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Quasi-Shuffle Products

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Abstract. Given a locally finite graded set A and a commutative, associative operation on A that adds degrees, we construct a commutative multiplication $*$ on the set of noncommutative polynomials in A which we call a quasi-shuffle product; it can be viewed as a generalization of the shuffle product $\sqcup\!\sqcup$. We extend this commutative algebra structure to a Hopf algebra $(\mathfrak{A}, *, \Delta)$; in the case where A is the set of positive integers and the operation on A is addition, this gives the Hopf algebra of quasi-symmetric functions. If rational coefficients are allowed, the quasi-shuffle product is in fact no more general than the shuffle product; we give an isomorphism exp of the shuffle Hopf algebra $(\mathfrak{A}, \sqcup\!\sqcup, \Delta)$ onto $(\mathfrak{A}, *, \Delta)$. Both the set L of Lyndon words on A and their images $\{\exp(w) \mid w \in L\}$ freely generate the algebra $(\mathfrak{A}, *)$. We also consider the graded dual of $(\mathfrak{A}, *, \Delta)$. We define a deformation $*_q$ of $*$ that coincides with $*$ when $q = 1$ and is isomorphic to the concatenation product when q is not a root of unity. Finally, we discuss various examples, particularly the algebra of quasi-symmetric functions (dual to the noncommutative symmetric functions) and the algebra of Euler sums.

Keywords: Hopf algebra, shuffle algebra, quasi-symmetric function, noncommutative symmetric function, quantum shuffle product

1. Introduction

Let k be a subfield of $\mathbf{C}$, and let A be a locally finite graded set. If we think of the graded noncommutative polynomial algebra $\mathfrak{A} = k\langle A\rangle$ as a vector space over k, we can make it commutative k-algebra by giving it the shuffle multiplication $\sqcup\!\sqcup$, defined inductively by

$$aw_1 \sqcup\!\sqcup\, bw_2 = a(w_1 \sqcup\!\sqcup\, bw_2) + b(aw_1 \sqcup\!\sqcup\, w_2)$$

for $a, b \in A$ and words w_1, w_2. The commutative k-algebra $(\mathfrak{A}, \sqcup\!\sqcup)$ is in fact a polynomial algebra on the Lyndon words in $\mathfrak{A}$ (as defined in §2 below). If we define

$$\Delta(w) = \sum_{uv=w} u \otimes v,$$

then $(\mathfrak{A}, \sqcup\!\sqcup, \Delta)$ becomes a commutative (but not cocommutative) Hopf algebra, usually called the shuffle Hopf algebra; and its graded dual is the concatenation Hopf algebra (see [14], Chapter 1).

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Recently another pair of dual Hopf algebras has inspired much interest. The Hopf algebra **Sym** of noncommutative symmetric functions, introduced in [7], has as its graded dual the Hopf algebra of quasi-symmetric functions [5, 13]. In a recent paper of the author [12], the algebra of quasi-symmetric functions arose via a modification of the shuffle product, which suggested a connection between the two pairs of Hopf algebras. In fact, the Hopf algebra of quasi-symmetric functions (over k) is known to be isomorphic to the shuffle Hopf algebra on a countably infinite set of generators (with one in each positive degree). It is the purpose of this paper to study this Hopf algebra isomorphism in a more general setting. (We emphasize that we are working over a subfield k of $\mathbf{C}$; if we instead work over $\mathbf{Z}$, there is no such isomorphism—the integral algebra of quasi-symmetric functions is a polynomial algebra [3, 16], but the integral shuffle algebra is not [11].)

More explicitly, our construction is as follows. Suppose also that for any generators $a, b \in A$ there is another generator $[a, b]$ so that the operation $[\cdot, \cdot]$ is commutative, associative, and adds degrees. If we define a "quasi-shuffle product" $*$ by

$$aw_1 * bw_2 = a(w_1 * bw_2) + b(aw_1 * w_2) + [a, b](w_1 * w_2),$$

then $(\mathfrak{A}, *)$ is a commutative and associative k-algebra (Theorem 2.1 below). In fact, as we show in §3, $(\mathfrak{A}, *, \Delta)$ is a Hopf algebra, which we call the quasi-shuffle Hopf algebra corresponding to A and $[\cdot, \cdot]$. This construction gives the quasi-symmetric functions in the case where A consists of one element z_i in each degree $i > 0$, with $[z_i, z_j] = z_{i+j}$; this and other examples are discussed in §6. We give an explicit isomorphism exp from the shuffle Hopf algebra on the generating set A onto any quasi-shuffle Hopf algebra with the same generating set (Theorems 2.5 and 3.3). This allows us to show that any quasi-shuffle algebra on A is the free polynomial algebra on Lyndon words in $\mathfrak{A}$ (Theorem 2.6). In §4 we take (graded) duals, giving an isomorphism $\exp^*$ from the concatenation Hopf algebra to the dual of $(\mathfrak{A}, *, \Delta)$.

In §5 we consider a q-deformation $*_q$ of the quasi-shuffle product, generalizing the quantum shuffle product as defined in [4] (see also [10, 15]). This product coincides with the quasi-shuffle product $*$ when $q = 1$, but is noncommutative when $q \neq 1$; when q is not a root of unity, we use the theorem of Varchenko [19] to prove that the algebra $(\mathfrak{A}, *_q)$ is isomorphic to the concatenation algebra on A (Theorem 5.4). In this case, if we declare the elements of A primitive, we get a Hopf algebra $(\mathfrak{A}, *_q, \Delta_q)$ isomorphic to the concatenation Hopf algebra.

A construction equivalent to the quasi-shuffle algebra, but (in effect) not assuming commutativity of the operation $[\cdot, \cdot]$, was developed independently by Fares [6]. The author thanks A. Joyal for bringing it to his attention.

2. The algebra structure

As above we begin with the graded noncommutative polynomial algebra $\mathfrak{A} = k\langle A\rangle$ over a subfield $k \subset \mathbf{C}$, where A is a locally finite set of generators (i.e. for each positive integer n the set A_n of generators in degree n is finite). We write $\mathfrak{A}_n$ for the vector space of homogeneous elements of $\mathfrak{A}$ of degree n. We shall refer to elements of A as letters, and to monomials in the letters as words. For any word w we write $\ell(w)$ for its length (the number of letters it contains) and $|w|$ for its degree (the sum of the degrees of its factors). The unique word of length 0 is 1, the empty word.

Now define a new multiplication $*$ on $\mathfrak{A}$ by requiring that $*$ distribute over addition, that $1 * w = w * 1 = w$ for any word w, and that, for any words w_1, w_2 and letters a, b,

$$aw_1 * bw_2 = a(w_1 * bw_2) + b(aw_1 * w_2) + [a, b](w_1 * w_2), \tag{1}$$

where $[\cdot, \cdot] : \bar{A} \times \bar{A} \to \bar{A}$ ($\bar{A} = A \cup \{0\}$) is a function satisfying

S0. $[a, 0] = 0$ for all $a \in \bar{A}$;
S1. $[a, b] = [b, a]$ for all $a, b \in \bar{A}$;
S2. $[[a, b], c] = [a, [b, c]]$ for all $a, b, c \in \bar{A}$; and
S3. Either $[a, b] = 0$ or $|[a, b]| = |a| + |b|$ for all $a, b \in A$.

Theorem 2.1 $(\mathfrak{A}, *)$ *is a commutative graded k-algebra.*

Proof: It is enough to show that the operation $*$ is commutative, associative, and adds degrees. For commutativity, it suffices to show $w_1 * w_2 = w_2 * w_1$ for any words w_1 and w_2. We proceed by induction on $\ell(w_1) + \ell(w_2)$. Since there is nothing to prove if either w_1 or w_2 is empty, we can assume there are letters a, b so that $w_1 = au$ and $w_2 = bv$. Then (1) together with the induction hypothesis gives

$$w_1 * w_2 - w_2 * w_1 = [a, b](u * v) - [b, a](v * u),$$

and the right-hand side is zero by the induction hypothesis and (S1). Similarly, for associativity it is enough to prove $w_1*(w_2*w_3) = (w_1*w_2)*w_3$ for any words w_1, w_2, and w_3: this follows from induction on $\ell(w_1)+\ell(w_2)+\ell(w_3)$ using (1) and (S2). Finally, to show $*$ adds degrees, induct on $\ell(w_1) + \ell(w_2)$ using (1) and (S3) to prove that $|w_1 * w_2| = |w_1| + |w_2|$ for any words w_1, w_2. $\square$

If $[a, b] = 0$ for all $a, b \in A$, then $(\mathfrak{A}, *)$ is the shuffle algebra as usually defined (see e.g. [14]) and we write ш for the multiplication instead of $*$. Suppose now that the set A of letters is totally ordered. Then lexicographic ordering gives a total order on the words: we put $u < uv$ for any nonempty word v, and $w_1 a w_2 < w_1 b w_3$ for any letters $a < b$ and words w_1, w_2, and w_3. We call a word $w \neq 1$ of $\mathfrak{A}$ Lyndon if $w < v$ for any nontrivial factorization $w = uv$. Then we have the following result from Chapter 6 of [14].

