Quasi-Randomness and Algorithmic Regularity for Graphs with General Degree Distributions*

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Abstract. We deal with two intimately related subjects: quasi-randomness and regular partitions. The purpose of the concept of quasi-randomness is to measure how much a given graph "resembles" a random one. Moreover, a regular partition approximates a given graph by a bounded number of quasi-random graphs. Regarding quasi-randomness, we present a new spectral characterization of low discrepancy, which extends to sparse graphs. Concerning regular partitions, we present a novel concept of regularity that takes into account the graph's degree distribution, and show that if $G = (V, E)$ satisfies a certain boundedness condition, then G admits a regular partition. In addition, building on the work of Alon and Naor [4], we provide an algorithm that computes a regular partition of a given (possibly sparse) graph G in polynomial time. As an application, we present a polynomial time approximation scheme for MAX CUT on (sparse) graphs without "dense spots".

Key words: quasi-random graphs, Laplacian eigenvalues, regularity lemma, Grothendieck's inequality.

1 Introduction and Results

This paper deals with quasi-randomness and regular partitions. Loosely speaking, a graph is quasi-random if the global distribution of the edges resembles the expected edge distribution of a random graph. Furthermore, a regular partition approximates a given graph by a constant number of quasi-random graphs; such partitions are of algorithmic importance, because a number of NP-hard problems can be solved in polynomial time on graphs that come with regular partitions. In this section we present our main results. References to related work can be found in Section 2, and the remaining sections contain the proofs and detailed descriptions of the algorithms.

Quasi-Randomness: discrepancy and eigenvalues. Random graphs are well known to have a number of remarkable properties (e.g., excellent expansion). Therefore, quantifying how much a given graph "resembles" a random graph is an important problem, both from a structural and an algorithmic point of view. Providing such measures is the purpose of the notion of *quasi-randomness*. While this concept is rather well developed for dense graphs (i.e., graphs $G = (V, E)$ with $|E| = \Omega(|V|^2)$), less is known in the sparse case, which we deal with in the present work. In fact, we shall actually deal with (sparse) graphs with *general degree distributions*, including but not limited to the ubiquitous power-law degree distributions (cf. [1]).

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We will mainly consider two types of quasi-random properties: low discrepancy and eigenvalue separation. The low discrepancy property concerns the global edge distribution and basically states that *every* set S of vertices approximately spans as many edges as we would expect in a random graph with the same degree distribution. More precisely, if $G = (V, E)$ is a graph, then we let d_v signify the degree of $v \in V$. Furthermore, the *volume* of a set $S \subset V$ is $\mathrm{vol}(S) = \sum_{v \in S} d_v$. In addition, $e(S)$ denotes the number of edges spanned by S.

$\mathrm{Disc}(\varepsilon)$: We say that G *has discrepancy at most* ε ("G has $\mathrm{Disc}(\varepsilon)$" for short) if

$$\forall S \subset V : \left| e(S) - \frac{\mathrm{vol}(S)^2}{2\mathrm{vol}(V)} \right| < \varepsilon \cdot \mathrm{vol}(V). \tag{1}$$

To explain (1), let $\boldsymbol{d} = (d_v)_{v \in V}$, and let $G(\boldsymbol{d})$ signify a uniformly distributed random graph with degree distribution $\boldsymbol{d}$. Then the probability p_{vw} that two vertices $v, w \in V$ are adjacent in $G(\boldsymbol{d})$ is proportional to the degrees of both v and w, and hence to their product. Further, as the total number of edges is determined by the sum of the degrees, we have $\sum_{(v,w) \in V^2} p_{vw} = \mathrm{vol}(V)$, whence $p_{vw} \sim d_v d_w / \mathrm{vol}(V)$. Therefore, in $G(\boldsymbol{d})$ the *expected* number of edges inside of $S \subset V$ equals $\frac{1}{2} \sum_{(v,w) \in S^2} p_{vw} \sim \frac{1}{2} \mathrm{vol}(S)^2 / \mathrm{vol}(V)$. Consequently, (1) just says that for *any* set S the actual number $e(S)$ of edges inside of S must not deviate from what we expect in $G(\boldsymbol{d})$ by more than an ε-fraction of the total volume.

An obvious problem with the bounded discrepancy property (1) is that it is quite difficult to check whether $G = (V, E)$ satisfies this condition. This is because one would have to inspect an exponential number of subsets $S \subset V$. Therefore, we consider a second property that refers to the eigenvalues of a certain matrix representing G. More precisely, we will deal with the *normalized Laplacian* $L(G)$, whose entries $(\ell_{vw})_{v,w \in V}$ are defined as

$$\ell_{vw} = \begin{cases} 1 & \text{if } v = w \text{ and } d_v \geq 1, \\ -(d_v d_w)^{-\frac{1}{2}} & \text{if } v, w \text{ are adjacent,} \\ 0 & \text{otherwise;} \end{cases}$$

Due to the normalization by the geometric mean $\sqrt{d_v d_w}$ of the vertex degrees, $L(G)$ turns out to be appropriate for representing graphs with general degree distributions. Moreover, $L(G)$ is well known to be positive semidefinite, and the multiplicity of the eigenvalue 0 equals the number of connected components of G (cf. [8]).

$\mathrm{Eig}(\delta)$: Letting $0 = \lambda_1(L(G)) \leq \cdots \leq \lambda_{|V|}(L(G))$ denote the eigenvalues of $L(G)$, we say that G *has* δ*-eigenvalue separation* ("G has $\mathrm{Eig}(\delta)$") if $1 - \delta \leq \lambda_2(L(G)) \leq \lambda_{|V|}(L(G)) \leq 1 + \delta$.

As the eigenvalues of $L(G)$ can be computed in polynomial time (within arbitrary numerical precision), we can essentially check efficiently whether G has $\mathrm{Eig}(\delta)$ or not.

It is not difficult to see that $\mathrm{Eig}(\delta)$ provides a *sufficient* condition for $\mathrm{Disc}(\varepsilon)$. That is, for any $\varepsilon > 0$ there is a $\delta > 0$ such that any graph G that has $\mathrm{Eig}(\delta)$ also has $\mathrm{Disc}(\varepsilon)$. However, while the converse implication is true if G is dense (i.e., $\mathrm{vol}(V) = \Omega(|V|^2)$), it is false for sparse graphs. In fact, providing a *necessary* condition for $\mathrm{Disc}(\varepsilon)$ in terms of eigenvalues has been an open problem in the area of sparse quasi-random graphs since the work of Chung and Graham [10]. Concerning this problem, we basically observe that the reason why $\mathrm{Disc}(\varepsilon)$ does in general not imply $\mathrm{Eig}(\delta)$ is the existence of a small set of "exceptional" vertices. With this in mind we refine the definition of Eig as follows.

$\text{ess-Eig}(\delta)$: We say that G *has essential* δ*-eigenvalue separation* ("G has $\text{ess-Eig}(\delta)$") if there is a set $W \subset V$ of volume $\mathrm{vol}(W) \geq (1 - \delta)\mathrm{vol}(V)$ such that the following is true. Let $L(G)_W = (\ell_{vw})_{v,w \in W}$ denote the minor of $L(G)$ induced on $W \times W$, and let $\lambda_1(L(G)_W) \leq \cdots \leq \lambda_{|W|}(L(G)_W)$ signify its eigenvalues; then we require that $1 - \delta < \lambda_2(L(G)_W) < \lambda_{|W|}(L(G)_W) < 1 + \delta$.

Theorem 1. *There is a constant* $\gamma > 0$ *such that the following is true for all graphs* $G = (V, E)$ *and all* $\varepsilon > 0$.

1. *If* G *has* $\text{ess-Eig}(\varepsilon)$*, then* G *satisfies* $\mathrm{Disc}(10\sqrt{\varepsilon})$.
2. *If* G *has* $\mathrm{Disc}(\gamma\varepsilon^2)$*, then* G *satisfies* $\text{ess-Eig}(\varepsilon)$.

The main contribution is the second implication. Its proof is based on Grothendieck's inequality and the duality theorem for semidefinite programs. In effect, the proof actually provides us with an efficient algorithm that computes a set W as in the definition of $\text{ess-Eig}(\varepsilon)$. The second part of Theorem 1 is best possible, up to the precise value of the constant γ (cf. Section 7).

The algorithmic regularity lemma. Loosely speaking, a regular partition of a graph $G = (V, E)$ is a partition of $(V_1, \ldots, V_t)$ of V such that for "most" index pairs i, j the bipartite subgraph spanned by V_i and V_j is quasi-random. Thus, a regular partition approximates G by quasi-random graphs. Furthermore, the number t of classes may depend on a parameter ε that rules the accuracy of the approximation, but it does *not* depend on the order of the graph G itself. Therefore, if for some class of graphs we can compute regular partitions in polynomial time, then this graph class will admit polynomial time algorithms for quite a few problems that are NP-hard in general.

In the sequel we introduce a new concept of regular partitions that takes into account the degree distribution of the graph. If $G = (V, E)$ is a graph and $A, B \subset V$ are disjoint, then the *relative density* of (A, B) in G is $\varrho(A, B) = \frac{e(A,B)}{\mathrm{vol}(A)\mathrm{vol}(B)}$. Further, we say that the pair (A, B) is ε*-volume regular* if for all $X \subset A$, $Y \subset B$ satisfying $\mathrm{vol}(X) \geq \varepsilon \mathrm{vol}(A)$, $\mathrm{vol}(Y) \geq \varepsilon \mathrm{vol}(B)$ we have

$$|e(X, Y) - \varrho(A, B)\mathrm{vol}(X)\mathrm{vol}(Y)| \leq \varepsilon \cdot \mathrm{vol}(A)\mathrm{vol}(B)/\mathrm{vol}(V), \tag{2}$$

where $e(X, Y)$ denotes the number of X-Y-edges in G. This condition essentially means that the bipartite graph spanned by A and B is quasi-random, given the degree distribution of G. Indeed, in a random graph the proportion of edges between X and Y should be proportional to both $\mathrm{vol}(X)$ and $\mathrm{vol}(Y)$, and hence to $\mathrm{vol}(X)\mathrm{vol}(Y)$. Moreover, $\varrho(A, B)$ measures the overall density of (A, B).

Finally, we state a condition that ensures the existence of regular partitions. While *every* dense graph G (of volume $\mathrm{vol}(V) = \Omega(|V|^2)$) admits a regular partition, such partitions do not necessarily exist for sparse graphs, the basic obstacle being extremely "dense spots". To rule out such dense spots, we consider the following notion.

(C, η)*-boundedness.* We say that a graph G is (C, η)*-bounded* if for all $X, Y \subset V$ with $\mathrm{vol}(X \cup Y) \geq \eta \mathrm{vol}(V)$ we have $\varrho(X, Y)\mathrm{vol}(V) \leq C$.

Now, we can state the following *algorithmic regularity lemma* for graphs with general degree distributions. which does not only ensure the *existence* of regular partitions, but also that such a partition can be computed efficiently.

Theorem 2. *For any two numbers $C \geq 1$ and $\varepsilon > 0$ there exist $\eta > 0$ and $n_0 > 0$ such that for all $n > n_0$ the following holds. If $G = (V, E)$ is a (C, η)-bounded graph on n vertices such that $\mathrm{vol}(V) \geq \eta^{-1}n$, then there is a partition $\mathcal{P} = \{V_i\colon\ 0 \leq i \leq t\}$ of V that enjoys the following two properties.*

REG1. *For all $1 \leq i \leq t$ we have $\eta \mathrm{vol}(V) \leq \mathrm{vol}(V_i) \leq \varepsilon \mathrm{vol}(V)$, and $\mathrm{vol}(V_0) \leq \varepsilon \mathrm{vol}(V)$.*
REG2. *Let $\mathcal{L}$ be the set of all pairs $(i, j) \in \{1, \ldots, t\}^2$ such that (V_i, V_j) is not ε-volume-regular. Then*

$$\sum_{(i,j)\in\mathcal{L}} \mathrm{vol}(V_i)\mathrm{vol}(V_j) \leq \varepsilon \mathrm{vol}^2(G).$$

Furthermore, for fixed $C > 0$ and $\varepsilon > 0$ such a partition $\mathcal{P}$ of V can be computed in time polynomial in n.

Condition **REG1** states that each of the classes $V_1, \ldots, V_t$ has some non-negligible volume, and that the "exceptional" class V_0 is not too big. Moreover, **REG2** requires that the share of edges of G that belongs to irregular pairs (V_i, V_j) is small. Thus, a partition $\mathcal{P}$ that satisfies **REG1** and **REG2** approximates G by a bounded number of bipartite quasi-random graphs, i.e., the number t of classes can be bounded solely in terms of ε and the boundedness parameter C.

We illustrate the use of Theorem 2 with the example of the MAX CUT problem. While approximating MAX CUT within a ratio better than $\frac{16}{17}$ is NP-hard on general graphs [17, 22], the following theorem provides a polynomial time approximation scheme for (C, η)-bounded graphs.

Theorem 3. *For any $\delta > 0$ and $C > 0$ there exist two numbers $\eta > 0$, n_0 and a polynomial time algorithm* `ApxMaxCut` *such that for all $n > n_0$ the following is true. If $G = (V, E)$ is a (C, η)-bounded graph on n vertices and $\mathrm{vol}(V) > \eta^{-1}n$, then* `ApxMaxCut`$(G)$ *outputs a cut $(S, \bar{S})$ of G that approximates the maximum cut within a factor of $1 - \delta$.*

The corresponding result for dense graphs was obtained by Frieze and Kannan [12].

