdots |b^{\beta_{|\alpha|}}|. \tag{51}$$

From this, one easily concludes that

$$|b_j^{(\alpha)}| < 3\, e^{2(d+|\alpha|)^2} \quad \text{for} \quad |\alpha| < 2d^2 \tag{52}$$

and in particular (8).

6. Appendix 2 (The periodic case)

In this case, the linearized operator $T = D + \varepsilon T_1$ on the (m, n)-lattice ($m \in \mathbf{Z}^d$, $n \in \mathbf{Z}$) is given by

$$D = \begin{pmatrix} -\lambda n + |m|^2 + \widehat{V}(m) & 0 \\ 0 & \lambda n + |m|^2 + \widehat{V}(-m) \end{pmatrix}$$

$$T_1 = \begin{pmatrix} S_{\frac{\partial^2 F}{\partial u\, \partial v}} & S_{\frac{\partial^2 F}{\partial v^2}} \\ S_{\frac{\partial^2 F}{\partial u^2}} & S_{\frac{\partial^2 F}{\partial u\, \partial v}} \end{pmatrix}$$

acting on pairs $(\hat{u}, \hat{v})$. To deal with its inverse, we use following known Fröhlich-Spencer type lemma (cf. [Pos]) when the singular sites of D appear in well-separated clusters. We will rely on a simple Fröhlich-Spencer type lemma to bound the inverses of certain linear operators of the form $D + \varepsilon\ T_1$, where the singular sites for the diagonal D_1 appear in well-separated clusters.

Lemma 1 *Fix some constants* $\frac{1}{10} > \varepsilon_1 > \varepsilon_2 > \varepsilon_3 > 0$ *and let* Ω *be a subset of the* M*-ball in* $\mathbf{Z}^{d+1}$ $(M \to \infty)$. *Assume* $\{\Omega_\alpha\}$ *a collection of subsets of* Ω *satisfying*

$$\operatorname{diam} \Omega_\alpha < M^{\varepsilon_1} \tag{2}$$

$$\operatorname{dist}(\Omega_\alpha, \Omega_\beta) > M^{\varepsilon_2} \quad \text{for} \quad \alpha \neq \beta. \tag{3}$$

Write $T = D + S \quad (D = \text{diagonal})$ *where*

$$\|S\| < \varepsilon \quad , \quad |S(x, y)| < \varepsilon\ e^{-|x-y|^c} \tag{4}$$

for some $c > 0$ *and*

$$|D(x)| > \rho \gg \varepsilon \quad \text{if} \quad x \in \Omega \backslash \cup \Omega_\alpha \tag{5}$$

$$\|(T \mid \widetilde{\Omega}_\alpha)^{-1}\| < M^{C_1} \quad \text{for all} \quad \alpha \tag{6}$$

denoting $\widetilde{\Omega}_\alpha$ *on* M^{ε_3}*-neighborhood of* Ω_α.

Then

$$\|(T \mid \Omega)^{-1}\| < M^{C_2} \tag{7}$$

and

$$|(T \mid \Omega)^{-1}(x, y)| < e^{-\frac{1}{10}|x-y|^c} \quad \text{if} \quad |x - y| > M^{2\varepsilon_1}. \tag{8}$$

Proof. The argument is essentially well-known. We repeat it here for completeness sake. Denote $\Omega_0 = \cup_\alpha \Omega_\alpha$.