Theorem 2.2 *The shuffle algebra* $(\mathfrak{A}, ш)$ *is the free polynomial algebra on the Lyndon words.*

We shall define an isomorphism $\exp : (\mathfrak{A}, ш) \to (\mathfrak{A}, *)$. To do so, we must first develop some notation relating to the operation $[\cdot, \cdot]$ and compositions. Define inductively $[S] \in \bar{A}$ for any finite sequence S of elements of A by setting $[a] = a$ for $a \in A$, and $[a, S] = [a, [S]]$ for any $a \in A$ and sequence S of elements of A.

Proposition 2.3

(i) *If* $[S] = 0$, *then* $[S'] = 0$ *whenever* S *is a subsequence of* S'*;*

(ii) $[S]$ *does not depend on the order of the elements of* S *(i.e., it depends only on the underlying multiset of* S*);*

(iii) *For any sequences* S_1 *and* S_2, $[S_1 \sqcup S_2] = [[S_1], [S_2]]$, *where* $S_1 \sqcup S_2$ *denotes the concatenation of sequences* S_1 *and* S_2*;*

(iv) *If* $[S] \neq 0$, *then the degree of* S *is the sum of the degrees of the elements of* S.

Proof: (i), (ii), (iii), (iv) follow from (S0), (S1), (S2), (S3) respectively. □

A composition of a positive integer n is a sequence $I = (i_1, i_2, \ldots, i_k)$ of positive integers such that $i_1 + i_2 + \cdots + i_k = n$. We call $n = |I|$ the weight of I and $k = \ell(I)$ its length; we write $\mathcal{C}(n)$ for the set of compositions of n, and $\mathcal{C}(n, k)$ for the set of compositions of n of length k. For $I \in \mathcal{C}(n, k)$ and $J \in \mathcal{C}(k, l)$, the composition $J \circ I \in \mathcal{C}(n, l)$ is given by

$$J \circ I = \left(i_1 + \cdots + i_{j_1}, i_{j_1+1} + \cdots + i_{j_1+j_2}, \ldots, i_{j_1+\cdots+j_{l-1}+1} + \cdots + i_k\right).$$

If $K = J \circ I$ for some J, we call I a refinement of K and write $I \succeq K$. Compositions act on words via $[\cdot, \cdot]$ as follows. For any word $w = a_1 a_2 \cdots a_n$ and composition $I = (i_1, \ldots, i_l) \in \mathcal{C}(n)$, set

$$I[w] = \left[a_1, \ldots, a_{i_1}\right]\left[a_{i_1+1}, \ldots, a_{i_1+i_2}\right] \cdots \left[a_{i_1+\cdots+i_{l-1}+1}, \ldots, a_n\right].$$

(This is really an action in the sense that $I[J[w]] = I \circ J[w]$.)

Now let $\exp : \mathfrak{A} \to \mathfrak{A}$ be the linear map with $\exp(1) = 1$ and

$$\exp(w) = \sum_{(i_1,\ldots,i_l) \in \mathcal{C}(\ell(w))} \frac{1}{i_1! \cdots i_l!} (i_1, \ldots, i_l)[w]$$

for any nonempty word w (so, e.g. $\exp(a_1 a_2 a_3) = a_1 a_2 a_3 + \frac{1}{2}[a_1, a_2]a_3 + \frac{1}{2}a_1[a_2, a_3] + \frac{1}{6}[a_1, a_2, a_3]$). There is an inverse log of exp given by

$$\log(w) = \sum_{(i_1,\ldots,i_l) \in \mathcal{C}(\ell(w))} \frac{(-1)^{\ell(w)-l}}{i_1 \cdots i_l} (i_1, \ldots, i_l)[w]$$

for any word w, and extended to $\mathfrak{A}$ by linearity; this follows by taking $f(t) = e^t - 1$ in the following lemma.

Lemma 2.4 *Let* $f(t) = a_1 t + a_2 t^2 + a_3 t^3 + \cdots$ *be a function analytic at the origin, with* $a_1 \neq 0$ *and* $a_i \in k$ *for all* i, *and let* $f^{-1}(t) = b_1 t + b_2 t^2 + b_3 t^3 + \cdots$ *be the inverse of* f. *Then the map* $\Psi_f : \mathfrak{A} \to \mathfrak{A}$ *given by*

$$\Psi_f(w) = \sum_{I \in \mathcal{C}(\ell(w))} a_{i_1} a_{i_2} \cdots a_{i_l} I[w]$$

for words w, and extended linearly, has inverse $\Psi_f^{-1} = \Psi_{f^{-1}}$ given by

$$\Psi_{f^{-1}}(w) = \sum_{I \in \mathcal{C}(\ell(w))} b_{i_1} b_{i_2} \cdots b_{i_l} I[w].$$

Proof: It suffices to show that $\Psi_{f^{-1}}(\Psi_f(w)) = w$ for any word w of length $n \geq 1$ (Note that $\Psi_f(\Psi_{f^{-1}}(w)) = w$ is then automatic, since Ψ_f and $\Psi_{f^{-1}}$ can be thought of as linear maps of the vector space with basis $\{I[w] \mid I \in \mathcal{C}(n)\}$.) Now for any $K = (k_1, \ldots, k_l) \in \mathcal{C}(n)$, the coefficient of $K[w]$ in $\Psi_{f^{-1}}(\Psi_f(w))$ is

$$\sum_{J \circ I = K} b_{j_1} b_{j_2} \cdots b_{j_l} a_{i_1} a_{i_2} \cdots a_{i_{|J|}}. \tag{2}$$

We must show that (2) is 1 if K is a sequence of n 1's, and 0 otherwise. To see this, let $t_1, t_2, \ldots$ be commuting variables. Then (2) is the coefficient of $t_1^{k_1} t_2^{k_2} \cdots t_l^{k_l}$ in

$$t_1 t_2 \cdots t_l = f^{-1}(f(t_1)) f^{-1}(f(t_2)) \cdots f^{-1}(f(t_l)). \qquad \square$$

Theorem 2.5 exp *is an isomorphism of* $(\mathfrak{A}, ш)$ *onto* $(\mathfrak{A}, *)$ *(as graded k-algebras).*

Proof: From the lemma, exp is invertible. Also, it follows from 2.3(iv) that exp preserves degree. To show exp a homomorphism it suffices to show $\exp(w ш v) = \exp(w) * \exp(v)$ for any words w, v. Let $w = a_1 \cdots a_n$ and $v = b_1 \cdots b_m$. Evidently both $\exp(w ш v)$ and $\exp(w) * \exp(v)$ are sums of rational multiples of terms

$$[S_1 \sqcup T_1][S_2 \sqcup T_2] \cdots [S_l \sqcup T_l] \tag{3}$$

where the S_i and T_i are subsequences of $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ respectively such that

i. for each i, at most one of S_i, T_i is empty; and
ii. the concatenation $S_1 \sqcup S_2 \sqcup \cdots \sqcup S_l$ is the sequence $a_1, \ldots, a_n$, and similarly the T_i concatenate to give the sequence $b_1, \ldots, b_m$.

Now the term (3) arises in $\exp(w) * \exp(v)$ in only one way, and its coefficient is

$$\frac{1}{(\text{length } S_1)!(\text{length } S_2)! \cdots (\text{length } S_l)!(\text{length } T_1)!(\text{length } T_2)! \cdots (\text{length } T_l)!}.$$

On the other hand, (3) can arise in $\exp(w ш v)$ from

$$\binom{\text{length } S_1 \sqcup T_1}{\text{length } S_1} \binom{\text{length } S_2 \sqcup T_2}{\text{length } S_2} \cdots \binom{\text{length } S_l \sqcup T_l}{\text{length } S_l}$$
$$= \frac{(\text{length } S_1 \sqcup T_1)! \cdots (\text{length } S_l \sqcup T_l)!}{(\text{length } S_1)! \cdots (\text{length } S_l)!(\text{length } T_1)! \cdots (\text{length } T_l)!}$$

distinct terms of the shuffle product $w \sqcup\!\sqcup\, v$, and after application of exp each such term acquires a coefficient of

$$\frac{1}{(\text{length } S_1 \sqcup T_1)! \cdots (\text{length } S_l \sqcup T_l)!}. \qquad \square$$

It follows from Theorems 2.2 and 2.5 that $(\mathfrak{A}, *)$ is the free polynomial algebra on the elements $\{\exp(w) \mid w \text{ is a Lyndon word}\}$. In fact the following is true.