2 Related Work

Quasi-random graphs. Quasi-random graphs with general degree distributions were first studied by Chung and Graham [9]. They considered the properties $\mathrm{Disc}(\varepsilon)$ and $\mathrm{Eig}(\delta)$, and a number of further related ones (e.g., concerning weighted cycles). Chung and Graham observed that $\mathrm{Eig}(\delta)$ implies $\mathrm{Disc}(\varepsilon)$, and that the converse is true in the case of *dense* graphs (i.e., $\mathrm{vol}(V) = \Omega(|V|^2)$).

Regarding the step from discrepancy to eigenvalue separation, Butler [7] proved that any graph G such that for all sets $X, Y \subset V$ the bound

$$|e(X,Y) - \mathrm{vol}(X)\mathrm{vol}(Y)/\mathrm{vol}(V)| \leq \varepsilon\sqrt{\mathrm{vol}(X)\mathrm{vol}(Y)} \tag{3}$$

holds, satisfies $\mathrm{Eig}(O(\varepsilon(1-\ln\varepsilon)))$. His proof builds upon the work of Bilu and Linial [5], who derived a similar result for regular graphs, and on the earlier related work of Bollobás and Nikiforov [6].

Butler's result relates to the second part of Theorem 1 as follows. The r.h.s. of (3) refers to the volumes of the sets X, Y, and may thus be significantly smaller than $\varepsilon\mathrm{vol}(V)$. By contrast, the second part of Theorem 1 just requires that the "original" discrepancy condition $\mathrm{Disc}(\delta)$ is true, i.e., we just need to bound $|e(S) - \frac{1}{2}\mathrm{vol}(S)^2/\mathrm{vol}(V)|$ in terms of the *total* volume $\mathrm{vol}(V)$. Hence, Butler shows that the "original" eigenvalue separation condition Eig follows from a stronger version of the discrepancy property. By contrast, Theorem 1 shows that the "original" discrepancy condition Disc implies a weak form of eigenvalue separation ess-Eig, thereby answering a question posed by Chung and Graham [9, 10]. Furthermore, relying on Grothendieck's inequality and SDP duality, the proof of Theorem 1 employs quite different techniques than the methods used in [5–7].

In the present work we consider a concept of quasi-randomness that takes into account the graph's degree sequence. Other concepts that do not refer to the degree sequence (and are therefore restricted to approximately regular graphs) were studied by Chung, Graham and Wilson [11] (dense graphs) and by Chung and Graham [10] (sparse graphs). Also in this setting it has been an open problem to derive eigenvalue separation from low discrepancy, and concerning this simpler concept of quasi-randomness, our techniques yield a similar result as Theorem 1 as well. The proof is similar and we omit the details here.

Regular partitions. Szemerédi's original regularity lemma [21] shows that any *dense* graph $G = (V, E)$ (with $|E| = \Omega(|V|^2)$) can be partitioned into a bounded number of sets $V_1, \ldots, V_t$ such that almost all pairs (V_i, V_j) are quasi-random. This statement has become an important tool in various areas, including extremal graph theory and property testing. Furthermore, Alon, Duke, Lefmann, Rödl, and Yuster [3] presented an algorithmic version, and showed how this lemma can be used to provide polynomial time approximation schemes for dense instances of NP-hard problems (see also [19] for a faster algorithm). Moreover, Frieze and Kannan [12] introduced a different algorithmic regularity concept, which yields better efficiency in terms of the desired approximation guarantee.

A version of the regularity lemma that applies to sparse graphs was established independently by Kohayakawa [18] and Rödl (unpublished). This result is of significance, e.g., in the theory of random graphs, cf. Gerke and Steger [13]. The regularity concept of Kohayakawa and Rödl is related to the notion of quasi-randomness from [10] and shows that any graph that satisfies a certain boundedness condition has a regular partition.

In comparison to the Kohayakawa-Rödl regularity lemma, the new aspect of Theorem 2 is that it takes into account the graph's degree distribution. Therefore, Theorem 2 applies to graphs with very irregular degree distributions, which were not covered by prior versions of the sparse regularity lemma. Further, Theorem 2 yields an efficient algorithm for computing a regular partition (see e.g., [14] for a non-polynomial time algorithm in the sparse setting). To achieve this algorithmic result, we build upon the algorithmic version of Grothendieck's inequality due to Alon and Naor [4]. Besides, our approach can easily be modified to obtain a polynomial time algorithm for computing a regular partition in the sense of Kohayakawa and Rödl, which was not known previously.

3 Preliminaries

3.1 Notation

If $S \subset V$ is a subset of some set V, then we let $\mathbf{1}_S \in \mathbf{R}^V$ denote the vector whose entries are 1 on the components corresponding to elements of S, and 0 otherwise. More generally, if $\xi \in \mathbf{R}^V$ is a vector, then $\xi_S \in \mathbf{R}^V$ signifies the vector obtained from ξ by replacing all components with indices in $V \setminus S$ by 0. Moreover, if $A = (a_{vw})_{v,w \in V}$ is a matrix, then $A_S = (a_{vw})_{v,w \in S}$ denotes the minor of A induced on $S \times S$. Further, for a vector $\xi \in \mathbf{R}^V$ we let $\|\xi\|$ signify the ℓ_2-norm, and for a matrix we let $\|M\| = \sup_{0 \neq \xi \in \mathbf{R}^V} \frac{\|M\xi\|}{\|\xi\|}$ denote the spectral norm.

If $\xi = (\xi_v)_{v \in V}$ is a vector, then $\mathrm{diag}(\xi)$ signifies the $V \times V$ matrix with diagonal ξ and off-diagonal entries equal to 0. In particular, $\boldsymbol{E} = \mathrm{diag}(\mathbf{1})$ denotes the identity matrix (of any size). Moreover, if M is a $\nu \times \nu$ matrix, then $\mathrm{diag}(M) \in \mathbf{R}^\nu$ signifies the vector comprising the diagonal entries of M. If both $A = (a_{ij})_{1 \leq i,j \leq \nu}, B = (b_{ij})_{1 \leq i,j \leq \nu}$ are $\nu \times \nu$ matrices, then we let $\langle A, B \rangle = \sum_{i,j=1}^{\nu} a_{ij} b_{ij}$.

If M is a symmetric $\nu \times \nu$ matrix, then $\lambda_1(M) \leq \cdots \leq \lambda_\nu(M) = \lambda_{\max}(M)$ denote the eigenvalues of M. Recall that a symmetric matrix M is positive semidefinite if $\lambda_1(M) \geq 0$; in this case we write $M \geq 0$. Furthermore, M positive definite if $\lambda_1(M) > 0$, denoted as $M > 0$. If M, M' are symmetric, then $M \geq M'$ (resp. $M > M'$) denotes the fact that $M - M' \geq 0$ (resp. $M - M' > 0$).

3.2 Grothendieck's inequality

An important ingredient to our proofs and algorithms is Grothendieck's inequality. Let $M = (m_{ij})_{i,j \in \mathcal{I}}$ be a matrix. Then the *cut-norm* of M is

$$\|M\|_{\mathrm{cut}} = \max_{I,J \subset \mathcal{I}} \left| \sum_{i \in I, j \in J} m_{ij} \right|.$$

In addition, consider the following optimization problem:

$$\mathrm{SDP}(M) = \max \sum_{i,j \in \mathcal{I}} m_{ij} \langle x_i, y_j \rangle \text{ s.t. } \|x_i\| = \|y_i\| = 1. \tag{4}$$

While we allow x_i, y_i to be elements of any Hilbert space, one can always assume without loss of generality that $x_i, y_i \in \mathbf{R}^{2|\mathcal{I}|}$ (because the space spanned by the vectors x_i, y_i has dimension $\leq 2|\mathcal{I}|$). Therefore, $\mathrm{SDP}(M)$ can be reformulated as a *linear* optimization problem over the cone of positive semidefinite $2|\mathcal{I}| \times 2|\mathcal{I}|$ matrices, i.e., as a semidefinite program (cf. Alizadeh [2]).

Lemma 4. *For any $\nu \times \nu$ matrix M we have*

$$\mathrm{SDP}(M) = \frac{1}{2} \max \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes M, X \right\rangle \textit{ s.t. } \mathrm{diag}(X) = \mathbf{1},\ X \geq 0,\ X \in \mathbf{R}^{2\nu \times 2\nu}. \tag{5}$$

Proof. Let $x_1, \ldots, x_{2\nu} \in \mathbf{R}^{2\nu}$ be a family of unit vectors such that $\mathrm{SDP}(M) = \sum_{i,j=1}^{\nu} m_{ij} \langle x_i, x_{j+\nu} \rangle$. Then we obtain a positive semidefinite matrix $X = (x_{ij})_{1 \leq i,j \leq 2\nu}$ by setting $x_{ij} = \langle x_i, x_j \rangle$. Since $x_{ii} = \|x_i\|^2 = 1$ for all i, this matrix satisfies $\mathrm{diag}(X) = \mathbf{1}$. Moreover,

$$\left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes M, X \right\rangle = 2 \sum_{i,j=1}^{\nu} m_{ij} x_{ij+\nu} = 2 \sum_{i,j=1}^{\nu} m_{ij} \langle x_i, x_{j+\nu} \rangle. \tag{6}$$

Hence, the optimization problem on the r.h.s. of (5) yields an upper bound on $\mathrm{SDP}(M)$.

Conversely, if $X = (x_{ij})$ is a feasible solution to (5), then there exist vectors $x_1, \ldots, x_{2\nu} \in \mathbf{R}^{2\nu}$ such that $x_{ij} = \langle x_i, x_j \rangle$, because X is positive semidefinite. Moreover, since $\mathrm{diag}(X) = \mathbf{1}$, we have $1 = x_{ii} = \|x_i\|^2$. Thus, $x_1, \ldots, x_{2\nu}$ is a feasible solution to (5), and (6) shows that the resulting objective function values coincide. □

Since by Lemma 4 $\mathrm{SDP}(M)$ can be stated as a semidefinite program, an optimal solution to $\mathrm{SDP}(M)$ can be approximated within any numerical precision, e.g., via the ellipsoid method [16].

Grothendieck [15] established the following relation between $\mathrm{SDP}(M)$ and $\|M\|_{\mathrm{cut}}$.

Theorem 5. *There is a constant $\theta > 1$ such that for all matrices M we have $\|M\|_{\mathrm{cut}} \leq \mathrm{SDP}(M) \leq \theta \cdot \|M\|_{\mathrm{cut}}$.*

The best current bounds on the above constant are $\frac{\pi}{2} \le \theta \le \frac{\pi}{2\ln(1+\sqrt{2})}$ [15, 20]. Furthermore, by applying an appropriate rounding procedure to a near-optimal solution to $\mathrm{SDP}(M)$, Alon and Naor [4] obtained the following algorithmic result.

Theorem 6. *There are a constant $\theta' > 0$ and a polynomial time algorithm* `ApxCutNorm` *that computes on input M two sets $I, J \subset \mathcal{I}$ such that $\theta' \cdot \|M\|_{\mathrm{cut}} \le \left|\sum_{i\in I, j\in J} m_{ij}\right|$.*

Alon and Naor presented a randomized algorithm that guarantees an approximation ration $\theta' > 0.56$, and a deterministic one with $\theta' \ge 0.03$.

To facilitate the proof of Theorem 1, we point out the following simple fact.

Lemma 7. *Let $M = (m_{ij})_{i,j\in\mathcal{I}}$ be a matrix, and let $\mathcal{J} \subset \mathcal{I}$. Then $\mathrm{SDP}(M_{\mathcal{J}}) \le \mathrm{SDP}(M)$.*

Proof. Let $(x_i)_{i\in\mathcal{J}}, (y_j)_{j\in\mathcal{J}}$ be an optimal solution to $\mathrm{SDP}(M_{\mathcal{J}})$; that is, x_i, y_j are unit vectors such that $\mathrm{SDP}(M_{\mathcal{J}}) = \sum_{i,j\in\mathcal{J}} m_{ij} \langle x_i, y_j \rangle$. Without loss of generality we may assume that $x_i, y_j \in \mathbf{R}^{2|\mathcal{I}|}$. Since the subspace of $\mathbf{R}^{2|\mathcal{I}|}$ spanned by the vectors $\{x_i, y_j : i, j \in \mathcal{J}\}$ has dimension $\le 2|\mathcal{J}|$, there is a family $\{x_i, y_j : i, j \in \mathcal{I} \setminus \mathcal{J}\}$ of mutually perpendicular unit vectors such that the space spanned by $\{x_i, y_j : i, j \in \mathcal{I} \setminus \mathcal{J}\}$ is perpendicular to the space spanned by $\{x_i, y_j : i, j \in \mathcal{J}\}$. Therefore, we obtain

$$\mathrm{SDP}(M) \ge \sum_{i,j\in\mathcal{I}} m_{ij} \langle x_i, y_j \rangle = \sum_{i,j\in\mathcal{J}} m_{ij} \langle x_i, y_j \rangle = \mathrm{SDP}(M_{\mathcal{J}}),$$

as desired. □

4 Quasi-Randomness: Proof of Theorem 1

4.1 From Essential Eigenvalue Separation to Low Discrepancy

Here we prove the first part of Theorem 1. Suppose that $G = (V, E)$ is a graph that admits a set $W \subset V$ of volume $\mathrm{vol}(W) \ge (1-\varepsilon)\mathrm{vol}(V)$ such that the eigenvalues of the minor L_W of the normalized Laplacian satisfy

$$1 - \varepsilon \le \lambda_2(L_W) \le \lambda_{\max}(L_W) \le 1 + \varepsilon. \tag{7}$$

We may assume without loss of generality that $\varepsilon < 10^{-6}$. Our goal is to show that G has $\mathrm{Disc}(10\sqrt{\varepsilon})$.