(i) Let $\overline{\Omega} \subset \Omega \backslash \Omega_0$ and write the restriction

$$T_{\overline{\Omega}} = D_{\overline{\Omega}} + S_{\overline{\Omega}} = \left(I + S_{\overline{\Omega}}\ D_{\overline{\Omega}}^{-1}\right)\ D_{\overline{\Omega}} \tag{9}$$

which inverse may be written as a Neumann series

$$T_{\overline{\Omega}}^{-1} = D_{\overline{\Omega}}^{-1} \sum_{j \geq 0} (-1)^j \left(S_{\overline{\Omega}}\ D_{\overline{\Omega}}^{-1}\right)^j \tag{10}$$

by (4), (5). In fact $\|T_{\overline{\Omega}}^{-1}\| < 2\rho^{-1}$ and analyzing (10) using (4) yields an off diagonal bound, say

$$\left|T_{\overline{\Omega}}^{-1}(x, y)\right| < e^{-\frac{1}{2}|x-y|^c} \quad \text{for} \quad x \neq y. \tag{11}$$

(ii) Let $x, y \in \Omega$. Denote M^{ε_3} by K.

Case I. Assume $\operatorname{dist}(x, \Omega_0) > K$.

We then write $\Omega = \Omega_1 + \Omega_2$, where $\Omega_1 = \Omega \cap B(x, K)$, hence $\Omega_1 \cap \Omega_0 = \emptyset$. From the identity for inverses, one finds

$$T_{\Omega}^{-1}(x, y) = T_{\Omega_1}^{-1}(x, y) + \sum_{z \in \Omega_1, w \in \Omega_2} T_{\Omega_1}^{-1}(x, z)\ S(z, w)\ T_{\Omega}^{-1}(w, y). \tag{12}$$

From (i) we have thus $|T_{\Omega_1}^{-1}(x,y)| < \rho^{-1}\ e^{-\frac{1}{2}|x-y|^c}$ and hence, by (12)

$$\left|T_\Omega^{-1}(x,y)\right| < \rho^{-1} + \sum_{z\in\Omega_1, w\in\Omega_2} e^{-\frac{1}{2}|x-z|^c - |z-w|^c} \left|T_\Omega^{-1}(w,y)\right|$$

$$< \rho^{-1} + \max_{|z-x|>K} e^{-\frac{1}{3}|x-z|^c} \left|T_\Omega^{-1}(z,y)\right|$$

$$= \rho^{-1} + \max_{R>K} e^{-\frac{1}{3}R^c} \cdot \max_{|z-y|>|x-y|-R} \left|T_\Omega^{-1}(z,y)\right| . \tag{13}$$

Clearly the first term vanishes in (13) for $|x-y|>K$ and hence

$$\left|T_\Omega^{-1}(x,y)\right| < \max_{R>K} e^{-\frac{1}{3}R^c} \max_{|z-y|>|x-y|-R} \left|T_\Omega^{-1}(z,y)\right| \quad \text{if} \quad |x-y|>K. \tag{14}$$

Case II. Assume $\operatorname{dist}(x,\Omega_0) < K$.

We then write $\Omega = \Omega_1 + \Omega_2$, with $\Omega_1 = \widetilde{S}_\alpha$ = K-neighborhood of S_α, if $\operatorname{dist}(x, S_\alpha) < K$. From (6) and (12)

$$\left|T_\Omega^{-1}(x,y)\right| < M^{C_1} + M^{C_1+1} \max_{z\in\Omega_1} \sum_{w\in\Omega_2} e^{-|z-w|^c} \left|T_\Omega^{-1}(w,y)\right| . \tag{15}$$

We distinguish further the cases

($\mathbf{II}_1$) $\operatorname{dist}(w, S_\alpha) > 2K$.

Then $|z-w| > 2K - K = K$ and

$$(2.15) < M^{C_1} + M^{C_1+1} \max_{R>K} e^{-\frac{1}{2}R^c} \max_{|w-y|>|x-y|-R-2K-M^{\epsilon_1}} \left|T_\Omega^{-1}(w,y)\right| \tag{16}$$

$$< M^{C_1} + \max_{R>K} e^{-\frac{1}{3}R^c} \max_{|z-y|>|x-y|-R-2M^{\epsilon_1}} \left|T_\Omega^{-1}(z,y)\right| . \tag{17}$$

since $|w-x| \le |w-z| + |z-x| \le R + 2K + \operatorname{diam} S_\alpha$.