Theorem 2.6 $(\mathfrak{A}, *)$ *is the free polynomial algebra on the Lyndon words.*

Proof: It suffices to show that any word w can be written as a $*$-polynomial of Lyndon words. We proceed by induction on $\ell(w)$. If $\ell(w) = 1$ the result is immediate, since every letter is a Lyndon word. Now let $\ell(w) > 1$: by Theorem 2.5 there are Lyndon words $w_1, \ldots, w_n$ and a polynomial P so that

$$w = P(\exp(w_1), \ldots, \exp(w_n))$$

in $(\mathfrak{A}, *)$. Note that since $\log(w) = P(w_1, \ldots, w_n)$ in $(\mathfrak{A}, \sqcup\!\sqcup)$, we can assume every term of $P(w_1, \ldots, w_n)$ (as a $\sqcup\!\sqcup$-polynomial) has length at most $\ell(w)$, since the shuffle product preserves lengths. But then in $(\mathfrak{A}, *)$,

$$w - P(w_1, \ldots, w_n) = P(\exp(w_1), \ldots, \exp(w_n)) - P(w_1, \ldots, w_n)$$

must consist of terms of length less than $\ell(w)$, and so is expressible in terms of Lyndon words by the induction hypothesis. $\square$

By the preceding result, the number of generators of $(\mathfrak{A}, *)$ in degree n is the number L_n of Lyndon words of degree n. This number can be calculated from Poincaré series

$$A(x) = \sum_{n \geq 0} (\dim \mathfrak{A}_n) x^n = \frac{1}{1 - \sum_{n \geq 1} (\text{card } A_n) x^n}$$

of $\mathfrak{A}$ as follows.

Proposition 2.7 *The number L_n of Lyndon words in $\mathfrak{A}_n$ is given by*

$$L_n = \frac{1}{n} \sum_{d \mid n} \mu\left(\frac{n}{d}\right) c_d,$$

where the numbers c_n are defined by

$$x \frac{d}{dx} \log A(x) = \sum_{n \geq 1} c_n x^n$$

for $A(x)$ as above.

Proof: In view of Theorems 2.2 and 2.6, we must have

$$A(x) = \prod_{n \geq 1} (1 - x^n)^{-L_n}.$$

The conclusion then follows from taking logarithms, differentiating, and using the Möbius inversion formula. □

3. The Hopf algebra structure

For basic definitions and facts about Hopf algebras see [17]. We define a comultiplication $\Delta : \mathfrak{A} \otimes \mathfrak{A} \to \mathfrak{A}$ and counit $\epsilon : \mathfrak{A} \to k$ by

$$\Delta(w) = \sum_{uv=w} u \otimes v$$

and

$$\epsilon(w) = \begin{cases} 1, & w = 1 \\ 0, & \text{otherwise} \end{cases}$$

for any word w of $\mathfrak{A}$. Then $(\mathfrak{A}, \Delta, \epsilon)$ is evidently a (non-cocommutative) coalgebra. In fact the following result holds.

Theorem 3.1 *$\mathfrak{A}$ with the $*$-multiplication and Δ-comultiplication is a bialgebra.*

Proof: It suffices to show that ϵ and Δ are $*$-homomorphisms. The statement for ϵ is obvious; to show $\Delta(w_1) * \Delta(w_2) = \Delta(w_1 * w_2)$ for any words w_1, w_2 use induction on $\ell(w_1) + \ell(w_2)$. Since the result is immediate if w_1 or w_2 is 1, we can write $w_1 = au$ and $w_2 = bv$ for letters a, b and words u, v. Adopting Sweedler's sigma notation [17], we write

$$\Delta(u) = \sum u_{(1)} \otimes u_{(2)}, \quad \text{and} \quad \Delta(v) = \sum v_{(1)} \otimes v_{(2)}.$$

Then from the definition of Δ,

$$\Delta(w_1) = \sum au_{(1)} \otimes u_{(2)} + 1 \otimes au \quad \text{and} \quad \Delta(w_2) = \sum bv_{(1)} \otimes v_{(2)} + 1 \otimes bv,$$

so that $\Delta(w_1) * \Delta(w_2)$ is

$$\sum \left(au_{(1)} * bv_{(1)}\right) \otimes \left(u_{(2)} * v_{(2)}\right) + \sum au_{(1)} \otimes \left(u_{(2)} * bv\right) \\ + \sum bv_{(1)} \otimes \left(au * v_{(2)}\right) + 1 \otimes (au * bv).$$

Using (1), this is

$$\sum a\big(u_{(1)} * bv_{(1)}\big) \otimes \big(u_{(2)} * v_{(2)}\big) + \sum b\big(au_{(1)} * v_{(1)}\big) \otimes \big(u_{(2)} * v_{(2)}\big) + \sum [a,b]\big(u_{(1)} * v_{(1)}\big) \otimes \big(u_{(2)} * v_{(2)}\big) + \sum au_{(1)} \otimes \big(u_{(2)} * bv\big) + \sum bv_{(1)} \otimes \big(au * v_{(2)}\big) + 1 \otimes a(u * w_2) + 1 \otimes b(w_1 * v) + 1 \otimes [a,b](u * v),$$

or, applying the induction hypothesis,

$$(a \otimes 1)(\Delta(u) * \Delta(w_2)) + 1 \otimes a(u * w_2) + (b \otimes 1)(\Delta(w_1) * \Delta(v)) + 1 \otimes b(w_1 * v) + ([a,b] \otimes 1)\Delta(u * v) + 1 \otimes [a,b](u * v),$$

which can be recognized as $\Delta(w_1 * w_2) = \Delta(a(u * w_2) + b(w_1 * v) + [a,b](u * v))$. $\square$

Since both $*$ and Δ respect the grading, it follows automatically that $\mathfrak{A}$ is a Hopf algebra (cf. Lemma 2.1 of [5]). In fact there are two explicit formulas for the antipode, whose agreement is of some interest.

Theorem 3.2 *The bialgebra $\mathfrak{A}$ has antipode S given by*

$$S(w) = \sum_{(i_1,\dots,i_l)\in\mathcal{C}(n)} (-1)^l a_1 \cdots a_{i_1} * a_{i_1+1} \cdots a_{i_1+i_2} * \cdots * a_{i_1+\cdots+i_{l-1}+1} \cdots a_n = (-1)^n \sum_{I\in\mathcal{C}(n)} I[a_n a_{n-1} \cdots a_1]$$

for any word $w = a_1 a_2 \cdots a_n$ of $\mathfrak{A}$.

Proof: We can compute S recursively from $S(1) = 1$ and

$$S(w) = -\sum_{k=0}^{n-1} S(a_1 \cdots a_k) * a_{k+1} \cdots a_n \tag{4}$$

for a word $w = a_1 \cdots a_n$. The first formula for S then follows easily by induction on n. For the the second formula, we also proceed by induction on n, following the proof of Proposition 3.4 of [5]. For $w = a_1 \cdots a_n$, $n > 0$, the induction hypothesis and (4) give $S(w)$ as

$$\sum_{k=0}^{n-1} \sum_{(i_1,\dots,i_l)\in\mathcal{C}(k)} (-1)^{k+1} (i_1,\dots,i_l)[a_k a_{k-1} \cdots a_1] * a_{k+1} \cdots a_n = \sum_{k=0}^{n-1} \sum_{(i_1,\dots,i_l)\in\mathcal{C}(k)} (-1)^{k+1} \big[a_k, a_{k-1},\dots,a_{k-i_1+1}\big] \cdots \big[a_{i_l},\dots,a_1\big] * a_{k+1} \cdots a_n$$

Now the first factor of each term of the $*$-product in the inner sum is, from consideration of (1), one of three generators: $[a_k,\dots,a_{k-i_1+1}]$, $[a_{k+1}, a_k,\dots,a_{k-i_1+1}]$, or a_{k+1}. We say the term is of type k in the first case, and of type $k+1$ in the latter two cases. Now consider a word that appears in the expansion of $S(w)$. If it has type $i \le n-1$, then it occurs for both $k=i$ and $k=i-1$, and the two occurrences will cancel. The only words that do not cancel are those of type n, which occur only for $k=n-1$: these will all carry the coefficient $(-1)^n$, and give the second formula for $S(w)$. □

Remark In the case of the shuffle algebra (i.e., where $[\cdot,\cdot]$ is identically zero), the second formula for the antipode is simply $S(w) = (-1)^{\ell(w)}\bar{w}$. Cf. [14, p. 35].