Let $\Delta = (\sqrt{d_v})_{v\in W} \in \mathbf{R}^W$, and let $\mathcal{L}_W$ denote the matrix whose vw'th entry is $(d_v d_w)^{-\frac{1}{2}}$ if v, w are adjacent, and 0 otherwise ($v, w \in W$), so that $L_W = \boldsymbol{E} - \mathcal{L}_W$. Further, let $\mathcal{M}_W = \mathrm{vol}(V)^{-1}\Delta\Delta^T - \mathcal{L}_W$. Then for all unit vectors $\xi \perp \Delta$ we have

$$L_W\xi - \xi = -\mathcal{L}_W\xi = \mathcal{M}_W\xi. \tag{8}$$

Moreover, for all $S \subset W$

$$|\langle \mathcal{M}_W \Delta_S, \Delta_S \rangle| = \left| \frac{\mathrm{vol}(S)^2}{\mathrm{vol}(V)} - 2e(S) \right|. \tag{9}$$

We will derive the following bound on the operator norm of $\mathcal{M}_W$.

Lemma 8. *We have $\|\mathcal{M}_W\| \le 10\sqrt{\varepsilon}$.*

The Lemma easily implies that G has $\mathrm{Disc}(10\sqrt{\varepsilon})$; for let $R \subset V$ be arbitrary. Set $S = R \cap W$ and $T = R \setminus W$. Since $\|\Delta_S\|^2 = \mathrm{vol}(S) \le \mathrm{vol}(V)$, Lemma 8 and (9) imply that

$$\left| \frac{\mathrm{vol}(S)^2}{2\mathrm{vol}(V)} - e(S) \right| \le 5\sqrt{\varepsilon}\mathrm{vol}(V). \tag{10}$$

Furthermore, as $\mathrm{vol}(W) \geq (1-\varepsilon)\mathrm{vol}(V)$,

$$e(R) - e(S) \leq e(T) + e(S,T) \leq \mathrm{vol}(T) \leq \mathrm{vol}(V \setminus W) \leq \varepsilon \mathrm{vol}(V), \text{ and} \tag{11}$$

$$\frac{\mathrm{vol}(R)^2 - \mathrm{vol}(S)^2}{2\mathrm{vol}(V)} \leq \frac{\mathrm{vol}(T)^2}{2\mathrm{vol}(V)} + \frac{\mathrm{vol}(S)\mathrm{vol}(T)}{\mathrm{vol}(V)} \leq \frac{\mathrm{vol}(V \setminus W)^2}{2\mathrm{vol}(V)} + \mathrm{vol}(V \setminus W) \leq 2\varepsilon \mathrm{vol}(V). \tag{12}$$

Finally, combining (10)–(12), we see that $\left|\frac{\mathrm{vol}(R)^2}{2\mathrm{vol}(V)} - e(R)\right| < 10\sqrt{\varepsilon}\mathrm{vol}(V)$, whence G satisfies $\mathrm{Disc}(10\sqrt{\varepsilon})$.

Proof of Lemma 8. Although the smallest eigenvalue of L equals 0 and the corresponding eigenvector is Δ, the smallest eigenvalue $\lambda_1(L_W)$ of the minor L_W may be strictly positive. Let ζ be an eigenvector of L_W with eigenvalue $\lambda_1(L_W)$ of unit length. Then we have a decomposition $\Delta = \|\Delta\| \cdot (s\zeta + t\chi)$, where $s^2 + t^2 = 1$ and $\chi \perp \zeta$ is a unit vector. Since $\langle L_W \Delta, \Delta \rangle = e(W, V \setminus W) \leq \mathrm{vol}(V \setminus W) \leq \varepsilon \mathrm{vol}(V)$ and $\|\Delta\|^2 = \mathrm{vol}(W) \geq 0.99\mathrm{vol}(V)$, (7) entails that

$$2\varepsilon \geq \|\Delta\|^{-2} \langle L_W \Delta, \Delta \rangle = s^2 \langle L_W \zeta, \zeta \rangle + t^2 \langle L_W \chi, \chi \rangle \geq t^2 \lambda_2(L_W) \geq \frac{t^2}{2}.$$

Consequently,

$$t^2 \leq 4\varepsilon, \text{and} \quad s^2 \geq 1 - 4\varepsilon. \tag{13}$$

Now, let $\xi \perp \Delta$ be a unit vector, and decompose $\xi = x\zeta + y\eta$, where $\eta \perp \zeta$ is a unit vector. Because $\zeta = s^{-1}\left(\frac{\Delta}{\|\Delta\|} - t\chi\right)$, we have $x = \langle \zeta, \xi \rangle = s^{-1}\left\langle \frac{\Delta}{\|\Delta\|}, \xi \right\rangle - \frac{t}{s}\langle \chi, \xi \rangle = -\frac{t}{s}\langle \chi, \xi \rangle$. Hence, (13) entails

$$x^2 \leq 5\varepsilon, \quad y^2 \geq 1 - 5\varepsilon. \tag{14}$$

Combining (7), (8) and (14), we conclude that $\|\mathcal{M}_W \xi\| = \|L_W \xi - \xi\| \leq x(1 - \lambda_1(L_W)) + y\|L_W \eta - \eta\| \leq 3\sqrt{\varepsilon}$. Hence, we have established that

$$\sup_{0 \neq \xi \perp \Delta} \frac{\|\mathcal{M}_W \xi\|}{\|\xi\|} \leq 3\sqrt{\varepsilon}. \tag{15}$$

Furthermore, as by assumption $\mathrm{vol}(W) \geq (1-\varepsilon)\mathrm{vol}(V)$,

$$\begin{aligned} \frac{|\langle \mathcal{M}_W \Delta, \Delta \rangle|}{\|\Delta\|^2} &= \left|\frac{\|\Delta\|^2}{\mathrm{vol}(V)} - \frac{2e(W)}{\|\Delta\|^2}\right| = \left|\frac{\mathrm{vol}(W)}{\mathrm{vol}(V)} - \frac{2e(W)}{\mathrm{vol}(W)}\right| \\ &= \frac{\mathrm{vol}(V \setminus W)}{\mathrm{vol}(V)} + \frac{e(W, V \setminus W)}{\mathrm{vol}(W)} \leq \frac{3\mathrm{vol}(V \setminus W)}{\mathrm{vol}(V)} < 3\varepsilon. \end{aligned} \tag{16}$$

Finally, combining (15) and (16), we conclude that $\|\mathcal{M}_W\| \leq 10\sqrt{\varepsilon}$. □

4.2 From Low Discrepancy to Essential Eigenvalue Separation

In this section we establish the second part of Theorem 1. Assume that $G = (V, E)$ is a graph that has $\mathrm{Disc}(\gamma\varepsilon^2)$, where $\gamma > 0$ signifies some small enough constant (e.g., $\gamma = (6400\theta)^{-1}$, where θ is the constant from Theorem 5). We may assume that $\varepsilon < 0.001$. Moreover, let d_v denote the degree of $v \in V$, $n = |V|$, and $\bar{d} = n^{-1}\sum_{v \in V} d_v$. Our goal is to show that G has ess-Eig(ε). To this end, we need to introduce an additional property.

$\mathrm{Cut}(\varepsilon)$: We say G has $\mathrm{Cut}(\varepsilon)$, if the matrix $M = (m_{vw})_{v,w \in V}$ with entries

$$m_{vw} = \frac{d_v d_w}{\mathrm{vol}(V)} - e(v, w)$$

has cut norm $\|M\|_{\mathrm{cut}} < \varepsilon \cdot \mathrm{vol}(V)$; here $e(v, w) = 1$ if $\{v, w\} \in E$ and 0 otherwise.

Since for any $S \subset V$ we have $\langle M\mathbf{1}_S, \mathbf{1}_S \rangle = \frac{\mathrm{vol}(S)^2}{\mathrm{vol}(V)} - 2e(S)$, one can easily derive the following.

Proposition 9. *If G satisfies $\mathrm{Disc}(0.01\delta)$, then G enjoys $\mathrm{Cut}(\delta)$.*

Proof. Suppose that $G = (V, E)$ has $\mathrm{Disc}(0.01\delta)$. We shall prove below that for any two $S, T \subset V$

$$|\langle M\mathbf{1}_S, \mathbf{1}_T\rangle| \le 0.06\delta \mathrm{vol}(V) \text{ if } S \cap T = \emptyset, \tag{17}$$
$$|\langle M\mathbf{1}_S, \mathbf{1}_T\rangle| \le 0.02\delta \mathrm{vol}(V) \text{ if } S = T. \tag{18}$$

To see that (17) and (18) imply the assertion, consider two arbitrary subsets $X, Y \subset V$. Letting $Z = X \cap Y$ and combining (17) and (18), we obtain

$$\begin{aligned}|\langle M\mathbf{1}_X, \mathbf{1}_Y\rangle| &\le \left|\langle M\mathbf{1}_{X\setminus Z}, \mathbf{1}_{Y\setminus Z}\rangle\right| + \left|\langle M\mathbf{1}_Z, \mathbf{1}_{Y\setminus Z}\rangle\right| + \left|\langle M\mathbf{1}_Z, \mathbf{1}_{X\setminus Z}\rangle\right| + 2\left|\langle M\mathbf{1}_Z, \mathbf{1}_Z\rangle\right| \\ &\le \delta \mathrm{vol}(V).\end{aligned}$$

Since this bound holds for any X, Y, we conclude that $\|M\|_{\mathrm{cut}} \le \delta \mathrm{vol}(V)$.

To prove (17), we note that $\mathrm{Disc}(0.01\delta)$ implies that

$$\left|e(S) - \frac{\mathrm{vol}(S)^2}{2\mathrm{vol}(V)}\right| \le 0.01\delta \mathrm{vol}(V), \tag{19}$$
$$\left|e(T) - \frac{\mathrm{vol}(T)^2}{2\mathrm{vol}(V)}\right| \le 0.01\delta \mathrm{vol}(V), \tag{20}$$
$$\left|e(S \cup T) - \frac{(\mathrm{vol}(S) + \mathrm{vol}(T))^2}{2\mathrm{vol}(V)}\right| \le 0.01\delta \mathrm{vol}(V). \tag{21}$$

As S and T are disjoint, (19)–(21) yield

$$\begin{aligned}|\langle M\mathbf{1}_S, \mathbf{1}_T\rangle| &= 2\left|e(S,T) - \frac{\mathrm{vol}(S)\mathrm{vol}(T)}{2\mathrm{vol}(V)}\right| \\ &= 2\left|e(S \cup T) - e(S) - e(T) - \frac{(\mathrm{vol}(S) + \mathrm{vol}(T))^2 - \mathrm{vol}(S)^2 - \mathrm{vol}(T)^2}{2\mathrm{vol}(V)}\right| \\ &\le 2\left|e(S) - \frac{\mathrm{vol}(S)^2}{2\mathrm{vol}(V)}\right| + 2\left|e(T) - \frac{\mathrm{vol}(T)^2}{2\mathrm{vol}(V)}\right| + 2\left|e(S \cup T) - \frac{(\mathrm{vol}(S) + \mathrm{vol}(T))^2}{2\mathrm{vol}(V)}\right| \\ &\le 0.06\delta \mathrm{vol}(V).\end{aligned}$$

Finally, as $|\langle M\mathbf{1}_S, \mathbf{1}_S\rangle| = 2\left|e(S) - \frac{\mathrm{vol}(S)^2}{2\mathrm{vol}(V)}\right|$, (18) follows from (19). □

To show that $\mathrm{Disc}(\gamma\varepsilon^2)$ implies ess-Eig(ε), we proceed as follows. By Proposition 9, $\mathrm{Disc}(\gamma\varepsilon^2)$ implies $\mathrm{Cut}(100\gamma\varepsilon^2)$. Moreover, if G satisfies $\mathrm{Cut}(100\gamma\varepsilon^2)$, then Theorem 5 entails that not only the cut norm of M is small, but even the semidefinite relaxation $\mathrm{SDP}(M)$ satisfies $\mathrm{SDP}(M) < \beta\varepsilon^2\mathrm{vol}(V)$, for some constant $0 < \beta \le 100\theta\gamma$. This bound on $\mathrm{SDP}(M)$ can be rephrased in terms of an eigenvalue minimization problem for a matrix closely related to M. More precisely, using the duality theorem for semidefinite programs, we can infer the following.

Lemma 10. *For any symmetric $n \times n$ matrix Q we have*

$$\mathrm{SDP}(Q) = n \cdot \min_{z \in \mathbf{R}^n,\, z \perp \mathbf{1}} \lambda_{\max}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix} \otimes Q - \mathrm{diag}\begin{pmatrix} z \\ z\end{pmatrix}\right].$$

We defer the proof of Lemma 10 to Section 4.3. Let $D = \mathrm{diag}(d_v)_{v \in V}$ be the matrix with the vertex degrees on the diagonal. Establishing the following lemma is the key step in the proof.

Lemma 11. *Suppose that $\mathrm{SDP}(M) < \varepsilon^2\mathrm{vol}(V)/64$. Then there exists a subset $W \subset V$ of volume $\mathrm{vol}(W) \ge (1-\varepsilon)\cdot\mathrm{vol}(V)$ such that the matrix $\mathcal{M} = D^{-\frac{1}{2}}MD^{-\frac{1}{2}}$ satisfies $\|\mathcal{M}_W\| < \varepsilon$.*

Observe that vw'th the entry of $\mathcal{M}$ if $\frac{\sqrt{d_v d_w}}{\mathrm{vol}(V)} - (d_v d_w)^{-1/2}$ if v, w are adjacent, and $\frac{\sqrt{d_v d_w}}{\mathrm{vol}(V)}$ otherwise.