Again the first term in (17) may be dropped if $|x-y| > 2M^{\epsilon_1}$.

($\mathbf{II}_2$) $\operatorname{dist}(w, S_\alpha) \le 2K$.

From the separation hypothesis (3), it follows that $\operatorname{dist}(w,\Omega_0) > K$, since $\operatorname{dist}(w, S_\alpha) > K$. Thus $T_\Omega^{-1}(w,y)$ may be estimated as in Case I. Substituting (13) in (15) yields

$$(15) < M^{C_1+1}\ \rho^{-1} + \max_{R>K} e^{-\frac{1}{4}R^c} \max_{|z-y|>|x-y|-2M^{\epsilon_1}-R} \left|T_\Omega^{-1}(z,y)\right| . \tag{18}$$

Since in this case with w as in (15), $|w-y| > |x-y| - 3K - \operatorname{diam} S_\alpha$, it follows that the first term in (18) may be dropped if $|x-y| > 2M^{\epsilon_1}$.

Summarizing (13), (17), (18), it follows that $|T_\Omega^{-1}(x,y)|$ is bounded by (18) in all cases and the first term may be ignored if $|x-y| > 2M^{\epsilon_1}$. Hence $|T_\Omega^{-1}(x,y)| < M^{C_1+1}\ \rho^{-1}$ and (7). If we assume $|x-y| > M^{2\epsilon_1}$ say, an iteration of (18) yields

$$\left|T_\Omega^{-1}(x,y)\right| < \sum_{R_1+2M^{\epsilon_1}+R_2+2M^{\epsilon_1}+\cdots<|x-y|} e^{-\frac{1}{4}(R_1^c+R_2^c+\cdots)} < e^{-\frac{1}{10}|x-y|^c} \tag{19}$$

assuming $2(1-c) > 1$ (considering decay functions of the form $e^{-\eta^c}$ for $c < 1$ simplifies iteration here in fact).

We need to satisfy the hypothesis of Lemma 1.

First make the following observation related to condition (6). Let $\Omega \subset B(x_0, \frac{|x_0|}{10}) \subset \mathbf{Z}^{d+1}$ and consider the selfadjoint operator $\lambda_1 T_\Omega$, where $\lambda_1 = (\lambda)^{-1}$. Clearly

$$\partial_{\lambda_1} (\lambda_1 T_\Omega) = \begin{pmatrix} -|m|^2 + \widehat{V}(m) & 0 \\ 0 & -|m|^2 + \widehat{V}(m) \end{pmatrix} + 0(\varepsilon) \tag{20}$$

(T_1 has a bounded dependence on λ_0, λ). It follows then from first order variational calculus that $|\partial_{\lambda_1}(\mu)| \approx |x_0|^2$ for $|x_0|$ large enough for the eigenvalues μ of T_Ω. On the other hand, $\partial_{\lambda_0}(\mu)$ is bounded. This enables us to make an admissible restriction of (λ_0, λ) to a set of small complementary measure such that

$$\|T_\Omega^{-1}\| < M^{c'} \tag{21}$$

for Ω ranging in a given collection of M^C subsets of $B(0, M)$, as above.

We consider T as a (2×2)-block matrix $\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$, $T_{12} = T_{21}^*$. Keeping this in mind (alternatively, we duplicate each element from the index set), the decay assumption (2.4) will be fulfilled provided the symbols $\varphi = \partial^2_{uv} F$, $\partial^2_u F$, $\partial^2_v F$ satisfy the corresponding Fourier transform decay property. Considering for instance a polynomial function F in u, v, such property results from the iterative construction of the sequence of approximative solutions u^j using the Newton scheme and will be fulfilled at each stage (with a fixed constant c), cf. [B$_1$]. Our next task is to describe the Ω_α-sets which is a simple arithmetical issue.