Theorem 3.3 $\exp : \mathfrak{A} \to \mathfrak{A}$ *is a Hopf algebra isomorphism of* $(\mathfrak{A}, ш, \Delta)$ *onto* $(\mathfrak{A}, *, \Delta)$.

Proof: We have already shown that exp is an algebra homomorphism. It suffices to show that $\exp \circ \epsilon(w) = \epsilon \circ \exp(w)$ and $\Delta \circ \exp(w) = (\exp \otimes \exp) \circ \Delta(w)$ for any word w. The first equation is immediate, and the second follows since both sides are equal to

$$\sum_{uv=w} \sum_{\substack{(i_1,\dots,i_k)\in\mathcal{C}(\ell(u))\\(j_1,\dots,j_l)\in\mathcal{C}(\ell(v))}} \frac{1}{i_1!\cdots i_k!} I[u] \otimes \frac{1}{j_1!\cdots j_l!} J[v].$$ □

4. Duality

The graded dual $\mathfrak{A}^* = \bigoplus_{n\ge 0} \mathfrak{A}_n^*$ has a basis consisting of elements w^*, where w is a word of $\mathfrak{A}$: the pairing $(\cdot,\cdot) : \mathfrak{A} \otimes \mathfrak{A}^* \to k$ is given by

$$(u, v^*) = \begin{cases} 1 & \text{if } u = v \\ 0 & \text{otherwise.} \end{cases}$$

Then the transpose of Δ is the concatenation product $\mathrm{conc}(u^* \otimes v^*) = (uv)^*$, and the transpose of ш is the comultiplication δ defined by

$$\delta(w^*) = \sum_{\text{words } u,\, v \text{ of } \mathfrak{A}} (u \,ш\, v, w^*) u^* \otimes v^*.$$

Since $(\mathfrak{A}, ш, \Delta)$ is a Hopf algebra, so is its graded dual $(\mathfrak{A}^*, \mathrm{conc}, \delta)$, which is called the concatenation Hopf algebra in [14]. Dualizing $(\mathfrak{A}, *, \Delta)$, we also have a Hopf algebra $(\mathfrak{A}^*, \mathrm{conc}, \delta')$, where δ' is the comultiplication defined by

$$\delta'(w^*) = \sum_{\text{words } u,\, v \text{ of } \mathfrak{A}} (u * v, w^*) u^* \otimes v^*.$$

Then from our earlier results we have the following.

Theorem 4.1 *There is a Hopf algebra isomorphism* $\exp^*$ *from* $(\mathfrak{A}^*, \mathrm{conc}, \delta')$ *to* $(\mathfrak{A}^*, \mathrm{conc}, \delta)$.

$\exp^*$ is the transpose of exp: explicitly, $\exp^*$ is the endomorphism of $(\mathfrak{A}^*, \text{conc})$ with

$$\exp^*(a^*) = \sum_{n\geq 1} \frac{1}{n!} \sum_{(n)[w]=a} w^* = \sum_{n\geq 1} \sum_{[a_1,\ldots,a_n]=a} \frac{1}{n!}(a_1\cdots a_n)^*$$

for $a \in A$. It has inverse $\log^*$ given by

$$\log^*(a^*) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \sum_{(n)[w]=a} w^*, \quad a \in A. \tag{5}$$

The set of Lie polynomials in $\mathfrak{A}^*$ is the smallest sub-vector-space of $\mathfrak{A}^*$ containing the set of generators $\{a^* \mid a \in A\}$ and closed under the Lie bracket

$$[P, Q]_{\text{Lie}} = PQ - QP.$$

Since the Lie polynomials are exactly the primitives for δ [14, Theorem 1.4], we have the following result.

Theorem 4.2 *The primitives for δ' are elements of the form $\log^* P$, where P is a Lie polynomial.*

We note that $(\mathfrak{A}^*, \text{conc}, \delta')$ has antipode

$$S^*(w^*) = \sum_{v\in\mathcal{P}(\bar{w})} (-1)^{\ell(v)} v^*,$$

where $\bar{w}$ is the reverse of w (i.e. $\bar{w} = a_n a_{n-1}\cdots a_1$ if $w = a_1 a_2 \cdots a_n$) and $\mathcal{P}(w) = \{v \mid I[v] = w \text{ for some } I \in \mathcal{C}(\ell(v))\}$.

5. *q*-deformation

We now define a deformation of $(\mathfrak{A}, *)$. We again start with the noncommutative polynomial algebra $\mathfrak{A} = k\langle A\rangle$ and define, for $q \in k$, a new multiplication $*_q$ by requiring that $*_q$ distribute over addition, that $w *_q 1 = 1 *_q w = w$ for any word w and that

$$aw_1 *_q bw_2 = a(w_1 *_q bw_2) + q^{|aw_1||b|} b(aw_1 *_q w_2) + q^{|w_1||b|}[a, b](w_1 *_q w_2) \tag{6}$$

for any words w_1, w_2 and letters a, b.

Theorem 5.1 *$(\mathfrak{A}, *_q)$ is a graded k-algebra, which coincides with $(\mathfrak{A}, *)$ when $q = 1$.*

Proof: The argument is similar to that for Theorem 2.1. It is easy to show that $|w_1 *_q w_2| = |w_1| + |w_2|$ for any words w_1, w_2 by induction on $\ell(w_1) + \ell(w_2)$. To show the operation $*_q$ associative, it suffices to show that $w_1 *_q (w_2 *_q w_3) = (w_1 *_q w_2) *_q w_3$ for any words

w_1, w_2, and w_3, which we do by induction on $\ell(w_1)+\ell(w_2)+\ell(w_3)$. We can assume $w_i = a_i u_i$ for letters a_i and words u_i, $i = 1, 2, 3$. Then $w_1 *_q (w_2 *_q w_3)$ is

$$\begin{aligned}
&a_1(u_1 *_q a_2(u_2 *_q w_3)) + q^{|w_1||a_2|}a_2(w_1 *_q (u_2 *_q w_3))\\
&\quad + q^{|u_1||a_2|}[a_1, a_2](u_1 *_q (u_2 *_q w_3)) + q^{|w_2||a_3|}a_1(u_1 *_q a_3(w_2 *_q u_3))\\
&\quad + q^{|w_2||a_3|+|w_1||a_3|}a_3(w_1 *_q (w_2 *_q u_3)) + q^{|w_2||a_3|+|u_1||a_3|}[a_1, a_3](u_1 *_q (w_2 *_q u_3))\\
&\quad + q^{|u_2||a_3|}a_1(u_1 *_q [a_2, a_3](u_2 *_q u_3)) + q^{|u_2||a_3|+|w_1||a_2a_3|}[a_2, a_3](w_1 *_q (u_2 *_q u_3))\\
&\quad + q^{|u_2||a_3|+|u_1||a_2a_3|}[a_1, a_2, a_3](u_1 *_q (u_2 *_q u_3)),
\end{aligned}$$

while $(w_1 *_q w_2) *_q w_3$ is

$$\begin{aligned}
&a_1((u_1 *_q w_2) *_q w_3) + q^{|w_1w_2||a_3|}a_3(a_1(u_1 *_q w_2) *_q u_3)\\
&\quad + q^{|u_1w_2||a_3|}[a_1, a_3]((u_1 *_q w_2) *_q u_3) + q^{|w_1||a_2|}a_2((w_1 *_q u_2) *_q w_3)\\
&\quad + q^{|w_1||a_2|+|w_1w_2||a_3|}a_3(a_2(w_1 *_q u_2) *_q u_3)\\
&\quad + q^{|w_1||a_2|+|w_1u_2||a_3|}[a_2, a_3]((w_1 *_q u_2) *_q u_3)\\
&\quad + q^{|u_1||a_2|}[a_1, a_2]((u_1 *_q u_2) *_q w_3)\\
&\quad + q^{|u_1||a_2|+|w_1w_2||a_3|}a_3([a_1, a_2](u_1 *_q u_2) *_q u_3)\\
&\quad + q^{|u_1||a_2|+|u_1u_2||a_3|}[a_1, a_2, a_3]((u_1 *_q u_2) *_q u_3).
\end{aligned}$$