Before we get to the proof of Lemma 11, we show that the lemma implies that G has ess-Eig(ε). Combining Theorem 5, Proposition 9, and Lemma 11, we conclude if G has $\mathrm{Disc}(\gamma\varepsilon^2)$, then there is a set W such that $\mathrm{vol}(W) \geq (1-\varepsilon)\mathrm{vol}(V)$ and $\|\mathcal{M}_W\| < \varepsilon$. Furthermore, $\mathcal{M}_W$ relates to the minor L_W of the Laplacian as follows. Let

$$\mathcal{L}_W = \boldsymbol{E} - L_W$$

be the matrix whose vw'th entry is $(d_v d_w)^{-1/2}$ if $v, w \in W$ are adjacent, and 0 otherwise. Moreover, let $\varDelta = (\sqrt{d_v})_{v \in W} \in \mathbf{R}^W$. Then

$$\mathcal{M}_W = \mathrm{vol}(V)^{-1} \varDelta \varDelta^T - \mathcal{L}_W.$$

Therefore, for all unit vectors $\xi \perp \varDelta$ we have

$$|\langle L_W \xi, \xi \rangle - 1| = |\langle \mathcal{L}_W \xi, \xi \rangle| = |\langle \mathcal{M}_W \xi, \xi \rangle| \leq \|\mathcal{M}_W\| < \varepsilon. \tag{22}$$

Combining (22) with the Rayleigh characterization of $\lambda_2(L_W)$, we obtain

$$\lambda_2(L_W) = \max_{0 \neq \zeta \in \mathbf{R}^W} \min_{\xi \perp \zeta,\, \|\xi\|=1} \langle L_W \xi, \xi \rangle \geq \min_{\xi \perp \varDelta,\, \|\xi\|=1} \langle L_W \xi, \xi \rangle \geq 1 - \varepsilon. \tag{23}$$

In addition, since $\|\varDelta\|^2 = \mathrm{vol}(W) \geq \frac{1}{2}\mathrm{vol}(V)$, we have

$$\frac{\|L_W \varDelta\|^2}{\|\varDelta\|^2} = \sum_{v \in W} \frac{(e(v,W) - d_v)^2}{d_v \cdot \mathrm{vol}(W)} \leq 2 \sum_{v \in W} \frac{d_v - e(v,W)}{\mathrm{vol}(V)} \leq \frac{2\mathrm{vol}(V \setminus W)}{\mathrm{vol}(V)} < 2\varepsilon. \tag{24}$$

Further, decomposing any unit vector $\eta \in \mathbf{R}^W$ as $\eta = \alpha \|\varDelta\|^{-1} \varDelta + \beta \xi$ with $\xi \perp \varDelta$ and $\alpha^2 + \beta^2 = 1$, we get

$$\begin{aligned} \langle L_W \eta, \eta \rangle &= \alpha^2 \|\varDelta\|^{-2} \langle L_W \varDelta, \varDelta \rangle + 2\alpha\beta \|\varDelta\|^{-1} \langle L_W \varDelta, \xi \rangle + \beta^2 \langle L_W \xi, \xi \rangle \\ &\overset{(24)}{\leq} 4\alpha^2 \varepsilon^2 + 4\alpha\beta\varepsilon + \beta^2 \langle L_W \xi, \xi \rangle \overset{(22)}{\leq} 4\alpha^2 \varepsilon^{1/2} + 4\alpha\beta\varepsilon^{1/2} + \beta^2(1+\varepsilon) \leq 1 + \varepsilon, \end{aligned}$$

because we are assuming that $\varepsilon < 0.001$. Hence,

$$\lambda_{\max}(L_W) = \max_{\|\eta\|=1} \langle L_W \eta, \eta \rangle \leq 1 + \varepsilon. \tag{25}$$

Thus, (23) and (25) imply that G has ess-Eig(ε).

Proof of Lemma 11. Let $U = \{v \in V : d_v > \varepsilon \bar{d}/8\}$. Then

$$\mathrm{vol}(V \setminus U) \leq \varepsilon \bar{d} |V \setminus U|/8 \leq \varepsilon \mathrm{vol}(V)/2. \tag{26}$$

Since $\mathrm{SDP}(M_U) \leq \mathrm{SDP}(M)$ by Lemma 7, Lemma 10 entails that there is a vector $\mathbf{1} \perp z \in \mathbf{R}^U$ such that

$$\lambda_{\max}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes M_U - \mathrm{diag}\begin{pmatrix} z \\ z \end{pmatrix}\right] < \varepsilon^2 \bar{d}/64. \tag{27}$$

Consequently, as all entries of the diagonal matrix D_U exceed $\varepsilon\bar{d}/8$, for $y = D_U^{-1} z$ we have

$$\begin{aligned} &\lambda_{\max}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \mathcal{M}_U - \mathrm{diag}\begin{pmatrix} y \\ y \end{pmatrix}\right] \\ &\quad = \lambda_{\max}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes D_U^{-\frac{1}{2}} \cdot \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes M_U - \mathrm{diag}\begin{pmatrix} z \\ z \end{pmatrix}\right] \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes D_U^{-\frac{1}{2}}\right] \\ &\quad \leq 8\left(\varepsilon\bar{d}\right)^{-1} \lambda_{\max}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes M_U - \mathrm{diag}\begin{pmatrix} z \\ z \end{pmatrix}\right] < \varepsilon/8. \end{aligned} \tag{28}$$

Moreover, as $z \perp \mathbf{1}$,

$$\langle y, D_U \mathbf{1} \rangle = \langle D_U y, \mathbf{1} \rangle = \langle z, \mathbf{1} \rangle = 0. \tag{29}$$

Now, let $W = \{v \in U : |y_v| < \varepsilon/8\}$ consist of all vertices v on which the "correcting vector" y is small. Since on W all entries of the diagonal matrix $\mathrm{diag}\binom{y}{y}$ are smaller than $\varepsilon/8$ in absolute value, we have $\|\mathrm{diag}\binom{y_W}{y_W}\| < \varepsilon/8$. Therefore, (28) yields

$$\lambda_{\max}\left[\begin{pmatrix}0\,1\\1\,0\end{pmatrix}\otimes\mathcal{M}_W\right] \leq \lambda_{\max}\left[\begin{pmatrix}0\,1\\1\,0\end{pmatrix}\otimes\mathcal{M}_W - \mathrm{diag}\begin{pmatrix}y_W\\y_W\end{pmatrix}\right] + \left\|\mathrm{diag}\begin{pmatrix}y_W\\y_W\end{pmatrix}\right\| \leq \varepsilon/4; \qquad (30)$$

in other words, on W the effect of y is negligible.

Further, (30) entails that $\|\mathcal{M}_W\| < \varepsilon$. To see this, consider a pair $\xi, \eta \in \mathbf{R}^W$ of unit vectors. Since $\mathcal{M}_W$ is symmetric, (30) implies that

$$\begin{aligned}2\varepsilon > 2\lambda_{\max}\left[\begin{pmatrix}0\,1\\1\,0\end{pmatrix}\otimes\mathcal{M}_W\right] &\geq \left\langle\begin{pmatrix}0\,1\\1\,0\end{pmatrix}\otimes\mathcal{M}_W\cdot\begin{pmatrix}\xi\\\eta\end{pmatrix},\begin{pmatrix}\xi\\\eta\end{pmatrix}\right\rangle\\ = \left\langle\begin{pmatrix}\mathcal{M}_W\eta\\\mathcal{M}_W\xi\end{pmatrix},\begin{pmatrix}\xi\\\eta\end{pmatrix}\right\rangle &= \langle\mathcal{M}_W\eta,\xi\rangle + \langle\mathcal{M}_W\xi,\eta\rangle = 2\langle\mathcal{M}_W\xi,\eta\rangle.\end{aligned}$$

Since this holds for any pair ξ, η, we conclude that $\|\mathcal{M}_W\| < \varepsilon$.

Finally, we need to show that $\mathrm{vol}(W)$ is large. To this end, we consider the set $S = \{v \in U : y_v < 0\}$. Then (27) yields

$$\begin{aligned}\varepsilon^2 d|S|/32 = \frac{\varepsilon^2 d}{64}\left\|\begin{pmatrix}\mathbf{1}_S\\\mathbf{1}_S\end{pmatrix}\right\|^2 &\geq \left\langle\left[\begin{pmatrix}0\,1\\1\,0\end{pmatrix}\otimes M_U - \mathrm{diag}\begin{pmatrix}z\\z\end{pmatrix}\right]\cdot\begin{pmatrix}\mathbf{1}_S\\\mathbf{1}_S\end{pmatrix},\begin{pmatrix}\mathbf{1}_S\\\mathbf{1}_S\end{pmatrix}\right\rangle\\ &= 2\langle M_U\mathbf{1}_S,\mathbf{1}_S\rangle - 2\sum_{v\in S} z_v = 2\langle M_U\mathbf{1}_S,\mathbf{1}_S\rangle - 2\sum_{v\in S} d_v y_v, \qquad (31)\end{aligned}$$

because $z = D_U y$. Further, Theorem 5 and Lemma 7 entail that

$$|\langle M_U\mathbf{1}_S,\mathbf{1}_S\rangle| \leq \|M_U\|_{\mathrm{cut}} \leq \mathrm{SDP}(M_U) \leq \mathrm{SDP}(M) \leq \varepsilon^2\mathrm{vol}(V)/64.$$

Plugging this bound into (31) and recalling that $y_v < 0$ for all $v \in S$, we conclude that

$$\sum_{v\in S} d_v|y_v| \leq (\varepsilon^2|S|d + \varepsilon^2\mathrm{vol}(V))/64 \leq \varepsilon^2\mathrm{vol}(V)/32. \qquad (32)$$

Combining (29) and (32), we get

$$\sum_{v\in U} d_v|y_v| \leq \varepsilon^2\mathrm{vol}(V)/16.$$

As $|y_v| \geq \varepsilon/8$ for all $v \in U \setminus W$ (by the definition of W), we thus obtain $\mathrm{vol}(U \setminus W) \leq \varepsilon\mathrm{vol}(V)/2$. Hence, (26) yields $\mathrm{vol}(V \setminus W) < \varepsilon\mathrm{vol}(V)$, as desired. □

4.3 Proof of Lemma 10

Let Q be a symmetric $n \times n$ matrix, and set $\mathcal{Q} = \frac{1}{2}\begin{pmatrix}0\,1\\1\,0\end{pmatrix}\otimes Q$. Furthermore, let

$$\mathrm{DSDP}(Q) = \min\,\langle\mathbf{1}, y\rangle \text{ s.t. } \mathcal{Q} \leq \mathrm{diag}(y), \quad y \in \mathbf{R}^{2n}.$$

Lemma 12. *We have* $\mathrm{SDP}(Q) = \mathrm{DSDP}(Q)$.

Proof. By Lemma 4 we can rewrite the vector program $\mathrm{SDP}(Q)$ in the standard form of a semidefinite program:

$$\mathrm{SDP}(Q) = \max\,\langle\mathcal{Q}, X\rangle \text{ s.t. } \mathrm{diag}(X) = \mathbf{1},\ X \geq 0,\ X \in \mathbf{R}^{(2n)\times(2n)}.$$

Since $\mathrm{DSDP}(Q)$ is the dual of $\mathrm{SDP}(Q)$, the lemma follows directly from the SDP duality theorem as stated in [23, Corollary 2.2.6]. □

To infer Lemma 10, we shall simplify DSDP and reformulate this semidefinite program as an eigenvalue minimization problem. First, we show that it suffices to optimize over $y \in \mathbf{R}^n$ rather than $y \in \mathbf{R}^{2n}$.

Lemma 13. *Let* $\mathrm{DSDP}'(Q) = \min 2\langle \mathbf{1}, y'\rangle$ *s.t.* $\mathcal{Q} \leq \mathrm{diag}\binom{1}{1} \otimes y',\ y' \in \mathbf{R}^n$. *Then* $\mathrm{DSDP}(Q) = \mathrm{DSDP}'(Q)$.

Proof. Since for any feasible solution y' to $\mathrm{DSDP}'(Q)$ the vector $y = \binom{1}{1} \otimes y'$ is a feasible solution to $\mathrm{DSDP}(Q)$, we conclude that $\mathrm{DSDP}(Q) \leq \mathrm{DSDP}'(Q)$. Thus, we just need to establish the converse inequality $\mathrm{DSDP}'(Q) \leq \mathrm{DSDP}(Q)$.