We will show that the region $\{x \in \mathbf{Z}^d \mid |x| < M\}$ may be partionned in set $\{\Omega'_\alpha\}$ satisfying

$$\text{diam } \Phi(\Omega'_\alpha) < M^\delta \tag{22}$$

$$\text{dist}\,(\Phi(\Omega'_\alpha), \Phi(\Omega'_\beta)) > M^\varepsilon \quad \text{for} \quad \alpha \neq \beta \tag{23}$$

for some small $\varepsilon = \varepsilon(d) > 0$, $\delta = \delta(d) > 0$, where $\Phi(x) = (x, |x|^2)$. A partionning of the near-resonant sites Ω_0 $\{(x,n) \in \mathbf{Z}^{d+1} \mid |x| < M\ ,\ \big||x|^2 - \lambda n\big| < C\}$ may then be obtained defining

$$\Omega_\alpha = \{(x,n) \in \mathbf{Z}^{d+1} \mid x \in \Omega'_\alpha\ ,\ \big||x|^2 - \lambda n\big| < C\}. \tag{24}$$

The existence of such partition $\{\Omega'_\alpha\}$ will clearly result from following fact

Lemma 25. *Given $x_0 \in \mathbf{Z}^d$, there is a set $\Omega \subset \mathbf{Z}^d$, $x_0 \in \Omega$ satisfying*

$$\text{diam } \Phi(\Omega) < M^\delta \tag{26}$$

$$|\Phi(x) - \Phi(y)| > M^\varepsilon \quad \text{if} \quad x \in \Omega\ ,\ y \notin \Omega. \tag{27}$$

Here $\delta = C(d)\ \varepsilon$.

Lemma (25) is a consequence of

Lemma 28. *Let $\{x_j\}_{j=0,1,\dots,k}$ be a sequence in $\mathbf{Z}^d$ of distinct elements such that $|\Phi(x_j) - \Phi(x_{j+1})| < B$. Then $k < B^C$, with $C = C(d)$.*

Indeed, given $x_0 \in \mathbf{Z}^d$, define $\Omega_1 = \{x \in \mathbf{Z}^d \mid |\Phi(x) - \Phi(x_0)| \leq M^\varepsilon\}$ and in general let $\Omega_{j+1} = \{x \in \mathbf{Z}^d \mid \text{dist}(\Phi(x), \Phi(\Omega_j)) \leq M^\varepsilon\}$. Put $\Omega = \cup_j \Omega_j$. Thus clearly there is for each $x \in \Omega$ a chain $(x_j)_{j=0,\dots,k}$ with $x_k = x$ and $|\Phi(x_{j-1}) - \Phi(x_j)| \leq M^\varepsilon$, $j = 1, \dots, k$. Hence $k < M^{C\varepsilon}$ by Lemma 28 and $|\Phi(x) - \Phi(x_0)| \leq k \cdot M^\varepsilon < M^{(1+C)\varepsilon}$. Thus the lemma holds with $\delta = (1+C)\varepsilon$, proving Lemma 25.

Proof of Lemma 28.

We perform an inductive construction, based on the same ideas as in the $d = 2$ case.

Define $j_0 < \frac{k}{2}$ such that $|x_{j_0}| > c(d)\ k^{1/d}$ (using the fact that the points x_j are distinct lattice points). Denote $\Delta_0\ x_j = x_j - x_{j_0}$ for $j_0 \leq j \leq k$. From assumption, $|\Phi(x_j) - \Phi(x_{j_0})| < (j - j_0)B$, while $\Phi(x_j) - \Phi(x_{j_0}) = (\Delta_0\ x_j\ ,\ 2\langle x_{j_0}, \Delta_0\ x_j\rangle + |\Delta_0\ x_j|^2)$. Consequently

$$|\langle x_{j_0}\ ,\ \Delta_0\ x_j\rangle| < (j - j_0)^2\ B^2. \tag{29}$$

Take $j_0 \leq j \leq j_0 + J_1 < k$. If $\dim[\Delta_0\ x_j \mid j_0 \leq j \leq j_0 + J_1] = d$, it easily follows from (29) that $|x_{j_0}| < C(d)\ (J_1\ B)^{d+1}$, which is a contradiction if