Applying the induction hypothesis, the difference is

$$\begin{aligned}
&a_1(u_1 *_q (a_2(u_2 *_q w_3) + q^{|w_2||a_3|}a_3(w_2 *_q u_3) + q^{|u_2||a_3|}[a_2, a_3](u_2 *_q u_3)))\\
&\quad + q^{(|w_2|+|w_1|)|a_3|}a_3(w_1 *_q (w_2 *_q u_3)) - a_1((u_1 *_q w_2) *_q w_3)\\
&\quad - q^{|w_1w_2||a_3|}a_3((a_1(u_1 *_q w_2) + q^{|w_1||a_2|}a_2(w_1 *_q u_2)\\
&\quad + q^{|u_1||a_2|}[a_1, a_2](u_1 *_q u_2)) *_q u_3),
\end{aligned}$$

which by application of (6) and the induction hypothesis is seen to be zero. □

Remark The author arrived at the definition (6) as follows. Knowing the first two terms on the right-hand side from the definition of the quantum shuffle product, he tried an arbitrary power of q on the third term, and found that the resulting product was only associative when the exponent is as in (6). Shortly afterward he discussed this with J.-Y. Thibon, who directed him to [18], where the rule (6) appears in the special case of the quasi-symmetric functions (see Example 1 below).

Of course, for $q \neq 1$ the algebra $(\mathfrak{A}, *_q)$ is no longer commutative. For each fixed q, there is a homomorphism Φ_q of graded associative k-algebras from the concatenation algebra $(\mathfrak{A}, \text{conc})$ to $(\mathfrak{A}, *_q)$ defined by

$$\Phi_q(a_1a_2\cdots a_n) = a_1 *_q a_2 *_q \cdots *_q a_n$$

for letters $a_1, a_2, \dots, a_n$; we call q generic if Φ_q is an isomorphism (i.e., if it is surjective). To give an explicit formula for Φ_q, we introduce some notation. For a permutation σ of

$\{1, 2, \ldots, n\}$, let $\iota(\sigma) = \{(i, j) \mid 1 \le i < j \le n$ and $\sigma(i) > \sigma(j)\}$ be the set of inversions of σ, and let $C(\sigma)$ be the descent composition of σ, i.e. the composition $(i_1, i_2, \ldots, i_l) \in \mathcal{C}(n)$ with

$$\sigma(i_1 + \cdots + i_{j-1} + 1) < \sigma(i_1 + \cdots + i_{j-1} + 2) < \cdots < \sigma(i_1 + \cdots + i_j)$$

for $j = 1, 2, \ldots, l$ and l minimal. (Equivalently, $C(\sigma) = (i_1, \ldots, i_l)$ is the composition such that the associated subset $\{i_1, i_1 + i_2, \ldots, i_1 + \cdots + i_{l-1}\}$ of $\{1, 2, \ldots, n-1\}$ is the descent set of σ, i.e. the set of $1 \le i \le n-1$ such that $\sigma(i) > \sigma(i+1)$.)

Lemma 5.2 *For any letters $a_1, a_2, \ldots, a_n$,*

$$\Phi_q(a_1 a_2 \cdots a_n) = \sum_{\text{permutations } \sigma} q^{\sum_{(i,j)\in\iota(\sigma)} |a_i||a_j|} \sum_{I \succeq C(\sigma)} I\big[a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)}\big].$$

Proof: We proceed by induction on n, the case $n = 2$ being immediate. Assuming the induction hypothesis, we have

$$\Phi_q(a_1 \cdots a_{n+1}) = \sum_{(\sigma, I)\in P(n)} q^{\sum_{(i,j)\in\iota(\sigma)} |a_i||a_j|} I[a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)}] * a_{n+1}$$

where $P(n)$ is the set of ordered pairs (σ, I) such that σ is a permutation of $\{1, 2, \ldots, n\}$ and $I \succeq C(\sigma)$. For $(\sigma, I) \in P(n)$ with $I = (i_1, i_2, \ldots, i_l)$ and $0 \le k \le l$, let σ'_k be the permutation of $\{1, 2, \ldots, n+1\}$ given by

$$\sigma'_k(j) = \begin{cases} \sigma(j), & j \le i_1 + \cdots + i_k \\ n+1, & j = i_1 + \cdots + i_k + 1 \\ \sigma(j-1), & j > i_1 + \cdots + i_k + 1. \end{cases}$$

Also, for $0 \le k \le l$ let $I'_k = (i_1, \ldots, i_k, 1, i_{k+1}, \ldots, i_l)$, and for $1 \le k \le l$ let $I''_k = (i_1, \ldots, i_{k-1}, i_k + 1, i_{k+1}, \ldots, i_l)$; evidently $(\sigma'_k, I'_k), (\sigma'_k, I''_k) \in P(n+1)$ for all k. By iterated application of (6) we have

$$\begin{aligned} I\big[a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)}\big] * a_{n+1} &= q^{\sum_{i=1}^{n} |a_i||a_{n+1}|} a_{n+1} I\big[a_{\sigma(1)} \cdots a_{\sigma(n)}\big] \\ &\quad + \sum_{k=1}^{l} q^{\sum_{j=i_1+\cdots+i_k+1}^{n} |a_{\sigma(j)}||a_{n+1}|} \big(I'_k\big[a_{\sigma'_k(1)} \cdots a_{\sigma'_k(n+1)}\big] \\ &\quad + I''_k\big[a_{\sigma'_k(1)} \cdots a_{\sigma'_k(n+1)}\big]\big). \end{aligned}$$

Hence $\Phi_q(a_1 \cdots a_{n+1})$ is the sum over $(\sigma, I) \in P(n)$ of

$$\begin{aligned} &q^{\sum_{(i,j)\in\iota(\sigma'_0)} |a_i||a_j|} I'_0\big[a_{\sigma'_0(1)} \cdots a_{\sigma'_0(n+1)}\big] + \sum_{k=1}^{l} q^{\sum_{(i,j)\in\iota(\sigma'_k)} |a_i||a_j|} \big(I'_k\big[a_{\sigma'_k(1)} \cdots a_{\sigma'_k(n+1)}\big] \\ &\quad + I''_k\big[a_{\sigma'_k(1)} \cdots a_{\sigma'_k(n+1)}\big]\big) \end{aligned}$$

and the conclusion follows by noting that every $(\tau, J) \in P(n+1)$ can be written uniquely as one of (σ'_k, I'_k) or (σ'_k, I''_k) for some $(\sigma, I) \in P(n)$. □

In the case $q = 0$, our formula for $\Phi_q(w)$ reduces to

$$\Phi_0(w) = \sum_{I \in \mathcal{C}(\ell(w))} I[w] = (-1)^{\ell(w)} S(\bar{w}),$$

and by applying Lemma 2.4 with $f(t) = t/(1-t)$ we see that Φ_0 has inverse Φ_0^{-1} given by

$$\Phi_0^{-1}(w) = \sum_{k=1}^{\ell(w)} \sum_{I \in \mathcal{C}(\ell(w),k)} (-1)^{\ell(w)-k} I[w].$$

For any word $w = a_1 a_2 \cdots a_n$, let V_w be the vector space over k with basis $\{a_{\tau(1)} \cdots a_{\tau(n)} \mid$ permutations $\tau\}$, and let $\phi_{w,q} : V_w \to V_w$ be Φ_q followed by projection onto V_w. Then $\phi_{w,q}$ is given by

$$\phi_{w,q}\big(a_{\tau(1)} \cdots a_{\tau(n)}\big) = \sum_{\text{permutations } \sigma} q^{\sum_{(i,j)\in\iota(\sigma)} |a_{\tau(i)}||a_{\tau(j)}|} a_{\sigma\tau(1)} \cdots a_{\sigma\tau(n)},$$

and we have the following result.