To this end, let $\mathcal{F}(Q) \subset \mathbf{R}^{2n}$ signify the set of all feasible solutions y to $\mathrm{DSDP}(Q)$. We shall prove that $\mathcal{F}(Q)$ is closed under the linear operator

$$\mathcal{I} : \mathbf{R}^{2n} \to \mathbf{R}^{2n}, \quad (y_1, \dots, y_n, y_{n+1}, \dots, y_{2n}) \mapsto (y_{n+1}, \dots, y_{2n}, y_1, \dots, y_n),$$

i.e., $\mathcal{I}(\mathcal{F}(Q)) \subset \mathcal{F}(Q)$; note that $\mathcal{I}$ just swaps the first and the last n entries of y. To see that this implies the assertion, consider an optimal solution $y = (y_i)_{1 \leq i \leq 2n} \in \mathcal{F}(Q)$. Then $\frac{1}{2}(y + \mathcal{I}y) \in \mathcal{F}(Q)$, because $\mathcal{F}(Q)$ is convex. Now, let $y' = (y'_i)_{1 \leq i \leq n}$ be the projection of $\frac{1}{2}(y + \mathcal{I}y)$ onto the first n coordinates. Since $\frac{1}{2}(y + \mathcal{I}y)$ is a fixed point of $\mathcal{I}$, we have $\frac{1}{2}(y + \mathcal{I}y) = \binom{1}{1} \otimes y'$. Hence, the fact that $\frac{1}{2}(y + \mathcal{I}y)$ is feasible to $\mathrm{DSDP}(Q)$ implies that y' is feasible to $\mathrm{DSDP}'(Q)$. Thus, we conclude that

$$\mathrm{DSDP}'(Q) \leq 2\langle \mathbf{1}, y'\rangle = \langle \mathbf{1}, y\rangle = \mathrm{DSDP}(Q).$$

To show that $\mathcal{F}(Q)$ is closed under $\mathcal{I}$, consider a vector $y \in \mathcal{F}(Q)$. Since $\mathrm{diag}(y) - \mathcal{Q}$ is positive semidefinite, we have

$$\forall \eta \in \mathbf{R}^{2n} : \langle (\mathrm{diag}(y) - \mathcal{Q})\eta, \eta\rangle \geq 0. \tag{33}$$

Furthermore, our objective is to show that $\mathrm{diag}(\mathcal{I}y) - \mathcal{Q}$ is positive semidefinite, i.e.,

$$\forall \xi \in \mathbf{R}^{2n} : \langle (\mathrm{diag}(\mathcal{I}y) - \mathcal{Q})\xi, \xi\rangle \geq 0. \tag{34}$$

To derive (34) from (33), we decompose y into its two halfs $y = \binom{u}{v}$ $(u, v \in \mathbf{R}^n)$. Then $\mathcal{I}y = \binom{v}{u}$. Moreover, let $\xi = \binom{\alpha}{\beta} \in \mathbf{R}^{2n}$ be any vector, and set $\eta = \mathcal{I}\xi = \binom{\beta}{\alpha}$. As Q is symmetric, we obtain

$$\langle (\mathrm{diag}(\mathcal{I}y) - \mathcal{Q})\xi, \xi\rangle = \langle \mathrm{diag}(v)\alpha, \alpha\rangle + \langle \mathrm{diag}(u)\beta, \beta\rangle - 2\langle Q\alpha, \beta\rangle = \langle (\mathrm{diag}(y) - \mathcal{Q})\eta, \eta\rangle \overset{(33)}{\geq} 0,$$

thereby proving (34). □

Proof of Lemma 10. Let

$$\mathrm{DSDP}''(Q) = n \cdot \min_{z \in \mathbf{R}^n,\, z \perp \mathbf{1}} \lambda_{\max}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes Q + \mathrm{diag}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes z\right].$$

By Lemmas 12 and 13, it suffices to prove that $\mathrm{DSDP}'(Q) = \mathrm{DSDP}''(Q)$.

To see that $\mathrm{DSDP}''(Q) \leq \mathrm{DSDP}'(Q)$, consider an optimal solution y' to $\mathrm{DSDP}'(Q)$. Let $\lambda = n^{-1}\langle \mathbf{1}, y'\rangle$ and $z = 2(\lambda\mathbf{1} - y')$. Then $\langle z, \mathbf{1}\rangle = 2(n\lambda - \langle \mathbf{1}, y'\rangle) = 0$, whence z is a feasible solution to $\mathrm{DSDP}''(Q)$. Furthermore, as y' is a feasible solution to $\mathrm{DSDP}'(Q)$, we have

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes Q = 2\mathcal{Q} \leq 2\mathrm{diag}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes y' = 2\lambda \boldsymbol{E} - \mathrm{diag}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes z,$$

where $\boldsymbol{E}$ is the identity matrix. Consequently, $\lambda_{\max}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes Q + \mathrm{diag}\binom{1}{1} \otimes z\right) \leq 2\lambda$, and thus

$$\mathrm{DSDP}''(Q) \leq n\lambda_{\max}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes Q + \mathrm{diag}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes z\right] \leq 2n\lambda = 2\langle \mathbf{1}, y'\rangle = \mathrm{DSDP}'(Q).$$

Conversely, consider an optimal solution z to $\mathrm{DSDP}''(Q)$. Set

$$\mu = \lambda_{\max}\left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\otimes Q - \mathrm{diag}\begin{pmatrix}1\\1\end{pmatrix}\otimes z\right] = n^{-1}\mathrm{DSDP}''(Q), \quad y' = \frac{1}{2}(\mu\mathbf{1}+z).$$

Then the definition of μ implies that $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\otimes Q - \mathrm{diag}\binom{1}{1}\otimes z \leq \mu\boldsymbol{E}$, whence

$$\mathcal{Q} = \frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\otimes Q \leq \frac{1}{2}\left(\mu\boldsymbol{E} + \mathrm{diag}\begin{pmatrix}1\\1\end{pmatrix}\otimes z\right) = \mathrm{diag}\begin{pmatrix}1\\1\end{pmatrix}\otimes y'.$$

Hence, y' is a feasible solution to $\mathrm{DSDP}'(Q)$. Furthermore, since $z \perp \mathbf{1}$ we obtain

$$\mathrm{DSDP}'(Q) \leq 2\langle \mathbf{1}, y'\rangle = \mu n = \mathrm{DSDP}''(Q),$$

as desired. □

5 The Algorithmic Regularity Lemma: Proof of Theorem 2

In this section we present a polynomial time algorithm Regularize that computes for a given graph $G = (V, E)$ a partition satisfying **REG1** and **REG2**, provided that G satisfies the assumptions of Theorem 2. In particular, this will show that such a partition exists and thus prove Theorem 2 We will outline Regularize in Section 5.1. The crucial ingredient is a subroutine Witness for checking whether a given pair (A, B) of subsets of V is ε-volume regular. This subroutine is the content of Section 5.2.

Throughout this section, we let $\varepsilon > 0$ be an arbitrarily small but fixed and $C > 0$ an arbitrarily large but fixed number. In addition, we define a sequence $(t_k)_{k\geq 1}$ by letting

$$t_1 = \lceil 2/\varepsilon \rceil \text{ and } t_{k+1} = t_k 2^{t_k}. \tag{35}$$

Moreover, let

$$k^* = \lceil C\varepsilon^{-3} \rceil,\ \eta = t_{k^*}^{-6}\varepsilon^{-8k^*}, \tag{36}$$

and choose $n_0 = n_0(C, \varepsilon) > 0$ big enough.

We always assume that $G = (V, E)$ is a graph on $n = |V| > n_0$ vertices that is (C, η)-bounded, and that $\mathrm{vol}(V) \geq \eta^{-1}n$.

5.1 The Algorithm `Regularize`

In order to compute the desired regular partition of its input graph G, the algorithm Regularize proceeds as follows. In its first step, Regularize computes any initial partition $\mathcal{P}^1 = \{V_i^1 : 0 \leq i \leq s_1\}$ such that each class V_i $(1 \leq i \leq s_1)$ has a decent volume.

Algorithm 14. Regularize(G)
Input: A graph $G = (V, E)$. ***Output:*** A partition of V.

1. Compute an initial partition $\mathcal{P}^1 = \{V_0^1 : 0 \leq i \leq s_1\}$ such that $\frac{1}{4}\varepsilon\mathrm{vol}(V) \leq \mathrm{vol}(V_i^1) \leq \frac{3}{4}\varepsilon\mathrm{vol}(V)$ for all $1 \leq i \leq s_1$; thus, $s_1 \leq 4\varepsilon^{-1}$. Set $V_0^1 = \emptyset$.

Then, in the subsequent steps, Regularize computes a sequence $\mathcal{P}^k$ of partitions such that $\mathcal{P}^{k+1}$ is a "more regular" refinement of $\mathcal{P}^k$ $(k \geq 1)$. As soon as Regularize can verify that $\mathcal{P}^k$ satisfies both **REG1** and **REG2**, the algorithm stops.

To check whether the current partition $\mathcal{P}^k = \{V_i^k : 1 \leq i \leq s_1\}$ satisfies **REG2**, Regularize employs the subroutine Witness (which is the subject of the next section). Given a pair (V_i^k, V_j^k), Witness tries to check whether (V_i^k, V_j^k) is ε-volume-regular. Recall that the relative density of $A, B \subset V$ is

$$\varrho(A, B) = \frac{e(A, B)}{\mathrm{vol}(A)\mathrm{vol}(B)}.$$

Lemma 15. *There is a polynomial time algorithm* `Witness` *that satisfies the following. Let* $A, B \subset V$ *be disjoint.*

1. *If* `Witness`(G, A, B) *answers "yes", then the pair* (A, B) *is* ε*-volume regular.*
2. *On the other hand, if the answer is "no", then* (A, B) *is not* $\varepsilon/200$*-volume regular. In this case* `Witness` *outputs a pair* (X^*, Y^*) *of subsets* $X^* \subset A$*,* $Y^* \subset B$ *such that* $\mathrm{vol}(X^*) \geq \frac{\varepsilon}{200}\mathrm{vol}(A)$*,* $\mathrm{vol}(Y^*) \geq \frac{\varepsilon}{200}\mathrm{vol}(B)$*, and* $|e(X^*, Y^*) - \varrho(A, B)\mathrm{vol}(X^*)\mathrm{vol}(Y^*)| > \frac{\varepsilon \mathrm{vol}(A)\mathrm{vol}(B)}{200\mathrm{vol}(V)}$.

We call a pair (X^*, Y^*) as in 2. an $\frac{\varepsilon}{200}$-*witness* for (A, B).

By applying `Witness` to each pair (V_i^k, V_j^k) of the partition $\mathcal{P}^k$, `Regularize` can single out a set $\mathcal{L}^k$ such that all pairs V_i, V_j with $(i, j) \notin \mathcal{L}^k$ are ε-volume regular. Hence, if

$$\sum_{(i,j)\in\mathcal{L}^k} \mathrm{vol}(V_i^k)\mathrm{vol}(V_j^k) < \varepsilon \mathrm{vol}(V)^2, \tag{37}$$

then $\mathcal{P}^k$ satisfies **REG2**. Indeed, if (37) holds, then `Regularize` stops and outputs the desired regular partition, as we will see below that by construction $\mathcal{P}^k$ satisfies **REG1** for all k.

2. For $k = 1, 2, 3, \ldots, k^*$ do
3. Initially, let $\mathcal{L}^k = \emptyset$.
   For each pair (V_i^k, V_j^k) $(i < j)$ of classes of the previously partition $\mathcal{P}^k$
4. call the procedure `Witness`$(G, V_i^k, V_j^k, \varepsilon)$.
   If it answers "no" and hence outputs an $\frac{\varepsilon}{200}$-witness (X_{ij}^k, X_{ji}^k) for (V_i^k, V_j^k), then add (i, j) to $\mathcal{L}^k$.
5. If $\sum_{(i,j)\in\mathcal{L}^k} \mathrm{vol}(V_i^k)\mathrm{vol}(V_j^k) < \varepsilon\mathrm{vol}(V)^2$, then output the partition $\mathcal{P}^k$ and halt.

If Step 5 does not halt, `Regularize` constructs a refinement $\mathcal{P}^{k+1}$ of $\mathcal{P}^k$. To this end, the algorithm decomposes each class V_i^k of $\mathcal{P}^k$ into up to 2^{s_k} pieces, where s_k is the number of classes of $\mathcal{P}^k$. Consider the sets X_{ij} with $(i, j) \in \mathcal{L}^k$ and define an equivalence relation $\equiv_i^k$ on V_i by letting $u \equiv_i^k v$ iff for all j such that $(i, j) \in \mathcal{L}_k$ it is true that $u \in X_{ij} \leftrightarrow v \in X_{ij}$. Thus, the equivalence classes of $\equiv_i^k$ are the regions of the Venn diagram of the sets V_i and X_{ij} with $(i, j) \in \mathcal{L}^k$. Then `Regularize` obtains $\mathcal{P}^{k+1}$ as follows.

6. Let $\mathcal{C}^k$ be the set of all equivalence classes of the relations $\equiv_i^k$ $(1 \leq i \leq s_k)$. Moreover, let $\mathcal{C}_*^k = \{V_1^{k+1}, \ldots, V_{s_{k+1}}^{k+1}\}$ be the set of all classes $W \in \mathcal{C}$ such that $\mathrm{vol}(W) > \varepsilon^{4(k+1)}\mathrm{vol}(V)/(15t_{k+1}^3)$. Finally, let $V_0^{k+1} = V_0^k \cup \bigcup_{W\in\mathcal{C}^k\setminus\mathcal{C}_*^k} W$, and set $\mathcal{P}^{k+1} = \{V_i^{k+1} : 0 \leq i \leq s_{k+1}\}$.

Since for each i there are at most s_k indices j such that $(i, j) \in \mathcal{L}^k$, in $\mathcal{P}^{k+1}$ every class V_i^k gets split into at most 2^{s_k} pieces. Hence, $s_{k+1} \leq s_k 2^{s_k}$. Thus, as $s_1 \leq t_1$, (35) implies that that $s_k \leq t_k$ for all k. Therefore, our choice (36) of η ensures that

$$\mathrm{vol}(V_i^{k+1}) \geq \eta\mathrm{vol}(V) \text{ for all } 1 \leq i \leq s_{k+1} \tag{38}$$

(because Step 6 puts all equivalence classes $W \in \mathcal{C}^k$ of "extremely small" volume into the exceptional class). Moreover, it is easily seen that $\mathrm{vol}(V_0^{k+1}) \leq \varepsilon\mathrm{vol}(V)$. In effect, $\mathcal{P}^{k+1}$ satisfies **REG1**.

Thus, to complete the proof of Theorem 2 it just remains to show that Step 5 of `Regularize` will actually output a partition $\mathcal{P}^k$ for some $k \leq k^*$. To show this, we define the *index* of a partition $\mathcal{P} = \{V_i : 0 \leq i \leq s\}$ as

$$\mathrm{ind}(\mathcal{P}) = \sum_{1\leq i<j\leq s} \varrho(V_i, V_j)^2\mathrm{vol}(V_i)\mathrm{vol}(V_j) = \sum_{1\leq i<j\leq s} \frac{e(V_i, V_j)^2}{\mathrm{vol}(V_i)\mathrm{vol}(V_j)}.$$

Note that we do *not* take into account the (exceptional) class V_0 here. Using the boundedness-condition, we derive the following.