$$(J_1\ B)^{d+1} < c(d)\ k^{1/d}. \tag{30}$$

Thus, if (30) holds, there is a subspace H_1 of $\mathbf{R}^d$, $\dim H_1 = d-1$ such that $x_j \in x_{j_0} + H_1$ for $j_0 \leq j \leq j_0 + J_1$. Choose next $j_0 \leq j_1 \leq j_0 + \frac{1}{2}\ J_1$ satisfying

$$|P_{H_1}\ x_{j_1}| > c(d)\ J_1^{1/d-1} \tag{31}$$

and consider $j_1 \leq j \leq j_1 + \frac{1}{2}\ J_1$. One has again $|\langle x_{j_1}, \Delta_1\ x_j\rangle| < (j - j_1)^2\ B^2$ if $\Delta_1\ x_j = x_j - x_{j_1}$. Hence

$$|\langle P_{H_1}\ x_{j_1}\ ,\ \Delta_1\ x_j\rangle| < (j - j_1)^2\ B^2. \tag{32}$$

Restricting $j_1 \leq j \leq j_1 + J_2$, $J_2 < \frac{1}{2}\ J_1$ and assuming $\dim[\Delta_1\ x_j \mid j_1 \leq j \leq j_1 + J_2] = d - 1$, it follows that $|P_{H_1}\ x_{j_1}| < C(d)\ (J_2\ B)^d$. Thus we take J_2 satisfying

$$(J_2\ B)^d < c\ J_1^{1/d-1}. \tag{33}$$

If (33) holds, there is a subspace H_2 of $\mathbf{R}^d$, $\dim H_2 = d - 2$ such that $x_j \in x_{j_1} + H_2$ for $j_1 \leq j \leq j_1 + J_2$. Continuing this process d times yields a contradiction, provided $B^2 < J_{d-1}$. This proves the lemma for $k < B^{d^{2d}}$, taking into account the conditions (30), (33) etc.
This proves the lemma.

Remark: The existence of partitions as above was observed earlier by A. Granville and T. Spencer.

We now return to applying Lemma 1 to our problem.

Fix $\kappa > 0$ and write $\Omega = \overline{\Omega} + \overline{\overline{\Omega}}$ where $\overline{\Omega} = B(0, M^\kappa)$ and $B(0, M) \subset \Omega \subset B(0, 2M)$. The preceding allows to partition $\overline{\overline{\Omega}} \cap \Omega_0$ in sets Ω_α satisfying

$$\text{diam}\,\Omega_\alpha < M^\rho\ ,\ \text{dist}(\Omega_\alpha, \Omega_\beta) > M^{\rho/2} \quad \text{for} \quad \alpha \neq \beta. \tag{34}$$

Define

$$\Omega_{-1} = (\overline{\Omega} \cap \Omega_0) \cup \bigcup_{\text{dist}(\Omega_\alpha, \overline{\Omega}) < M^\rho} \Omega_\alpha \tag{35}$$

and partition $\Omega_0 \cap \Omega$ as $\Omega_{-1}, \{\Omega_\alpha \mid \text{dist}(\Omega_\alpha, \overline{\Omega}) \geq M^\rho\}$. These sets are of diameter $\leq M^\kappa$ (assuming $\kappa > \rho$) and at least $M^{\rho/2}$-separated. Thus in (2), (3), we let $\varepsilon_1 = \kappa$, $\varepsilon_2 = \frac{\rho}{2}, \varepsilon_3 = \frac{\rho}{4}$. To satisfy (6), apply (21) taking for Ω the sets $\tilde{\Omega}_\alpha$. Considering $\tilde{\Omega}_{-1}$ of a smaller size-scale, we assume a bound on $(T|\tilde{\Omega}_{-1})^{-1}$ obtained at previous stage of the construction. We may then apply Lemma 1.

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