Lemma 5.3 *The linear map $\phi_{w,q}$ as defined above has determinant*

$$\prod_{m=2}^{n} \prod_{\substack{m\text{-sets} \\ S \subset \{1,\ldots,n\}}} \left(1 - q^{2\sum_{i,j\in S} |a_i||a_j|}\right)^{(n-m+1)!(m-2)!}.$$

Proof: Following [4], we use Varchenko's theorem [19] on determinants of bilinear forms on hyperplane arrangements. To apply the result of [19], we consider the set of hyperplanes in $\mathbf{R}^\mathbf{n}$ given by $\mathcal{H}_{ij} = \{(x_1, \ldots, x_n) \mid x_i = x_j\}$. To the hyperplane $\mathcal{H}_{ij}$ we assign the weight $\operatorname{wt} \mathcal{H}_{ij} = q^{|a_i||a_j|}$. The edges (nontrivial intersections) of this arrangement are indexed by subsets $S \subset \{1, 2, \ldots, n\}$ with two or more elements: the edge E_S corresponding to the set S is

$$\bigcap \{\mathcal{H}_{ij} \mid i, j \in S\} = \{(x_1, \ldots, x_n) \mid x_i = x_j \text{ for all } i, j \in S\}.$$

The edge E_S has weight

$$\operatorname{wt} E_S = \prod_{i,j\in S} \operatorname{wt} \mathcal{H}_{ij} = q^{\sum_{i,j\in S} |a_i||a_j|}.$$

The domains (connected components) for this hyperplane arrangement are indexed by permutations: $C_\sigma = \{(x_{\sigma(1)}, \dots, x_{\sigma(n)}) \mid x_1 < x_2 < \dots < x_n\}$. Then the quadratic form B on this arrangement given by

$$B(C_\sigma, C_\tau) = \prod_{\substack{\text{hyperplanes } \mathcal{H}_{ij} \\ \text{separating } C_\sigma \text{ and } C_\tau}} \operatorname{wt} \mathcal{H}_{ij} = \prod_{(i,j)\in\iota(\sigma\tau^{-1})} q^{|a_{\tau(i)}||a_{\tau(j)}|}$$

has the same matrix as $\phi_{w,q}$. Hence, by Theorem 1.1 of [19] we have

$$\det \phi_{w,q} = \prod_{\text{edges } E} (1 - \operatorname{wt}(E)^2)^{n(E)p(E)},$$

where the product is over the edges of the hyperplane arrangement, and $n(E)$ and $p(E)$ are numbers defined in §2 of [19]. It is easy to see from the definitions that $n(E_S) = (n-m+1)!$ and $p(E_S) = (m-2)!$ for the edge E_S corresponding to a m-set $S \subset \{1, \dots, n\}$, so the conclusion follows. □

Theorem 5.4 *Any $q \in k$ that is not a root of unity is generic (i.e., Φ_q is an isomorphism when q is not a root of unity).*

Proof: Suppose q is not a root of unity. We shall show that $\Phi_q^{-1}(w)$ exists for any word w by induction on $\ell(w)$. Using Lemma 5.2 and the induction hypothesis, to find $\Phi_q^{-1}(a_1 \cdots a_n)$ it suffices to find an element u such that

$$\Phi_q(u) = a_1 a_2 \cdots a_n + \text{terms of length} < n.$$

But we can do this by taking $u = \phi_{w,q}^{-1}(w)$, and $\phi_{w,q}$ is invertible by Lemma 5.3. □

If q is generic, we can define a comultiplication Δ_q on $\mathfrak{A}$ by requiring that all letters be Δ_q-primitives and that Δ_q be a $*_q$-homomorphism, i.e. that $\Delta_q(a) = a \otimes 1 + 1 \otimes a$ for all $a \in A$ and $\Delta_q(u *_q v) = \Delta_q(u) *_q \Delta_q(v)$ for all $u, v \in \mathfrak{A}$. This makes $(\mathfrak{A}, *_q, \Delta_q)$ a Hopf algebra. In fact, as we see in the next result, it is isomorphic to the concatenation Hopf algebra $(\mathfrak{A}, \text{conc}, \delta)$, where

$$\delta(w) = \sum_{\text{words } u,v \text{ of } \mathfrak{A}} (u \sqcup\!\sqcup\, v, w^*) u \otimes v.$$

Theorem 5.5 *For generic q, Φ_q is a Hopf algebra isomorphism from $(\mathfrak{A}, \text{conc}, \delta)$ to $(\mathfrak{A}, *_q, \Delta_q)$.*

Proof: Since q is generic, Φ_q is an algebra isomorphism. It suffices to show that $(\Phi_q \otimes \Phi_q) \circ \delta = \Delta_q \circ \Phi_q$ on a set of generators: but this follows because they agree on the primitives (elements of A), which generate $\mathfrak{A}$ under conc. □

In the next result we record a formula for $\Delta_q(ab)$ when q is generic. This may be compared with the corresponding formula in Example 5.2 of [4].

Proposition 5.6 *Let $a, b, c \in A$. For q generic,*

$$\Delta_q(ab) = ab \otimes 1 + 1 \otimes ab + \frac{1}{1+q^{|a||b|}}(a \otimes b + b \otimes a).$$

Proof: Apply Δ_q to the equation

$$ab = \left(1 - q^{2|a||b|}\right)^{-1}\left(a *_q b - q^{|a||b|} b *_q a\right) - \left(1 - q^{|a||b|}\right)^{-1}[a, b]. \qquad \square$$

A formula for $\Delta_q(abc)$ can be derived by applying Δ_q to

$$abc = \left(\phi^{-1}_{abc,q}\right)_{\mathrm{id},\mathrm{id}} a *_q b *_q c + \left(\phi^{-1}_{abc,q}\right)_{\mathrm{id},(12)} b *_q a *_q c + \cdots + \text{terms of length} \le 2,$$

but it is too complicated to give here (it contains twenty terms).

For the cases $q = 1$ and q not a root of unity, we have defined a Hopf algebra $(\mathfrak{A}, *_q, \Delta_q)$ with all elements of A primitive. It would be of interest to define such a Hopf algebra structure for all q.

6. Examples

As we have already remarked, if $[a, b] = 0$ for all generators $a, b \in A$ then $(\mathfrak{A}, *) = (\mathfrak{A}, ш)$ is the shuffle algebra as described in Chapter 1 of [14] (Note, however, that the grading may be different). The q-shuffle product $\odot_q$ as defined in [4, §4] is the operation $*_q = ш_q$ in this case. This algebra may also be obtained as a special case of the constructions of Green [10] and Rosso [15] involving quantum groups. To identify Green's "quantized shuffle algebra" with our construction, take the "datum" to be our generating set A, with bilinear form $a \cdot b = |a||b|$ for $a, b \in A$; then Green's algebra $G(k, q, A, \cdot)$ [10, p. 284], is our $(\mathfrak{A}, ш_q)$, except that Green's algebra is **NA**-graded rather than **N**-graded. To obtain our algebra from Rosso's "exemple fondamental" of [15, §2.1], take V to be the vector space over k generated by $A = \{e_1, e_2, \ldots\}$, and let $q_{ij} = q^{|e_i||e_j|}$. Here are some other examples.

Example 1 Let $A_n = \{z_n\}$ for all $n \ge 1$ and $[z_i, z_j] = z_{i+j}$. Then $(\mathfrak{A}, *)$ is just the algebra $\mathfrak{H}^1$ as presented in [12]. As is proved there (Theorem 3.4 ff.), the map ϕ defined by

$$\phi\left(z_{i_1} z_{i_2} \cdots z_{i_k}\right) = \sum_{n_1 > n_2 > \cdots > n_k \ge 1} t_{n_1}^{i_1} t_{n_2}^{i_2} \cdots t_{n_k}^{i_k}$$

is an isomorphism of $\mathfrak{H}^1$ onto the algebra of quasi-symmetric functions over k (denoted QSym_k in [13]). For each $n \ge 0$, the monomial quasi-symmetric functions $M_{(i_1,\ldots,i_k)} = \phi(z_{i_k} \cdots z_{i_1})$, where $(i_1, \ldots, i_k) \in \mathcal{C}(n)$, form a vector-space basis for $\mathfrak{A}_n$. For our purposes it is more convenient to identify $M_{(i_1,\ldots,i_k)}$ with $z_{i_1} \cdots z_{i_k}$: under this identification (which is also an isomorphism), the notation used above is simplified by the observation that, for compositions $I \in \mathcal{C}(n, k)$ and $J \in \mathcal{C}(k)$, $J[M_I] = M_{J \circ I}$. So, e.g., $S(M_I) = (-1)^{\ell(I)} \sum_{\bar{I} \succeq J} M_J$, where $\bar{I}$ is the reverse of I. If we let $\mathcal{L}$ denote the set of I such that M_I corresponds to a Lyndon word, then Theorem 2.6 says that $\{M_I \mid I \in \mathcal{L}\}$ generates $\mathfrak{A} = \mathrm{QSym}_k$ as an algebra. The Hopf algebra structure is that described in [5, 13]; the two formulas for its antipode are discussed in [5, §3].