Proposition 16. *If* $G = (V, E)$ *is a* (C, η)*-bounded graph and* $\mathcal{P} = \{V_i \colon 0 \leq 1 \leq t\}$ *is a partition of* V *with* $\mathrm{vol}(V_i) \geq \eta\mathrm{vol}(V)$ *for all* $i \in \{1, \ldots, t\}$*, then* $\mathrm{ind}(\mathcal{P}) \leq C$.

Proof. ¿From $\mathrm{vol}(V_i) \geq \eta \mathrm{vol}(V)$ we derive for all $i \in \{1, \dots, t\}$

$$\mathrm{ind}(\mathcal{P}) = \sum_{1 \leq i < j \leq s} \frac{e(V_i, V_j)^2}{\mathrm{vol}(V_i)\mathrm{vol}(V_j)} \leq \sum_{1 \leq i < j \leq s} \frac{Ce(V_i, V_j)}{\mathrm{vol}(V)} \leq C.$$

□

Proposition 16 and (38) entail that $\mathrm{ind}(\mathcal{P}^k) \leq C$ for all k. In addition, since `Regularize` obtains $\mathcal{P}^{k+1}$ by refining $\mathcal{P}^k$ according to the witnesses of irregularity computed by `Witness`, the index of $\mathcal{P}^{k+1}$ is actually considerably larger than the index of $\mathcal{P}^k$. More precisely, the following is true.

Lemma 17. *If* $\sum_{(i,j) \in \mathcal{L}^k} \mathrm{vol}(V_i^k)\mathrm{vol}(V_j^k) \geq \varepsilon \mathrm{vol}(V)^2$, *then* $\mathrm{ind}(\mathcal{P}^{k+1}) \geq \mathrm{ind}(\mathcal{P}^k) + \varepsilon^3/8$.

To prove the Lemma 17 we follow the lines of the original proof of Szemerédi [21]. First we need the following observation.

Proposition 18. *Let* $\mathcal{P}' = \{V_j' \colon 0 \leq j \leq s\}$ *and* $\mathcal{P} = \{V_i \colon 0 \leq i \leq t\}$ *be two partitions of* V. *If* $\mathcal{P}'$ *refines* $\mathcal{P}$ *then* $\mathrm{ind}(\mathcal{P}') \geq \mathrm{ind}(\mathcal{P})$.

Proof. For $V_i \in \mathcal{P}$, $i \in [t]$ let $I_i = \{j \colon V_j' \in \mathcal{P}', V_j' \subset V_i\}$. Then, using the Cauchy-Schwarz-inequality, we conclude

$$\begin{aligned}
\mathrm{ind}(\mathcal{P}') = \sum_{1 \leq i < j \leq s} \frac{e^2(V_i', V_j')}{\mathrm{vol}(V_i')\mathrm{vol}(V_j')} &\leq \sum_{1 \leq k < l \leq t} \sum_{\substack{i \in I_k \\ j \in I_l}} \frac{e^2(V_i', V_j')}{\mathrm{vol}(V_i')\mathrm{vol}(V_j')} \\
&\geq \sum_{1 \leq k < l \leq t} \frac{\left(\sum_{\substack{i \in I_k \\ j \in I_l}} e(V_i', V_j')\right)^2}{\sum_{\substack{i \in I_k \\ j \in I_l}} \mathrm{vol}(V_i')\mathrm{vol}(V_j')} = \sum_{1 \leq k < l \leq t} \frac{e^2(V_k, V_l)}{\mathrm{vol}(V_k)\mathrm{vol}(V_l)} = \mathrm{ind}(\mathcal{P}).
\end{aligned}$$

□

Furthermore the proof will use the following defect-form of the Cauchy-Schwarz-Lemma.

Lemma 19 (Defect form of Cauchy-Schwarz-inequality). *For all* $i \in I$ *let* σ_i, d_i *be positive real numbers satisfying* $\sum_{i \in I} \sigma_i = 1$. *Furthermore let* $J \subset I$, $\varrho = \sum_{i \in I} \sigma_i \varrho_i$ *and* $\sigma_J = \sum_{j \in J} \sigma_j$. *If*

$$\sum_{j \in J} \sigma_j \varrho_j = \sigma_J(\varrho + \nu)$$

then

$$\sum_{i \in I} \sigma_i \varrho_i^2 \geq \varrho^2 + \nu^2 \sigma_J.$$

Lastly, for technical reasons we state the following proposition. Its proof is straightforward and we omit it here.

Proposition 20. *Let* $1/5 > \delta > 0$, $G = (V, E)$ *and* $A, B \subset V$ *be disjoint subsets of* V. *Furthermore let* $A' \subset A$ *and* $B' \subset B$ *with* $\mathrm{vol}(A \setminus A') < \delta \mathrm{vol}(A)$ *and* $\mathrm{vol}(B \setminus B') < \delta \mathrm{vol}(B)$. *Then the following inequalities hold*

$$\left| \frac{e(A, B)}{\mathrm{vol}(A)\mathrm{vol}(B)} - \frac{e(A', B')}{\mathrm{vol}(A')\mathrm{vol}(B')} \right| \leq \frac{5\delta}{\min\{\mathrm{vol}(A), \mathrm{vol}(B)\}} \tag{39}$$

$$\left| \frac{e^2(A, B)}{\mathrm{vol}(A)\mathrm{vol}(B)} - \frac{e^2(A', B')}{\mathrm{vol}(A')\mathrm{vol}(B')} \right| \leq 15\delta. \tag{40}$$

Proof of the Lemma 17. Without loss of generality we assume $\varepsilon \leq 1/8$. Moreover, we let $K \subset V$ be the union of the equivalence classes with a negligible volume size, more precisely

$$K = \bigcup_{W \in \mathcal{C}_*^k \setminus \mathcal{C}^k} \mathrm{vol}(W) = \bigcup \left\{ W \in \mathcal{C}^k : \ \mathrm{vol}(W) \leq \frac{\varepsilon^4 (k+1) \mathrm{vol}(V)}{15 t_{k+1}^3} \right\}.$$

Now let $\mathcal{P}' = \{V_i' : \ 0 \leq i \leq s_k\}$ be an auxiliary partition given by

$$V_i' = \begin{cases} V_0^k \cup K & \text{if } i = 0, \\ V_i^k \setminus K & \text{otherwise.} \end{cases}$$

To show the index increment $\mathrm{ind}(\mathcal{P}^{k+1}) \geq \mathrm{ind}(\mathcal{P}^k) + \varepsilon^3/8$ we will proceed in two steps. In the first step we will compare the index of $\mathcal{P}'$ to the index of $\mathcal{P}^k$. This will yield the following.

Claim 1 $|\mathrm{ind}(\mathcal{P}^k) - ind(\mathcal{P}')| \leq \varepsilon^4$.

The second step will reveal the index increment of $\mathcal{P}^{k+1}$ compared to $\mathcal{P}'$.

Claim 2 $\mathrm{ind}(\mathcal{P}^{k+1}) \geq \mathrm{ind}(\mathcal{P}) + \varepsilon^3/4$.

Together, with $\varepsilon \leq 1/8$, this yields an index increment

$$\mathrm{ind}(\mathcal{P}^{k+1}) \geq \mathrm{ind}(\mathcal{P}^k) + \varepsilon^3/8.$$

□

Proof of Claim 1. Let (V_i^k, V_j^k) be a pair of partition classes of $\mathcal{P}^k$ and let $V_i' = V_i^k \setminus K$ and $V_j' = V_j^k \setminus K$. Note that $\mathrm{vol}(V_i^k) \geq \varepsilon^{4k} \mathrm{vol}(V)/15 t_k^3$. Thus we have

$$\mathrm{vol}(V_i') \geq \mathrm{vol}(V_i^k) - \mathrm{vol}(K) \geq \mathrm{vol}(V_i^k) - \varepsilon^4 \left(\frac{\varepsilon^{4k}}{15} \frac{\mathrm{vol}(G)}{t_{k+1}^2} \right) \geq \left(1 - \frac{\varepsilon^4}{15 t_k^2} \right) \mathrm{vol}(V_i^k).$$

Analogously $\mathrm{vol}(V_j') \geq \left(1 - \varepsilon^4/(15 t_k^2)\right) \mathrm{vol}(V_j^k)$ holds. In effect, using the Proposition 20 we get

$$\left| \frac{e^2(V_i', V_j')}{\mathrm{vol}(V_i')\mathrm{vol}(V_j')} - \frac{e^2(V_i^k, V_j^k)}{\mathrm{vol}(V_i^k)\mathrm{vol}(V_j^k)} \right| \leq \frac{\varepsilon^4}{t_k^2}.$$

Consequently

$$|\mathrm{ind}(\mathcal{P}^k) - \mathrm{ind}(\mathcal{P}')| \leq \sum_{1 \leq i < j \leq s_k} \left| \frac{e^2(V_i^k, V_j^k)}{\mathrm{vol}(V_i^k)\mathrm{vol}(V_j^k)} - \frac{e^2(V_i', V_j')}{\mathrm{vol}(V_i')\mathrm{vol}(V_j')} \right| \leq \varepsilon^4.$$

□

Proof of Claim 2. Let (V_i^k, V_j^k) be an irregular pair and $(A, B) = (V_i^k \setminus K, V_j^k \setminus K)$. Furthermore let (X_{ij}^k, X_{ji}^k) be the witness of irregularity. Then, for $X = X_{ij}^k \setminus K \subset A$ and $Y = X_{ji}^k \setminus K \subset B$, we have

$$\begin{aligned} \left| \frac{e(X,Y)}{\mathrm{vol}(X)\mathrm{vol}(Y)} - \frac{e(A,B)}{\mathrm{vol}(A)\mathrm{vol}(B)} \right| &= \varepsilon \frac{\mathrm{vol}(A)\mathrm{vol}(B)}{\mathrm{vol}(X_{ij}^k)\mathrm{vol}(X_{ji}^k)\mathrm{vol}(G)} - \frac{10\varepsilon^4}{t_{k+1}\mathrm{vol}(B)} \\ &\geq \frac{\varepsilon}{2} \frac{\mathrm{vol}(A)\mathrm{vol}(B)}{\mathrm{vol}(X)\mathrm{vol}(Y)\mathrm{vol}(G)} \end{aligned}$$

due to Proposition 20. Thus, (X, Y) witnesses that (A, B) is not $\varepsilon/2$-volume-regular.

Now we will use the Lemma 19 to prove $\mathrm{ind}(\mathcal{P}^{k+1}) \geq \mathrm{ind}(\mathcal{P}') + \varepsilon^3/4$. So let $I = (A \times B)$ and for all $(u, v) \in I$ let

$$\sigma_{uv} = \frac{\deg(u)\deg(v)}{\mathrm{vol}(A)\mathrm{vol}(B)} \quad \text{and} \quad d_{uv} = \varrho(V^{k+1}(u), V^{k+1}(y))$$

where $V^{k+1}(x)$ denote the partition class $V_i^{k+1} \in \mathcal{P}^{k+1}$ such that $x \in V_i^{k+1}$. Then

$$\sum_{(u,v)\in I} \sigma_{uv} = 1 \quad \text{and} \quad d = \sum_{(u,v)\in I} \sigma_{uv} d_{uv} = \varrho(A,B).$$

Moreover, let $J = (X \times Y)$ and $\sigma_J = \sum_{(u,v)\in J} \sigma_{uv} = \frac{\mathrm{vol}(X)\mathrm{vol}(Y)}{\mathrm{vol}(A)\mathrm{vol}(B)}$. Then we have

$$\begin{aligned}\frac{1}{\sigma_J} \sum_{(u,v)\in J} \sigma_{uv} d_{uv} &= \frac{\mathrm{vol}(A)\mathrm{vol}(B)}{\mathrm{vol}(X)\mathrm{vol}(Y)} \sum_{\substack{V_i^{k+1}\subset A \\ V_j^{k+1}\subset B}} \sum_{\substack{u\in V_i^{k+1} \\ v\in V_j^{k+1}}} \frac{\deg(u)\deg(v)}{\mathrm{vol}(A)\mathrm{vol}(B)} \varrho(V_i^{k+1}, V_j^{k+1}) \\ &= \frac{e(X,Y)}{\mathrm{vol}(X)\mathrm{vol}(Y)} = \varrho(X,Y) = \varrho(A,B) + \nu\end{aligned}$$

for some $|\nu| \geq \frac{\varepsilon}{2} \frac{\mathrm{vol}(A)\mathrm{vol}(B)}{\mathrm{vol}(X)\mathrm{vol}(Y)\mathrm{vol}(G)}$ due to (41).

Hence, from the Cauchy-Schwarz-inequality (Lemma 19) we deduce

$$\sum_{(u,v)\in I} \sigma_{uv} d_{uv}^2 = \sum_{u,v\in I} \frac{\deg(u)\deg(v)}{\mathrm{vol}(A)\mathrm{vol}(B)} \varrho^2(V^{k+1}(u), V^{k+1}(v)) \tag{41}$$

$$= \frac{1}{\mathrm{vol}(A)\mathrm{vol}(B)} \sum_{\substack{V_i^{k+1}\subset A \\ V_j^{k+1}\subset B}} \varrho^2(V_i^{k+1}, V_j^{k+1}) \mathrm{vol}(V_i^{k+1}) \mathrm{vol}(V_j^{k+1}) \tag{42}$$

$$\geq \varrho^2(A,B) + \left(\frac{\varepsilon \mathrm{vol}(A)\mathrm{vol}(B)}{2\mathrm{vol}(X)\mathrm{vol}(Y)\mathrm{vol}(G)}\right)^2 \times \frac{\mathrm{vol}(X)\mathrm{vol}(Y)}{\mathrm{vol}(A)\mathrm{vol}(B)} \tag{43}$$

$$\geq \frac{1}{\mathrm{vol}(A)\mathrm{vol}(B)} \left(\mathrm{ind}(A,B) + \frac{\varepsilon^2 \mathrm{vol}(A)\mathrm{vol}(B)}{4\mathrm{vol}^2(G)}\right). \tag{44}$$

¿From (42) and (44) we infer the amount of the index increment on the irregular pair (A,B). So, summing over all irregular pairs we get

$$\mathrm{ind}(\mathcal{P}^{k+1}) - \mathrm{ind}(\mathcal{P}') \geq \sum_{(i,j)\in L} \frac{\varepsilon^2}{4} \frac{\mathrm{vol}(A)\mathrm{vol}(B)}{\mathrm{vol}^2(G)} - \varepsilon^4 \geq \frac{\varepsilon^3}{4}.$$

□

Since the index of the initial partition $\mathcal{P}^1$ is non-negative, Propositions 16 and Lemma 17 readily imply that `Regularize` will terminate and output a feasible partition $\mathcal{P}^k$ for some $k < k^*$.

Finally, we point out that the overall running time of `Regularize` is polynomial. For the running time of Steps 1–3 and 5–6 is $O(\mathrm{vol}(V))$, and the running time of Step 4 is polynomial due to Lemma 15.

5.2 The Procedure `Witness`: Proof of Lemma 15

The subroutine `Witness` for Lemma 15 employs the algorithm `ApxCutNorm` from Theorem 6 for approximating the cut norm as follows.

Algorithm 21. `Witness`(G,A,B)
Input: A graph $G = (V,E)$, disjoint sets $A, B \subset V$, and a number $\varepsilon > 0$.
Output: A partition of V.

1. Set up the matrix $M = (m_{vw})_{(v,w)\in A\times B}$ with entries $m_{vw} = 1 - \varrho(A,B)d_v d_w$ if v, w are adjacent in G, and $m_{vw} = -\varrho(A,B)d_v d_w$ otherwise.
   Call `ApxCutNorm`(M) to compute sets $X \subset A$, $Y \subset B$ such that $|\langle M\mathbf{1}_X, \mathbf{1}_Y\rangle| \geq \frac{3}{100}\|M\|_{\mathrm{cut}}$.

2. If $|\langle M\mathbf{1}_X, \mathbf{1}_Y\rangle| < \frac{3\varepsilon}{100}\frac{\mathrm{vol}(A)\mathrm{vol}(B)}{\mathrm{vol}(G)}$, then return "yes".
3. Otherwise, pick an arbitrary set $X' \subset A \setminus X$ of volume $\frac{3\varepsilon}{100}\mathrm{vol}(A) \le \mathrm{vol}(X')$.

   - If $\mathrm{vol}(X) \ge \frac{3\varepsilon}{100}\mathrm{vol}(A)$, then let $X^* = X$.
   - If $\mathrm{vol}(X) < \frac{3\varepsilon}{100}\mathrm{vol}(A)$ and $|e(X', Y) - \varrho(A, B)\mathrm{vol}(X')\mathrm{vol}(Y)| > \frac{\varepsilon\mathrm{vol}(A)\mathrm{vol}(B)}{100\mathrm{vol}(V)}$, set $X^* = X'$.
   - Otherwise, set $X^* = X \cup X'$.

4. Pick a further set $Y' \subset B \setminus Y$ of volume $\frac{\varepsilon}{200}\mathrm{vol}(B) \le \mathrm{vol}(Y')$.

   - If $\mathrm{vol}(Y) \ge \frac{\varepsilon}{200}\mathrm{vol}(B)$, then let $Y^* = Y$.
   - If $\mathrm{vol}(Y) < \frac{\varepsilon}{200}\mathrm{vol}(B)$ and $|e(X^*, Y') - \varrho(A, B)\mathrm{vol}(X^*)\mathrm{vol}(Y')| > \frac{\varepsilon\mathrm{vol}(A)\mathrm{vol}(B)}{200\mathrm{vol}(V)}$, let $Y^* = Y'$.
   - Otherwise, set $Y^* = Y \cup Y'$.

5. Answer "no" and output (X^*, Y^*) as an $\varepsilon/200$-witness.

Proof of Lemma 15. Note that for any two subsets $S \subset A$ and $T \subset B$ we have

$$\langle M\mathbf{1}_S, \mathbf{1}_T\rangle = e(S, T) - \varrho(A, B)\mathrm{vol}(S)\mathrm{vol}(T).$$

Therefore, if the sets $X \subset A$ and $Y \subset B$ computed by `ApxCutNorm` are such that

$$|\langle M\mathbf{1}_X, \mathbf{1}_Y\rangle| < \frac{3\varepsilon}{100}\frac{\mathrm{vol}(A)\mathrm{vol}(B)}{\mathrm{vol}(G)}$$

then by Theorem 6 we have

$$|e(S, T) - \varrho(A, B)\mathrm{vol}(S)\mathrm{vol}(T)| \le \|M\|_{\mathrm{cut}} \le \frac{100}{3}|\langle M\mathbf{1}_X, \mathbf{1}_Y\rangle| < \varepsilon\frac{\mathrm{vol}(A)\mathrm{vol}(B)}{\mathrm{vol}(G)}$$

for all $S \subset A$ and $T \subset B$. Thus, if `Witness` answers "yes" then the pair (A, B) is ε-volume regular.

One the other hand, if `ApxCutNorm` yields sets X, Y such that $\langle M\mathbf{1}_X, \mathbf{1}_Y\rangle \ge \frac{3\varepsilon}{100}\frac{\mathrm{vol}(A)\mathrm{vol}(B)}{\mathrm{vol}(G)}$ then `Witness` has to guarantee that the output pair (X^*, Y^*) is an $\varepsilon/200$-witness.

Indeed, if $\mathrm{vol}(X) \ge \frac{3\varepsilon}{100}\mathrm{vol}(A)$ and $\mathrm{vol}(Y) \ge \frac{\varepsilon}{200}\mathrm{vol}(B)$ then (X, Y) actually is an $\varepsilon/200$-witness. However, as `ApxCutNorm` does not guarantee any lower bound on $\mathrm{vol}(X)$ and $\mathrm{vol}(Y)$ let assume first that $\mathrm{vol}(X) < \frac{3\varepsilon}{100}\mathrm{vol}(A)$ and $\mathrm{vol}(Y) \ge \frac{\varepsilon}{200}\mathrm{vol}(B)$. Then according to step 3. `Witness` picks a set $X' \subset A \setminus X$ of volume $\mathrm{vol}(X') \ge \frac{3}{100}\mathrm{vol}(A)$. If X' itself satisfies $|e(X', Y) - \varrho(A, B)\mathrm{vol}(X')\mathrm{vol}(Y)| > \frac{\varepsilon\mathrm{vol}(A)\mathrm{vol}(B)}{100\mathrm{vol}(V)}$ then (X', Y) obviously is an $\varepsilon/200$-witness. Otherwise, by triangle inequality, we deduce

$$\left|e(X \cup X', Y) - e(A, B)\frac{\mathrm{vol}(X \cup X')\mathrm{vol}(Y)}{\mathrm{vol}(A)\mathrm{vol}(B)}\right| \ge \frac{2\varepsilon}{100}\frac{\mathrm{vol}(A)\mathrm{vol}(B)}{\mathrm{vol}(G)}$$

and thus, $(X \cup X', Y)$ is an $\varepsilon/200$-witness.

In the case $\mathrm{vol}(X) < \frac{3\varepsilon}{100}\mathrm{vol}(A)$ and $\mathrm{vol}(Y) < \frac{\varepsilon}{200}\mathrm{vol}(B)$ we simply repeat the argument for Y, and hence `Witness` outputs an $\varepsilon/200$-witness for (A, B). □

6 An Application: MAX CUT

As an application of Theorem 2 and, in particular, the polynomial time algorithm `Regularize` for computing a regular partition, we obtain the following algorithm for approximating the max cut of a graph $G = (V, E)$ that satisfies the assumptions of Theorem 3.

Algorithm 22. `ApxMaxCut`(G)
Input: A (C, η)-bounded graph $G = (V, E)$ and $\delta > 0$.
Output: A cut $(S, \bar{S})$ of G that approximates the maximum cut of G within a factor of $1 - \delta$.

1. Use `Regularize` to compute an $\varepsilon = \frac{\delta}{400C}$-volume regular partition $\mathcal{P} = \{V_i : 0 \le i \le t\}$ of G.
2. Determine an optimal solution $(c_1^*, \ldots, c_t^*)$ to the optimization problem

$$\max \sum_{i \ne j} \varepsilon c_i (1 - \varepsilon c_j) e(V_i, V_j) \text{ s.t. } \forall 1 \le j \le t : 0 \le c_j \le \varepsilon^{-1},\ c_j \in \mathbb{Z}.$$

3. For each $1 \le i \le t$ let $S_i \subset V_i$ be a subset such that $|\text{vol}(S_i) - c_i^* \varepsilon \text{vol}(V_i)| \le 2\varepsilon \text{vol}(V_i)$. Output $S = \bigcup_{i=1}^t S_i$ and $\bar{S} = V \setminus S$.

The basic insight behind `ApxMaxCut` is the following. If (V_i, V_j) is an ε-volume regular pair of $\mathcal{P}$, then for any subsets $X, X' \subset V_i$ and $Y, Y' \subset V_j$ such that $\text{vol}(X) = \text{vol}(X')$ and $\text{vol}(Y) = \text{vol}(Y')$ the condition **REG2** ensures that $|e(X,Y) - e(X',Y')| \le \frac{2\varepsilon \text{vol}(V_i)\text{vol}(V_j)}{\text{vol}(V)}$. That is, the difference between $e(X,Y)$ and $e(X',Y')$ is negligible. In other words, as far as the number of edges is concerned, subsets that have the same volume are "interchangeable".

Therefore, to compute a good cut $(S, \bar{S})$ of G we just have to optimize the *proportion of volume* of each V_i that is to be put into S or into $\bar{S}$, but it does not matter which subset of V_i of this volume we choose. However, determining the optimal fraction of volume is still a somewhat involved (essential continuous) optimization problem. Hence, in order to discretize this problem, we chop each V_i into at most ε^{-1} chunks of volume $\varepsilon \text{vol}(V_i)$. Then, we just have to determine the number c_i of chunks of each V_i that we join to S. This is exactly the optimization problem detailed in Step 2 of `ApxMaxCut`.

Observe that the time required to solve this problem is *independent* of n, i.e., Step 2 has a *constant* running time. For the number t of classes of $\mathcal{P}$ is bounded by a number independent of n, and the number $\lceil \varepsilon^{-1} \rceil + 1$ of choices for each c_i does not depend on n either. In addition, Step 3 can be implemented so that it runs in linear time, because $S_i \subset V_i$ can be *any* subset that satisfies the volume condition stated in Step 3. Thus, the total running time of `ApxMaxCut` is polynomial.

To prove that `ApxMaxCut` does indeed guarantee an approximation ratio of $1 - \delta$, we compare the maximum cut of G with the optimal solution μ^* of the optimization problem from Step 2, i.e.,

$$\mu^* = \max \sum_{i,j} \varepsilon c_i (1 - \varepsilon c_j) e(V_i, V_j) \text{ s.t. } \forall 1 \le j \le t : 0 \le c_j \le \varepsilon^{-1},\ c_j \in \mathbb{Z}. \tag{45}$$

To this end, we say that a cut $(T, \bar{T})$ of G is *compatible* with a feasible solution $(c_1, \ldots, c_t)$ to the optimization problem (45) if $|\text{vol}(T \cap V_i) - c_i \varepsilon \text{vol}(V_i)| \le 2\varepsilon \text{vol}(V_i)$.

Lemma 23. *Suppose that $(T, \bar{T})$ is compatible with the feasible solution $(c_1, \ldots, c_t)$ of (45). Moreover, let*

$$\mu = \sum_{i,j} \varepsilon c_i (1 - \varepsilon c_j) e(V_i, V_j)$$

be the objective function value corresponding to $(c_1, \ldots, c_t)$. Then $|e(T, \bar{T}) - \mu| \le \frac{\delta}{8} \text{vol}(V)$.