For the *integral* Hopf algebra QSym of quasi-symmetric functions, $\{M_I \mid I \in \mathcal{C}(n)\}$ is a **Z**-module basis for the elements of degree n, but $\{M_I \mid I \in \mathcal{L}\}$ is not an algebra basis. Nevertheless, from [3, 16] QSym has an algebra basis $\{M_I \mid I \in \mathcal{L}^{\mathrm{mod}}\}$, where $\mathcal{L}^{\mathrm{mod}}$ is the set of "modified Lyndon" or "elementary unreachable" compositions, i.e. concatenation powers of elements of $\mathcal{L}$ whose parts have greatest common factor 1. (There is a bijection of $\mathcal{L}$ onto $\mathcal{L}^{\mathrm{mod}}$ given by sending $(i_1, \dots, i_l)$ to the dth concatenation power of $(\frac{i_1}{d}, \dots, \frac{i_l}{d})$, where d is the greatest common factor of $i_1, \dots, i_k$.) Of course exp cannot be defined over **Z** because of denominators.

Another algebra basis for QSym_k is given by $\{P_I \mid I \in \mathcal{L}\}$, where $P_I = \exp(M_I)$. (These are exactly the elements whose duals $P_I^* = \log^*(M_I^*)$ are introduced in [13, §2] as a basis for the dual QSym_k^*; cf. equations (2.12) of [13] and (5) above.) Since exp is a Hopf algebra isomorphism, we have the formulas

$$P_I * P_J = \sum_{K \in I \,\text{ш}\, J} P_K, \quad \Delta(P_K) = \sum_{I \sqcup J = K} P_I \otimes P_J, \quad \text{and} \quad S(P_I) = (-1)^{\ell(I)} P_{\bar{I}},$$

where, for compositions I and J, $I \,\text{ш}\, J$ is the multiset of compositions obtained by "shuffling" I and J (e.g. $(1,2) \,\text{ш}\, (2) = \{(2,1,2), (1,2,2), (1,2,2)\}$), and $I \sqcup J$ is the concatenation of I and J.

Following Gessel [8], there is still another basis $\{F_I \mid I \in \mathcal{L}\}$ for QSym_k, where $F_I = \sum_{J \succeq I} M_J$. (Then $M_I = \sum_{J \succeq I} (-1)^{\ell(J)-\ell(I)} F_J$, and since the coefficients are integral $\{F_I \mid I \in \mathcal{L}^{\mathrm{mod}}\}$ is a basis for QSym). The expansion of the product $F_I * F_J$ in terms of the F_K can be described using permutations and their descent compositions; see [18] or [13]. Dualizing Proposition 3.13 and Corollary 3.16 of [7] (see below), we have

$$\Delta(F_K) = \sum_{I \sqcup J = K} F_I \otimes F_J + \sum_{I \vee J = K} F_I \otimes F_J \quad \text{and} \quad S(F_I) = (-1)^{|I|} F_{I^\sim},$$

where $I \vee J = (i_1, \dots, i_{k-1}, i_k + j_1, j_2, \dots, j_l)$ for nonempty compositions $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_l)$, and $I^\sim$ is the conjugate composition of I (as defined in [7, §3.2]). By dualizing Corollary 4.28 of [7] we have a formula for F_I in terms of the P_I:

$$F_I = \sum_{|J|=|I|} \mathrm{phr}(I, J) \frac{P_J}{\Pi(J)}.$$

Here $\Pi(I)$ is the product of the parts of the composition I, and $\mathrm{phr}(I, J)$ is as defined in [7, §4.9]: for compositions I and $J = (j_1, \dots, j_s)$ of the same weight, let $I = I_1 \bullet I_2 \bullet \cdots \bullet I_s$ be the unique decomposition of I such that $|I_i| = j_i$ for $1 \le i \le s$ and each symbol $\bullet$ is either $\sqcup$ or $\vee$; then

$$\mathrm{phr}(I, J) = \prod_{i=1}^{s} \frac{(-1)^{\ell(I_i)-1}}{\binom{|I_i|-1}{\ell(I_i)-1}}.$$

The dual Hopf algebra QSym_k^* is described in [13, §2]; it is also the algebra **Sym** of noncommutative symmetric functions as defined in [7]. (The coproduct δ' of §4 corresponds to the coproduct denoted γ in [13] and [7].) The M_I are dual to the "products of complete homogeneous symmetric functions" S^I (i.e., $(M_I, S^J) = \delta_{IJ}$), while the "products of power sums of the second kind" Φ^I are dual to the elements $P_I/\Pi(I)$ (see [7, §3] for definitions). The F_I are dual to the "ribbon Schur functions" R_I [7, Theorem 6.1].

The deformation $(\mathfrak{A}, *_q)$ is the algebra of quantum quasi-symmetric functions as defined in [18]. The multiplication rule for "quantum quasi-monomial functions" as given in [18, p. 7345] can be recognized as (6).

Example 2 For a fixed positive integer r, let $A_n = \{z_{n,i} \mid 0 \le i \le r-1\}$ and $[z_{n,i}, z_{m,j}] = z_{n+m,i+j}$, where the second subscript is to be understood mod r. By Theorem 2.6, $(\mathfrak{A}, *)$ is the polynomial algebra on the Lyndon words in the $z_{i,j}$; by Proposition 2.7, the number of Lyndon words in $\mathfrak{A}_n$ is

$$L_n = \frac{1}{n}\sum_{d|n} \mu\left(\frac{n}{d}\right)(r+1)^d$$

for $n \ge 2$ (and $L_1 = r$). In this case, we call the Hopf algebra $(\mathfrak{A}, *, \Delta)$ the Euler algebra $\mathfrak{E}_r$. Of course $\mathfrak{E}_1$ is the preceding example (We write z_i for $z_{i,0}$ if $r = 1$); in general there is a homomorphism $\pi_r : \mathfrak{E}_r \to \mathfrak{E}_1$ given by $\pi_r(z_{i,j}) = z_i$. The map $\phi : \mathfrak{E}_r \to \mathbf{C}[[t_1, t_2 \ldots]]$ with

$$\phi(z_{i_1,j_1} z_{i_2,j_2} \cdots z_{i_k,j_k}) = \sum_{n_1 > n_2 > \cdots > n_k \ge 1} e^{\frac{2\pi i}{r}(n_1 j_1 + \cdots + n_k j_k)} t_{n_1}^{i_1} \cdots t_{n_k}^{i_k} \tag{7}$$

is an isomorphism of $\mathfrak{E}_r$ onto a subring of $\mathbf{C}[[t_1, t_2 \ldots]]$ (for proof see §7 below.) If we define $\psi_r : \mathbf{C}[[t_1, t_2 \ldots]] \to \mathbf{C}[[t_1, t_2 \ldots]]$ by

$$\psi_r(t_i) = \begin{cases} 0, & r \nmid i \\ t_j, & i = rj \end{cases}$$

(Note ψ_r takes $\mathrm{QSym}_k \subset \mathbf{C}[[t_1, t_2 \ldots]]$ isomorphically onto itself!), then $\psi_r \circ \phi = \phi \circ \pi_r$. The sets L of Lyndon words in the $z_{i,j}$ and $\{\exp(w) \mid w \in L\}$ are both algebra bases for $\mathfrak{E}_r$, corresponding to the bases $\{M_I \mid I \in \mathcal{L}\}$ and $\{P_I \mid I \in \mathcal{L}\}$, respectively, of Example 1. If we set $\hat{w} = \sum_{v \in \mathcal{P}(w)} v$, where $\mathcal{P}(w)$ is as defined at the end of §4, then there is a a basis $\{\hat{w} \mid w \in L\}$ corresponding to $\{F_I \mid I \in \mathcal{L}\}$. Note, however, that while π_r maps words to the M_I and exponentials of words to the P_I (exp commutes with π_r), in general $\pi_r(\hat{w})$ is not of the form F_I.

The dual $\mathfrak{E}_r^*$ of the Euler algebra is the concatenation algebra on elements $z_{i,j}^*$, with coproduct δ' as described in §4. The transpose of π_r is the homomorphism $\pi_r^* : \mathfrak{E}_1^* \to \mathfrak{E}_r^*$ with $\pi_r^*(z_i^*) = \sum_{j=1}^{r-1} z_{i,j}^*$.