Proof. Set $T_i = T \cap V_i$ and $\bar{T}_i = V_i \setminus T_i$, so that $e(T, \bar{T}) = \sum_{i \ne j} e(T_i, \bar{T}_j) + \sum_{i=0}^t e(T_i, \bar{T}_i)$, and let $\mu_{ij} = \varepsilon c_i (1 - \varepsilon c_j) e(V_i, V_j)$ $(1 \le i, j \le t)$. Moreover, let $\mathcal{L}$ be the set of all pairs (i,j) such that the pair (V_i, V_j) is not ε-volume-regular. Then **REG 2** and the (C, η)-boundedness of G imply that

$$\sum_{(i,j) \in \mathcal{L}} \mu_{ij} \le \sum_{(i,j) \in \mathcal{L}} e(V_i, V_j) \le \sum_{(i,j) \in \mathcal{L}} \frac{C \text{vol}(V_i) \text{vol}(V_j)}{\text{vol}(V)} \le C\varepsilon \text{vol}(V) = \frac{\delta}{400} \text{vol}(V), \tag{46}$$

$$\sum_{(i,j) \in \mathcal{L}} e(T_i, \bar{T}_j) \le \sum_{(i,j) \in \mathcal{L}} e(V_i, V_j) \le \frac{\delta}{400} \text{vol}(V). \tag{47}$$

Furthermore, since $\text{vol}(V_0) \le \varepsilon \text{vol}(V)$ and $C \ge 1$ we have

$$e(T_0, \bar{T}) + e(\bar{T}_0, T) \le \text{vol}(V_0) \le \varepsilon \text{vol}(V) \le \frac{\delta}{400} \text{vol}(V), \tag{48}$$

and as $\mathrm{vol}(V_i) \le \varepsilon \mathrm{vol}(V)$ for all i, the (C,η)-boundedness condition yields

$$\sum_{i=1}^{t} e(T_i, \bar{T}_i) \le \sum_{i=1}^{t} \frac{C\mathrm{vol}(V_i)^2}{\mathrm{vol}(V)} \le C\varepsilon \mathrm{vol}(V) = \frac{\delta}{400}\mathrm{vol}(V). \tag{49}$$

In addition, let

$$\mathcal{S} = \{(i,j) : i,j > 0,\ i \ne j \wedge (i,j) \notin \mathcal{L} \wedge (\mathrm{vol}(T_i) < \varepsilon\mathrm{vol}(V_i) \vee \mathrm{vol}(\bar{T}_j) < \varepsilon\mathrm{vol}(V_j))\}.$$

We shall prove below that

$$\left|\mu_{ij} - e(T_i, \bar{T}_j)\right| < \frac{\delta}{10} e(V_i, V_j) \qquad \text{for all } (i,j) \notin (\mathcal{L} \cup \mathcal{S}),\ i,j > 0,\ i \ne j, \text{ and} \tag{50}$$

$$\sum_{(i,j)\in\mathcal{S}} \mu_{ij} + e(T_i, \bar{T}_j) < 6\varepsilon\mathrm{vol}(V). \tag{51}$$

Combining (46)–(51), we thus obtain

$$\begin{aligned}
&\left|e(T,\bar{T}) - \mu\right| \\
&\quad \le \sum_{\substack{(i,j)\notin(\mathcal{L}\cup\mathcal{S}) \\ i,j>0,\, i\ne j}} \left|\mu_{ij} - e(T_i,\bar{T}_j)\right| + \sum_{(i,j)\in(\mathcal{L}\cup\mathcal{S})} (\mu_{ij} + e(T_i, T_j)) + e(T_0, \bar{T}) + e(\bar{T}_0, T) + \sum_{i=1}^{t} e(T_i, \bar{T}_i) \\
&\qquad \le \frac{\delta}{10}\mathrm{vol}(V) + \frac{\delta}{200}\mathrm{vol}(V) + 6\varepsilon\mathrm{vol}(V) + \frac{\delta}{400}\mathrm{vol}(V) + \frac{\delta}{400}\mathrm{vol}(V) \le \frac{\delta}{8}\mathrm{vol}(V),
\end{aligned}$$

as desired.

To establish (50), consider a pair $(i,j) \notin (\mathcal{L} \cup \mathcal{S})$, $i \ne j$. Since $\mathrm{vol}(T_i) \ge \varepsilon\mathrm{vol}(V_i)$ and $\mathrm{vol}(\bar{T}_j) \ge \varepsilon\mathrm{vol}(V_j)$ and (V_i, V_j) is ε-volume-regular, we have

$$\left| e(T_i, \bar{T}_j) - \frac{\mathrm{vol}(T_i)\mathrm{vol}(\bar{T}_j)}{\mathrm{vol}(V_i)\mathrm{vol}(V_j)} e(V_i, V_j) \right| < \frac{\varepsilon\mathrm{vol}(V_i)\mathrm{vol}(V_j)}{\mathrm{vol}(V)}. \tag{52}$$

Moreover, as $(T, \bar{T})$ is compatible with $(c_1, \ldots, c_t)$,

$$\left| \frac{\mathrm{vol}(T_i)}{\mathrm{vol}(V_i)} - \varepsilon c_i \right| < 2\varepsilon, \qquad \left| \frac{\mathrm{vol}(\bar{T}_j)}{\mathrm{vol}(V_j)} - (1 - \varepsilon c_j) \right| < 2\varepsilon, \tag{53}$$

and combining (52) and (53) yields (50).

Finally, to prove (51), consider an index i such that $\mathrm{vol}(T_i) < \varepsilon\mathrm{vol}(V_i)$. Then $\sum_{j=1}^{t} e(T_i, \bar{T}_j) \le \mathrm{vol}(T_i) < \varepsilon\mathrm{vol}(V_i)$. Similarly, if $\mathrm{vol}(\bar{T}_j) < \varepsilon\mathrm{vol}(V_j)$, then $\sum_{i=1}^{t} e(T_i, \bar{T}_j) < \varepsilon\mathrm{vol}(V_j)$. Therefore,

$$\sum_{(i,j)\in\mathcal{S}} e(T_i, \bar{T}_j) < 2\varepsilon\mathrm{vol}(V). \tag{54}$$

Further, if $\mathrm{vol}(T_i) < \varepsilon\mathrm{vol}(V_i)$, then $c_i \le 2$, because $(T, \bar{T})$ is compatible with $(c_1, \ldots, c_t)$. Thus $\sum_{j=1}^{t} \mu_{ij} \le 2\varepsilon \sum_j e(V_i, V_j) \le 2\varepsilon\mathrm{vol}(V_i)$. Analogously, if $\mathrm{vol}(\bar{T}_j) < \varepsilon\mathrm{vol}(V_j)$, then $\sum_{i=1}^{t} \mu_{ij} \le 2\varepsilon\mathrm{vol}(V_j)$. Consequently,

$$\sum_{(i,j)\in\mathcal{S}} \mu_{ij} < 4\varepsilon\mathrm{vol}(V). \tag{55}$$

Hence, (51) follows from (54) and (55). □

Proof of Theorem 3. Step 3 of `ApxMaxCut` ensures that $(S, \bar{S})$ is compatible with $(c_1^*, \ldots, c_t^*)$. Therefore, Lemma 23 yields

$$e(S, \bar{S}) \geq \mu^* - \frac{\delta}{8}\mathrm{vol}(V). \tag{56}$$

Further, let $(T, \bar{T})$ be a maximum cut of G. Then we can construct a feasible solution to (45) that is compatible with $(T, \bar{T})$ by letting

$$c_i = \left\lfloor \frac{\mathrm{vol}(T \cap V_i)}{\varepsilon \mathrm{vol}(V_i)} \right\rfloor \qquad (1 \leq i \leq t).$$

Let $\mu = \sum_{i,j} \varepsilon c_i(1 - \varepsilon c_j)e(V_i, V_j)$ be the corresponding objective function value. Then Lemma 23 entails that

$$e(T, \bar{T}) \leq \mu + \frac{\delta}{8}\mathrm{vol}(V). \tag{57}$$

As μ^* is the optimal value of (45), we have $\mu^* \geq \mu$, and thus (56) and (57) yield $e(S, \bar{S}) \geq e(T, \bar{T}) - \frac{\delta}{4}\mathrm{vol}(V) \geq (1 - \delta)e(T, \bar{T})$. Consequently, `ApxMaxCut` provides the desired approximation guarantee. □

7 Conclusion

1. Theorem 1 states that $\mathrm{Disc}(\gamma\varepsilon^2)$ implies $\mathrm{ess\text{-}Eig}(\varepsilon)$, where $\gamma > 0$ is a constant. This statement is best possible, up to the precise value of γ. To see this, we describe a (probabilistic) construction of a graph $G = (V, E)$ on n vertices that has $\mathrm{Disc}(100\varepsilon)$ but does not have $\mathrm{ess\text{-}Eig}(0.01\sqrt{\varepsilon})$. Assume that $\varepsilon > 0$ is a sufficiently small number, and choose $n = n(\varepsilon)$ sufficiently large. Moreover, let $X = \{1, \ldots, \sqrt{\varepsilon}n\}$ and $\bar{X} = \{\sqrt{\varepsilon}n + 1, \ldots, n\}$. Set $x = \sqrt{\varepsilon}n$ and $\bar{x} = (1 - \sqrt{\varepsilon})n$. Further, let $d = n^{1/4}$ and set

$$p_X = \frac{2d}{n}, \quad p_{X\bar{X}} = p_{\bar{X}X} = \frac{d}{n} \cdot \frac{1 - 2\sqrt{\varepsilon}}{1 - \sqrt{\varepsilon}}, \quad p_{\bar{X}} = \frac{d}{n} \cdot \frac{1 - 2\sqrt{\varepsilon} + 2\varepsilon}{(1 - \sqrt{\varepsilon})^2}.$$

   Finally, let G be the random graph with vertex set $V = \{1, \ldots, n\}$ obtained as follows: any two vertices in X are connected with probability p_X independently; any two vertices in $\bar{X}$ are connected with probability $p_{\bar{X}}$ independently; and each possible X-$\bar{X}$ edge is present with probability $p_{X\bar{X}}$ independently. Then the expected degree of each vertex is d. Moreover, the expected number of neighbors that a vertex $v \in X$ has inside of X equals $\sqrt{\varepsilon}p_X n = 2\sqrt{\varepsilon}d$. Thus, $\mathrm{vol}(X) \sim \varepsilon n^2 p_X \sim \varepsilon \mathrm{vol}(G)$. Hence, X is a fairly small but densely connected set of vertices. It is not difficult to see that G satisfies $\mathrm{Disc}(100\varepsilon)$, and standard results on random matrices show that G violates $\mathrm{ess\text{-}Eig}(0.01\sqrt{\varepsilon})$.
2. In the conference version of this paper we stated erroneously that the implication "$\mathrm{Disc}(\gamma\varepsilon^3) \Rightarrow \mathrm{ess\text{-}Eig}(\varepsilon)$" is best possible.
3. The techniques presented in Section 4 can be adapted easily to obtain a similar result as Theorem 1 with respect to the concepts of discrepancy and eigenvalue separation from [10]. More precisely, let $G = (V, E)$ be a graph on n vertices, let $p = 2|E|n^{-2}$ be the edge density of G, and let $\gamma > 0$ denote a small enough constant. If for any subset $X \subset V$ we have $|2e(X) - |X|^2 p| < \gamma\varepsilon^2 n^2 p$, then there exists a set $W \subset V$ of size $|W| \geq (1 - \varepsilon)n$ such that the following is true. Letting $A = A(G)$ signify the adjacency matrix of G, we have $\max\{-\lambda_1(A_W), \lambda_{|W|-1}(A_W)\} \leq \varepsilon np$. That is, all eigenvalues of the minor A_W except for the largest are at most εnp in absolute value. The same example as under 1. shows that this result is best possible up to the precise value of γ.
4. The methods from Section 5 yield an algorithmic version of the "classical" sparse regularity lemma of Kohayakawa [18] and Rödl (unpublished), which does not take into account the degree distribution.

References

1. Albert, R., Barabási, A.L.: Statistical mechanics of complex networks. Reviews of modern physics **74** (2002) 47–97

2. Alizadeh, F.: Interior point methods in semidefinite programming with applications to combinatorial optimization. SIAM J. Optimization **5** (1995) 13–51
3. Alon, N., Duke, R.A., Lefmann, H., Rödl, V., Yuster, R.: The algorithmic aspects of the regularity lemma. J. of Algorithms **16** (1994) 80–109
4. Alon, N, Naor, A.: Approximating the cut-norm via Grothendieck's inequality. Proc. 36th STOC (2004) 72–80
5. Bilu, Y., Linial, N.: Lifts, discrepancy and nearly optimal spectral gap. Combinatorica, to appear
6. B. Bollobás and V. Nikiforov: Graphs and Hermitian matrices: discrepancy and singular values, Discrete Math. **285** (2004) 17–32
7. Butler, S.: On eigenvalues and the discrepancy of graphs. preprint
8. Chung, F.: Spectral graph theory. American Mathematical Society (1997).
9. Chung, F., Graham, R.: Quasi-random graphs with given degree sequences. Random Structures and Algorithms, to appear.
10. Chung, F., Graham, R.: Sparse quasi-random graphs. Combinatorica **22** (2002) 217–244
11. Chung, F., Graham, R., Wilson, R.M.: Quasi-random graphs. Combinatorica **9** (1989) 345–362
12. Frieze, A., Kannan, R.: Quick approximation to matrices and applications. Combinatorica **19** (1999) 175–200
13. Gerke, S., Steger, A.: The sparse regularity lemma and its applications. In: Surveys in Combinatorics, Proc. 20th British Combinatorial Conference, London Mathematical Society Lecture Notes Series 327, ed. Bridget S. Webb, Cambridge University Press (2005) 227–258
14. Gerke, S., Steger, A.: A characterization for sparse ε-regular pairs. The Electronic J. Combinatorics **14** (2007), R4, 12pp
15. Grothendieck, A.: Résumé de la théorie métrique des produits tensoriels topologiques. Bol. Soc. Mat. Sao Paulo **8** (1953) 1–79
16. Grötschel, M., Lovász, L., Schrijver, A.: Geometric algorithms and combinatorial optimization. Springer (1988)
17. Håstad, J.: Some optimal inapproximability results. J. of the ACM **48** (2001) 798–859
18. Kohayakawa, Y.: Szemerédi's regularity lemma for sparse graphs. In Cucker, F., Shub, M. (eds.): Foundations of computational mathematics (1997) 216–230.
19. Kohayakawa, Y., Rödl, V., Thoma, L.: An optimal algorithm for checking regularity. SIAM J. Comput., **32** (2003) 1210–1235
20. Krivine, J.L.: Sur la constante de Grothendieck. C. R. Acad. Sci. Paris Ser. A-B **284** (1977) 445–446
21. Szemerédi, E.: Regular partitions of graphs. Problémes Combinatoires et Théorie des Graphes Colloques Internationaux CNRS **260** (1978) 399-401
22. Trevisan, L., Sorkin, G., Sudan, M., Williamson, D.: Gadgets, approximation, and linear programming. SIAM J. Computing **29** (2000) 2074–2097
23. Helmberg, C.: Semidefinite programming for combinatorial optimization. Habilitation thesis. Report ZR-00-34, Zuse Institute Berlin (2000)