The motivation for the Euler algebra $\mathfrak{E}_r$ comes from numerical series of the form

$$\sum_{n_1 > n_2 > \cdots > n_k \ge 1} \frac{\epsilon_1^{n_1} \epsilon_2^{n_2} \cdots \epsilon_k^{n_k}}{n_1^{i_1} n_2^{i_2} \cdots n_k^{i_k}}, \tag{8}$$

where the ϵ_i are rth roots of unity and $i_1, i_2, \ldots, i_k$ are positive integers (with $\epsilon_1 i_1 \neq 1$, for convergence). In fact (8) is $\lim_{n\to\infty} \phi_n(z_{i_1,j_1} \cdots z_{i_k,j_k})(1, 2, \ldots, \frac{1}{n})$, where ϕ_n is as defined in §7 and the j_s are chosen appropriately, so the algebra of such series can be seen as a homomorphic image of (a subalgebra of) $\mathfrak{E}_r$. These series are called "Euler sums" in [1, 2] and "values of multiple polylogarithms at roots of unity" in [9]; in the case $r = 1$ the corresponding series are known as "multiple harmonic series" [12] or "multiple zeta values" [20].

Example 3 Fix a positive integer m and let $A_n = \{z_n\}$ for $n \leq m$ and $A_n = \emptyset$ for $n > m$. Define

$$[z_i, z_j] = \begin{cases} z_{i+j} & \text{if } i + j \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(\mathfrak{A}, *)$ is the algebra of quasi-symmetric functions on variables $t_1, t_2, \ldots$ subject to the relations $t_i^{m+1} = 0$ for all i.

Example 4 Let $P(n)$ be the set of partitions of n and let $A_n = \{z_\lambda \mid \lambda \in P(n)\}$. Define $[z_\lambda, z_\mu] = z_{\lambda\cup\mu}$, where $\lambda \cup \mu$ is the union λ and μ as multisets. Then $(\mathfrak{A}, *)$ can be thought of as the algebra of quasi-symmetric functions in the variables $t_{i,j}$, where $|t_{i,j}| = j$, in the following sense. For a partition $\lambda = (n_1, \ldots, n_l)$, let $t_i^\lambda = t_{i,n_1} \cdots t_{i,n_l}$. Then any monomial in the $t_{i,j}$ can be written in the form $t_{i_1}^{\lambda_1} \cdots t_{i_k}^{\lambda_k}$, and we call a formal power series quasi-symmetric when the coefficients of any two monomials $t_{i_1}^{\lambda_1} \cdots t_{i_k}^{\lambda_k}$ and $t_{j_1}^{\lambda_1} \cdots t_{j_k}^{\lambda_k}$ with $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$ are the same.

7. The Euler algebra as power series

Fix a positive integer r, and let $\mathfrak{E}_r$ and $\pi_r : \mathfrak{E}_r \to \mathfrak{E}_1$ be as in Example 2. We shall show $\mathfrak{E}_r$ can be imbedded in the formal power series ring $\mathbf{C}[[t_1, t_2, \ldots]]$. For positive integers n, define a map $\phi_n : \mathfrak{E}_r \to \mathbf{C}[t_1, \ldots, t_n]$ as follows. Let ϕ_n send $1 \in \mathfrak{E}_r$ to $1 \in \mathbf{C}[t_1, \ldots, t_n]$ and any nonempty word $w = z_{i_1,j_1} z_{i_2,j_2} \ldots z_{i_k,j_k}$ to the polynomial

$$\sum_{n\geq n_1>n_2>\cdots>n_k\geq 1} \omega^{j_1n_1+j_2n_2+\cdots+j_kn_k} t_{n_1}^{i_1} t_{n_2}^{i_2} \cdots t_{n_k}^{i_k},$$

where $\omega = e^{\frac{2\pi i}{r}}$ (If $k > n$, the sum is empty and we assign it the value 0). Extend ϕ_n to $\mathfrak{E}_r$ by linearity. If we make $\mathbf{C}[t_1, \ldots, t_n]$ a graded algebra by setting $|t_i| = 1$, then ϕ_n preserves the grading. Also, it is immediate from the definition that

$$\phi_n(z_{p,i} w) = \sum_{n\geq m>1} \omega^{im} t_m^p \phi_{m-1}(w) \tag{9}$$

for any nonempty word w.

Theorem 7.1 *For any n, $\phi_n : \mathfrak{E}_r \to \mathbf{C}[t_1, \ldots, t_n]$ is a homomorphism of graded k-algebras.*

Proof: It suffices to show $\phi_n(w_1 * w_2) = \phi_n(w_1)\phi_n(w_2)$ for words w_1, w_2. This can be done by induction on $\ell(w_1) + \ell(w_2)$, following the argument of [12, Theorem 3.2] (and using Eq. (9) above in place of equation (*) of [12]). □

Lemma 7.2 *For* $0 \le j_1, j_2, \dots, j_m \le r-1$, *let* $c_{j_1,j_2,\dots,j_m} \in \mathbf{Q}$ *be such that*

$$\sum_{j_1=0}^{r-1}\sum_{j_2=0}^{r-1}\cdots\sum_{j_m=0}^{r-1} c_{j_1,j_2,\dots,j_m}\omega^{n_1j_1+n_2j_2+\cdots+n_mj_m} = 0$$

for all $mr \ge n_1 > n_2 > \cdots > n_m \ge 1$, *where* $\omega = e^{\frac{2\pi i}{r}}$ *as above. Then all the* $c_{j_1,j_2,\dots,j_m}$ *are zero.*

Proof: We use induction on m. For $m = 1$ the hypothesis is

$$\sum_{j=1}^{r-1} c_j\omega^{nj} = 0 \quad \text{for all } 1 \le n \le r,$$

which is evidently equivalent to having the equality for $0 \le n \le r-1$. But then the conclusion follows from the nonsingularity of the Vandermonde determinant of the quantities $1, \omega, \omega^2, \dots, \omega^{r-1}$.

Now let $m > 1$, and fix $(m-1)r \ge n_2 > n_3 > \cdots > n_m \ge 1$. Then the hypothesis says

$$\sum_{j_1=0}^{r-1}\left(\sum_{j_2=0}^{r-1}\cdots\sum_{j_m=0}^{r-1} c_{j_1,j_2,\dots,j_m}\omega^{n_2j_2+\cdots+n_mj_m}\right)\omega^{n_1j_1} = 0 \quad \text{for } (m-1)r < n_1 \le mr.$$

This is evidently equivalent to having the equality hold for all $1 \le n_1 \le r$: but then we are in the situation of the preceding paragraph and so

$$\sum_{j_2=0}^{r-1}\cdots\sum_{j_m=0}^{r-1} c_{j_1,j_2,\dots,j_m}\omega^{n_2j_2+\cdots+n_mj_m} = 0,$$

from which the conclusion follows by the induction hypothesis. □

Theorem 7.3 *The homomorphism* ϕ_{nr} *is injective through degree* n.

Proof: Suppose $u \in \ker\phi_{nr}$ has degree $\le n$. Without loss of generality we can assume u is homogeneous, and in fact that $\pi_r(u)$ is a multiple of $z_{i_1}z_{i_2}\cdots z_{i_m}$ for $m \le n$. Then u has the form

$$u = \sum_{j_1=0}^{r-1}\sum_{j_2=0}^{r-1}\cdots\sum_{j_m=0}^{r-1} c_{j_1,j_2,\dots,j_m} z_{i_1,j_1}z_{i_2,j_2}\cdots z_{i_m,j_m},$$

and $u \in \ker\phi_{nr}$ implies that

$$\sum_{j_1=0}^{r-1}\sum_{j_2=0}^{r-1}\cdots\sum_{j_m=0}^{r-1} c_{j_1,j_2,\dots,j_m}\omega^{n_1j_1+n_2j_2+\cdots+n_mj_m} = 0$$

for all $nr \ge n_1 > n_2 > \cdots > n_m \ge 1$. But then $u = 0$ by the lemma. □

For $m \geq n$, there is a restriction map $\rho_{m,n} : \mathbf{C}[t_1, \ldots, t_m] \to \mathbf{C}[t_1, \ldots, t_n]$ sending t_i to t_i for $1 \leq i \leq n$ and t_i to zero for $i > n$. Let $\mathfrak{P}$ be the inverse limit of the $\mathbf{C}[t_1, \ldots, t_n]$ with respect to these maps (in the category of graded algebras); $\mathfrak{P}$ is a subring of $\mathbf{C}[[t_1, t_2, \ldots]]$. The ϕ_n define a homomorphism $\phi : \mathfrak{E}_r \to \mathfrak{P}$, and the following result is evident.

Theorem 7.4 *The homomorphism ϕ is injective, and satisfies Eq. (7).*

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